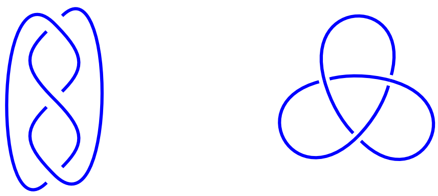
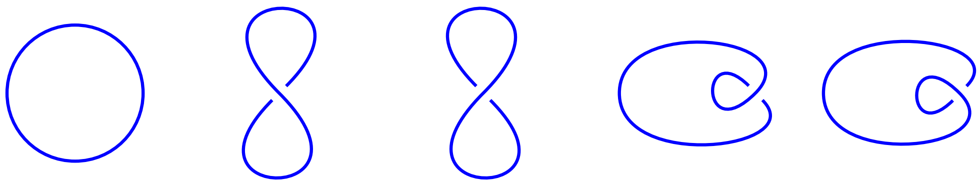
KNOTS AND SURFACES PROBLEMS

Exercise 1. Show, using a piece of string, that the following two pictures represent the same knot:

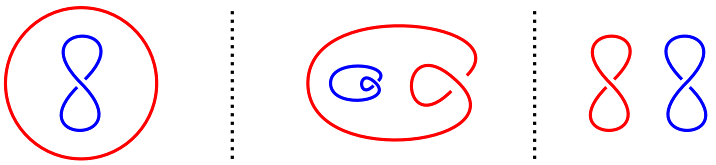


Solution: See the video on the course web page (`http://goo.gl/cU443p`).

Exercise 2. There are only five link diagrams with one component and at most one crossing:

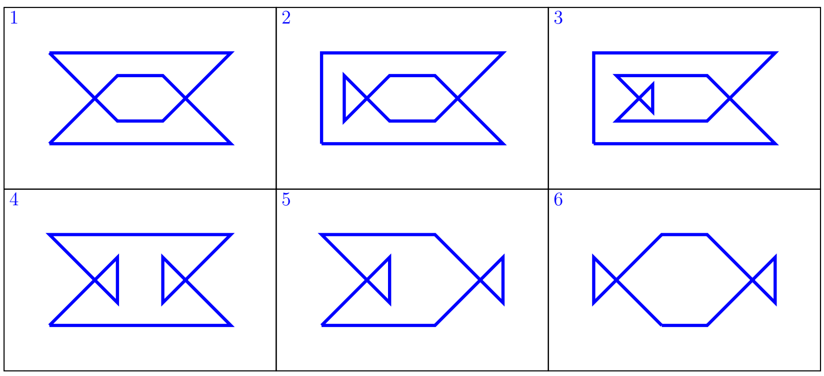


They are all R-equivalent, so they all represent the unknot. Now consider link diagrams with precisely two crossings. There are a large number of diagrams which can be divided into two separate pieces with no crossings in common. A few of these are shown below:



Draw all possible link diagrams that **cannot** be split in this uninteresting way. How many different knots or links do these represent?

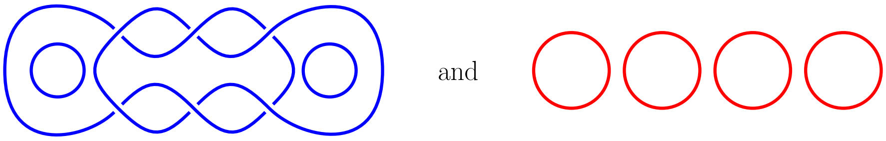
Solution: This is surprisingly hard to get right. There are six possible link universes, as follows:

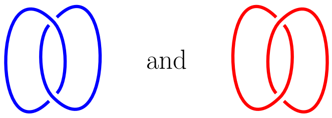


To see this, note that the arcs can either join a crossing to itself, or they can bridge between the two crossings. If there are no bridges, then the universe disconnects into two pieces, and we were told to ignore that case. There cannot be an odd number of bridges, because it would then be impossible to match up the remaining loose ends to make loops. If there are four bridges, then the only possibility is picture 1. If there are two bridges, then there must also be two loops. The loops might both be small (pictures 4,5 and 6), or one of them they might wrap all the way around the outside (pictures 2 and 3). Both of the loops might be nestled in between the pair of bridges, as in picture 4. Alternatively, one might be between the bridges, and the other might be outside (pictures 3 and 5) or both might be outside (pictures 2 and 6). We can get additional pictures by rotating pictures 2, 3 and 5 through 180 degrees, but that rotation is an ambient isotopy, so we regard the rotated diagram as being equivalent to the original one.

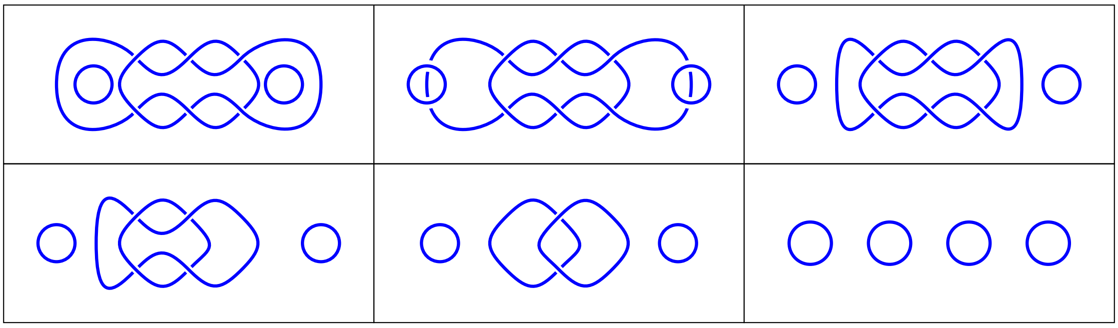
Each of pictures 2 to 6 has a small triangle. If we choose overcrossings and undercrossings to make the universe into a link diagram, we can immediately remove the small triangle by a Reidemeister move of type 1. This will leave a diagram with only one crossing, which can again be removed by a Reidemeister move of type 1, leaving the unknot. On the other hand, there are four possible ways to specify the crossings in picture 1, each of which gives two component circles. Two of the four possibilities give a pair of unlinked circles; the other two give a Hopf link. Thus, there are three possible outcomes altogether: the unknot, the unlink and the Hopf link.

Exercise 3. Show that the following pairs of link diagrams are R-equivalent.



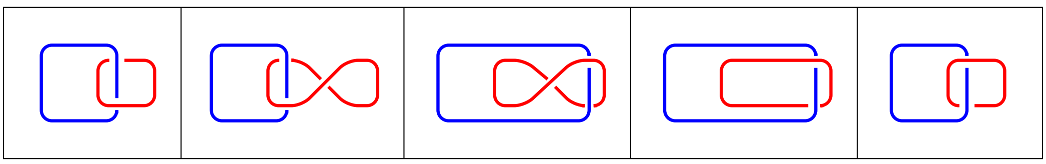


Solution: For the first pair of diagrams, we can use the following sequence of moves:



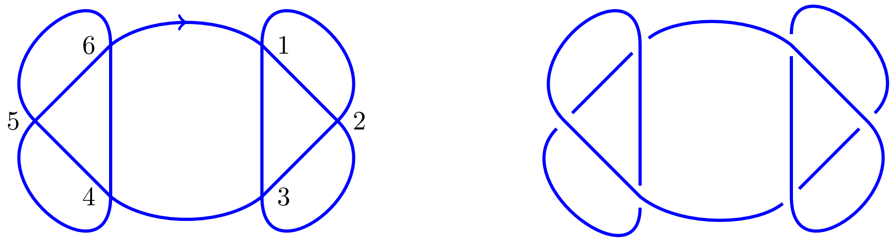
All the Reidemeister moves here are of type 2. In the first and second steps we have performed two Reidemeister moves at the same time, but the remaining three steps just consist of a single Reidemeister move.

For the second pair of diagrams, we use a Reidemeister move of type 1, then a move of type 3, then another move of type 1.



Exercise 4. Show that given any knot universe you can choose the crossings so that the resulting knot diagram represents the unknot. (This will probably take some experimentation. If you fail to do it with pen and paper, I highly recommend getting a piece of string and trying to do it with that.) Can you prove that your method always gives the unknot?

Solution: Start with a knot universe, so none of the crossings are marked as undercrossings or overcrossings. Choose a point a on one of the arcs, and start walking around from there. Each time you get to a crossing that has not already been marked, mark it so that your current strand is the upper one. Eventually you will get back to a, and all the crossings will be marked. For example, consider the following picture:



Starting from the position marked with an arrow, we visit crossings 1, 2 and 3, and we mark them as overcrossings. We then revisit crossings 1, 2 and 3, this time passing underneath. We then cross over to the left hand part of the picture, and visit crossings 4, 5 and 6, marking them as overcrossings. We then revisit crossings 4, 5 and 6, passing underneath. This brings us back to the start.

The above procedure gives a diagram which corresponds to the unknot. To see why, imagine the knot lying close to the xy plane, and just bending up or down a little bit at the crossings. Imagine an unknotted circle C lying in the plane $z = 1$. Stretch and pull a short section of string near a to move a onto C. Then work around the knot, pulling the string up to C. The first time we get to each crossing, we will be on top, so there is nothing to stop us from moving the string. The second time we get to each crossing we will be on the bottom strand, but the top strand will already have been removed, so there will again be nothing in the way.

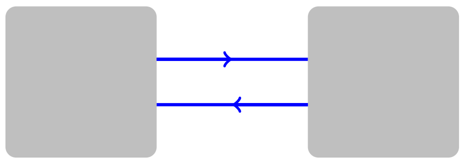
There is a video to illustrate this argument at `http://goo.gl/cU443p`.

Exercise 5. Have a look at some of the websites listed on the course webpage. Have a read of the wikipedia webpage on knot theory. How many different knots can you draw using diagrams with at most 10 crossings? Where can you find pictures of all of them?

Solution: A table of knots with up to 10 crossings (originally due to Rolfsen) can be found here:

`http://katlas.math.toronto.edu/wiki/The_Rolfsen_Knot_Table`

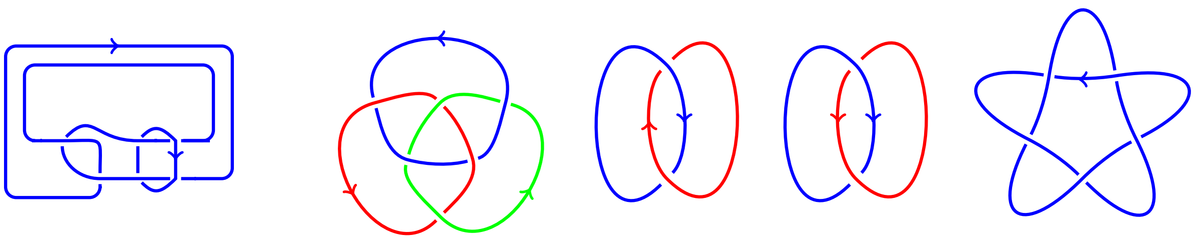
Note, however, that this table only includes *prime* knots. Many knots can be obtained by splicing together two nontrivial knots, like this:



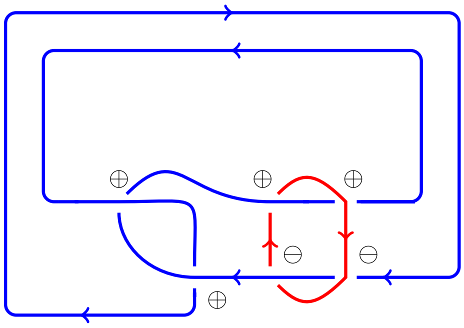
Knots of this form are considered *composite*, and they are not listed in the Rolfsen table. Any knot can be decomposed into prime knots, so this is no great loss. Moreover, many prime knots are different from their mirror images, and only one of each mirror pair appears in the Rolfsen table, so this is another source of extra knots. The number of knots in the table can be tabulated as follows:

crossings	0	1	2	3	4	5	6	7	8	9	10
knots	1	0	0	1	1	2	3	7	21	49	165

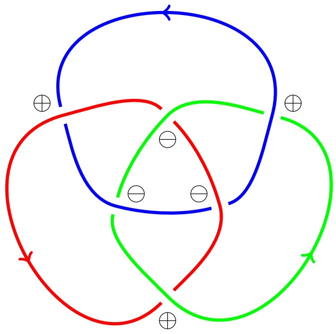
Exercise 6. For each of the following link diagrams, find the number of components, and mark the crossings positive or negative.



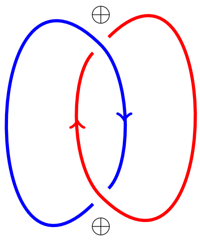
Solution:



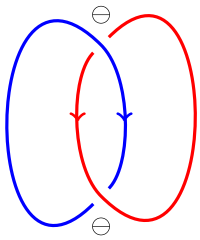
2 components



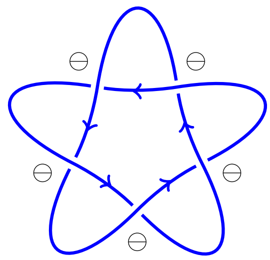
3 components



2 components



2 components



1 component

Exercise 7. Given a link diagram k, define the total linking number $\mathrm{lk}(k)$ to be

$$\mathrm{lk}(k) := \frac{1}{2}\sum_{c} \epsilon(c),$$

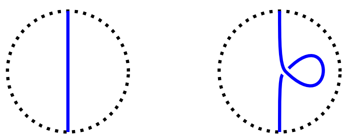
where the sum is taken over all crossing *between different components*.

(1) Calculate the total linking number for each of the links in the previous question.
(2) Prove that the total linking number is invariant under each of the Reidemeister moves, and deduce that it is a link invariant.

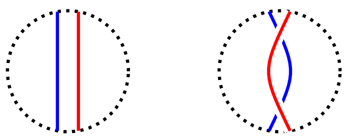
Solution: The linking numbers for the above pictures are as follows.

- In the first picture, there are six crossings. The two leftmost crossings only involve one component (the blue one) so they do not contribute to the linking number. The other four crossings involve both the red component and the blue component, so they do contribute. Two of them are positive and two are negative, so the linking number is zero.
- In the second picture (the Borromean rings) we have six crossings. All of these involve two different components, and so contribute to the linking number. Three are positive and three are negative, so again the linking number is zero.
- In the third picture we count two positive crossings, giving a linking number of $+2/2 = 1$.
- In the fourth picture we count two negative crossings, giving a linking number of $-2/2 = -1$.
- In the last picture, there is only one component, so the linking number is zero.

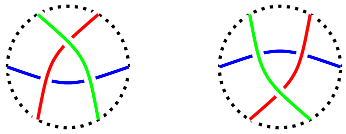
Now consider what happens when we perform a Reidemeister move on an oriented link diagram D. A type 1 move adds or subtracts a crossing which only involves one component; this does not change the linking number.



A type 2 move adds or subtracts two crossings.

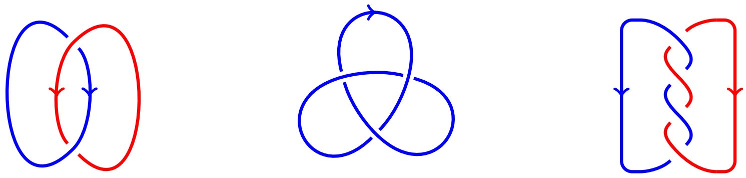


It may be that the two strands connect together somewhere outside the dotted circle, so they are part of the same component. In this case, it is clear that the linking number is unchanged. If two strands are part of different components, then we need to think about the signs of the two extra crossings. If both strands run in the same direction, then the top crossing is positive and the bottom crossing is negative, so they cancel. If the two strands run in opposite directions then the top crossing is negative and the bottom one is positive, so again they cancel. Thus, this kind of Reidemeister move does not change the linking number. Finally, consider a move of type 3.



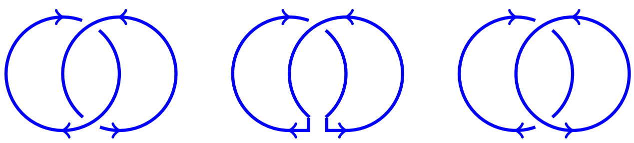
Here the two pictures have the same components, the same strands and the same crossings. The only thing that is different is the relative position of the crossings, but that does not enter into the definition of the linking number, so the linking number is unchanged.

Exercise 8. Calculate the Jones polynomials of the following knots or links:



positive Hopf link　　　positive trefoil

Solution: The first picture shows the positive Hopf link H_+. For this we have a skein relation for the following diagrams:



H_+　　　U_1　　　U_2

We recall that $f(U_1) = 1$ and $f(U_2) = -A^2 - A^{-2}$, so the skein relation is

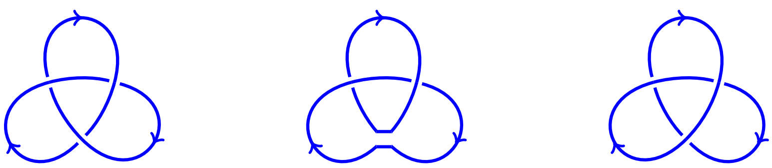
$$A^4 f(H_+) - A^{-4}(-A^2 - A^{-2}) = (A^{-2} - A^2).$$

Rearranging this gives

$$A^4 f(H_+) = -A^{-2} - A^{-6} + A^{-2} - A^2 = -A^2 - A^{-6},$$

and so $f(H_+) = -A^{-2} - A^{-10} = -A^{-2}(1 + A^{-8})$.

Next, the second picture involves the positive trefoil T_+, which can be analysed using the following skein relation:



T_+　　　H_+　　　U_1

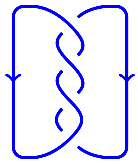
We already know that $f(H_+) = -A^{-2} - A^{-10}$ and $f(U_1) = 1$, so we get

$$A^4 f(T_+) - A^{-4} = (A^{-2} - A^2)(-A^{-2} - A^{-10}).$$

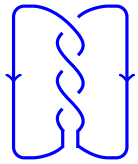
This gives

$$\begin{aligned} A^4 f(T_+) &= A^{-4} + (A^{-2} - A^2)(-A^{-2} - A^{-10}) \\ &= A^{-4} - A^{-4} - A^{-12} + 1 + A^{-8} = 1 + A^{-8} - A^{-12} \\ f(T_+) &= A^{-4} + A^{-12} - A^{-16}. \end{aligned}$$

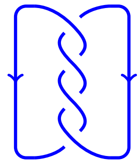
For the third picture, we start by considering the following diagrams.



$D_1 \sim H_-$



D_2

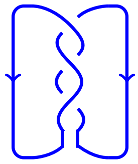


D_3

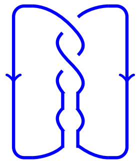
The thing that we are trying to calculate is $f(D_3)$. We have a skein relation $A^4 f(D_1) - A^{-4} f(D_3) = (A^{-2} - A^2) f(D_2)$, so

$$f(D_3) = A^8 f(D_1) - A^4(A^{-2} - A^2) f(D_2).$$

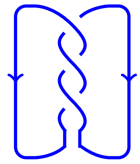
It is easy to see that $D_1 \sim H_-$, so $f(D_1) = -A^2 - A^{10}$. This just leaves $f(D_2)$. One approach is to notice that D_2 can be converted too T_- by pulling the leftmost strand all the way over to the right. Alternatively, we can use a second skein relation as follows:



$D_4 \sim U_1$



$D_5 \sim H_-$



D_2

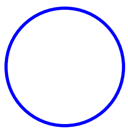
(You should check carefully that this is correct, with a positive crossing in the first diagram replaced by a negative crossing in the third diagram, and the strands in the middle diagram reconnected in a way that is compatible with the orientation.)

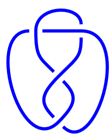
Note that D_4 can be converted to the unknot U_1 by a Reidemeister move of type 2 followed by a move of type 1. Similarly, D_5 is H_-. We therefore have $f(D_4) = 1$ and $f(D_5) = -A^2 - A^{10}$, so the skein relation $A^4 f(D_4) - A^{-4} f(D_2) = (A^{-2} - A^2) f(D_5)$ gives

$$\begin{aligned}
f(D_2) &= A^4(A^4 f(D_4) - (A^{-2} - A^2) f(D_5)) = A^8 f(D_4) - (A^2 - A^6) f(D_5) \\
&= A^8 - (A^2 - A^6)(-A^2 - A^{10}) = A^8 + A^4 + A^{12} - A^8 - A^{16} \\
&= A^4 + A^{12} - A^{16} \\
f(D_3) &= A^8 f(D_1) - A^4(A^{-2} - A^2) f(D_2) \\
&= A^8(-A^2 - A^{10}) - A^4(A^{-2} - A^2)(A^4 + A^{12} - A^{16}) \\
&= -A^{10} - A^{18} - A^4(A^2 + A^{10} - A^{14} - A^6 - A^{14} + A^{18}) \\
&= -A^{10} - A^{18} - A^6 - A^{14} + 2A^{18} + A^{10} - A^{22} \\
&= -A^6 - A^{14} + A^{18} - A^{22}.
\end{aligned}$$

Exercise 9. A *3-colouring* of a link diagram is an assignment of one of the colours red, green and blue to each of the arcs in the diagram such that at each crossing the colours of the three incident arcs are either all the same or all different. (So in other words you can't have exactly two colours used at a crossing.) A diagram is said to be *trichromatic* if it has a 3-colouring which uses all three colours.

(1) (a) Pick a diagram of the trefoil and show that it is trichromatic.
  (b) Show that the standard unknot and figure-eight diagrams (shown below) are not trichromatic.



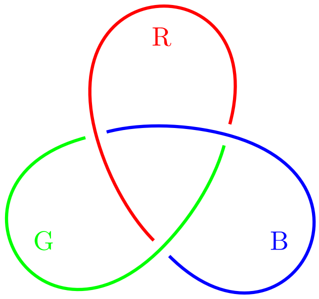


  (c) Show that if a *knot* diagram is obtained from another by a Reidemeister move then either both diagrams are trichromatic or neither is.
  (d) Deduce that the property of being trichromatic is a knot invariant.
  (e) Deduce that the trefoil knot and the unknot are not equivalent knots.
  (f) What can you deduce about the figure-eight diagram?

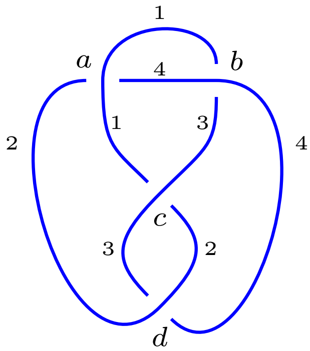
(2) (Optional.) More generally, for a link diagram l define $\tau(l)$ to be the *number* of different 3-colourings of l.
  (a) Show that τ is a knot invariant — this is very similar to the proof above.
  (b) Calculate τ for some knots and links. Make some conjectures about the possible values of τ and try to prove them.

Solution:

(1) (a) Here is a trichromatic colouring of the trefoil:



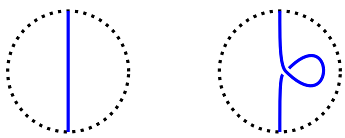
(b) The unknot has only one arc, so we can only use one colour. By definition, a 3-colouring must use all three colours, so the unknot has no 3-colouring. Now consider the figure eight. We can label the crossings a, b, c, d and the arcs $1, 2, 3, 4$ as follows.



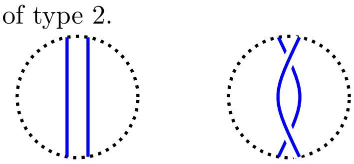
crossing	arcs
a	$1, 2, 4$
b	$1, 3, 4$
c	$1, 2, 3$
d	$2, 3, 4$

At crossing c, we are allowed to give arcs 1, 2 and 3 the same colour, or three different colours. However, if we give them the same colour, then we would then be using at most one other colour for arc 4, so we would not use all 3 colours, which is required for a 3-colouring. Thus, arcs 1, 2 and 3 must have three different colours. It will be harmless to assume that arc 1 is red, arc 2 is green and arc 3 is blue. By applying the same analysis to crossing a, we see that arcs 1, 2 and 4 must have different colours. As arc 1 is red and arc 2 is green, we see that arc 4 is also blue. Now at crossing d the arcs 2, 3 and 4 are red, blue and blue, which is not allowed. Thus, there is no 3-colouring.

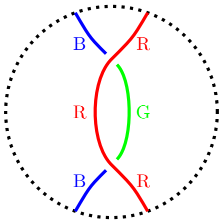
(c) Consider a pair of knot diagrams related by a Reidemeister move. First consider a move of type 1.



If the left hand diagram has a 3-colouring with the arc in the circle coloured blue (say), then we can make a 3-colouring on the right hand diagram by colouring everything in the circle blue, and leaving everything outside the circle the same. Conversely, suppose we have a 3-colouring of the right hand diagram. Two of the arcs at the crossing are the same, so there cannot be three different colours there, so everything in the circle must be the same colour, say blue. We can then colour the arc in the left hand diagram blue, and nothing outside the circle will be affected, so we get a 3-colouring of the left hand diagram. Thus, we see that the left diagram is trichromatic if and only if the right hand diagram is trichromatic. Now consider instead a move of type 2.

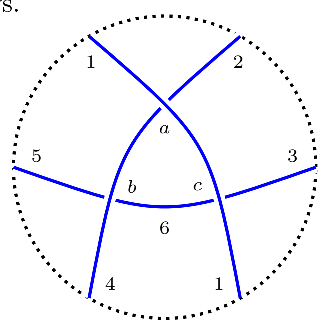


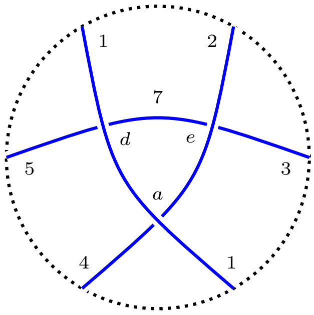
Suppose we have a 3-colouring of the left diagram. First consider the case where the two arcs in the left diagram have the same colour, say blue. We can then make a 3-colouring on the right hand diagram by colouring everything in the circle blue, and leaving everything outside the circle the same. Suppose instead that the two arcs in the left diagram have different colours, say blue and red. We can then make a 3-colouring of the right hand diagram as follows:



Conversely, suppose that we have a 3-colouring of the right hand diagram. If everything in the circle has the same colour, we can just use that colour in the left hand diagram. Otherwise, the upper strand will have one colour, say red; the middle section of the lower strand will have a different colour, say green; and then the remaining sections of the lower strand must both be blue, to ensure that each crossing has all three colours. We can thus make a 3-colouring of the left hand diagram by making the two strands blue and red respectively. The only problem with this is that we have lost a green arc. If there were no more green arcs elsewhere in the diagram, then we would no longer have all three colours. However, in that situation the blue arcs could never cross the red arcs, so the blue arcs and red arcs would be separate components of the diagram. However, we assumed that our diagram was a knot diagram, with only one component, so that scenario is impossible. Thus, we again see that the left diagram is trichromatic if and only if the right hand diagram is trichromatic.

Finally, we can consider a Reidemeister move of type 3. We can label the arcs and crossings as follows.





In the left diagram, arcs 1 to 5 connect to the outside, but arc 6 stays wholly inside. In the right diagram, arcs 1 to 5 connect to the outside in the same way as the corresponding arcs in the left diagram, but there is again an additional arc that stays wholly inside. We have labelled this 7, because it does not correspond in an obvious way to arc 6. Both diagrams have a crossing labelled a, which involves arcs 1, 2 and 4. The remaining crossings on the left do not correspond in an obvious way to the crossings on the right, so we have used labels b and c on the left, and d and e on the right.

Suppose we have a 3-colouring of the left diagram, and we want to transfer it to a 3-colouring of the right diagram. To maintain compatibility with whatever is happening outside the dotted circle, we must colour arcs 1 to 5 on the right the same colour as arcs 1 to 5 on the left. The only problem is how to colour arc 7. If arcs 1 and 5 have the same colour, then we use that colour for arc 7 as well; otherwise, arcs 1 and 5 have two different colours, and we use the remaining colour for arc 7. This means that the 3-colouring rule is satisfied at crossing d. It is also satisfied at crossing a, because crossing a on the right involves the same arcs with the same colours as on the left. The only problem is to prove that the 3-colouring rule is also satisfied at crossing e.

Suppose that arcs 2 and 3 have the same colour, say red. If arc 1 is also red then it is not hard to see that everything in the left diagram is red and then that everything in the right diagram is red so there is no problem. Suppose instead that arcs 2 and 3 are red, but that arc 1 is different, say green. Then we can use crossings a and c to see that arcs 4 and 6 are blue, and then use crossing b to see that arc 5 is also blue. Now arcs 1 and 5 are green and blue, so our rule for arc 7 says that we should make it red. Now arcs 2, 3 and 7 are all red, so the 3-colouring condition is satisfied at crossing e.

We still need to deal with the case where arcs 2 and 3 have different colours, say red and green. We split this into three subcases depending on the colour of arc 1. The possibilities are as follows:

2	3	1	4	6	5	7
R	G	R	R	B	G	B
R	G	G	B	G	R	B
R	G	B	G	R	B	B

To explain in more detail: we are considering the case where arc 2 is red and arc 3 is green, so we always have R and G in the first two columns. We consider three subcases, where the colour of arc 1 is as listed in the third column. The 3-colouring rule for crossing a lets us work out the colour of arc 4 from the colours of arcs 1 and 2; we use this to fill in the next column. The 3-colouring rule for crossing c lets us work out the colour of arc 6 from the colours of arcs 1 and 3; we use this to fill in the next column. The 3-colouring rule for crossing b lets us work out the colour of arc 5 from the colours of arcs 4 and 6; we use this to fill in the next column. We can now use our rule for arc 7 to fill in the last column, and we see that arc 7 always gets coloured blue. As arcs 2 and 3 are red and green, we see that the 3-colouring rule is satisfied at crossing e, as required.

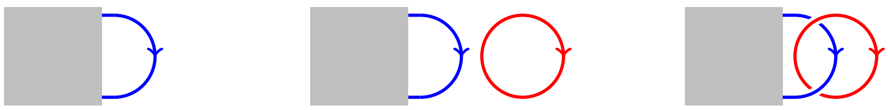
By reviewing the above analysis, we also see that we will never lose the property of using all three colours. Thus, if the left hand diagram is trichromatic, then so is the right hand diagram. A similar argument proves the converse: if the right hand diagram is trichromatic, then so is the left hand diagram.

(2) Let $T(D)$ denote the set of 3-colourings of D, so $\tau(D) = |T(D)|$. In part (1)(c), we considered diagrams D and D' related by single Reidemeister move. In each case, we considered a 3-colouring $\alpha \in T(D)$, and we showed how to construct a 3-colouring $\alpha' \in T(D')$. The construction gives a function $f\colon T(D) \to T(D')$. We also gave a converse construction $g\colon T(D') \to T(D)$. In each case it is not hard to check that if we combine the two constructions we get back to where we started, so $f \circ g = 1$ and $g \circ f = 1$. Thus, f and g are bijections, so $\tau(D) = \tau(D')$.

Next, if you calculate $\tau(D)$ in a few cases you should see that it always has the form $3^d - 3$ for some $d \geq 0$. The reason is as follows. We can identify the set of colours with the set $\mathbb{Z}/3 = \{0, 1, 2\}$, with red corresponding to 0, green corresponding to 1 and blue corresponding to 2. Suppose we have arcs numbered 1 to n, with colours $c_1, \dots, c_n \in \mathbb{Z}/3$. If there is a crossing involving arcs i, j and k, then c_i, c_j and c_k must be all the same or all different, and it is not hard to see that this is equivalent to the equation $c_i + c_j + c_k = 0$ in $\mathbb{Z}/3$. Thus, the 3-colouring conditions are a system of homogeneous linear equations in the variables c_i, which can be solved in the usual way by row-reduction or gaussian elimination. If there are d independent variables, then the number of solutions will be 3^d. However, 3 of these solutions will be ones where all the variables c_i are the same, so not all three colours are used. Thus, only $3^d - 3$ of the solutions count as 3-colourings. (It is not hard to see that there are no solutions involving precisely two colours, so we do not need to adjust for that possibility.)

Note here that the usual methods of linear algebra often involve dividing by a nonzero scalar, so it is important that we can do that in $\mathbb{Z}/3$. This is not a problem, because $2 = -1 \pmod 3$ so the nonzero scalars in $\mathbb{Z}/3$ are just ± 1 and we can certainly divide by these. In other words, $\mathbb{Z}/3$ is a field.

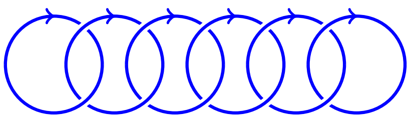
Exercise 10. (1) Suppose we have a link diagram D_1 as shown on the left (with lots of strands and crossings hidden in the gray square). Let D_2 be the diagram obtained by adding an extra unlinked circle to D_1, and let D_3 be obtained by adding an extra linked circle.



D_1 D_2 D_3

Prove that $f(D_2) = -(A^2 + A^{-2})f(D_1)$ and $f(D_3) = -A^2(1 + A^8)f(D_1)$.

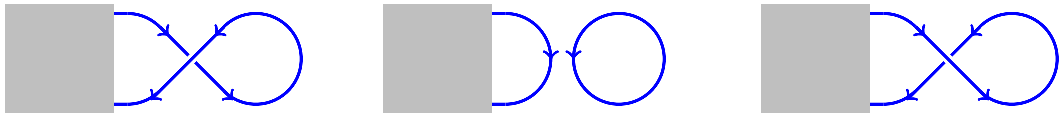
(2) Now let C_n^- denote the n-component negative chain. For example, we have

$C_6 =$ 

Deduce that $f(C_n^-) = (-A^2(1 + A^8))^{n-1}$ for $n \geq 1$.

Solution:

(1) For D_2, we use the skein relation for the following diagrams:



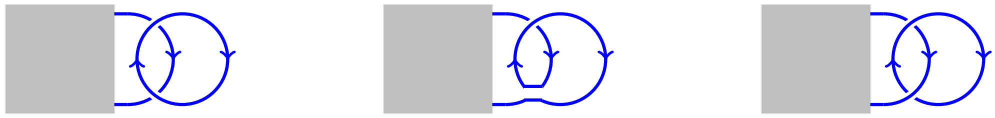
D_1 D_2 D_1

The first and last diagrams can be untwisted to give D_1, and the middle diagram can be converted to D_2 by turning over the circle. We therefore have a skein relation $(A^4 - A^{-4})f(D_1) = (A^{-2} - A^2)f(D_2)$, which gives

$$f(D_2) = \frac{A^4 - A^{-4}}{A^{-2} - A^2} f(D_1) = -(A^2 + A^{-2})f(D_1),$$

as claimed.

We next consider D_3. This can be fitted into a skein relation as follows.



D_2 D_1 D_3

In the first picture, the two components can be moved apart to give D_2. In the second picture, the loop can be untwisted to give the original diagram D_1. The third picture is D_3. We thus have a skein relation

$$A^4 f(D_2) - A^{-4} f(D_3) = (A^{-2} - A^2)f(D_1),$$

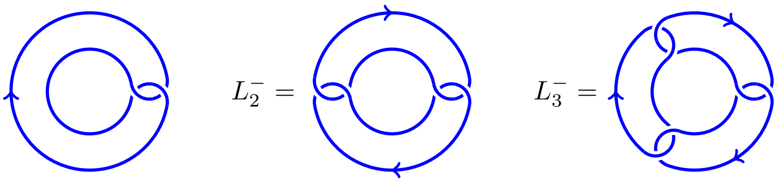
which rearranges to give

$$\begin{aligned} A^{-4} f(D_3) &= A^4 f(D_2) - (A^{-2} - A^2)f(D_1) \\ &= A^4(-A^2 - A^{-2})f(D_1) - (A^{-2} - A^2)f(D_1) \\ &= (-A^6 - A^2 - A^{-2} + A^2)f(D_1) = -(A^{-2} + A^6)f(D_1) \\ f(D_3) &= -(A^2 + A^{10})f(D_1). \end{aligned}$$

(2) By taking $D_1 = C_n^-$ in part (1) we get $f(C_n^-) = -A^2(1 + A^8)f(C_{n-1})$, and it follows by induction that $f(C_n^-) = (-A^2(1 + A^8))^{n-1} f(C_1^-)$. However, C_1^- is an unknot so $f(C_1^-) = 1$, so $f(C_n^-) = (-A^2(1 + A^8))^{n-1}$ for all n.

Exercise 11. Now let L_n^- denote the n-link negative loop, obtained by linking together the first and last circles in C_n^-.

$L_1^- =$ $L_2^- =$ $L_3^- =$



Show that for $n \geq 1$ we have $f(L_n^-) = A^8 \dfrac{X^n - Y^n}{X - Y} + Y^{n-1}(1 - A^8)$, where $X = -A^2(1 + A^8)$ and $Y = -A^2(1 - A^4)$.

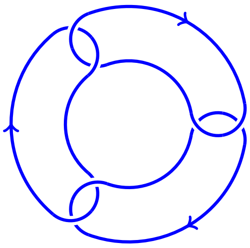
Solution: Put $X = -A^2(1 + A^8)$ and $Y = -A^2(1 - A^4)$ and

$$p(n) = A^8 \frac{X^n - Y^n}{X - Y} + Y^{n-1}(1 - A^8),$$

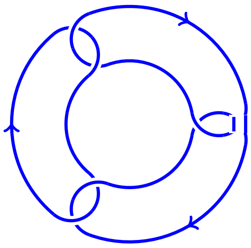
so the claim is that $f(C_n^-) = p(n)$ for $n \geq 1$. Note that L_1^- can be untwisted (by two Reidemeister moves of type 1) to give the unknot, so $f(L_1^-) = 1$. On the other hand, we have

$$p(1) = A^8 \frac{X - Y}{X - Y} + (1 - A^8) = A^8 + 1 - A^8 = 1.$$

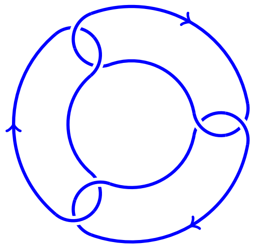
Thus, the claim is true for $n = 1$. Now suppose that $f(L_{n-1}^-) = p(n-1)$, and consider $f(L_n^-)$. We will draw pictures for $n = 3$, but it should be clear that everything works the same way for all n. Consider the skein relation obtained by resolving the rightmost crossing:



C_n^-



L_{n-1}^-



L_n^-

The right hand picture is L_n^-. In the first picture, we have disconnected two of the links so we can straighten out the chain to get a copy of C_n^-. In the middle picture, we can untwist the small loop to get a copy of L_{n-1}^-. The previous exercise tells us that $f(C_n^-) = X^{n-1}$, and we are assuming that $f(L_{n-1}^-) = p(n-1)$. The skein relation therefore gives

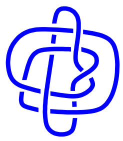
$$A^4 X^{n-1} - A^{-4} f(L_n^-) = (A^{-2} - A^2)p(n-1),$$

or

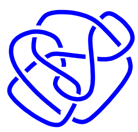
$$\begin{aligned}
f(L_n^-) &= A^4 \left(A^4 X^{n-1} - (A^{-2} - A^2)p(n-1)\right) = A^8 X^{n-1} - Y p(n-1) \\
&= A^8 X^{n-1} + Y \left(A^8 \frac{X^{n-1} - Y^{n-1}}{X - Y} + Y^{n-1}(1 - A^8)\right) \\
&= A^8 \left(X^{n-1} + Y \frac{X^{n-1} - Y^{n-1}}{X - Y}\right) + Y^n (1 - A^8) \\
&= \frac{A^8}{X - Y} \left(X^n - X^{n-1}Y + X^{n-1}Y - Y^n\right) + Y^n(1 - A^8) = A^8 \frac{X^n - Y^n}{X - Y} + Y^n(1 - A^8) \\
&= p(n).
\end{aligned}$$

It follows by induction that $f(L_n^-) = p(n)$ for all n, as claimed.

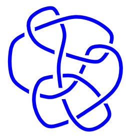
Exercise 12. Consider the following knots:



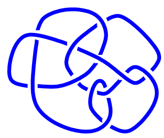
K_0



K_1



K_2



K_3

It can be shown that

$$\begin{aligned}
f(K_0) = f(K_1) &= A^{-12} + A^{-24} - A^{-28} + A^{-32} - A^{-36} + A^{-40} - A^{-44} \\
f(K_2) = f(K_3) &= -A^{16} + 2A^{12} - 2A^8 + 2A^3 + A^{-8} - 2A^{-12} + 2A^{-16} - 2A^{-20} + A^{-24}.
\end{aligned}$$

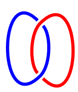
What can you deduce from this?

Solution: We deduce that

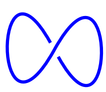
- K_0 is not equivalent to K_2
- K_0 is not equivalent to K_3
- K_1 is not equivalent to K_2
- K_1 is not equivalent to K_3.

We do not have enough information to decide whether K_0 is equivalent to K_1, or whether K_2 is equivalent to K_3. (In fact, K_0 and K_1 are equivalent, but K_2 and K_3 are not; however, the methods needed to prove that are beyond the scope of this course.)

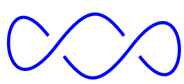
Exercise 13. Write down all states S of the underlying universes for the following link diagrams D. For each state calculate $\langle D|S\rangle$ and $|S|$. Hence write down the unnormalized bracket $\langle\langle D\rangle\rangle$.



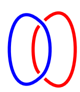
D_1



D_2

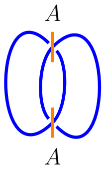
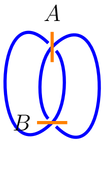
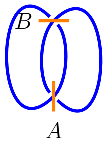
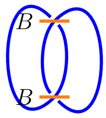
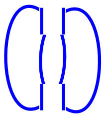
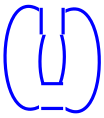
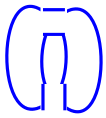
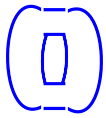


D_3



D_4

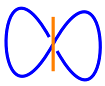
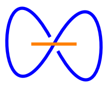
Solution: The states for the D_1 are as follows

S				
D/S				
$\|S\|$	2	1	1	2
$\langle D\|S\rangle$	A^2	AB	AB	B^2

From this we get

$$\langle\langle D_1\rangle\rangle = A^2C + 2AB + B^2C.$$

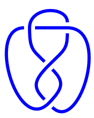
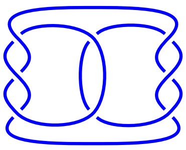
For D_2, the states are as follows:

S		
$\|S\|$	2	1
$\langle D\|S\rangle$	A	B

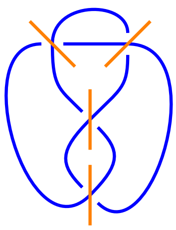
This gives $\langle\langle D_2\rangle\rangle = AC + B$. The other two diagrams can be analysed in a similar way, giving

$$\langle\langle D_3\rangle\rangle = A^2C^2 + 2ABC + B^2$$
$$\langle\langle D_4\rangle\rangle = A^2 + 2ABC + B^2.$$

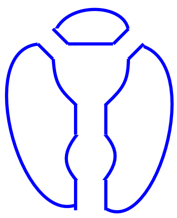
Exercise 14. For each of the following diagrams write splitting markers for the "all A" state S_A and the "all B" state S_B. Calculate $|S_A|$ and $|S_B|$.

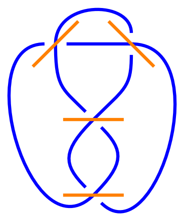
Solution: For the first diagram, we have $|S_A| = |S_B| = 3$.



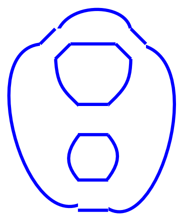
S_A



D/S_A

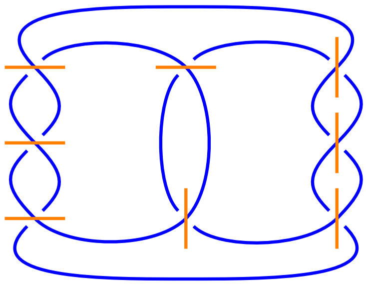


S_B

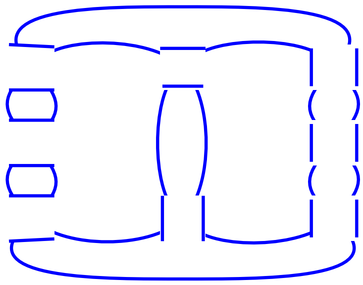


D/S_B

For the second diagram, these pictures show that $|S_A| = 3$.



S_A

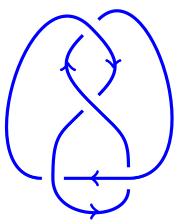


D/S_A

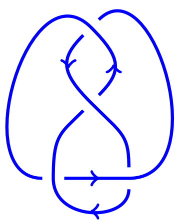
A very similar picture also shows that $|S_B| = 3$.

Exercise 15. For the standard diagram of the negative Hopf link, calculate the Kauffman bracket and hence the Jones polynomial. (You calculated the unnormalized bracket in Sheet 3 Exercise 4.) Check that your answer agrees with the answer obtained in lectures using the skein relation.

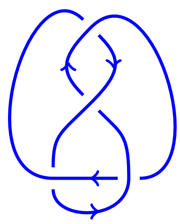
Exercise 16. Consider the following four diagrams related to the figure eight knot:



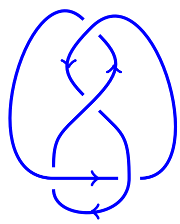
D



D^*



$\overline{D}$

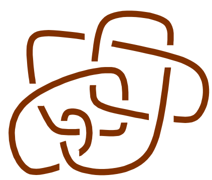


$\overline{D}^*$

Show that they are all R-equivalent, so the figure 8 is both reversible and amphicheiral. (It is not too hard to give an R-equivalence $D \sim D^*$ by imagining what would happen if you rotated D around a vertical line. The same approach gives $\overline{D} \sim \overline{D}^*$. However, it is harder to show that $D \sim \overline{D}$, and you will probably need to experiment with string.)

Now calculate $f(D)$ using the Kauffman bracket, and check that you get the same answer as the one obtained in the lecture using the skein relation.

Exercise 17. The following two knots respectively have Jones polynomials $-A^{32} + A^{20} + A^{12}$ and $A^{12} - A^8 + A^4 - 1 + A^{-4} - A^{-8} + A^{-12}$.



What can you deduce about the symmetry properties of these knots?

Exercise 18. Let $f_1(D)$ be the number obtained by substituting $A = 1$ in $f(D)$.

(a) Deduce from Exercise 4 on Problem Sheet 1 that any link can be converted to an unlink by switching crossings.

(b) By thinking about the skein relation, show that $f_1(D)$ only depends on the underlying universe of D, not on the overcrossings or undercrossings.

(c) Now deduce that $f_1(D) = 2^{c-1}$, where c is the number of components in D. Thus, we can find the number of components in D by inspecting the Jones polynomial.

(d) Deduce that the sum of coefficients of the Jones polynomial of a *knot* is always equal to 1.

(e) (Optional) What is the sum of coefficients of the *derivative* of the Jones polynomial of a knot?

Exercise 19. For each of the following sets, decide whether is is a closed surface, a surface with boundary, or not a surface.

$$X_0 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$$
$$X_1 = \{(w, x, y, z) \in \mathbb{R}^4 \mid w^2 + x^2 = y^2 + z^2 = 1\}$$
$$X_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \geq 0,\ x + y + z = 1\}$$
$$X_3 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$
$$X_4 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1,\ xyz = 0\}$$
$$X_5 = \{(w, x, y, z) \in \mathbb{R}^4 \mid w + z = 0,\ w^2 + x^2 + y^2 + z^2 = 1\}.$$

Solution:

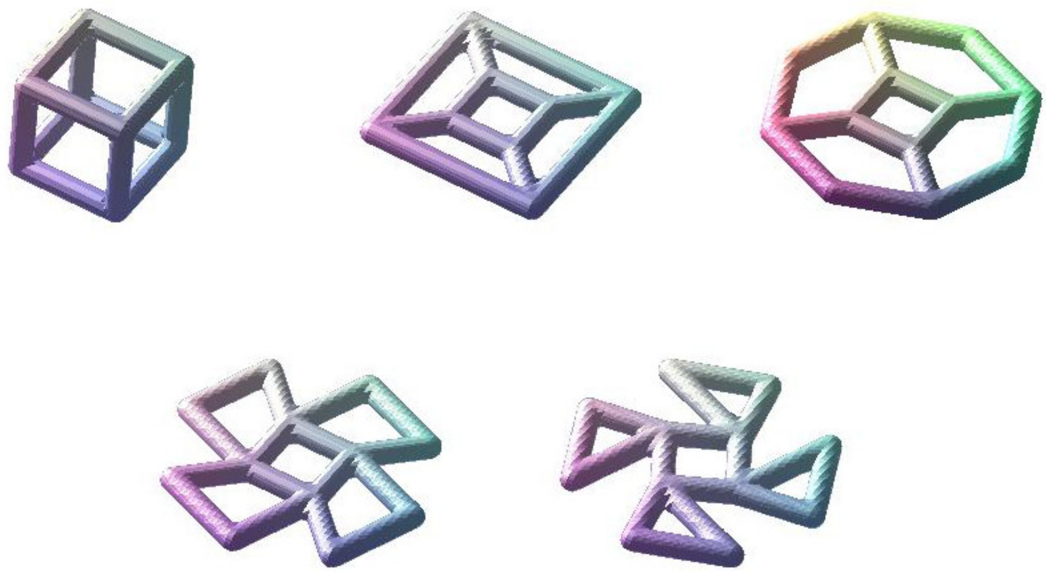
- The space X_0 is not a surface, because it is solid. The point $(0, 0, 0) \in X_0$ has an open neighbourhood homeomorphic to the 3-dimensional open ball, but has no neighbourhood homeomorphic to the 2-dimensional open disc.
- The space X_1 is a surface, and in fact is homeomorphic to a torus. Indeed, we can define
$$f(s, t) = (\cos(2\pi s), \sin(2\pi s), \cos(2\pi t), \sin(2\pi t)),$$
and then X_1 is precisely the image of the map f. Given any point $a \in X_1$, we can choose (s, t) such that $a = f(s, t)$, then we can define a map g from the open unit disc to X_1 by $g(x, y) = f(s + x, t + y)$. This identifies the open unit disc with a neighbourhood of a in X_1. Alternatively, we can say that every point in X_1 has the form $f(s, t)$ for some point (s, t) in the unit square, and this point is unique except that $f(s, 0) = f(s, 1)$ and $f(0, t) = f(1, t)$. In other words, the left and right edges of the square have the same image in X_1, as do the top and bottom edges. This is precisely the gluing pattern for the torus.
- The space X_2 is just a triangle, with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Any triangle is homeomorphic to a closed disc which is a surface with boundary, so X_2 is a also surface with boundary.
- The space X_3 is not compact, because it contains the sequence of points $(1 - 1/n, 0)$, which converge to the point $(1, 0)$, which is not in X_3. Thus, X_3 is not a surface according to our definitions. (However, every point has a full disc neighbourhood, so it is a surface according to some other people's definitions.)
- Note that the condition $xyz = 0$ means that $x = 0$ or $y = 0$ or $z = 0$. This means that the space X_4 can be thought of as the union of three closed unit discs, one in the xy-plane, one in the xz-plane and one in the yz-plane. If a point lies on the intersection of two of these discs, then it will not have a disc of half-disc neighbourhood in X_4. Thus, X_4 is not a surface.
- For X_5, note that the condition $w+z = 0$ gives $w = -z$ and so $w^2 = z^2$. Given this, the condition $w^2 + x^2 + y^2 + z^2 = 1$ becomes $x^2 + y^2 + (\sqrt{2}z)^2 = 1$. Thus, the map $(w, x, y, z) \mapsto (x, y, \sqrt{2}z)$ gives a homeomorphism from X_5 to S^2, showing that X_5 is a surface.

Exercise 20. Consider the space
$$X = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, -0.9 \leq x, y, z \leq 0.9\}.$$
This is homeomorphic to $D^{\#n}$ for some n. What is n?

Solution: The condition $x^2 + y^2 + z^2 = 1$ describes a sphere. The equation $x = 0.9$ describes a plane, which passes through the sphere close to the point $(1, 0, 0)$. The inequality $x \leq 0.9$ cuts off a small disc around that point, leaving a hole in the sphere. Similarly, the remaining inequalities $-0.9 \leq x, y, z \leq 0.9$ cut out holes near the points $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$ and $(0, 0, -1)$. That gives six holes in total. We showed in the notes that $Z\#D$ is the same as Z with a disc removed. Thus, we have $X \simeq S^2\#D^{\#6}$. On the other hand, $S^2\#Z$ is the same as Z, so $X \simeq D^{\#6}$. In other words, $n = 6$.

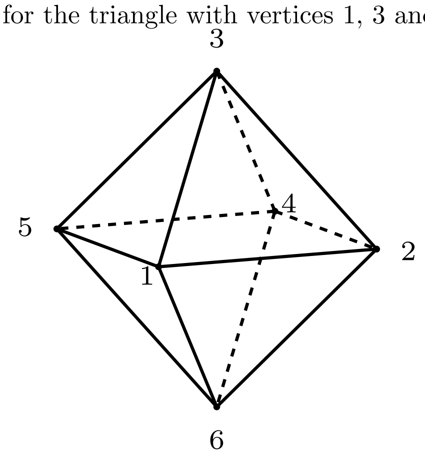
Exercise 21. The space below is homeomorphic to $T^{\#n}$ for some n (where T is the torus). What is n?

Solution: We can flatten out and distort the surface as shown in the following series of pictures.



The last picture is the connected sum of a (square) torus in the middle with four (triangular) tori around the outside. Thus, we have $X \simeq T^{\#5}$ and so $n = 5$.

Exercise 22. Here is a picture of an octahedron, with vertices labelled 1 to 6. We write 26 for the edge joining vertices 2 and 6, and 135 for the triangle with vertices 1, 3 and 5 and so on.



The eight triangles give a closed linear surface triangulation of the surface of the octahedron, as defined in Definition 11.2. If we just consider the top four triangles, then they give a linear surface triangulation with boundary of the top half of the octahedron. Explain carefully why these statements are true, with examples.

Solution: For a closed linear surface triangulation, the first condition is that the intersection any pair of distinct triangles should be empty, a common vertex or a common edge. For example:

- The triangles 123 and 456 have empty intersection.
- The triangles 123 and 246 meet only at point 2, which is a vertex of both triangles.
- The intersection of triangles 123 and 126 is the line segment 12, which is an edge of both of the tiangles.

One can check that all other pairs of triangles also follow one of the above patterns.

The second condition is that each edge should be an edge of precisely two triangles. For example:

- The edge 12 is an edge of the triangles 123 and 126 (and no others).
- The edge 13 is an edge of the triangles 123 and 135 (and no others).
- The edge 56 is an edge of the triangles 156 and 456 (and no others).

One can check all twelve edges in the same way.

The last condition concerns the pattern of triangles containing any given vertex. For example, the triangles containing point 1 can be listed as follows:

$$T_1 = 123 \qquad T_2 = 135 \qquad T_3 = 156 \qquad T_4 = 126.$$

Here

- T_1 shares the edge 13 with T_2;
- T_2 shares the edge 15 with T_3;
- T_3 shares the edge 16 with T_4;
- T_4 shares the edge 11 with T_1;
- there are no other shared edges.

This verifies the required condition for vertex 1, and one can do the same for all other vertices.

Exercise 23. Using Exercise or otherwise, prove that the sphere S^2 is homeomorphic to a PL closed surface.

Solution: If we place our octahedron X with the origin at the centre, then we can define a homeomorphism $f\colon X \to S^2$ by $f(u) = u/\|u\|$.

Exercise 24. For each of the following words W_k, describe the surface $\Sigma(W_k)$. In particular, you should state whether $\Sigma(W_k)$ is orientable, whether it has a boundary, and which of the surfaces in the notes it is homeomorphic to.

$$W_1 = ba\bar{d}\,\bar{c}\,dc\bar{b}\,\bar{a} \qquad W_2 = ba\bar{d}\,\bar{c}\,dcba \qquad W_3 = abb\bar{a} \qquad W_4 = abac \qquad W_5 = abc\bar{c}\,\bar{a}\,\bar{b} \qquad W_6 = abc\bar{c}a\bar{b}.$$

Solution:

- In W_2, every letter occurs once barred and once unbarred, so the corresponding surface is closed and orientable. The word can be rotated to give $\bar{d}\bar{c}dc\bar{b}\bar{a}ba$, which can be relabelled as $uv\bar{u}\bar{v}wx\bar{w}\bar{x}$. This is two copies of the torus word T, and we have seen that $TV \sim T\#V$ for any V, so $W_1 \sim T\#T$. Thus, $\Sigma(W_1)$ is the connected sum of two tori. In particular, it is closed and orientable.
- In W_2 every letter occurs twice, but both copies of a are unbarred. This means that the surface is closed but not orientable. We have surface moves as follows:
$$W_2 = b[a\bar{d}\,\bar{c}\,dc]ba \sim bb\bar{c}\bar{d}cd(\bar{a}a) \sim bb\bar{c}\bar{d}cd \cong PT \sim P\#T.$$
Thus, $\Sigma(W_2)$ is the connected sum of a torus and a projective plane.
- The moves $W_3 = abb|\bar{a} \sim (\bar{a}a)bb \sim bb \cong P$ show that $\Sigma(W_3)$ is a projective plane, which is again closed and not orientable.
- W_4 is the standard word for the Möbius strip; it is neither closed nor orientable.
- We can cancel $c\,\bar{c}$ to see that $W_5 \sim T$, so $\Sigma(W_5)$ is a torus, which is closed and orientable.
- We have word moves as follows:
$$W_6 = ab(c\bar{c})a\bar{b} \sim a[b]a\bar{b} \sim aa\overline{bb} \cong PP \sim P\#P,$$
so $\Sigma(W_6)$ is the connected sum of two projective planes.

Exercise 25. How many different surfaces (up to homeomorphism) can be constructed as $\Sigma(W)$ with W a word of length 4?

Solution: I claim that the possibilities are as follows:
$$\begin{aligned}
D &\simeq \Sigma(abcd) \\
D\#D &\simeq \Sigma(abc\bar{b}) \\
P\#D &\simeq \Sigma(aabc) \\
P\#P &\simeq \Sigma(aabb) \\
P &\simeq \Sigma(aab\bar{b}) \\
T &\simeq \Sigma(ab\bar{a}\bar{b}) \\
S &\simeq \Sigma(a\bar{a}b\bar{b}).
\end{aligned}$$

To justify this, we first need to check that $\Sigma(abcd)$ really is homeomorphic to D, and similarly for the other equations above. This is true by merger moves for D, by definition for $D\#D$ and T, and by cancellation for P and S. For the remaining cases $P\#D$ and $P\#P$, we need the general fact that $P\#V \sim PV$ for all V.

Next, we need to check that any word W of length four is equivalent to one of those listed above.

- If W has four unmatched letters, then after relabelling it is the same as $abcd$ and so gives D.
- Suppose next that W has two unmatched letters, and one letter that occurs twice, once barred and once unbarred. If the matched letters are adjacent then they can be cancelled and we get D again. This also works (after rotation) if one copy is at the beginning and the other copy is at the end. In the only remaining case, we can relabel and rotate if necessary to get $abc\bar{b}$, which gives $D\#D$.

- Suppose next that W has two unmatched letters, and one letter that occurs twice with the same barring. After a crosscap move, we can assume that the matched letters are adjacent. After rotation, we can assume that they occur at the beginning. After relabelling, we now have $aabc$. After merging bc this becomes $aab = PD \sim P\#D$.
- Now suppose we have two matched pairs. If one or the other pair is nonorientable, then we can perform a crosscap move to make that pair adjacent, and then rotate it to the beginning. This will give us either $aabb = PP \sim P\#P$ or $aab\overline{b} \sim aa = P$.
- Suppose instead that we have two matched pairs, both orientable. If either pair is adjacent then we can cancel it, and then the other pair will become adjacent, so we can cancel it as well, giving S.
- Finally, suppose we have two matched orientable pairs, neither of which is adjacent. After relabelling this must just be $ab\overline{a}\overline{b}$, which gives T.

Exercise 26. Show by example that if U and V are non-closed surface words, then UV need not be equivalent to $U\#V$.

Solution: The simplest possible example will do: we just take $U = x$ and $V = y$, so $UV \sim U$ by a merger move, and so $\Sigma(UV)$ is a disk. However, $\Sigma(U\#V)$ is a connected sum of two discs, which is homeomorphic to a cylinder.

Exercise 27. Show (directly from the definition of word equivalence) that an orientable word can never be equivalent to a non-orientable word, and a closed word can never be equivalent to a non-closed word.

Solution: It will be sufficient to consider a single word move. We first consider orientability. For the purposes of this discussion, a "precisely repeated letter" means a letter which occurs twice barred or twice unbarred.

- As UV and VU have exactly the same letters in a different order, we see that UV has a precisely repeated letter if and only if VU does. Thus, a rotation does not affect orientability. Neither does an inner rotation, by the same argument.
- Now consider a cancellation $Ux\overline{x}V \to UV$. If UV has a precisely repeated letter, then that must also be present in $Ux\overline{x}V$. Conversely, suppose that $Ux\overline{x}V$ contains a precisely repeated letter. That letter cannot be x, because x is only allowed to occur twice, so we still have the precisely repeated letter in UV. Thus, cancellation does not affect orientability.
- In a crosscap move $TxUxV \to Txx\overline{U}V$, both of the words are visibly nonorientable, because of the repeated x; we do not need to worry about what might be present in T, U or V.
- In a merger move $UxyV \to UxV$, the letters x and y are assumed not to be repeated. From this it is clear that $UxyV$ has a precisely repeated letter if and only if UxV does.
- It is also clear from the definitions that relabelling does not affect orientability.

Similar arguments work for closedness.

- As UV and VU have exactly the same letters in a different order, we see that UV has an unmatched letter if and only if VU does. Thus, a rotation does not affect closedness. Neither does an inner rotation, by the same argument.
- Essentially the same argument works for a crosscap move. That kind of move will add or subtract some bars as well as changing the order of the letters, but that does not affect the presence or absence of unmatched letters.
- Now consider a cancellation $Ux\overline{x}V \to UV$. This just removes a matched pair, so UV has the same unmatched letters as $Ux\overline{x}V$ (if any).
- In a merger move $UxyV \to UxV$, it is clear by assumption that both of the words are non-closed; we do not need to worry about what might be present in U or V.
- It is also clear from the definitions that relabelling does not affect closedness.

Exercise 28.

(a) Show that $a_1a_2U\overline{a}_1\overline{a}_2V \sim a_1a_2\overline{a}_1\overline{a}_2VU$.

(b) Now put

$$W_n = a_1a_2a_3\cdots a_n\overline{a}_1\overline{a}_2\overline{a}_3\cdots\overline{a}_n.$$

Prove that $W_{2m} \sim W_{2m+1} \sim T^{\#m}$ for all $m \geq 0$.

(c) Give similar results for the words

$$U_n = a_1a_2a_3\cdots a_na_1a_2a_3\cdots a_n$$
$$V_n = a_1a_2a_3\cdots a_na_n\cdots a_3a_2a_1.$$

Solution:

(a) $a_1a_2(U|\overline{a}_1)\overline{a}_2V \sim a_1a_2\overline{a}_1(U|\overline{a}_2V) \sim a_1a_2\overline{a}_1\overline{a}_2VU$.

(b) The word W_0 is empty, and the word $W_1 = a_1\overline{a}_1$ cancels down to the empty word, so $W_0 \sim W_1 \sim T^{\#0}$. Now suppose that $n \geq 2$. Put $U = a_3 \cdots a_n$ and $V = \overline{a}_3 \cdots \overline{a}_n$, so $W_n = a_1a_2U\overline{a}_1\overline{a}_2V$. By part (a) we have $W_n \sim a_1a_2\overline{a}_1\overline{a}_2VU = TVU \sim T\#VU$. However, if we relabel a_{i+2} as $\overline{a}_i$ (and $\overline{a}_{i+2}$ as a_i), then VU becomes W_{n-2}. We therefore see that $W_n \sim T\#W_{n-2}$. It now follows easily by induction that $W_{2m} \sim W_{2m+1} \sim T^{\#m}$ for all $m \geq 0$.

(c) The case of U_n is very simple: a single crosscap move gives

$$U_n \sim a_1a_1\overline{a}_n \cdots \overline{a}_3\overline{a}_2a_2a_3 \cdots a_n,$$

and then we can cancel repeatedly to get $U_n \sim a_1a_1 \cong P$.

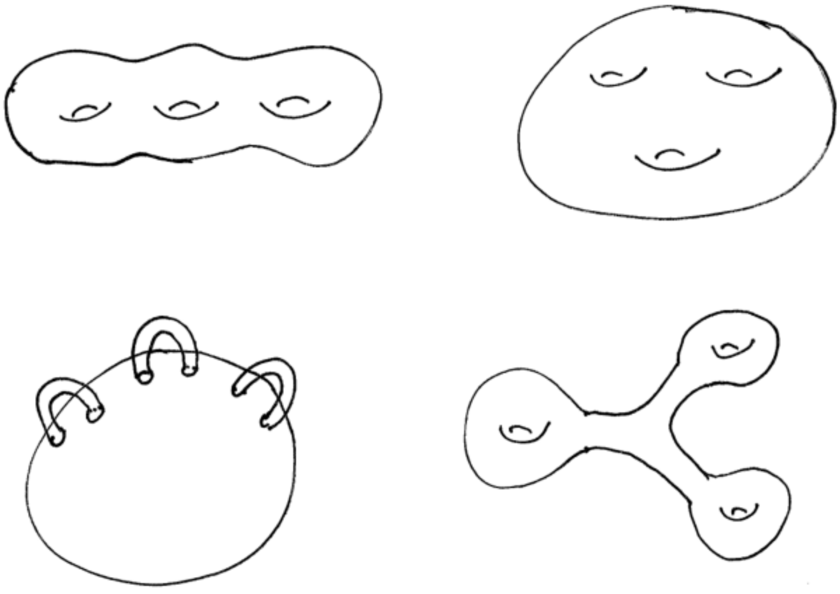
On the other hand, a crosscap move on V_n produces

$$V_n \sim a_1a_1\overline{a}_2\overline{a}_3 \cdots \overline{a}_n\overline{a}_n \cdots \overline{a}_3\overline{a}_2.$$

The first pair of letters is a relabelled P, and the rest is a relabelled V_{n-1}, so $V_n \sim PV_{n-1} \sim P\#V_{n-1}$. It is also clear that $V_0 = S$ and $V_1 = P$, so it follows inductively that $V_n \sim P^{\#n}$ for all n.

Exercise 29. Draw several different surfaces that are all homeomorphic to the connected sum of three tori.

Solution: Here are some possibilities:



Exercise 30. For each of the following surfaces, write down a corresponding word. Try to make it as short as possible.

(a) A torus with two discs removed.
(b) The connected sum of three tori.
(c) The connected sum of a Klein bottle and a sphere.
(d) A dodecahedron with two opposite faces removed.

Solution:

(a) Removing a disc is the same as taking the connected sum with D. Thus, the obvious answer is

$$T\#(D\#D) = (ab\overline{a}\,\overline{b})\#(def\overline{d}) = ab\overline{a}\,\overline{b}cdef\overline{d}\,\overline{c}.$$

However, this can be shortened slightly, because we know that $T\#U \sim TU$ for any U. We can thus use the word

$$T(D\#D) = ab\overline{a}\overline{b}def\overline{d}$$

instead.

(b) The obvious answer is just

$$TTT = ab\overline{a}\,\overline{b}cd\overline{c}\,\overline{d}ef\overline{e}\,\overline{f}.$$

(c) Taking the connected sum with the sphere does not change anything, so we just want the standard Klein bottle word, which is $xy\overline{x}y$.

(d) The dodecahedron is just a sphere, and removing one face just leaves a disc. To remove another hole, we just take the connected sum with a disc. Thus, the word we want is just $D\#D = xyz\overline{y}$. Alternatively, one can see geometrically that removing two opposite faces leaves a surface which is homeomorphic to a cylinder, and the standard word for a cylinder is $xy\overline{x}z$. This is equivalent by rotation and relabelling to the word $D\#D = xyz\overline{y}$.

Exercise 31. Suppose that U, V and W are surface words, and that $U\#V \sim U\#W$. Give an example to show that V need not be equivalent to W. Investigate this question in more detail, and formulate a clear and concise proposition about when we can or cannot conclude that $V \sim W$.

Solution: First, take $U = P$ and $V = T$ and $W = P\#P$. By Proposition 13.21, we have $U\#V \sim U\#W$. However, V is orientable and W is not, so V and W are not equivalent.

We have seen that any word is equivalent to $T^{\#i}\#D^{\#j}$ or $P^{\#(i+1)}\#D^{\#j}$ (for some $i, j \geq 0$), and none of these words are equivalent to each other. Connected sums are as follows:

$$\begin{aligned}
(T^{\#i}\#D^{\#j})\#(T^{\#n}\#D^{\#m}) &\sim T^{\#(i+n)}\#D^{\#(j+m)} \\
(T^{\#i}\#D^{\#j})\#(P^{\#(n+1)}\#D^{\#m}) &\sim P^{\#(2i+n+1)}\#D^{\#(j+m)} \\
(P^{\#(i+1)}\#D^{\#j})\#(T^{\#n}\#D^{\#m}) &\sim P^{\#(i+2n+1)}\#D^{\#(j+m)} \\
(P^{\#(i+1)}\#D^{\#j})\#(P^{\#(n+1)}\#D^{\#m}) &\sim P^{\#(i+n+2)}\#D^{\#(j+m)}.
\end{aligned}$$

From this table, we can deduce the following:

Proposition. Suppose that U, V and W are surface words, and that $V \not\sim W$ but $U\#V \sim U\#W$. Then U must be nonorientable, and there must exist $i, j \geq 0$ such that either

(a) $V \sim T^{\#(i+1)}\#D^{\#j}$ and $W \sim P^{\#(2i+2)}\#D^{\#j}$; or
(b) $V \sim P^{\#(2i+2)}\#D^{\#j}$ and $W \sim T^{\#(i+1)}\#D^{\#j}$.