Linear Mathematics for Applications
All exam questions

Question 1

(1.1; 2013-14) You may assume the following row reductions. Some of them are relevant, and some of them are not. Some questions can be done more easily without row-reduction.

$$\begin{bmatrix} 0 & 2 & 0 & -1 & 10 \\ 0 & -1 & 1 & 1 & 5 \\ -1 & -3 & 2 & 2 & -7 \\ -1 & 3 & 3 & 0 & 36 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & -1 & -1 \\ 2 & -1 & -3 & 3 \\ 0 & 1 & 2 & 3 \\ -1 & 1 & 2 & 0 \\ 10 & 5 & -7 & 36 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Give examples of the following: **(6 marks)**

  (i) A 3×3 RREF matrix A such that A^T is also in RREF.

  (ii) A 2×4 RREF matrix B that is no longer in RREF if you delete the second column.

  (iii) A 3×3 RREF matrix C in which four of the entries are not zero.

(b) Find the general solution for the following system of linear equations, or prove that there is no solution. **(4 marks)**

$$\begin{aligned} 2b &= 10 + d \\ c + d &= b + 5 \\ 2c + 2d &= a + 3b - 7 \\ 3b + 3c &= a + 36. \end{aligned}$$

(c) Consider the vectors

$$v = \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 8 \\ 7 \\ 6 \\ 5 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

Either express v as a linear combination of u_1, u_2, u_3 and u_4, or prove that that is impossible. **(4 marks)**

(d) Do the vectors u_i in part (c) form a basis for $\mathbb{R}^4$? Justify your answer. **(2 marks)**

(e) By performing row operations on $C - tI$ or otherwise, evaluate the characteristic polynomial of the following matrix: **(4 marks)**

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

(1.2; 2012-13 resit) You may assume the following row reductions. Some of them are relevant, and some of them are not.

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 10 \\ 0 & 1 & 0 & 12 \\ 1 & 11 & -11 & 11 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 11 \\ 0 & 1 & 0 & -11 \\ 1 & 10 & 12 & 11 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 3 & -3 \\ 1 & 2 & 0 \\ 3 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 1 & 3 \\ 3 & 3 & 2 & 2 \\ 1 & -3 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1/2 & 3/2 \\ 0 & 1 & 1/6 & -5/6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -2 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 & 1 \\ 1 & 2 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Explain what it means for a matrix to be in reduced row echelon form (RREF). **(4 marks)**

(b) List all the 2×2 matrices in RREF for which every entry is 0 or 1. **(4 marks)**

(c) Find the general solution for the following system of linear equations, or prove that there is no solution. **(3 marks)**

$$\begin{aligned} r + s &= 2q + p + 2 \\ 2r + s &= 1 \\ 2r + p + 2q &= 1. \end{aligned}$$

(d) Consider the vectors

$$u_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 3 \\ 3 \\ 2 \\ 2 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 4 \end{bmatrix}.$$

Are they linearly independent? Justify your answer. **(3 marks)**

(e) Either find the inverse of the following matrix, or show that it has no inverse: **(4 marks)**

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 10 \\ 0 & 1 & 0 & 12 \\ 1 & 11 & -11 & 11 \end{bmatrix}$$

(f) Is zero an eigenvalue of A? Justify your answer. **(2 marks)**

(1.3; 2012-13) You may assume the following row reductions. Some of them are relevant, and some of them are not.

$$\begin{bmatrix} 2 & -1 & -1 & 3 \\ -2 & 2 & 3 & 7 \\ -2 & 12 & 23 & 107 \\ 3 & -2 & -3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & -2 & -2 & 3 \\ -1 & 2 & 12 & -2 \\ -1 & 3 & 23 & -3 \\ 3 & 7 & 107 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 10 & 0 \\ 0 & 1 & 11 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 3 \\ 3 & -3 & 1 & 0 & 11 \\ -2 & 2 & 0 & -3 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & -2 \\ -1 & -3 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \\ 3 & 11 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Explain what it means for a matrix to be in reduced row echelon form (RREF). **(4 marks)**

(b) Give an example of a 4×4 RREF matrix with pivots in columns 1 and 3, and precisely five nonzero entries. **(2 marks)**

(c) Find the general solution for the following system of linear equations, or prove that there is no solution. **(4 marks)**

$$\begin{aligned} a + d &= b + 3 \\ 3a + c &= 3b + 11 \\ -2a + 2b &= 3d - 7. \end{aligned}$$

(d) Consider the vectors

$$v = \begin{bmatrix} 3 \\ -2 \\ -3 \\ -2 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 3 \end{bmatrix} \qquad u_2 = \begin{bmatrix} -2 \\ 2 \\ 3 \\ 7 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -2 \\ 12 \\ 23 \\ 107 \end{bmatrix}$$

Either express v as a linear combination of u_1, u_2 and u_3, or prove that that is impossible. **(3 marks)**

(e) By performing row operations or otherwise, evaluate the determinant of the following matrix: **(4 marks)**

$$A = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 4 & 3 & 0 \\ 2 & 4 & 6 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

(f) Do the columns of A span $\mathbb{R}^4$? Justify your answer. **(3 marks)**

(1.4; 2011-12 resit)

(a) Which of the following matrices are in reduced row echelon form (RREF)? Explain your answers. **(3 marks)**

$$A = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Row-reduce the following matrix. **(6 marks)**

$$D = \begin{bmatrix} 7 & 14 & 3 & 15 & -12 \\ 5 & 10 & 5 & 25 & 0 \\ 3 & 6 & 7 & 35 & 12 \end{bmatrix}$$

(c) You may assume the row-reduction

$$\begin{bmatrix} 1 & 2 & -3 & 7 & 7 \\ -3 & 1 & 2 & 14 & -14 \\ 2 & -3 & 1 & 7 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 & 5 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Solve the following two systems of equations (the first system on the left, and the second system on the right) :

$$\begin{aligned} x + 2y - 3z &= 7 \\ -3x + y + 2z &= 14 \\ 2x - 3y + z &= 7 \end{aligned} \qquad\qquad \begin{aligned} x + 2y - 3z &= 7 \\ -3x + y + 2z &= -14 \\ 2x - 3y + z &= 7 \end{aligned}$$

In each case say whether the system has a unique solution, an infinite family of solutions, or no solution. **(6 marks)**

(d) Find the determinant of the following matrix: **(3 marks)**

$$E = \begin{bmatrix} 0 & a & 0 & b \\ c & d & 0 & e \\ f & 0 & 0 & g \\ h & 0 & i & j \end{bmatrix}$$

(e) State, with justification, which of the following matrices are invertible. **(7 marks)**

$$F = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 11 & 21 & 31 & 41 \\ 101 & 201 & 301 & 401 \\ 1001 & 2001 & 3001 & 4001 \end{bmatrix} \qquad G = \begin{bmatrix} 6 & 2 & 1 & 5 \\ 7 & 3 & 1 & 4 \\ 9 & 4 & 4 & 3 \end{bmatrix} \qquad H = \begin{bmatrix} 2 & 1 & -2 \\ 3 & 1 & -3 \\ -2 & -1 & 1 \end{bmatrix} \qquad J = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(1.5; 2011-12)

(a) Which of the following matrices are in reduced row echelon form (RREF)? Explain your answers. **(3 marks)**

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Row-reduce the following matrix. **(6 marks)**

$$D = \begin{bmatrix} 11 & 10 & 1 & 1 & 11 \\ 11 & 1 & 10 & 10 & 1 \\ 1 & 1 & 0 & 0 & 10 \end{bmatrix}$$

(c) You may assume the row-reduction

$$\begin{bmatrix} 7 & -3 & 1 & -1 & 1 \\ 3 & 2 & 7 & 16 & 16 \\ 4 & -1 & 2 & 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Solve the following two systems of equations (the first system on the left, and the second system on the right) :

$$\begin{aligned} 7x - 3y + z &= -1 \\ 3x + 2y + 7z &= 16 \\ 4x - y + 2z &= 3 \end{aligned} \qquad\qquad \begin{aligned} 7x - 3y + z &= 1 \\ 3x + 2y + 7z &= 16 \\ 4x - y + 2z &= -3 \end{aligned}$$

In each case say whether the system has a unique solution, an infinite family of solutions, or no solution. **(6 marks)**

(d) Find the determinant of the following matrix: **(3 marks)**

$$E = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 4 \\ 0 & 4 & 0 & 5 \end{bmatrix}$$

(e) State, with justification, which of the following matrices are invertible. **(7 marks)**

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \\ 9 & 9 & 9 & 9 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & 2 & 5 \\ 6 & 4 & 3 \\ 5 & 1 & 2 \\ 7 & 9 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} -2 & -2 & -1 \\ -1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(1.6; Mock 1)

(a) Which of the following matrices are in reduced row echelon form (RREF)? Explain your answers. **(3 marks)**

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(b) Row-reduce the following matrix. **(6 marks)**

$$D = \begin{bmatrix} 2 & 2 & 2 & 4 & 6 \\ 3 & 3 & 3 & 5 & 7 \\ 5 & 5 & 5 & 8 & 9 \end{bmatrix}$$

(c) You may assume the row-reduction

$$\begin{bmatrix} 1 & -2 & 0 & 1 & -3 & 2 \\ -2 & 4 & 0 & -2 & 6 & -4 \\ 1 & -2 & -1 & 5 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -4 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find the general solution of the system of equations

$$\begin{aligned} v - 2w + y - 3z &= 2 \\ -2v + 4w - 2y + 6z &= -4 \\ v - 2w - x + 5y - 4z &= 0 \end{aligned}$$

Then find a specific solution (with no free variables) where $x = 0$. **(6 marks)**

(d) Find the determinant of the following matrix: **(3 marks)**

$$E = \begin{bmatrix} 0 & a & 0 & 0 \\ b & c & d & e \\ f & g & 0 & h \\ i & j & 0 & k \end{bmatrix}$$

(e) State, with justification, which of the following matrices are invertible. **(7 marks)**

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 2 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} -3 & 1 & -2 \\ 1 & 2 & 2 \\ 0 & -1 & -1 \end{bmatrix} \qquad J = \begin{bmatrix} 100 & 20 & 3 & 123 \\ 300 & 10 & 7 & 317 \\ 500 & 70 & 1 & 571 \\ 200 & 60 & 9 & 269 \end{bmatrix}$$

(1.7; Mock 2)

(a) Which of the following matrices are in reduced row echelon form (RREF)? Explain your answers. **(4 marks)**

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Row-reduce the following matrix. **(6 marks)**

$$E = \begin{bmatrix} 2 & 2 & 2 & 4 & 6 \\ 3 & 3 & 3 & 5 & 7 \\ 5 & 5 & 5 & 8 & 9 \end{bmatrix}$$

(c) Consider the vectors

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ 7 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 2 \end{bmatrix} \quad v_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 2 \end{bmatrix} \quad v_5 = \begin{bmatrix} 7 \\ 6 \\ 4 \\ 5 \end{bmatrix} \quad v_6 = \begin{bmatrix} 7 \\ 4 \\ 6 \\ 5 \end{bmatrix}$$

You may assume the row-reduction

$$\begin{bmatrix} 1 & 4 & 2 & 1 & 7 & 7 \\ 3 & 1 & 1 & 2 & 6 & 4 \\ 7 & 0 & 1 & 5 & 4 & 6 \\ 2 & 2 & 1 & 2 & 5 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(i) Either express v_5 as a linear combination of v_1, v_2, v_3, v_4, or explain why that is not possible. **(3 marks)**

(ii) Either express v_6 as a linear combination of v_1, v_2, v_3, v_4, or explain why that is not possible. **(3 marks)**

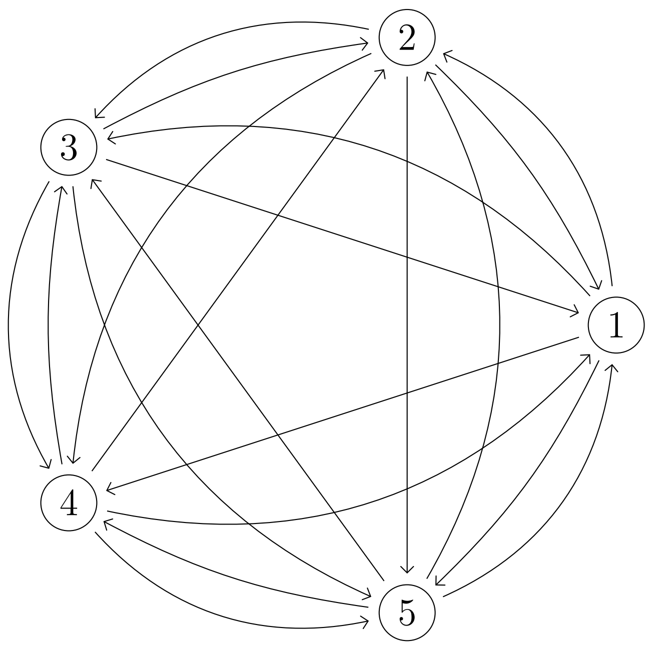
(d) Find inverses for the matrices F, G and H below, and explain why J does not have an inverse. **(9 marks)**

$$F = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad G = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \qquad H = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad J = \begin{bmatrix} 1 & 100 & 10 & 111 \\ 10 & 1 & 100 & 111 \\ 10 & 100 & 1 & 111 \\ 100 & 1 & 10 & 111 \end{bmatrix}$$

Hint: for H, trial and error may be easier than a systematic method.

Question 2

(2.1; 2013-14) Consider the following Markov chain:



There is an arrow from every state to every other state, but no arrows from any state to itself. All the arrows have the same probability p.

(a) What must p be? **(2 marks)**

(b) Write down the transition matrix P. **(2 marks)**

(c) Calculate P^2, and thus find constants α and β such that $P^2 = \alpha P + \beta I_5$. **(4 marks)**

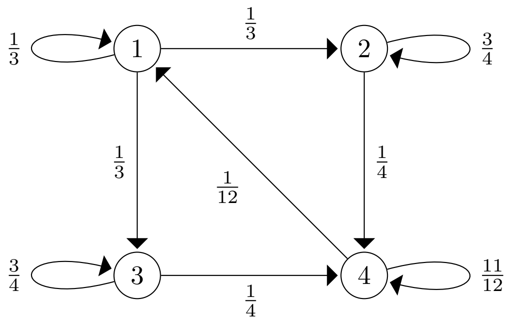
(d) Show that if v is an eigenvector for P with eigenvalue λ, then $\lambda^2 = \alpha\lambda + \beta$. **(2 marks)**

(e) Use (d) to find the eigenvalues of P. **(3 marks)**

(f) You may assume that P has a unique stationary distribution. What is it? **(3 marks)**
Hint: you could use row-reduction, but other methods are much easier.

(g) Find a basis for $\mathbb{R}^5$ consisting of eigenvectors for P. **(4 marks)**

(2.2; 2012-13 resit) Consider the following Markov chain:



(a) Write down the transition matrix P. **(2 marks)**

(b) Consider the following vectors:

$$u_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ -2 \\ -2 \\ 3 \end{bmatrix}$$

Show that these are eigenvectors for P, and find the corresponding eigenvalues. **(4 marks)**

Note: it is not necessary to calculate the characteristic polynomial.

(c) Find another eigenvector of the form

$$u_3 = \begin{bmatrix} 0 \\ x \\ y \\ 0 \end{bmatrix},$$

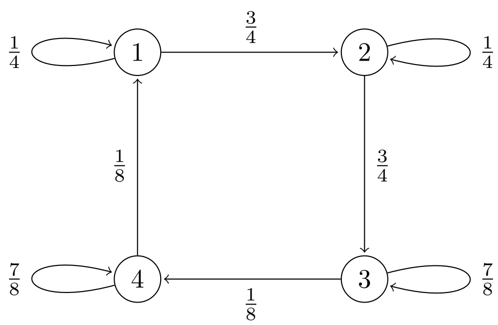
and state the corresponding eigenvalue. **(4 marks)**

(c) Using the general theory of Markov chains, write down one more eigenvalue; then find a corresponding eigenvector. **(5 marks)**

(d) Write down an eigenvector of P^T. **(2 marks)**

(e) What is the long-run average probability of being in state 3? **(3 marks)**

(2.3; 2012-13) Consider the following Markov chain:



(a) Write down the transition matrix P. **(2 marks)**

(b) Consider the following vectors:

$$u_1 = \begin{bmatrix} 1 \\ -6 \\ 6 \\ -1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 3 \\ -6 \\ 2 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 2 \\ -6 \\ 3 \end{bmatrix}$$

Show that these are all eigenvectors for P, and find the corresponding eigenvalues. **(6 marks)**

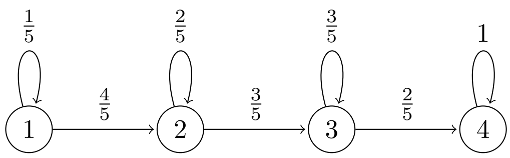
Note: it is not necessary to calculate the characteristic polynomial.

(c) Using the general theory of Markov chains, write down one more eigenvalue; then find a corresponding eigenvector. **(6 marks)**

(d) Give an invertible matrix U and a diagonal matrix D such that $P = UDU^{-1}$. Explain how this can be used to calculate P^n. **(3 marks)**

(e) What is the long run average probability of being in state 1? **(3 marks)**

(2.4; 2011-12 resit) Consider the following Markov chain:



(a) Write down the associated transition matrix P. **(2 marks)**

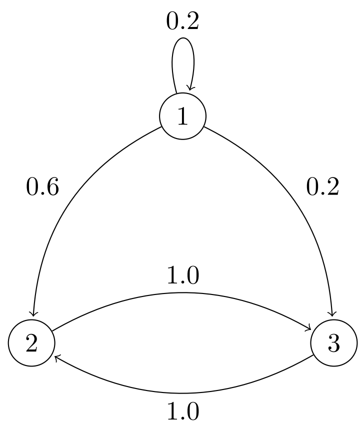
(b) Show that the following vectors are eigenvectors for P, and find the corresponding eigenvalues. **(4 marks)**

$$u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(c) Find all the other eigenvalues and eigenvectors. **(9 marks)**

(d) At time $t = 0$ the system is in state 2. What is the probability that it is in state 3 at $t = 8$? **(10 marks)**

(2.5; 2011-12) Consider the following Markov chain:

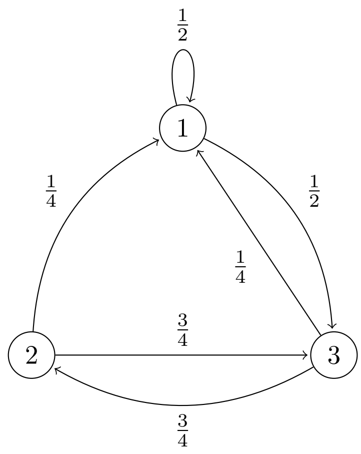


(a) Write down the associated transition matrix. **(2 marks)**

(b) Find a stationary distribution for the system. **(6 marks)**

(c) If the system is in state 1 at $t = 0$, what is the probability that it is in state 2 at $t = 4$? **(17 marks)**

(2.6; Mock 1) Consider the following Markov chain:



(a) Write down the associated transition matrix P. **(2 marks)**

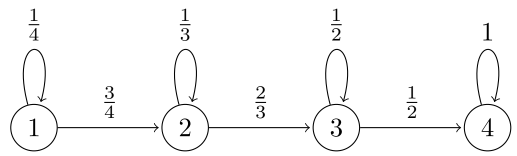
(b) You may assume that the eigenvalues of P are $\lambda_1 = 1$ and $\lambda_2 = -3/4$ and $\lambda_3 = 1/4$. Find corresponding eigenvectors u_1, u_2 and u_3. **(9 marks)**

(c) Find a stationary distribution for the system. **(2 marks)**

(d) Show that the vector $v = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ can be expressed as a linear combination of u_1 and u_2. **(4 marks)**

(e) At time $t = 0$ the system has an equal probability of being in any of the three states. What is the probability of being in state 1 at time $t = 10$? **(8 marks)**

(2.7; Mock 2) Consider the following Markov chain:



(a) Write down the associated transition matrix P. **(2 marks)**

(b) Show that the following vectors are eigenvectors for P, and find the corresponding eigenvalues. **(4 marks)**

$$u_1 = \begin{bmatrix} 1 \\ -9 \\ 24 \\ -16 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ -4 \\ 3 \end{bmatrix}$$

(c) Find all the other eigenvalues and eigenvectors. **(9 marks)**

(e) At time $t = 0$ the system is in state 1. What is the probability that it is in state 4 at $t = 5$? **(10 marks)**

Question 3

(3.1; 2013-14)

(a) Are the following statements true or false? Justify your answers. **(10 marks)**

(i) There are subspaces $V, W \leq \mathbb{R}^6$ with $\dim(V) = \dim(W) = 4$ and $\dim(V \cap W) = 1$.

(ii) There are subspaces $V, W \leq \mathbb{R}^6$ with $\dim(V) = \dim(W) = 4$ and $\dim(V \cap W) = 2$.

(iii) The following list is linearly independent:

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \begin{bmatrix}1\\2\\4\\8\end{bmatrix}, \quad \begin{bmatrix}1\\3\\9\\27\end{bmatrix}, \quad \begin{bmatrix}1\\4\\16\\81\end{bmatrix}, \quad \begin{bmatrix}1\\5\\25\\125\end{bmatrix}.$$

(iv) If A is a 3×3 matrix with only 2 distinct eigenvalues, then it cannot be diagonalised.

(v) There is a 4×4 symmetric matrix with characteristic polynomial t^4+1.

(b) Give examples of the following: **(10 marks)**

(i) A list $u_1,\dots,u_4$ of vectors in $\mathbb{R}^2$ such that u_1,u_2 is a basis and u_2,u_3 is a basis and u_3,u_4 is a basis but u_4,u_1 is not a basis.

(ii) A pair of subspaces $V,W\leq\mathbb{R}^4$ such that $\dim(V)=\dim(W)=2$ and

$$V\cap W=\{\begin{bmatrix}w & x & y & z\end{bmatrix}^T \mid w+x=x+y=y+z=0\}.$$

(iii) A non-diagonalisable 3×3 matrix whose only eigenvalue is 111.

(iv) A stochastic matrix with eigenvalues 1, 1/2 and 1/3.

(v) A 3×2 matrix of rank 1 that is not in RREF.

(3.2; 2012-13 resit)

(a) Are the following statements true or false? Justify your answers. **(10 marks)**

(i) If V and W are subspaces of $\mathbb{R}^n$, then $\dim(V+W)\geq\dim(V)+\dim(W)$.

(ii) If the list $v_1,\dots,v_5$ spans $\mathbb{R}^4$, then it is also linearly independent.

(iii) If w_1 cannot be expressed as a linear combination of w_2, w_3 and w_4, then the list w_1,w_2,w_3,w_4 is linearly independent.

(iv) If $a_1,a_2,a_3,b\in\mathbb{R}^4$ and the matrix $[a_1|a_2|a_3|b]$ row-reduces to the identity matrix, then b is a linear combination of a_1, a_2 and a_3.

(v) If M is a 3×3 symmetric matrix and u and v are vectors in $\mathbb{R}^3$ with $Mu=2u$ and $Mv=3v$, then $u.v=0$.

(b) Give examples of the following: **(10 marks)**

(i) A linearly independent list of 3 vectors in $\mathbb{R}^4$.

(ii) A pair of subspaces $V,W\leq\mathbb{R}^4$ such that $\dim(V)=\dim(W)=2$ and $V\cap W=\operatorname{span}(e_1+e_2+e_3+e_4)$.

(iii) A two-dimensional subspace $U\leq\mathbb{R}^4$ that contains the vector $\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}^T$ but not the vector $\begin{bmatrix}4 & 3 & 2 & 1\end{bmatrix}^T$.

(iv) A matrix whose characteristic polynomial is t^2+2.

(v) A 3×3 matrix of rank 2 that is not in RREF.

(3.3; 2012-13)

(a) Are the following statements true or false? Justify your answers. **(10 marks)**

(i) If V and W are subspaces of $\mathbb{R}^n$, then $\dim(V+W)\leq\dim(V)+\dim(W)$.

(ii) If the list $v_1,\dots,v_4$ spans $\mathbb{R}^4$, then it is also linearly independent.

(iii) If w_1 can be expressed as a linear combination of w_2, w_3 and w_4, then the list w_1,w_2,w_3,w_4 is linearly independent.

(iv) If $a_1,a_2,a_3,b\in\mathbb{R}^4$ and the matrix $[a_1|a_2|a_3|b]$ row-reduces to the identity matrix, then b is a linear combination of a_1, a_2 and a_3.

(v) If M is a square matrix with $M^T=M$, and u and v are vectors with $u+Mu=v-Mv=0$, then $u.v=0$.

(b) Give examples of the following: **(10 marks)**

(i) A spanning set for $\mathbb{R}^3$ that is not a basis.
(ii) A pair of subspaces $V, W \leq \mathbb{R}^4$ such that $\dim(V) = \dim(W) = 2$ and $\dim(V + W) = 3$.
(iii) A two-dimensional subspace $U \leq \mathbb{R}^4$ such that $w + x + y + z = 0$ for all vectors $\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in U$.
(iv) A non-diagonal matrix whose characteristic polynomial is $t^2 - 1$.
(v) A 2×3 matrix of rank 1 that is not in RREF.

(3.4; 2011-12 resit)

(1) Are the following statements true or false? Justify your answers carefully. **(10 marks)**

(a) Let A be a 4×4 matrix with exactly two nonzero rows. Then the rank of A is 2.
(b) No list of five vectors in $\mathbb{R}^6$ can span $\mathbb{R}^6$.
(c) The following vectors are linearly independent:

$$\begin{bmatrix} 1 \\ 7 \\ 9 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ 8 \\ 1 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 7 \\ 0 \\ 3 \\ 6 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ 1 \\ 5 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ -2 \\ 6 \\ -1 \end{bmatrix}$$

(d) Let V and W be subspaces of $\mathbb{R}^n$; then $\dim(V + W) = \dim(V) + \dim(W)$.
(e) Every subspace of $\mathbb{R}^3$ is either a line or a plane.

(2) Which of the following sets is a subspace of $\mathbb{R}^4$? Justify your answers. **(8 marks)**

$$\begin{aligned} V_1 &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid wxyz = 0\} \\ V_2 &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w^4 + z^4 = 0\} \\ V_3 &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w + x + y + z \geq 0\} \\ V_4 &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w = x = y = z\}. \end{aligned}$$

(3) Give examples of the following. **(7 marks)**

(a) A list of four vectors in $\mathbb{R}^3$ such that any two of them are linearly independent, but any three of them are linearly dependent.
(b) A linearly dependent list that spans $\mathbb{R}^2$.
(c) A 3×3 matrix that has only two distinct eigenvalues.
(d) A 2×2 matrix that is invertible but not orthogonal.

(3.5; 2011-12)

(1) Are the following statements true or false? Justify your answers carefully. **(9 marks)**

(a) Any list of four vectors in $\mathbb{R}^3$ spans $\mathbb{R}^3$.
(b) There exists a linearly dependent list of vectors that spans $\mathbb{R}^3$.
(c) The following vectors are linearly independent:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 3 \\ 3 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

(d) The following vectors form a basis of $\mathbb{R}^3$:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 10 \\ 100 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 11 \\ 101 \end{bmatrix}$$

(2) Which of the following sets is a subspace of $\mathbb{R}^4$? Justify your answers. **(9 marks)**

$$\begin{aligned}
V_1 &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w+x+y+z=0\} \\
V_2 &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w^2+x^2+y^2+z^2=0\} \\
V_3 &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w^3+x^3+y^3+z^3=0\} \\
V_4 &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w+x+y+z=1\}.
\end{aligned}$$

(3) Give examples of the following. **(7 marks)**

(a) A list of 4 vectors in $\mathbb{R}^3$ such that any three of them form a basis.

(b) A pair of subspaces $V, W \leq \mathbb{R}^6$ with $\dim(V) = \dim(W) = 3$ and $\dim(V+W) = 4$.

(c) A list of three subspaces $P, Q, R \leq \mathbb{R}^3$ such that $\dim(P) = \dim(Q) = \dim(R) = 2$ and $\dim(P \cap Q \cap R) = 1$.

(3.6; Mock 1)

(a) Are the following statements true or false? Justify your answers. **(9 marks)**

(i) No list of four vectors in $\mathbb{R}^5$ can span $\mathbb{R}^5$.

(ii) Every linearly independent list is a basis.

(iii) Let u and v be eigenvectors for a square matrix A, with different eigenvalues; then u and v are orthogonal.

(iv) Let V and W be subspaces of $\mathbb{R}^n$ with $V \cap W = \{0\}$; then $\dim(V+W) = \dim(V) + \dim(W)$.

(b) Which of the following sets is a subspace of $\mathbb{R}^4$? Justify your answers. **(9 marks)**

$$\begin{aligned}
V_1 &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w^2-x^2+y^2-z^2=0\} \\
V_2 &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid 2w-3x+4y-5z=0\} \\
V_3 &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid 2w-3x+4y-5z=1\} \\
V_4 &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w+x^2+y^3+z^4=0\}.
\end{aligned}$$

(c) Give examples of the following. **(7 marks)**

(i) A list of four vectors in $\mathbb{R}^2$ such that any two of them form a basis.

(ii) A list of four vectors in $\mathbb{R}^4$ such that any two of them are linearly independent, but any three of them are linearly dependent.

(iii) A 3×3 matrix that has only two distinct eigenvalues.

(3.7; Mock 2)

(a) Are the following statements true or false? You do not need to justify your answers. **(9 marks)**

(i) Let A be a 3×5 matrix with linearly independent rows. Then the rank of A is 5.

(ii) Let A be a 5×5 matrix with $A^T = A$, and let u and v be vectors with $Au = 2u$ and $Av = 3v$. Then $u.v = 0$.

(iii) Let U be a 4×4 matrix whose columns are all orthogonal to each other. Then $U^TU = I$.

(iv) The list $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ is linearly independent.

(v) The list $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ spans $\mathbb{R}^4$.

(vi) Let A be a 5×5 matrix with $\text{rank}(A) = 5$. Then A is invertible.

(vii) Let $u_1, u_2, u_3, u_4, u_5, u_6$ be a linearly independent list of vectors in $\mathbb{R}^6$. Then the list u_1, u_3, u_5 is also linearly independent.

(viii) Let $u_1, u_2, u_3, u_4, u_5, u_6$ be a list of vectors that spans $\mathbb{R}^6$. Then the list u_1, u_3, u_5 also spans $\mathbb{R}^6$.

(ix) Every subspace of $\mathbb{R}^3$ is either a line or a plane.

(b) Which of the following sets is a subspace of $\mathbb{R}^4$? Here you do need to justify your answers. **(8 marks)**

$$\begin{aligned} V_1 &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w \geq x\} \\ V_2 &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w = x = y = z\} \\ V_3 &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w + x + y + z \text{ is an integer } \} \\ V_4 &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid wxyz = 0\}. \end{aligned}$$

(c) Give examples of the following. **(8 marks)**

(i) A linearly dependent list that spans $\mathbb{R}^2$.

(ii) A 2×2 matrix that is invertible but not orthogonal.

(iii) A subspace of $\mathbb{R}^3$ that contains the vector $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$ and has dimension 2.

(iv) A 4×4 matrix A such that $A \neq I_4$ and $\chi_A(t) = (t-1)^4$.

Question 4

(4.1; 2013-14) Consider the vectors

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad a_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \quad v_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad c_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \quad c_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

and the subspaces

$$U = \operatorname{ann}(a_1, a_2) \qquad V = \operatorname{span}(v_1, v_2) \qquad W = \operatorname{ann}(c_1, c_2)$$

in $\mathbb{R}^4$.

(a) Find the canonical basis for U. **(4 marks)**

(b) Find the canonical basis for V. **(3 marks)**

(c) Find the canonical basis for W. **(4 marks)**

(d) Find the canonical basis for $U \cap V \cap W$. **(5 marks)**

(e) Find the canonical basis for $U + V + W$. **(4 marks)**

(4.2; 2012-13 resit) Consider the vectors

$$a_1 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ -1 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ -3 \end{bmatrix} \qquad b_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \qquad b_2 = \begin{bmatrix} -8 \\ 5 \\ 4 \\ 2 \end{bmatrix},$$

and put $V = \operatorname{ann}(a_1, a_2)$, and $W = \operatorname{ann}(b_1, b_2)$.

(a) Find the canonical basis for V. **(4 marks)**

(b) Find the canonical basis for W. **(4 marks)**

(c) Find the canonical basis for $V \cap W$. **(5 marks)**

(d) Find the canonical basis for $V+W$. **(5 marks)**

(e) State the standard equation relating the dimensions of the above four spaces, and verify that it holds in this case. **(2 marks)**

(4.3; 2012-13) Consider the vectors

$$a_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -2 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ -1 \end{bmatrix} \qquad b_1 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix} \qquad b_2 = \begin{bmatrix} -4 \\ 2 \\ -4 \\ 8 \end{bmatrix},$$

and put $V = \operatorname{ann}(a_1, a_2)$, and $W = \operatorname{ann}(b_1, b_2)$.

(a) Find the canonical basis for V. **(4 marks)**

(b) Find the canonical basis for W. **(4 marks)**

(c) Find the canonical basis for $V \cap W$. **(5 marks)**

(d) Find the canonical basis for $V+W$. **(5 marks)**

(e) Find a vector that lies in $V+W$ but does not lie in V or in W. **(2 marks)**

(4.4; 2011-12 resit) Put

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 0 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 2 \\ 10 \\ 5 \end{bmatrix}$$

and $V = \operatorname{span}(v_1, v_2)$ and $W = \operatorname{ann}(u_1, u_2)$.

(a) Find the canonical basis for V. **(4 marks)**

(b) Find the canonical basis for W. **(6 marks)**

(c) Find the canonical basis for $V+W$. **(5 marks)**

(d) Find vectors c_1 and c_2 such that $V = \operatorname{ann}(c_1, c_2)$. **(5 marks)**

(e) Find the canonical basis for $V \cap W$. **(5 marks)**

(4.5; 2011-12) Put

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 0 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

and $V = \operatorname{span}(v_1, v_2)$ and $W = \operatorname{ann}(u_1, u_2)$.

(a) Find the canonical basis for V. **(4 marks)**

(b) Find the canonical basis for W. **(6 marks)**

(c) Find the canonical basis for $V+W$. **(5 marks)**

(d) Find vectors c_1 and c_2 such that $V = \operatorname{ann}(c_1, c_2)$. **(5 marks)**

(e) Find the canonical basis for $V \cap W$. **(5 marks)**

(4.6; Mock 1) Let V be the set of all vectors in $\mathbb{R}^5$ of the form

$$x = \begin{bmatrix} p & p+q & 0 & -p-q & -p \end{bmatrix}^T$$

(where $p, q \in \mathbb{R}$ are arbitrary). Also, put

$$W = \{x \in \mathbb{R}^5 \mid x_1 + x_2 + x_3 + 3x_4 = x_1 + x_2 + x_3 - x_4 + 2x_5 = 0\}.$$

(a) Find the canonical basis for V. **(5 marks)**

(b) Find the canonical basis for W. **(6 marks)**

(c) Find the canonical basis for $V + W$. **(5 marks)**

(d) Write down a formula relating the dimensions of V, W, $V + W$ and $V \cap W$, and use it to determine $\dim(V \cap W)$. **(3 marks)**

(e) Find a basis for $V \cap W$ **(6 marks)**.

(4.7; Mock 2) Consider the vectors

$$a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 2 \\ 4 \\ -2 \\ -6 \end{bmatrix} \qquad a_3 = \begin{bmatrix} 3 \\ 6 \\ 0 \\ -6 \end{bmatrix} \qquad p = \begin{bmatrix} 11 \\ 22 \\ 22 \\ 0 \end{bmatrix} \qquad q = \begin{bmatrix} 222 \\ -111 \\ 0 \\ 0 \end{bmatrix}$$

Put $V = \mathrm{span}(a_1, a_2, a_3)$ and $W = \mathrm{ann}(a_1, a_2, a_3)$.

(a) Find the canonical basis for V. **(4 marks)**

(b) Find the canonical basis for W. **(8 marks)**

(c) Find the canonical basis for $V + W$. **(5 marks)**

(d) Use the dimension formula to determine $V \cap W$. **(4 marks)**

(e) Prove that $p \in V$ and $q \in W$. **(4 marks)**

Question 5

(5.1; 2013-14) Consider the matrix $A = \frac{1}{27}\begin{bmatrix} 9 & 8 & -8 \\ 8 & 23 & 0 \\ -8 & 0 & -5 \end{bmatrix}$.

(a) State the main results about eigenvalues and eigenvectors of symmetric matrices. **(4 marks)**

(b) Show that the following are eigenvectors of A: **(2 marks)**

$$u_1 = \begin{bmatrix} 7 \\ -4 \\ -4 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}.$$

(c) Find an orthogonal matrix U and a diagonal matrix D such that $A = UDU^T$. **(9 marks)**
Hint: For any square matrix, the sum of the eigenvalues is the same as the sum of the diagonal entries. Because of this, you do not need to calculate the characteristic polynomial.

(d) Find $\lim_{n\to\infty} A^n$. **(5 marks)**

(5.2; 2012-13 resit) Consider the matrix $M = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

(a) Find the eigenvalues and eigenvectors of M. To simplify the calculations you may use the following fact: for this particular matrix, if $\begin{bmatrix} w & x & y & z \end{bmatrix}^T$ is an eigenvector of eigenvalue λ, then $\begin{bmatrix} w & -x & y & -z \end{bmatrix}^T$ is an eigenvector of eigenvalue $-\lambda$. **(14 marks)**

(b) Find an orthogonal matrix P and a diagonal matrix D such that $M = PDP^T$. **(6 marks)**

(5.3; 2012-13) Consider the matrix $M = \begin{bmatrix} 8 & 2 & 2 \\ 2 & -4 & 5 \\ 2 & 5 & -4 \end{bmatrix}$.

(a) Find the eigenvalues and eigenvectors of M. **(14 marks)**

(b) Find an orthogonal matrix P and a diagonal matrix D such that $M = PDP^T$. **(6 marks)**

(5.4; 2011-12 resit) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 1 & 1 & 4 \\ 1 & 1 & 1 & 4 \\ 4 & 4 & 4 & 1 \end{bmatrix}.$$

You may assume that $\det(tI - A) = t^4 - 4t^3 - 45t^2$.

(a) Find the eigenvalues of A. **(2 marks)**

(b) Find an eigenvector u_1 of eigenvalue 0 with $\|u_1\| = 1$. **(3 marks)**

(c) Find another eigenvector u_2 of eigenvalue 0 with $u_1.u_2 = 0$ and $\|u_2\| = 1$. **(4 marks)**

(d) Find an orthonormal basis of $\mathbb{R}^4$ consisting of eigenvectors of A. **(10 marks)**

(e) Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$. **(3 marks)**

(f) Express the quadratic form

$$Q = w^2 + x^2 + y^2 + z^2 + 2(wx + wy + xy) + 8(wz + xz + yz)$$

as $Q = F^2 - G^2$, where F and G are linear forms. **(3 marks)**

(5.5; 2011-12) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix}.$$

You may assume that $\det(A - tI) = t^4 - 4t^3 - 12t^2$.

(a) Find the eigenvalues of A. **(2 marks)**

(b) Find an orthonormal basis of $\mathbb{R}^4$ consisting of eigenvectors of A. **(14 marks)**

(c) Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$. **(4 marks)**

(d) Express the quadratic form

$$Q = w^2 + x^2 + y^2 + z^2 + 2(wx + yz) + 4(wy + wz + xy + xz)$$

as $Q = F^2 - G^2$, where F and G are linear forms. Hence express Q as a product of two linear forms. **(5 marks)**

(5.6; Mock 1) Put

$$A = \begin{bmatrix} 9 & 6 & 2 & 3 \\ 6 & 0 & 0 & 2 \\ 2 & 0 & 0 & 6 \\ 3 & 2 & 6 & 9 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ -2 \\ -2 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

(a) Show that $\det(A) = 1024 = 2^{10}$. **(4 marks)**

(b) Show that the vectors u_i are eigenvectors for A, and determine the corresponding eigenvalues. **(6 marks)** **Note:** You do not need to find the characteristic polynomial or perform any row-reduction.

(c) Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$. **(6 marks)**

(d) Express the quadratic form

$$Q = 9(w^2 + z^2) + 12(wx + yz) + 6wz + 4(wy + xz)$$

as $Q = F^2 + G^2 - H^2 - J^2$, where F, G, H and J are linear forms. **(6 marks)**

(e) What are the rank and signature of Q? **(3 marks)**

(5.7; Mock 2) Put

$$A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & 2 & 3 \\ 7 & 3 & 2 \end{bmatrix}.$$

You may assume the following row-reductions. (Some of them are useful, and some of them are not.)

$$\begin{bmatrix} 2 & 3 & 7 \\ 3 & 2 & 3 \\ 7 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 7 & 3 & 7 \\ 3 & 7 & 3 \\ 7 & 3 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 9 & 3 & 7 \\ 3 & 9 & 3 \\ 7 & 3 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 3 & 7 \\ 3 & -2 & 3 \\ 7 & 3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -7 & 3 & 7 \\ 3 & -7 & 3 \\ 7 & 3 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -9 & 3 & 7 \\ 3 & -9 & 3 \\ 7 & 3 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Find the eigenvalues of A. **(6 marks)**

(b) Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$. **(10 marks)**

(c) Express the quadratic form

$$Q = 2x^2 + 2y^2 + 2z^2 + 6xy + 6yz + 14xz$$

as $Q = L^2 - M^2$, where L and M are linear forms. Hence find linear forms F and G such that $Q = FG$. **(7 marks)**

(d) What are the rank and signature of Q? **(2 marks)**