Linear mathematics for applications

Lecture 1

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- A few applications will be treated in more detail: solution of difference equations; solution of differential equations; long-term behaviour of random systems known as Markov chains.

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- $\mathbb{R}^n$ is the set of column vectors with n entries, so

$$\begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} \in \mathbb{R}^3 \qquad \begin{bmatrix} \pi \\ \pi^2 \\ \pi^3 \\ \pi^4 \end{bmatrix} \in \mathbb{R}^4 \qquad \begin{bmatrix} 12.38 \\ -9.14 \end{bmatrix} \in \mathbb{R}^2.$$

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- $M_{m\times n}(\mathbb{R})$ is the set of all $m \times n$ matrices (with m rows and n columns, ie height m and width n)

$$\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix} \in M_{2\times 3}(\mathbb{R}) \qquad \begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix} \in M_{3\times 2}(\mathbb{R})$$

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- $M_n(\mathbb{R}) = M_{n\times n}(\mathbb{R})$ is the set of all $n \times n$ square matrices. I_n is the $n \times n$ identity matrix.

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 3 & 3 & 2 \\ 2 & 3 & 3 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix} \in M_4(\mathbb{R}) \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in M_4(\mathbb{R})$$

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- The *transpose* of an $m \times n$ matrix A is the $n \times n$ matrix A^T obtained by flipping A over, so the (i, j)'th entry in A^T is the same as the (j, i)'th entry in A. For example, we have

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}^T = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{bmatrix}.$$

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- We will typically write column vectors in this way when it is convenient to lay things out horizontally.

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For example: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} . \begin{bmatrix} 1000 \\ 100 \\ 10 \\ 1 \end{bmatrix} = 1000 + 200 + 30 + 4 = 1234.$

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$$u_1 = \begin{bmatrix} a \\ d \end{bmatrix} \qquad u_2 = \begin{bmatrix} b \\ e \end{bmatrix} \qquad u_3 = \begin{bmatrix} c \\ f \end{bmatrix} \qquad t_1 = x \qquad t_2 = y \qquad t_3 = z$$

so

$$t_1u_1 + t_2u_2 + t_3u_3 = x\begin{bmatrix} a \\ d \end{bmatrix} + y\begin{bmatrix} b \\ e \end{bmatrix} + z\begin{bmatrix} c \\ f \end{bmatrix} = \begin{bmatrix} ax+by+cz \\ dx+ey+fz \end{bmatrix} = At$$

Product of a matrix and a vector

In the example

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax + by + cz \\ dx + ey + fz \end{bmatrix}$$

$$(2 \times 3 \text{ matrix})(\text{vector in } \mathbb{R}^3) = (\text{vector in } \mathbb{R}^2)$$

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 10 & 10 & 10 \\ 100 & 100 & 100 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 20 & 20 & 20 \\ 300 & 300 & 300 \end{bmatrix}$$

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So our system of equations is equivalent to the single matrix equation

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ 100 \end{bmatrix}.$$

Matrices and linear equations

Systems of linear equations can be rewritten as matrix equations.

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The *augmented matrix* for an equation $Au = v$ is $[A|v]$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 1 & 2 & 4 & 20 \\ 1 & 3 & 9 & 30 \\ 1 & 4 & 16 & 40 \\ 1 & 5 & 25 & 50 \end{array}\right]$$

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Sometimes we need to tidy up first.

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$$\begin{array}{rrrrr} p & -q & +0r & +7s & = 1 \\ p & -7q & +5r & +0s & = -1 \\ p & +q & -r & -s & = 0 \end{array}$$

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$$
\begin{aligned}
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\end{aligned}
\qquad
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p & -q & +0r & +7s & = 1 \\
p & -7q & +5r & +0s & = -1 \\
p & +q & -r & -s & = 0
\end{array}
$$

$$
\begin{bmatrix} 1 & -1 & 0 & 7 \\ 1 & -7 & 5 & 0 \\ 1 & 1 & -1 & -1 \end{bmatrix}
\begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}
=
\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.
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Tidying

Sometimes we need to tidy up first.

$$\begin{aligned} p+7s &= q+1 \\ 5r+1 &= 7q-p \\ r+s &= p+q \end{aligned}$$

$$\begin{array}{ccccr} p & -q & +0r & +7s & =1 \\ p & -7q & +5r & +0s & =-1 \\ p & +q & -r & -s & =0 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 0 & 7 \\ 1 & -7 & 5 & 0 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 7 & 1 \\ 1 & -7 & 5 & 0 & -1 \\ 1 & 1 & -1 & -1 & 0 \end{array}\right]$$

Lecture 2

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If RREF0, RREF1 and RREF2 are satisfied but not RREF3 then we say that A is in (unreduced) row-echelon form.

(Reduced) row-echelon form

Example 5.2:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & \mathbf{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \mathbf{1} & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & \mathbf{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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$$E = \begin{bmatrix} \mathbf{1} & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & \mathbf{1} & 5 & 0 & 6 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in RREF.

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$$\begin{aligned} x - z &= 1 \\ y &= 2 \end{aligned}$$

is in RREF because its augmented matrix is in RREF:

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The system of equations

$$\begin{aligned} x + y + z &= 1 \\ y + z &= 2 \\ z &= 3 \end{aligned}$$

is not in RREF because its augmented matrix is not in RREF:

$$B = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

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If a system of equations is in RREF, it can be solved very easily.

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Sometimes it is convenient to introduce new letters for the independent variables, say λ and μ. Then the solution is

$$\begin{aligned} w &= 10 - 2\lambda - 3\mu \\ x &= \lambda \\ y &= 20 - 4\mu \\ z &= \mu \end{aligned}$$

where λ and μ can take arbitrary values.

Solving RREF systems — degenerate cases

The augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 11 \\ 0 & 0 & 1 & 0 & 12 \\ 0 & 0 & 0 & 1 & 13 \end{array}\right]$$

has a pivot in every column to the left of the bar, so there are no independent variables.

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$$\begin{aligned} w + z &= 0 \qquad & x + y &= 0 \\ 0 &= 1 \qquad & 0 &= 0 \end{aligned}$$

so there is clearly no solution.

Row operations

Let A be a matrix. The following operations on A are called *elementary row operations*:

ERO1: Exchange two rows.

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Let A be a matrix.

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(g) If we put the first row back in, we have a matrix that is nearly in RREF, except that the first row may have nonzero entries above the pivots in the lower rows. This can easily be fixed by subtracting multiples of those lower rows.

Row reduction example

Consider the following sequence of reductions:

$$\begin{bmatrix} 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & -1 & 1 & -2 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & -1 & 1 & -2 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} -1 & -2 & -1 & 1 & -2 \\ 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \xrightarrow{3}$$

$$\begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & -2 & -1 & -13 \\ 0 & 0 & 1 & -2 & -6 \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1/2 & 13/2 \\ 0 & 0 & 1 & -2 & -6 \end{bmatrix} \xrightarrow{5} \begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1/2 & 13/2 \\ 0 & 0 & 0 & -5/2 & -25/2 \end{bmatrix} \xrightarrow{6}$$

$$\begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1/2 & 13/2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

Exchange the first two rows; Multiply the first row by -1; Add the first row to the third row; Divide the second row by -2; Subtract the second row from the third; Multiplying the third row by $-2/5$;

Row reduction example

Consider the following sequence of reductions:

$$\begin{bmatrix} 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & -1 & 1 & -2 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} -1 & -2 & -1 & 1 & -2 \\ 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \xrightarrow{3}$$

$$\begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & -2 & -1 & -13 \\ 0 & 0 & 1 & -2 & -6 \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1/2 & 13/2 \\ 0 & 0 & 1 & -2 & -6 \end{bmatrix} \xrightarrow{5} \begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1/2 & 13/2 \\ 0 & 0 & 0 & -5/2 & -25/2 \end{bmatrix} \xrightarrow{6}$$

$$\begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1/2 & 13/2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \xrightarrow{7} \begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

Exchange the first two rows; Multiply the first row by -1; Add the first row to the third row; Divide the second row by -2; Subtract the second row from the third; Multiplying the third row by $-2/5$; Subtract half the bottom row from the middle row;

Row reduction example

Consider the following sequence of reductions:

$$\begin{bmatrix} 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & -1 & 1 & -2 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} -1 & -2 & -1 & 1 & -2 \\ 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \xrightarrow{3}$$

$$\begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & -2 & -1 & -13 \\ 0 & 0 & 1 & -2 & -6 \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1/2 & 13/2 \\ 0 & 0 & 1 & -2 & -6 \end{bmatrix} \xrightarrow{5} \begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1/2 & 13/2 \\ 0 & 0 & 0 & -5/2 & -25/2 \end{bmatrix} \xrightarrow{6}$$

$$\begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1/2 & 13/2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \xrightarrow{7} \begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \xrightarrow{8} \begin{bmatrix} 1 & 2 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

Exchange the first two rows; Multiply the first row by -1; Add the first row to the third row; Divide the second row by -2; Subtract the second row from the third; Multiplying the third row by $-2/5$; Subtract half the bottom row from the middle row; Subtract the middle row from the top row;

Row reduction example

Consider the following sequence of reductions:

$$\begin{bmatrix} 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & -1 & 1 & -2 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} -1 & -2 & -1 & 1 & -2 \\ 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \xrightarrow{3}$$

$$\begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & -2 & -1 & -13 \\ 0 & 0 & 1 & -2 & -6 \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1/2 & 13/2 \\ 0 & 0 & 1 & -2 & -6 \end{bmatrix} \xrightarrow{5} \begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1/2 & 13/2 \\ 0 & 0 & 0 & -5/2 & -25/2 \end{bmatrix} \xrightarrow{6}$$

$$\begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1/2 & 13/2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \xrightarrow{7} \begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \xrightarrow{8} \begin{bmatrix} 1 & 2 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \xrightarrow{9} \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

Exchange the first two rows; Multiply the first row by -1; Add the first row to the third row; Divide the second row by -2; Subtract the second row from the third; Multiplying the third row by $-2/5$; Subtract half the bottom row from the middle row; Subtract the middle row from the top row; Add the bottom row to the top row.

Row reduction example

Consider the following sequence of reductions:

$$C = \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ -1 & -1 & 3 & 0 & 1 & 3 \\ 1 & 2 & 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & 4 & 5 & 4 \\ 1 & 2 & 1 & 7 & 6 & 8 \end{bmatrix}$$

Row reduction example

Consider the following sequence of reductions:

$$C = \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ -1 & -1 & 3 & 0 & 1 & 3 \\ 1 & 2 & 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & 4 & 5 & 4 \\ 1 & 2 & 1 & 7 & 6 & 8 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & -2 & -2 & 1 \\ 0 & 1 & -3 & 7 & 7 & 4 \\ 0 & 0 & 4 & 4 & 4 & 8 \end{bmatrix}$$

Add row 1 to rows 2 and 4, and subtract it from rows 3 and 5;

Row reduction example

Consider the following sequence of reductions:

$$C = \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ -1 & -1 & 3 & 0 & 1 & 3 \\ 1 & 2 & 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & 4 & 5 & 4 \\ 1 & 2 & 1 & 7 & 6 & 8 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & -2 & -2 & 1 \\ 0 & 1 & -3 & 7 & 7 & 4 \\ 0 & 0 & 4 & 4 & 4 & 8 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & -2 & -2 & 1 \\ 0 & 0 & -3 & 4 & 4 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{bmatrix}$$

Add row 1 to rows 2 and 4, and subtract it from rows 3 and 5; Subtract row 2 from row 4;

Row reduction example

Consider the following sequence of reductions:

$$C = \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ -1 & -1 & 3 & 0 & 1 & 3 \\ 1 & 2 & 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & 4 & 5 & 4 \\ 1 & 2 & 1 & 7 & 6 & 8 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & -2 & -2 & 1 \\ 0 & 1 & -3 & 7 & 7 & 4 \\ 0 & 0 & 4 & 4 & 4 & 8 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & -2 & -2 & 1 \\ 0 & 0 & -3 & 4 & 4 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{3}$$

$$\begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & -3 & 4 & 4 & 1 \\ 0 & 0 & 3 & -2 & -2 & 1 \end{bmatrix}$$

Add row 1 to rows 2 and 4, and subtract it from rows 3 and 5; Subtract row 2 from row 4; Exchange rows 3 and 5;

Row reduction example

Consider the following sequence of reductions:

$$C = \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ -1 & -1 & 3 & 0 & 1 & 3 \\ 1 & 2 & 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & 4 & 5 & 4 \\ 1 & 2 & 1 & 7 & 6 & 8 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & -2 & -2 & 1 \\ 0 & 1 & -3 & 7 & 7 & 4 \\ 0 & 0 & 4 & 4 & 4 & 8 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & -2 & -2 & 1 \\ 0 & 0 & -3 & 4 & 4 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{3}$$

$$\begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & -3 & 4 & 4 & 1 \\ 0 & 0 & 3 & -2 & -2 & 1 \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 7 & 7 & 7 \\ 0 & 0 & 0 & -5 & -5 & -5 \end{bmatrix}$$

Add row 1 to rows 2 and 4, and subtract it from rows 3 and 5; Subtract row 2 from row 4; Exchange rows 3 and 5; Add 3 times row 3 to row 4, and subtract 3 times row 3 from row 5;

Row reduction example

Consider the following sequence of reductions:

$$C = \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ -1 & -1 & 3 & 0 & 1 & 3 \\ 1 & 2 & 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & 4 & 5 & 4 \\ 1 & 2 & 1 & 7 & 6 & 8 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & -2 & -2 & 1 \\ 0 & 1 & -3 & 7 & 7 & 4 \\ 0 & 0 & 4 & 4 & 4 & 8 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & -2 & -2 & 1 \\ 0 & 0 & -3 & 4 & 4 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{3}$$

$$\begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & -3 & 4 & 4 & 1 \\ 0 & 0 & 3 & -2 & -2 & 1 \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 7 & 7 & 7 \\ 0 & 0 & 0 & -5 & -5 & -5 \end{bmatrix} \xrightarrow{5} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Add row 1 to rows 2 and 4, and subtract it from rows 3 and 5; Subtract row 2 from row 4; Exchange rows 3 and 5; Add 3 times row 3 to row 4, and subtract 3 times row 3 from row 5; Divide row 4 by 7, then add 5 times row 4 to row 5;

Row reduction example

Consider the following sequence of reductions:

$$C = \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ -1 & -1 & 3 & 0 & 1 & 3 \\ 1 & 2 & 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & 4 & 5 & 4 \\ 1 & 2 & 1 & 7 & 6 & 8 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & -2 & -2 & 1 \\ 0 & 1 & -3 & 7 & 7 & 4 \\ 0 & 0 & 4 & 4 & 4 & 8 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & -2 & -2 & 1 \\ 0 & 0 & -3 & 4 & 4 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{3}$$

$$\begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & -3 & 4 & 4 & 1 \\ 0 & 0 & 3 & -2 & -2 & 1 \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 7 & 7 & 7 \\ 0 & 0 & 0 & -5 & -5 & -5 \end{bmatrix} \xrightarrow{5} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{6}$$

$$\begin{bmatrix} 1 & 0 & -3 & -3 & -4 & -6 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Add row 1 to rows 2 and 4, and subtract it from rows 3 and 5; Subtract row 2 from row 4; Exchange rows 3 and 5; Add 3 times row 3 to row 4, and subtract 3 times row 3 from row 5; Divide row 4 by 7, then add 5 times row 4 to row 5; Subtract 2 times row 2 from row 1;

Row reduction example

Consider the following sequence of reductions:

$$
C = \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ -1 & -1 & 3 & 0 & 1 & 3 \\ 1 & 2 & 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & 4 & 5 & 4 \\ 1 & 2 & 1 & 7 & 6 & 8 \end{bmatrix}
\xrightarrow{1}
\begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & -2 & -2 & 1 \\ 0 & 1 & -3 & 7 & 7 & 4 \\ 0 & 0 & 4 & 4 & 4 & 8 \end{bmatrix}
\xrightarrow{2}
\begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & -2 & -2 & 1 \\ 0 & 0 & -3 & 4 & 4 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{bmatrix}
\xrightarrow{3}
$$

$$
\begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & -3 & 4 & 4 & 1 \\ 0 & 0 & 3 & -2 & -2 & 1 \end{bmatrix}
\xrightarrow{4}
\begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 7 & 7 & 7 \\ 0 & 0 & 0 & -5 & -5 & -5 \end{bmatrix}
\xrightarrow{5}
\begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\xrightarrow{6}
$$

$$
\begin{bmatrix} 1 & 0 & -3 & -3 & -4 & -6 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\xrightarrow{7}
\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

Add row 1 to rows 2 and 4, and subtract it from rows 3 and 5; Subtract row 2 from row 4; Exchange rows 3 and 5; Add 3 times row 3 to row 4, and subtract 3 times row 3 from row 5; Divide row 4 by 7, then add 5 times row 4 to row 5; Subtract 2 times row 2 from row 1; Add 3 times row 3 to row 1;

Row reduction example

Consider the following sequence of reductions:

$$C = \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ -1 & -1 & 3 & 0 & 1 & 3 \\ 1 & 2 & 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & 4 & 5 & 4 \\ 1 & 2 & 1 & 7 & 6 & 8 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & -2 & -2 & 1 \\ 0 & 1 & -3 & 7 & 7 & 4 \\ 0 & 0 & 4 & 4 & 4 & 8 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & -2 & -2 & 1 \\ 0 & 0 & -3 & 4 & 4 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{3}$$

$$\begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & -3 & 4 & 4 & 1 \\ 0 & 0 & 3 & -2 & -2 & 1 \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 7 & 7 & 7 \\ 0 & 0 & 0 & -5 & -5 & -5 \end{bmatrix} \xrightarrow{5} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{6}$$

$$\begin{bmatrix} 1 & 0 & -3 & -3 & -4 & -6 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{7} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{8} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Add row 1 to rows 2 and 4, and subtract it from rows 3 and 5; Subtract row 2 from row 4; Exchange rows 3 and 5; Add 3 times row 3 to row 4, and subtract 3 times row 3 from row 5; Divide row 4 by 7, then add 5 times row 4 to row 5; Subtract 2 times row 2 from row 1; Add 3 times row 3 to row 1; Subtract 3 times row 4 from row 2, and subtract row 4 from row 3.

Deleting columns

We previously saw the following row-reduction:

$$\begin{bmatrix} 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & -1 & 1 & -2 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \to \begin{bmatrix} -1 & -2 & -1 & 1 & -2 \\ 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

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We previously saw the following row-reduction:

$$\begin{bmatrix} 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & -1 & 1 & -2 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \to \begin{bmatrix} -1 & -2 & -1 & 1 & -2 \\ 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

We can delete the middle column and it still works the same way:

$$\begin{bmatrix} 0 & 0 & & -1 & -13 \\ -1 & -2 & & 1 & -2 \\ -1 & -2 & & -1 & -8 \end{bmatrix} \to \begin{bmatrix} -1 & -2 & & 1 & -2 \\ 0 & 0 & & -1 & -13 \\ -1 & -2 & & -1 & -8 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 2 & & 0 & 3 \\ 0 & 0 & & 0 & 4 \\ 0 & 0 & & 1 & 5 \end{bmatrix}$$

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(However, the final result is no longer in RREF; we need further row operations to fix that.)

Deleting columns

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$$\begin{bmatrix} 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & -1 & 1 & -2 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \to \begin{bmatrix} -1 & -2 & -1 & 1 & -2 \\ 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

We can delete the middle column and it still works the same way:

$$\begin{bmatrix} 0 & 0 & -1 & -13 \\ -1 & -2 & 1 & -2 \\ -1 & -2 & -1 & -8 \end{bmatrix} \to \begin{bmatrix} -1 & -2 & 1 & -2 \\ 0 & 0 & -1 & -13 \\ -1 & -2 & -1 & -8 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

(However, the final result is no longer in RREF; we need further row operations to fix that.)

In general: suppose that $A \to A'$, and that B is obtained by deleting some columns from A, and that B' is obtained by deleting the corresponding columns from A'. Then $B \to B'$.

Solution by row-reduction

Theorem 6.8: Let A be an augmented matrix, and let A' be obtained from A by a sequence of row operations.

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This should be fairly clear.

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This should be fairly clear. The three types of elementary row operations correspond to reordering our system of equations, multiplying both sides of one equation by a nonzero constant, and adding one equation to another one.

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Theorem 6.8: Let A be an augmented matrix, and let A' be obtained from A by a sequence of row operations. Then the system of equations corresponding to A has the same solutions (if any) as the system of equations corresponding to A'.

This should be fairly clear. The three types of elementary row operations correspond to reordering our system of equations, multiplying both sides of one equation by a nonzero constant, and adding one equation to another one. None of these operations changes the solution set.

Theorem 6.8: Let A be an augmented matrix, and let A' be obtained from A by a sequence of row operations. Then the system of equations corresponding to A has the same solutions (if any) as the system of equations corresponding to A'.

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(d) Read off the solutions (which is easy for a system in RREF).

Example solution by row-reduction

We will try to solve the following system:

$$\begin{array}{rcrcrcr} 2x & + & y & + & z & = & 1 \\ 4x & + & 2y & + & 3z & = & -1 \\ 6x & + & 3y & - & z & = & 11 \end{array}$$

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There is a pivot in the rightmost column, which means that there are no solutions for the original system.

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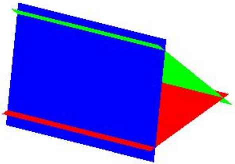
There is a pivot in the rightmost column, which means that there are no solutions for the original system.

Each of the equations defines a plane. These are arranged like the three faces of a Toblerone packet, so there is no point where they all meet.



Example solution by row-reduction

We will solve the equations

$$\begin{aligned} a + b + c + d &= 4 \\ a + b - c - d &= 0 \\ a - b + c - d &= 0 \\ a - b - c + d &= 0. \end{aligned}$$

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$$\begin{aligned} a+b+c+d &= 4 \\ a+b-c-d &= 0 \\ a-b+c-d &= 0 \\ a-b-c+d &= 0. \end{aligned}$$

The corresponding augmented matrix can be row-reduced as follows:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 1 & -1 & -1 & 1 & 0 \end{array}\right]$$

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Subtract row 1 from row 2, and row 3 from row 4;

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The corresponding augmented matrix can be row-reduced as follows:

$$\left[\begin{array}{cccc|c} 1&1&1&1&4\\ 1&1&-1&-1&0\\ 1&-1&1&-1&0\\ 1&-1&-1&1&0 \end{array}\right] \xrightarrow{1} \left[\begin{array}{cccc|c} 1&1&1&1&4\\ 0&0&-2&-2&-4\\ 1&-1&1&-1&0\\ 0&0&-2&2&0 \end{array}\right] \xrightarrow{2} \left[\begin{array}{cccc|c} 1&1&1&1&4\\ 0&0&1&1&2\\ 1&-1&1&-1&0\\ 0&0&1&-1&0 \end{array}\right]$$

Subtract row 1 from row 2, and row 3 from row 4; Multiply rows 2 and 4 by $-1/2$;

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$$\begin{aligned} a+b+c+d &= 4 \\ a+b-c-d &= 0 \\ a-b+c-d &= 0 \\ a-b-c+d &= 0. \end{aligned}$$

The corresponding augmented matrix can be row-reduced as follows:

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$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

Subtract row 1 from row 2, and row 3 from row 4; Multiply rows 2 and 4 by $-1/2$; Subtract row 2 from row 1, and row 4 from row 3;

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$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{4} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & -2 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 & -2 \end{array}\right]$$

Subtract row 1 from row 2, and row 3 from row 4; Multiply rows 2 and 4 by $-1/2$; Subtract row 2 from row 1, and row 4 from row 3; Subtract row 1 from row 3, and row 2 from row 4;

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$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{4} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & -2 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 & -2 \end{array}\right] \xrightarrow{5} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Subtract row 1 from row 2, and row 3 from row 4; Multiply rows 2 and 4 by $-1/2$; Subtract row 2 from row 1, and row 4 from row 3; Subtract row 1 from row 3, and row 2 from row 4; Multiply rows 3 and 4 by $-1/2$;

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$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Subtract row 1 from row 2, and row 3 from row 4; Multiply rows 2 and 4 by $-1/2$; Subtract row 2 from row 1, and row 4 from row 3; Subtract row 1 from row 3, and row 2 from row 4; Multiply rows 3 and 4 by $-1/2$; Subtract row 3 from row 1, and row 4 from row 2;

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We will solve the equations

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The corresponding augmented matrix can be row-reduced as follows:

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$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{4} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & -2 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 & -2 \end{array}\right] \xrightarrow{5} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{6}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{7} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Subtract row 1 from row 2, and row 3 from row 4; Multiply rows 2 and 4 by $-1/2$; Subtract row 2 from row 1, and row 4 from row 3; Subtract row 1 from row 3, and row 2 from row 4; Multiply rows 3 and 4 by $-1/2$; Subtract row 3 from row 1, and row 4 from row 2; Exchange rows 2 and 3.

Example solution by row-reduction

We will solve the equations

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The corresponding augmented matrix can be row-reduced as follows:

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Subtract row 1 from row 2, and row 3 from row 4; Multiply rows 2 and 4 by $-1/2$; Subtract row 2 from row 1, and row 4 from row 3; Subtract row 1 from row 3, and row 2 from row 4; Multiply rows 3 and 4 by $-1/2$; Subtract row 3 from row 1, and row 4 from row 2; Exchange rows 2 and 3.
The final matrix corresponds to the equations $a = 1$, $b = 1$, $c = 1$ and $d = 1$, which give the unique solution to the original system of equations.

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A system of linear equations is *homogeneous* if the values on the right hand side are all zero.

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$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & -1 & -1 \\ 3 & 3 & -1 & -1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

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$$a + b = 0 \qquad c + d = 0 \qquad e + f = 0.$$

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$$a+b=0 \qquad c+d=0 \qquad e+f=0.$$

Move the independent variables (from non-pivot columns) to the RHS:

$$a=-b \qquad c=-d \qquad e=-f.$$

A system of linear equations is *homogeneous* if the values on the right hand side are all zero. Example:

$$\begin{aligned} a+b+c+d+e+f &= 0 \\ 2a+2b+2c+2d-e-f &= 0 \\ 3a+3b-c-d-e-f &= 0 \end{aligned}$$

The last column of the augmented matrix is zero all through the row reduction, so we need not write it in; we can work with the unaugmented matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & -1 & -1 \\ 3 & 3 & -1 & -1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$a+b=0 \qquad c+d=0 \qquad e+f=0.$$

Move the independent variables (from non-pivot columns) to the RHS:

$$a=-b \qquad c=-d \qquad e=-f.$$

If we prefer we can introduce new variables λ, μ and ν, and say that the general solution is

$$\begin{aligned} a &= -\lambda & c &= -\mu & e &= -\nu \\ b &= \lambda & d &= \mu & f &= \nu \end{aligned}$$

for arbitrary values of λ, μ and ν.

Lecture 3

Linear combinations

Definition 7.1: Let $v_1, \ldots, v_k$ and w be vectors in $\mathbb{R}^n$. We say that w is a *linear combination* of $v_1, \ldots, v_k$ if there exist scalars $\lambda_1, \ldots, \lambda_k$ such that

$$w = \lambda_1 v_1 + \cdots + \lambda_k v_k.$$

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Example 7.2: Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \qquad w = \begin{bmatrix} 1 \\ 10 \\ 100 \\ -111 \end{bmatrix}$$

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If we take $\lambda_1 = 1$ and $\lambda_2 = 11$ and $\lambda_3 = 111$ we get

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 11 \\ -11 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 111 \\ -111 \end{bmatrix}$$

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Linear combinations example

w is a *linear combination* of $v_1, \dots, v_k$ if there exist scalars $\lambda_1, \dots, \lambda_k$ such that

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Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ 8 \\ 27 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 1 \\ 16 \\ 81 \end{bmatrix} \qquad w = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

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Thus, the first component of any such linear combination is zero.
(You should be able to see this without writing out the whole formula.)
As the first component of w is not zero, we see that w is *not* a linear combination of $v_1, \ldots, v_4$.

Linear combinations example

w is a *linear combination* of $v_1, \dots, v_k$ if there exist scalars $\lambda_1, \dots, \lambda_k$ such that

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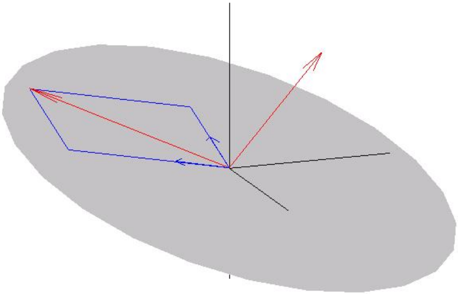
The first and last components of any such linear combination are the same. Again, you should be able to see this without writing the full formula. As the first and last components of w are different, we see that w is not a linear combination of $v_1, \ldots, v_5$.

Two vectors in $\mathbb{R}^3$ span a plane



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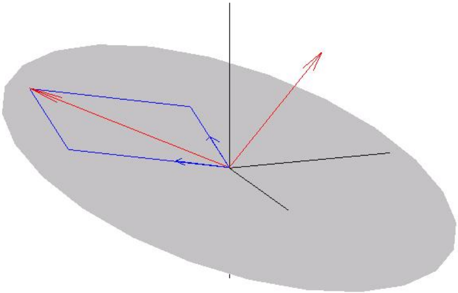

Any vector that lies in the grey plane can be expressed as a linear combination of the two blue vectors.

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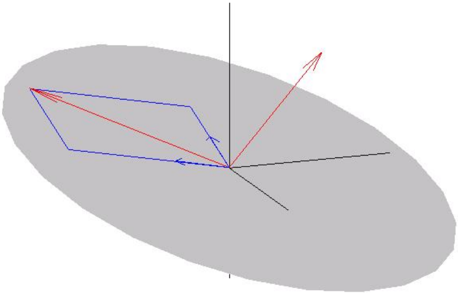

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Suppose we have vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ and another vector $w \in \mathbb{R}^n$

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Let A be the matrix whose columns are the vectors v_i:

$$A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_k \end{array}\right] \in M_{n\times k}(\mathbb{R}).$$

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Example of finding a linear combination

Is w a linear combination of v_1, v_2 and v_3?

$$v_1 = \begin{bmatrix} 11 \\ 11 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 11 \\ 11 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ 11 \\ 11 \end{bmatrix} \qquad w = \begin{bmatrix} 121 \\ 221 \\ 1211 \\ 1111 \end{bmatrix}.$$

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We write down the relevant augmented matrix and row-reduce it:

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$$\left[\begin{array}{ccc|c} 11 & 1 & 1 & 121 \\ 11 & 11 & 1 & 221 \\ 1 & 11 & 11 & 1211 \\ 1 & 1 & 11 & 1111 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 11 & 1111 \\ 11 & 1 & 1 & 121 \\ 11 & 11 & 1 & 221 \\ 1 & 11 & 11 & 1211 \end{array}\right]$$

Move the bottom row to the top;

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$$\left[\begin{array}{ccc|c} 11 & 1 & 1 & 121 \\ 11 & 11 & 1 & 221 \\ 1 & 11 & 11 & 1211 \\ 1 & 1 & 11 & 1111 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 11 & 1111 \\ 11 & 1 & 1 & 121 \\ 11 & 11 & 1 & 221 \\ 1 & 11 & 11 & 1211 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 11 & 1111 \\ 0 & -10 & -120 & -12100 \\ 0 & 0 & -120 & -12000 \\ 0 & 10 & 0 & 100 \end{array}\right]$$

Move the bottom row to the top; Subtract multiples of row 1 from the other rows;

Example of finding a linear combination

Is w a linear combination of v_1, v_2 and v_3?

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$$\to \left[\begin{array}{ccc|c} 1 & 1 & 11 & 1111 \\ 0 & 1 & 12 & 1210 \\ 0 & 0 & 1 & 100 \\ 0 & 1 & 0 & 10 \end{array}\right]$$

Move the bottom row to the top; Subtract multiples of row 1 from the other rows; Divide rows 2,3 and 4 by -10, -120 and 10;

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Move the bottom row to the top; Subtract multiples of row 1 from the other rows; Divide rows 2,3 and 4 by -10, -120 and 10; Subtract multiples of row 3 from the other rows;

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In particular, w can be expressed as a linear combination of v_1, v_2 and v_3.
We can check the above equation directly:

$$v_1 + 10v_2 + 100v_3 = \begin{bmatrix} 11 \\ 11 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 10 \\ 110 \\ 110 \\ 10 \end{bmatrix} + \begin{bmatrix} 100 \\ 100 \\ 1100 \\ 1100 \end{bmatrix} = \begin{bmatrix} 121 \\ 221 \\ 1211 \\ 1111 \end{bmatrix} = w.$$

Example of not finding a linear combination

Is b a linear combination of a_1, a_2 and a_3?

$$a_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

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We write down the relevant augmented matrix and row-reduce it:

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 1 \\ -1 & 0 & 3 & 2 \\ 0 & -1 & -2 & 3 \end{array}\right]$$

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We write down the relevant augmented matrix and row-reduce it:

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Move the top row to the bottom, and multiply the other two rows by -1;

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Move the top row to the bottom, and multiply the other two rows by -1;
Subtract 2 times row 1 from row 3;

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We write down the relevant augmented matrix and row-reduce it:

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$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 14 \end{array}\right]$$

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We write down the relevant augmented matrix and row-reduce it:

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Move the top row to the bottom, and multiply the other two rows by -1; Subtract 2 times row 1 from row 3; Subtract 3 times row 2 from row 3; Divide row 3 by 14;

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Move the top row to the bottom, and multiply the other two rows by -1; Subtract 2 times row 1 from row 3; Subtract 3 times row 2 from row 3; Divide row 3 by 14; Subtract multiples of row 3 from rows 1 and 2.

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The final matrix has a pivot in the rightmost column, corresponding to the equation $0 = 1$.

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We can also see this in a more direct but less systematic way, as follows.

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We can also see this in a more direct but less systematic way, as follows. It is easy to check that $b.a_1 = b.a_2 = b.a_3 = 0$, which means that $b.(\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3) = 0$ for all possible choices of λ_1, λ_2 and λ_3.

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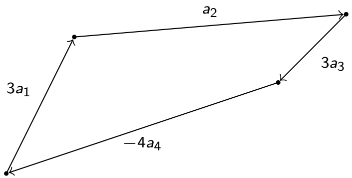

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As the only linear relation is the trivial one, we see that $\mathcal{U}$ is independent.

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Definition 8.6: Let B be a $p \times q$ matrix.
We say that B is *wide* if $p < q$, or *square* if $p = q$ or *tall* if $p > q$.

$$\underset{\text{wide}}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \qquad \underset{\text{square}}{\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}} \qquad \underset{\text{tall}}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}}$$

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(c) If B is tall then the only way for every column to contain a pivot is if B consists of I_q with $(p - q)$ rows of zeros added at the bottom.

$$B = \left[\begin{array}{c} I_q \\ \hline 0_{(p-q)\times q} \end{array} \right]$$

Checking dependence by row-reduction

Method 8.8: Let $\mathcal{V} = v_1, \ldots, v_m$ be a list of vectors in $\mathbb{R}^n$.

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(d) If some column of B has no pivot, then the list $\mathcal{V}$ is dependent. Moreover, we can find the coefficients λ_i in a nontrivial linear relation by solving the vector equation $B\lambda = 0$ (which is easy because B is in RREF).

Checking dependence by row-reduction

Method 8.8: Let $\mathcal{V} = v_1, \dots, v_m$ be a list of vectors in $\mathbb{R}^n$. We can check whether this list is dependent as follows.

(a) Form the $n \times m$ matrix $A = \left[\begin{array}{c|c|c} v_1 & \dots & v_m \end{array}\right]$ whose columns are the vectors v_i.

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Indeed, in this case B is wide, so it cannot have a pivot in every column. This only tells us that there **exists** a nontrivial relation $\lambda_1 v_1 + \dots + \lambda_m v_m = 0$

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Indeed, in this case B is wide, so it cannot have a pivot in every column. This only tells us that there **exists** a nontrivial relation $\lambda_1 v_1 + \dots + \lambda_m v_m = 0$, it does not tell us the coefficients λ_i. To find them we do need to go through the whole method as explained above.

Example of checking for (in)dependence

We previously considered the list

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

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We can write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

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We can write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix}1 & 0 & 1 & 0\\1 & 0 & 0 & 1\\0 & 1 & 0 & 1\\0 & 1 & 1 & 0\end{bmatrix} \xrightarrow{1} \begin{bmatrix}1 & 0 & 1 & 0\\0 & 0 & -1 & 1\\0 & 1 & 0 & 1\\0 & 0 & 1 & -1\end{bmatrix}$$

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$$\begin{bmatrix}1 & 0 & 1 & 0\\1 & 0 & 0 & 1\\0 & 1 & 0 & 1\\0 & 1 & 1 & 0\end{bmatrix} \xrightarrow{1} \begin{bmatrix}1 & 0 & 1 & 0\\0 & 0 & -1 & 1\\0 & 1 & 0 & 1\\0 & 0 & 1 & -1\end{bmatrix} \xrightarrow{2} \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\0 & 0 & 1 & -1\\0 & 0 & -1 & 1\end{bmatrix}$$

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The end result has no pivot in the last column

Example of checking for (in)dependence

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The end result has no pivot in the last column, so the original list is dependent.

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$$\begin{bmatrix}1&0&1&0\\1&0&0&1\\0&1&0&1\\0&1&1&0\end{bmatrix} \xrightarrow{1} \begin{bmatrix}1&0&1&0\\0&0&-1&1\\0&1&0&1\\0&0&1&-1\end{bmatrix} \xrightarrow{2} \begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&1&-1\\0&0&-1&1\end{bmatrix} \xrightarrow{3} \begin{bmatrix}1&0&0&1\\0&1&0&1\\0&0&1&-1\\0&0&0&0\end{bmatrix}$$

The end result has no pivot in the last column, so the original list is dependent. To find a specific linear relation, we solve the equation

$$\begin{bmatrix}1&0&0&1\\0&1&0&1\\0&0&1&-1\\0&0&0&0\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\\\lambda_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

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The end result has no pivot in the last column, so the original list is dependent. To find a specific linear relation, we solve the equation

$$\begin{bmatrix}1&0&0&1\\0&1&0&1\\0&0&1&-1\\0&0&0&0\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\\\lambda_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \qquad \lambda_1 = -\lambda_4$$

Example of checking for (in)dependence

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Example of checking for (in)dependence

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$$\begin{bmatrix}1&0&1&0\\1&0&0&1\\0&1&0&1\\0&1&1&0\end{bmatrix} \xrightarrow{1} \begin{bmatrix}1&0&1&0\\0&0&-1&1\\0&1&0&1\\0&0&1&-1\end{bmatrix} \xrightarrow{2} \begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&1&-1\\0&0&-1&1\end{bmatrix} \xrightarrow{3} \begin{bmatrix}1&0&0&1\\0&1&0&1\\0&0&1&-1\\0&0&0&0\end{bmatrix}$$

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Example of checking for (in)dependence

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$$\begin{bmatrix}1&0&1&0\\1&0&0&1\\0&1&0&1\\0&1&1&0\end{bmatrix} \xrightarrow{1} \begin{bmatrix}1&0&1&0\\0&0&-1&1\\0&1&0&1\\0&0&1&-1\end{bmatrix} \xrightarrow{2} \begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&1&-1\\0&0&-1&1\end{bmatrix} \xrightarrow{3} \begin{bmatrix}1&0&0&1\\0&1&0&1\\0&0&1&-1\\0&0&0&0\end{bmatrix}$$

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Taking $\lambda_4 = 1$ gives $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-1, -1, 1, 1)$

Example of checking for (in)dependence

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$$v_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \qquad v_2 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix} \qquad v_4 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}.$$

We can write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix}1&0&1&0\\1&0&0&1\\0&1&0&1\\0&1&1&0\end{bmatrix} \xrightarrow{1} \begin{bmatrix}1&0&1&0\\0&0&-1&1\\0&1&0&1\\0&0&1&-1\end{bmatrix} \xrightarrow{2} \begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&1&-1\\0&0&-1&1\end{bmatrix} \xrightarrow{3} \begin{bmatrix}1&0&0&1\\0&1&0&1\\0&0&1&-1\\0&0&0&0\end{bmatrix}$$

The end result has no pivot in the last column, so the original list is dependent. To find a specific linear relation, we solve the equation

$$\begin{bmatrix}1&0&0&1\\0&1&0&1\\0&0&1&-1\\0&0&0&0\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\\\lambda_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \qquad \begin{aligned}&\lambda_1 = -\lambda_4\\&\lambda_2 = -\lambda_4\\&\lambda_3 = \lambda_4\\&\lambda_4 \text{ arbitrary}\end{aligned}$$

Taking $\lambda_4 = 1$ gives $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-1, -1, 1, 1)$, corresponding to the relation $-v_1 - v_2 + v_3 + v_4 = 0$.

Example of checking for (in)dependence

We previously considered the list

$$a_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 12 \\ 1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad a_4 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Example of checking for (in)dependence

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Here we have 4 vectors in $\mathbb{R}^2$, so they must be dependent.

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Here we have 4 vectors in $\mathbb{R}^2$, so they must be dependent. Thus, there exist nontrivial linear relations $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4 = 0$.

Example of checking for (in)dependence

We previously considered the list

$$a_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 12 \\ 1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad a_4 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Here we have 4 vectors in $\mathbb{R}^2$, so they must be dependent. Thus, there exist nontrivial linear relations $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4 = 0$.
To actually find such a relation, we write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix} 1 & 12 & -1 & 3 \\ 2 & 1 & -1 & 1 \end{bmatrix}$$

Example of checking for (in)dependence

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To actually find such a relation, we write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix} 1 & 12 & -1 & 3 \\ 2 & 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 12 & -1 & 3 \\ 0 & -23 & 1 & -5 \end{bmatrix}$$

Example of checking for (in)dependence

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$$a_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 12 \\ 1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad a_4 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Here we have 4 vectors in $\mathbb{R}^2$, so they must be dependent. Thus, there exist nontrivial linear relations $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4 = 0$.
To actually find such a relation, we write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix} 1 & 12 & -1 & 3 \\ 2 & 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 12 & -1 & 3 \\ 0 & -23 & 1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 12 & -1 & 3 \\ 0 & 1 & -1/23 & 5/23 \end{bmatrix}$$

Example of checking for (in)dependence

We previously considered the list

$$a_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 12 \\ 1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad a_4 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Here we have 4 vectors in $\mathbb{R}^2$, so they must be dependent. Thus, there exist nontrivial linear relations $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4 = 0$.
To actually find such a relation, we write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix} 1 & 12 & -1 & 3 \\ 2 & 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 12 & -1 & 3 \\ 0 & 1 & -1/23 & 5/23 \end{bmatrix}$$

Example of checking for (in)dependence

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$$a_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 12 \\ 1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad a_4 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Here we have 4 vectors in $\mathbb{R}^2$, so they must be dependent. Thus, there exist nontrivial linear relations $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4 = 0$.
To actually find such a relation, we write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix} 1 & 12 & -1 & 3 \\ 2 & 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 12 & -1 & 3 \\ 0 & 1 & -1/23 & 5/23 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -11/23 & 9/23 \\ 0 & 1 & -1/23 & 5/23 \end{bmatrix}$$

Example of checking for (in)dependence

We previously considered the list

$$a_1 = \begin{bmatrix}1\\2\end{bmatrix} \qquad a_2 = \begin{bmatrix}12\\1\end{bmatrix} \qquad a_3 = \begin{bmatrix}-1\\-1\end{bmatrix} \qquad a_4 = \begin{bmatrix}3\\1\end{bmatrix}.$$

Here we have 4 vectors in $\mathbb{R}^2$, so they must be dependent. Thus, there exist nontrivial linear relations $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4 = 0$.
To actually find such a relation, we write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix}1 & 12 & -1 & 3\\ 2 & 1 & -1 & 1\end{bmatrix} \to \begin{bmatrix}1 & 12 & -1 & 3\\ 0 & 1 & -1/23 & 5/23\end{bmatrix} \to \begin{bmatrix}1 & 0 & -11/23 & 9/23\\ 0 & 1 & -1/23 & 5/23\end{bmatrix}$$

We now need to solve the matrix equation

$$\begin{bmatrix}1 & 0 & -11/23 & 9/23\\ 0 & 1 & -1/23 & 5/23\end{bmatrix} \begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Example of checking for (in)dependence

We previously considered the list

$$a_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 12 \\ 1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad a_4 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Here we have 4 vectors in $\mathbb{R}^2$, so they must be dependent. Thus, there exist nontrivial linear relations $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4 = 0$.
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$$\begin{bmatrix} 1 & 12 & -1 & 3 \\ 2 & 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 12 & -1 & 3 \\ 0 & 1 & -1/23 & 5/23 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -11/23 & 9/23 \\ 0 & 1 & -1/23 & 5/23 \end{bmatrix}$$

We now need to solve the matrix equation

$$\begin{bmatrix} 1 & 0 & -11/23 & 9/23 \\ 0 & 1 & -1/23 & 5/23 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives $\lambda_1 = \frac{11}{23}\lambda_3 - \frac{9}{23}\lambda_4$ and $\lambda_2 = \frac{1}{23}\lambda_3 - \frac{5}{23}\lambda_4$ with λ_3 and λ_4 arbitrary.

Example of checking for (in)dependence

We previously considered the list

$$a_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 12 \\ 1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad a_4 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Here we have 4 vectors in $\mathbb{R}^2$, so they must be dependent. Thus, there exist nontrivial linear relations $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4 = 0$.
To actually find such a relation, we write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix} 1 & 12 & -1 & 3 \\ 2 & 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 12 & -1 & 3 \\ 0 & 1 & -1/23 & 5/23 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -11/23 & 9/23 \\ 0 & 1 & -1/23 & 5/23 \end{bmatrix}$$

We now need to solve the matrix equation

$$\begin{bmatrix} 1 & 0 & -11/23 & 9/23 \\ 0 & 1 & -1/23 & 5/23 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives $\lambda_1 = \frac{11}{23}\lambda_3 - \frac{9}{23}\lambda_4$ and $\lambda_2 = \frac{1}{23}\lambda_3 - \frac{5}{23}\lambda_4$ with λ_3 and λ_4 arbitrary.
If we choose $\lambda_3 = 23$ and $\lambda_4 = 0$ we get $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (11, 1, 23, 0)$

Example of checking for (in)dependence

We previously considered the list

$$a_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 12 \\ 1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad a_4 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Here we have 4 vectors in $\mathbb{R}^2$, so they must be dependent. Thus, there exist nontrivial linear relations $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4 = 0$.
To actually find such a relation, we write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix} 1 & 12 & -1 & 3 \\ 2 & 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 12 & -1 & 3 \\ 0 & 1 & -1/23 & 5/23 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -11/23 & 9/23 \\ 0 & 1 & -1/23 & 5/23 \end{bmatrix}$$

We now need to solve the matrix equation

$$\begin{bmatrix} 1 & 0 & -11/23 & 9/23 \\ 0 & 1 & -1/23 & 5/23 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives $\lambda_1 = \frac{11}{23}\lambda_3 - \frac{9}{23}\lambda_4$ and $\lambda_2 = \frac{1}{23}\lambda_3 - \frac{5}{23}\lambda_4$ with λ_3 and λ_4 arbitrary. If we choose $\lambda_3 = 23$ and $\lambda_4 = 0$ we get $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (11, 1, 23, 0)$ so we have a relation $11a_1 + a_2 + 23a_3 + 0a_4 = 0$.

Example of checking for (in)dependence

We previously considered the list $\mathcal{U}$ given by

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Example of checking for (in)dependence

We previously considered the list $\mathcal{U}$ given by

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

We can write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Example of checking for (in)dependence

We previously considered the list $\mathcal{U}$ given by

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

We can write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Example of checking for (in)dependence

We previously considered the list $\mathcal{U}$ given by

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

We can write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Example of checking for (in)dependence

We previously considered the list $\mathcal{U}$ given by

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

We can write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right]$$

Example of checking for (in)dependence

We previously considered the list $\mathcal{U}$ given by

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

We can write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right]$$

The final matrix has a pivot in every column. It follows that the list $\mathcal{U}$ is independent.

Proof of correctness of the method

$$\text{Put } A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right] \text{ as in step (a)}$$

Proof of correctness of the method

Put $A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right]$ as in step (a), and let B be the RREF form of A.

Put $A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right]$ as in step (a), and let B be the RREF form of A.

Note that for any vector $\lambda = \begin{bmatrix}\lambda_1 & \dots & \lambda_m\end{bmatrix}^T \in \mathbb{R}^m$, we have

$$A\lambda = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right] \begin{bmatrix}\lambda_1 \\ \vdots \\ \lambda_m\end{bmatrix} = \lambda_1 v_1 + \cdots + \lambda_m v_m.$$

Proof of correctness of the method

Put $A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array} \right]$ as in step (a), and let B be the RREF form of A.

Note that for any vector $\lambda = \begin{bmatrix} \lambda_1 & \dots & \lambda_m \end{bmatrix}^T \in \mathbb{R}^m$, we have

$$A\lambda = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array} \right] \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix} = \lambda_1 v_1 + \cdots + \lambda_m v_m.$$

Thus, linear relations on our list are just the same as solutions to the homogeneous equation $A\lambda = 0$.

Proof of correctness of the method

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Thus, linear relations on our list are just the same as solutions to the homogeneous equation $A\lambda = 0$. We saw earlier that these are the same as solutions to the equation $B\lambda = 0$

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Thus, linear relations on our list are just the same as solutions to the homogeneous equation $A\lambda = 0$. We saw earlier that these are the same as solutions to the equation $B\lambda = 0$, which can be found by the standard method for RREF equations.

Proof of correctness of the method

Put $A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right]$ as in step (a), and let B be the RREF form of A.

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Thus, linear relations on our list are just the same as solutions to the homogeneous equation $A\lambda = 0$. We saw earlier that these are the same as solutions to the equation $B\lambda = 0$, which can be found by the standard method for RREF equations. If there is a pivot in every column then none of the variables λ_i is independent, so the only solution is $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$.

Proof of correctness of the method

Put $A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right]$ as in step (a), and let B be the RREF form of A.

Note that for any vector $\lambda = \begin{bmatrix}\lambda_1 & \dots & \lambda_m\end{bmatrix}^T \in \mathbb{R}^m$, we have

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Thus, linear relations on our list are just the same as solutions to the homogeneous equation $A\lambda = 0$. We saw earlier that these are the same as solutions to the equation $B\lambda = 0$, which can be found by the standard method for RREF equations. If there is a pivot in every column then none of the variables λ_i is independent, so the only solution is $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$. Thus, the only linear relation on $\mathcal{V}$ is the trivial one

Put $A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right]$ as in step (a), and let B be the RREF form of A.

Note that for any vector $\lambda = \begin{bmatrix}\lambda_1 & \dots & \lambda_m\end{bmatrix}^T \in \mathbb{R}^m$, we have

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Suppose instead that some column (the k'th one, say) does not contain a pivot.

Proof of correctness of the method

Put $A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right]$ as in step (a), and let B be the RREF form of A.

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Thus, linear relations on our list are just the same as solutions to the homogeneous equation $A\lambda = 0$. We saw earlier that these are the same as solutions to the equation $B\lambda = 0$, which can be found by the standard method for RREF equations. If there is a pivot in every column then none of the variables λ_i is independent, so the only solution is $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$. Thus, the only linear relation on $\mathcal{V}$ is the trivial one, which means that the list $\mathcal{V}$ is linearly independent.

Suppose instead that some column (the k'th one, say) does not contain a pivot. Then the variable λ_k will be independent, so we can choose $\lambda_k = 1$.

Proof of correctness of the method

Put $A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right]$ as in step (a), and let B be the RREF form of A.

Note that for any vector $\lambda = \begin{bmatrix} \lambda_1 & \dots & \lambda_m \end{bmatrix}^T \in \mathbb{R}^m$, we have

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Thus, linear relations on our list are just the same as solutions to the homogeneous equation $A\lambda = 0$. We saw earlier that these are the same as solutions to the equation $B\lambda = 0$, which can be found by the standard method for RREF equations. If there is a pivot in every column then none of the variables λ_i is independent, so the only solution is $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$. Thus, the only linear relation on $\mathcal{V}$ is the trivial one, which means that the list $\mathcal{V}$ is linearly independent.
Suppose instead that some column (the k'th one, say) does not contain a pivot. Then the variable λ_k will be independent, so we can choose $\lambda_k = 1$. This will give us a nonzero to solution to $B\lambda = 0$

Proof of correctness of the method

Put $A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right]$ as in step (a), and let B be the RREF form of A.

Note that for any vector $\lambda = \begin{bmatrix}\lambda_1 & \dots & \lambda_m\end{bmatrix}^T \in \mathbb{R}^m$, we have

$$A\lambda = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right] \begin{bmatrix}\lambda_1 \\ \vdots \\ \lambda_m\end{bmatrix} = \lambda_1 v_1 + \cdots + \lambda_m v_m.$$

Thus, linear relations on our list are just the same as solutions to the homogeneous equation $A\lambda = 0$. We saw earlier that these are the same as solutions to the equation $B\lambda = 0$, which can be found by the standard method for RREF equations. If there is a pivot in every column then none of the variables λ_i is independent, so the only solution is $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$. Thus, the only linear relation on $\mathcal{V}$ is the trivial one, which means that the list $\mathcal{V}$ is linearly independent.

Suppose instead that some column (the k'th one, say) does not contain a pivot. Then the variable λ_k will be independent, so we can choose $\lambda_k = 1$. This will give us a nonzero to solution to $B\lambda = 0$, or equivalently $A\lambda = 0$

Proof of correctness of the method

Put $A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right]$ as in step (a), and let B be the RREF form of A.

Note that for any vector $\lambda = \begin{bmatrix}\lambda_1 & \dots & \lambda_m\end{bmatrix}^T \in \mathbb{R}^m$, we have

$$A\lambda = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right] \begin{bmatrix}\lambda_1 \\ \vdots \\ \lambda_m\end{bmatrix} = \lambda_1 v_1 + \cdots + \lambda_m v_m.$$

Thus, linear relations on our list are just the same as solutions to the homogeneous equation $A\lambda = 0$. We saw earlier that these are the same as solutions to the equation $B\lambda = 0$, which can be found by the standard method for RREF equations. If there is a pivot in every column then none of the variables λ_i is independent, so the only solution is $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$. Thus, the only linear relation on $\mathcal{V}$ is the trivial one, which means that the list $\mathcal{V}$ is linearly independent.
Suppose instead that some column (the k'th one, say) does not contain a pivot. Then the variable λ_k will be independent, so we can choose $\lambda_k = 1$. This will give us a nonzero to solution to $B\lambda = 0$, or equivalently $A\lambda = 0$, corresponding to a nontrivial linear relation on $\mathcal{V}$.

Proof of correctness of the method

Put $A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right]$ as in step (a), and let B be the RREF form of A.

Note that for any vector $\lambda = \begin{bmatrix}\lambda_1 & \dots & \lambda_m\end{bmatrix}^T \in \mathbb{R}^m$, we have

$$A\lambda = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right] \begin{bmatrix}\lambda_1 \\ \vdots \\ \lambda_m\end{bmatrix} = \lambda_1 v_1 + \cdots + \lambda_m v_m.$$

Thus, linear relations on our list are just the same as solutions to the homogeneous equation $A\lambda = 0$. We saw earlier that these are the same as solutions to the equation $B\lambda = 0$, which can be found by the standard method for RREF equations. If there is a pivot in every column then none of the variables λ_i is independent, so the only solution is $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$. Thus, the only linear relation on $\mathcal{V}$ is the trivial one, which means that the list $\mathcal{V}$ is linearly independent.
Suppose instead that some column (the k'th one, say) does not contain a pivot. Then the variable λ_k will be independent, so we can choose $\lambda_k = 1$. This will give us a nonzero to solution to $B\lambda = 0$, or equivalently $A\lambda = 0$, corresponding to a nontrivial linear relation on $\mathcal{V}$. This shows that $\mathcal{V}$ is linearly dependent.

Lecture 4

Spanning

Definition 9.1: Suppose we have a list $\mathcal{V} = v_1, \ldots, v_m$ of vectors in $\mathbb{R}^n$.

Spanning

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Example 9.2: Consider the list $\mathcal{V} = v_1, v_2, v_3, v_4$, where

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Consider the list $\mathcal{U} = u_1, u_2, u_3$, where

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This expresses v as a linear combination of the list $\mathcal{U}$, as required.

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$$v = (y-3x)a_1 + (2x-2y)a_2 + ya_3.$$

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Method 9.7: Let $\mathcal{V} = v_1, \dots, v_m$ be a list of vectors in $\mathbb{R}^n$.
We can check whether this list spans $\mathbb{R}^n$ as follows.

(a) Form the $m \times n$ matrix $C = \left[\begin{array}{c} v_1^T \\ \hline \vdots \\ \hline v_m^T \end{array}\right]$ whose rows are the v_i^T.

(b) Row reduce C to get another $m \times n$ matrix D in RREF.

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Warning: transposing does not interact well with row-reduction, so the matrix D is **not** the transpose of B.

Example of spanning check

Consider the list

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ 8 \\ 27 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 1 \\ 16 \\ 81 \end{bmatrix}$$

Example of spanning check

Consider the list

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ 8 \\ 27 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 1 \\ 16 \\ 81 \end{bmatrix}$$

The relevant matrix is $C = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \\ 0 & 1 & 8 & 27 \\ 0 & 1 & 16 & 81 \end{bmatrix}$

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The first column is zero

Example of spanning check

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The first column is zero, and will remain zero no matter what row operations we perform.

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$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ 8 \\ 27 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 1 \\ 16 \\ 81 \end{bmatrix}$$

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The first column is zero, and will remain zero no matter what row operations we perform. Thus C cannot reduce to the identity matrix, so $\mathcal{V}$ does not span (as we already saw by a different method). In fact the row-reduction is

$$C \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example of spanning check

Consider the list

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ 8 \\ 27 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 1 \\ 16 \\ 81 \end{bmatrix}$$

The relevant matrix is $C = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \\ 0 & 1 & 8 & 27 \\ 0 & 1 & 16 & 81 \end{bmatrix}$

The first column is zero, and will remain zero no matter what row operations we perform. Thus C cannot reduce to the identity matrix, so $\mathcal{V}$ does not span (as we already saw by a different method). In fact the row-reduction is

$$C \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

but it is not really necessary to go through the whole calculation.

Example of spanning check

Consider the list $\mathcal{V}$ given by

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \qquad v_5 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}.$$

Example of spanning check

Consider the list $\mathcal{V}$ given by

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \qquad v_5 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}.$$

The relevant row-reduction is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 4 & 1 \\ 1 & 5 & 1 \end{bmatrix}$$

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Consider the list $\mathcal{V}$ given by

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \qquad v_5 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}.$$

The relevant row-reduction is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 4 & 1 \\ 1 & 5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 0 \end{bmatrix}$$

Example of spanning check

Consider the list $\mathcal{V}$ given by

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \qquad v_5 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}.$$

The relevant row-reduction is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 4 & 1 \\ 1 & 5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 0 \end{bmatrix} \to \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

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Consider the list $\mathcal{V}$ given by

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \qquad v_5 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}.$$

The relevant row-reduction is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 4 & 1 \\ 1 & 5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 0 \end{bmatrix} \to \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

At the end of the process the last column does not contain a pivot

Example of spanning check

Consider the list $\mathcal{V}$ given by

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \qquad v_5 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}.$$

The relevant row-reduction is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 4 & 1 \\ 1 & 5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 0 \end{bmatrix} \to \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

At the end of the process the last column does not contain a pivot (so the top 3×3 block is not the identity)

Example of spanning check

Consider the list $\mathcal{V}$ given by

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \qquad v_5 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}.$$

The relevant row-reduction is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 4 & 1 \\ 1 & 5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 0 \end{bmatrix} \to \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

At the end of the process the last column does not contain a pivot (so the top 3×3 block is not the identity), so $\mathcal{V}$ does not span $\mathbb{R}^3$.

Example of spanning check

Consider the list $\mathcal{V}$ given by

$$v_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\2\\1\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\3\\1\end{bmatrix} \qquad v_4 = \begin{bmatrix}1\\4\\1\end{bmatrix} \qquad v_5 = \begin{bmatrix}1\\5\\1\end{bmatrix}.$$

The relevant row-reduction is

$$\begin{bmatrix}1&1&1\\1&2&1\\1&3&1\\1&4&1\\1&5&1\end{bmatrix} \to \begin{bmatrix}1&1&1\\0&1&0\\0&2&0\\0&3&0\\0&4&0\end{bmatrix} \to \left[\begin{array}{ccc}1&0&1\\0&1&0\\0&0&0\\\hline 0&0&0\\0&0&0\end{array}\right]$$

At the end of the process the last column does not contain a pivot (so the top 3×3 block is not the identity), so $\mathcal{V}$ does not span $\mathbb{R}^3$. Again, we saw this earlier by a different method.

Example of spanning check

Example of spanning check

For the list

$$\mathcal{A} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

Example of spanning check

For the list

$$\mathcal{A} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

the relevant row-reduction is

$$\begin{bmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \\ 0 & 3 & -2 \end{bmatrix}$$

Example of spanning check

For the list

$$\mathcal{A} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

the relevant row-reduction is

$$\begin{bmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \\ 0 & 3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 3 & 0 & -1 \\ 0 & 3 & -2 \end{bmatrix}$$

Example of spanning check

For the list

$$\mathcal{A} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

the relevant row-reduction is

$$\begin{bmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \\ 0 & 3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 3 & 0 & -1 \\ 0 & 3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 3 & -2 \end{bmatrix}$$

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$$\mathcal{A} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

the relevant row-reduction is

$$\begin{bmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \\ 0 & 3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 3 & 0 & -1 \\ 0 & 3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 3 & -2 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

Example of spanning check

For the list

$$\mathcal{A} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

the relevant row-reduction is

$$\begin{bmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \\ 0 & 3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 3 & 0 & -1 \\ 0 & 3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 3 & -2 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}.$$

Example of spanning check

For the list

$$\mathcal{A} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

the relevant row-reduction is

$$\begin{bmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \\ 0 & 3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 3 & 0 & -1 \\ 0 & 3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 3 & -2 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}.$$

In the last matrix the third column has no pivot, so the list does not span.

Example of spanning check

Consider the list $\mathcal{U} = u_1, u_2, u_3$

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Example of spanning check

Consider the list $\mathcal{U} = u_1, u_2, u_3$

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

The relevant row-reduction is

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Example of spanning check

Consider the list $\mathcal{U} = u_1, u_2, u_3$

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The relevant row-reduction is

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Example of spanning check

Consider the list $\mathcal{U} = u_1, u_2, u_3$

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

The relevant row-reduction is

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

Example of spanning check

Consider the list $\mathcal{U} = u_1, u_2, u_3$

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

The relevant row-reduction is

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example of spanning check

Consider the list $\mathcal{U} = u_1, u_2, u_3$

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

The relevant row-reduction is

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example of spanning check

Consider the list $\mathcal{U} = u_1, u_2, u_3$

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

The relevant row-reduction is

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The end result is the identity matrix, so the list $\mathcal{U}$ spans $\mathbb{R}^3$.

Example of spanning check

Consider the list $\mathcal{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

Example of spanning check

Consider the list $\mathcal{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. The relevant row-reduction is

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 5 \end{bmatrix}$$

Example of spanning check

Consider the list $\mathcal{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. The relevant row-reduction is

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & -1 \end{bmatrix}$$

Example of spanning check

Consider the list $\mathcal{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. The relevant row-reduction is

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

Example of spanning check

Consider the list $\mathcal{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. The relevant row-reduction is

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \to \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right]$$

Example of spanning check

Consider the list $\mathcal{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. The relevant row-reduction is

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In the last matrix, the top 2×2 block is the identity. This means that the list $\mathcal{A}$ spans $\mathbb{R}^2$.

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Lemma 9.15: Let C be an $m \times n$ matrix, and let C' be obtained from C by a single elementary row operation. Let s be a row vector of length n. Then s can be expressed as a linear combination of the rows of C if and only if it can be expressed as a linear combination of the rows of C'.

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$C \in M_{m \times n}(\mathbb{R})$; C' obtained from C by a single row operation; s a row vector of length n. Claim: s is a linear combination of rows of C iff it is a linear combination of rows of C'.

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Corollary 9.16: Let C be an $m \times n$ matrix, and let D be obtained from C by a sequence of elementary row operation. Let s be a row vector of length n. Then s can be expressed as a linear combination of the rows of C if and only if it can be expressed as a linear combination of the rows of D.

Proof.

Just apply the lemma to each step in the row-reduction sequence. □

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Proof of (a): In this case the first n rows are the standard basis vectors

$$\begin{aligned} r_1 &= e_1^T = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \\ r_2 &= e_2^T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix} \cdots \\ r_n &= e_n^T = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \end{aligned}$$

and $r_i = 0$ for $i > n$. This means that any row vector $v = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ can be expressed as

$$\begin{aligned} v = &\begin{bmatrix} v_1 & 0 & 0 & \cdots & 0 \end{bmatrix} + \\ &\begin{bmatrix} 0 & v_2 & 0 & \cdots & 0 \end{bmatrix} + \cdots + \\ &\begin{bmatrix} 0 & 0 & 0 & \cdots & v_n \end{bmatrix} \\ = &v_1 r_1 + v_2 r_2 + v_3 r_3 + \cdots + v_n r_n, \end{aligned}$$

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Lemma: Let D be an $m \times n$ matrix in RREF.

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Example for proof of (b): Consider the matrix

$$D = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 6 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3 = \begin{bmatrix} 0 & \lambda_1 & 2\lambda_1 & 3\lambda_1 & \lambda_2 & 4\lambda_1 + 6\lambda_2 & 5\lambda_1 + 7\lambda_2 & \lambda_3 \end{bmatrix}.$$

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The entries in the pivot columns 2, 5 and 8 of s are just the coefficients λ_1, λ_2 and λ_3.

Lemma: Let D be an $m \times n$ matrix in RREF.

(b) Suppose instead that the k'th column of D does not contain a pivot. Then the k'th standard basis vector e_k **cannot** be expressed as a linear combination of the rows of D.

Example for proof of (b): Consider the matrix

$$D = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 6 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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For this to be $\lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3$ we need $\lambda_1 = 0$ and $\lambda_2 = 0$ and $\lambda_3 = 0$ (by looking in the pivot columns). But that means $\lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3 = 0 \neq e_6$.

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We conclude that in fact it is impossible to write e_k as $\lambda_1 r_1 + \cdots + \lambda_m r_m$, so e_k is not a linear combination of the rows of D.

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$$P = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array} \right] \in M_{n \times m}(\mathbb{R})$$

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In other words:

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- The vectors v_i span $\mathbb{R}^n$ if and only if $P^T \to \left[\begin{array}{c} I_n \\ \hline 0 \end{array}\right]$, if and only if the vectors w_j are linearly independent in $\mathbb{R}^m$.

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- The columns of P span if and only if the columns of P^T are independent.

Lecture 5

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Basis example

Consider the list $\mathcal{U} = (u_1, u_2, u_3)$, where

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

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which expresses v as a linear combination of u_1, u_2 and u_3.

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Consider the list $\mathcal{U} = (u_1, u_2, u_3)$, where

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For an arbitrary vector $v = \begin{bmatrix} x & y & z \end{bmatrix}^T$ we have

$$(a-b)u_1 + (b-c)u_2 + cu_3 = \begin{bmatrix} a-b \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} b-c \\ b-c \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ c \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = v,$$

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from which we read off that $\lambda_3 = 0$

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from which we read off that $\lambda_3 = 0$, then that $\lambda_2 = 0$

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from which we read off that $\lambda_3 = 0$, then that $\lambda_2 = 0$, then that $\lambda_1 = 0$. This means that the only linear relation on $\mathcal{U}$ is the trivial one

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Basis example

Consider the list $\mathcal{U} = (u_1, u_2, u_3)$, where

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Basis criterion

Proposition 10.4: Given $\mathcal{V} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$, put

$$A = \left[\begin{array}{c|c|c} v_1 & \ldots & v_n \end{array}\right] \in M_{n\times n}(\mathbb{R})$$

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By subtracting this from the earlier equation, we get

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This is a linear relation on the independent list $\mathcal{V}$

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This is a linear relation on the independent list $\mathcal{V}$, so it must be the trivial one

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so λ is a solution to $A\lambda = x$. Suppose that μ is also a solution, so

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This is a linear relation on the independent list $\mathcal{V}$, so it must be the trivial one, so the coefficients $\lambda_i - \mu_i$ are zero

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so λ is a solution to $A\lambda = x$. Suppose that μ is also a solution, so

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By subtracting this from the earlier equation, we get

$$(\lambda_1 - \mu_1)v_1 + \cdots + (\lambda_n - \mu_n)v_n = 0.$$

This is a linear relation on the independent list $\mathcal{V}$, so it must be the trivial one, so the coefficients $\lambda_i - \mu_i$ are zero, so $\lambda = \mu$. In other words, λ is the **unique** solution to $A\lambda = x$, as required.

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$$A = \left[\begin{array}{c|c|c} v_1 & \dots & v_n \end{array} \right] \in M_{n\times n}(\mathbb{R})$$

Then $\mathcal{V}$ is a basis iff $A\lambda = x$ has a **unique** solution for every $x \in \mathbb{R}^n$.

We now need to prove the converse.

Proposition 10.4: Given $\mathcal{V} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$, put

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We now need to prove the converse. Suppose that for every $x \in \mathbb{R}^n$, the equation $A\lambda = x$ has a unique solution.

Basis criterion

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We now need to prove the converse. Suppose that for every $x \in \mathbb{R}^n$, the equation $A\lambda = x$ has a unique solution. Equivalently, for every $x \in \mathbb{R}^n$, there is a unique sequence of coefficients $\lambda_1, \dots, \lambda_n$ such that $\lambda_1 v_1 + \dots + \lambda_n v_n = x$.

Basis criterion

Proposition 10.4: Given $\mathcal{V} = (v_1, \dots, v_n)$ in $\mathbb{R}^n$, put

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Then $\mathcal{V}$ is a basis iff $A\lambda = x$ has a **unique** solution for every $x \in \mathbb{R}^n$.

We now need to prove the converse. Suppose that for every $x \in \mathbb{R}^n$, the equation $A\lambda = x$ has a unique solution. Equivalently, for every $x \in \mathbb{R}^n$, there is a unique sequence of coefficients $\lambda_1, \dots, \lambda_n$ such that $\lambda_1 v_1 + \dots + \lambda_n v_n = x$. Firstly, we can temporarily ignore the uniqueness, and just note that every element $x \in \mathbb{R}^n$ can be expressed as a linear combination of $v_1, \dots, v_n$.

Basis criterion

Proposition 10.4: Given $\mathcal{V} = (v_1, \dots, v_n)$ in $\mathbb{R}^n$, put

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Then $\mathcal{V}$ is a basis iff $A\lambda = x$ has a **unique** solution for every $x \in \mathbb{R}^n$.

We now need to prove the converse. Suppose that for every $x \in \mathbb{R}^n$, the equation $A\lambda = x$ has a unique solution. Equivalently, for every $x \in \mathbb{R}^n$, there is a unique sequence of coefficients $\lambda_1, \dots, \lambda_n$ such that $\lambda_1 v_1 + \dots + \lambda_n v_n = x$. Firstly, we can temporarily ignore the uniqueness, and just note that every element $x \in \mathbb{R}^n$ can be expressed as a linear combination of $v_1, \dots, v_n$. This means that the list $\mathcal{V}$ spans $\mathbb{R}^n$.

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Method to check for a basis

Let $\mathcal{V} = (v_1, \ldots, v_m)$ be a list of vectors in $\mathbb{R}^n$.

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(c) If $A \to B \neq I_n$ then B cannot have a pivot in every column. By our method for checking independence, the list $\mathcal{V}$ is dependent and so is not a basis.

Basis example

Consider the vectors

$$v_1 = \begin{bmatrix}1\\2\\3\\2\\1\end{bmatrix} \qquad v_2 = \begin{bmatrix}3\\2\\1\\2\\3\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} \qquad v_4 = \begin{bmatrix}1\\3\\5\\3\\1\end{bmatrix} \qquad v_5 = \begin{bmatrix}5\\3\\1\\3\\5\end{bmatrix}$$

To decide whether they form a basis, we construct the corresponding matrix A and start row-reducing it:

$$\begin{bmatrix}1&3&1&1&5\\2&2&1&3&3\\3&1&1&5&1\\2&2&1&3&3\\1&3&1&1&5\end{bmatrix} \to \begin{bmatrix}1&3&1&1&5\\0&-4&-1&1&-7\\0&-8&-2&2&-14\\0&-4&-1&1&-7\\0&0&0&0&0\end{bmatrix} \to \begin{bmatrix}1&3&1&1&5\\0&-4&-1&1&-7\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

Already after the first step we have a row of zeros, and it is clear that we will still have a row of zeros after we complete the row-reduction, so A does not reduce to the identity matrix, so the vectors v_i do not form a basis.

Basis example

Consider the vectors

$$p_1 = \begin{bmatrix} 1 \\ 1 \\ 11 \\ 1 \end{bmatrix} \qquad p_2 = \begin{bmatrix} 1 \\ 11 \\ 1 \\ 11 \end{bmatrix} \qquad p_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 11 \end{bmatrix} \qquad p_4 = \begin{bmatrix} 1 \\ 11 \\ 11 \\ 11 \end{bmatrix}$$

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After a few more steps, we obtain the identity matrix.

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After a few more steps, we obtain the identity matrix. It follows that the list p_1, p_2, p_3, p_4 is a basis.

Coefficients in terms of a basis

Suppose that the list $\mathcal{V} = v_1, \ldots, v_n$ is a basis for $\mathbb{R}^n$, and that w is another vector in $\mathbb{R}^n$.

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Method 10.8: Let $\mathcal{V} = v_1, \dots, v_n$ be a basis for $\mathbb{R}^n$, and let w be another vector in $\mathbb{R}^n$.

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$$B = \left[\begin{array}{c|c|c|c} v_1 & \cdots & v_n & w \end{array}\right] \in M_{n\times(n+1)}(\mathbb{R}).$$

(b) Let B' be the RREF form of B. Then B' will have the form $[I_n|\lambda]$ for some column vector

$$\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}.$$

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It is clear from our recent discussion that this is valid.

Example of coefficients in terms of a basis

We will express $q = \begin{bmatrix} 0.9 & 0.9 & 0 & 10.9 \end{bmatrix}^T$ in terms of the basis p_1, p_2, p_3, p_4 in the previous example.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0.9 \\ 1 & 11 & 1 & 11 & 0.9 \\ 11 & 1 & 1 & 11 & 0 \\ 1 & 11 & 11 & 11 & 10.9 \end{array}\right]$$

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We will express $q = \begin{bmatrix} 0.9 & 0.9 & 0 & 10.9 \end{bmatrix}^T$ in terms of the basis p_1, p_2, p_3, p_4 in the previous example. We form the relevant augmented matrix, and apply the same row-reduction steps as before, except that we now have an extra column.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0.9 \\ 1 & 11 & 1 & 11 & 0.9 \\ 11 & 1 & 1 & 11 & 0 \\ 1 & 11 & 11 & 11 & 10.9 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0.9 \\ 0 & 10 & 0 & 10 & 0 \\ 0 & -10 & -10 & 0 & -9.9 \\ 0 & 10 & 10 & 10 & 10 \end{array}\right]$$

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The final result is $[I_4|\lambda]$, where $\lambda = \begin{bmatrix} -0.1 & -0.01 & 1 & 0.01 \end{bmatrix}^T$.

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The final result is $[I_4|\lambda]$, where $\lambda = \begin{bmatrix} -0.1 & -0.01 & 1 & 0.01 \end{bmatrix}^T$. This means that q can be expressed in terms of the vectors p_i as follows:

$$q = -0.1p_1 - 0.01p_2 + p_3 + 0.01p_4.$$

Example of coefficients in terms of a basis

One can check that the vectors u_1, u_2, u_3 and u_4 below form a basis for $\mathbb{R}^4$.

$$u_1 = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/4 \\ 1/5 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1/3 \\ 1/4 \\ 1/5 \\ 1/6 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 1/4 \\ 1/5 \\ 1/6 \\ 1/7 \end{bmatrix}$$

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One can check that the vectors u_1, u_2, u_3 and u_4 below form a basis for $\mathbb{R}^4$.

$$u_1 = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/4 \\ 1/5 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1/3 \\ 1/4 \\ 1/5 \\ 1/6 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 1/4 \\ 1/5 \\ 1/6 \\ 1/7 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

We would like to express v in terms of this basis.

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We would like to express v in terms of this basis. The matrix formed by the vectors u_i is called the *Hilbert matrix*; it is notoriously hard to row-reduce. We will therefore use Maple.

```
with(LinearAlgebra):
RREF := ReducedRowEchelonForm;
u[1] := <1,1/2,1/3,1/4>;
u[2] := <1/2,1/3,1/4,1/5>;
u[3] := <1/3,1/4,1/5,1/6>;
u[4] := <1/4,1/5,1/6,1/7>;
v    := <1,1,1,1>;
B    := <u[1]|u[2]|u[3]|u[4]|v>;
RREF(B);
```

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$$\left[\begin{array}{cccc|c} 1 & 1/2 & 1/3 & 1/4 & 1 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 60 \\ 0 & 0 & 1 & 0 & -180 \\ 0 & 0 & 0 & 1 & 140 \end{array}\right].$$

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We conclude that

$$v = -4u_1 + 60u_2 - 180u_3 + 140u_4.$$

Proposition 10.11: Let A be an $n \times n$ matrix. Then the columns of A form a basis for $\mathbb{R}^n$ if and only if the columns of A^T form a basis for $\mathbb{R}^n$.

Duality for bases

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Proof.

Recall:

$\square$

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Recall:

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- A list is a basis iff it is independent and also spans.

The claim is clear from this. $\square$

Proposition 10.12: Let $\mathcal{V}$ be a list of n vectors in $\mathbb{R}^n$ (so the number of vectors is the same as the number of entries in each vector).

(a) If the list is linearly independent then it also spans, and so is a basis.

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(b) Suppose instead that $\mathcal{V}$ (which is the list of columns of A) spans $\mathbb{R}^n$. By duality, we conclude that the columns of A^T are linearly independent. Now A^T has n columns, so we can apply part (a) to deduce that the columns of A^T form a basis. By duality again, the columns of A must form a basis as well.

$\square$

Lecture 6

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Example 11.2: In the case $n = 4$, we have

$$D_2(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$D_2(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_{24}(\mu) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \mu \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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(a) Suppose that $1 \leq p \leq n$ and that λ is a nonzero real number. We then let $D_p(\lambda)$ be the matrix that is the same as I_n except that $(D_p(\lambda))_{pp} = \lambda$.

(b) Supose that $1 \leq p, q \leq n$ with $p \neq q$, and that μ is an arbitrary real number. We then let $E_{pq}(\mu)$ be the matrix that is the same as I_n except that $(E_{pq}(\lambda))_{pq} = \mu$.

(c) Supose again that $1 \leq p, q \leq n$ with $p \neq q$. We let F_{pq} be the matrix that is the same as I_n except that $(F_{pq})_{pp} = (F_{pq})_{qq} = 0$ and $(F_{pq})_{pq} = (F_{pq})_{qp} = 1$.

Example 11.2: In the case $n = 4$, we have

$$D_2(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_{24}(\mu) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \mu \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad F_{24} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

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An *elementary matrix* is a matrix of one of these types.

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Elementary matrices and row operations

Proposition 11.3: Let A be an $n \times n$ matrix, and let A' be obtained from A by a single row operation. Then $A' = UA$ for some elementary matrix U.

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(a) Let A' be obtained from A by multiplying the p'th row by λ. Then $A' = D_p(\lambda)A$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ \lambda e & \lambda f & \lambda g & \lambda h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$$

Proposition 11.3: Let A be an $n \times n$ matrix, and let A' be obtained from A by a single row operation. Then $A' = UA$ for some elementary matrix U. In more detail:

(a) Let A' be obtained from A by multiplying the p'th row by λ. Then $A' = D_p(\lambda)A$.

(b) Let A' be obtained from A by adding μ times the q'th row to the p'th row. Then $A' = E_{pq}(\mu)A$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \mu \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e + \mu m & f + \mu n & g + \mu o & h + \mu p \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$$

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(a) Let A' be obtained from A by multiplying the p'th row by λ. Then $A' = D_p(\lambda)A$.

(b) Let A' be obtained from A by adding μ times the q'th row to the p'th row. Then $A' = E_{pq}(\mu)A$.

(c) Let A' be obtained from A by exchanging the p'th row and the q'th row. Then $A' = F_{pq}A$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ m & n & o & p \\ i & j & k & l \\ e & f & g & h \end{bmatrix}$$

Corollary 11.4: Let A and B be $n \times n$ matrices, and suppose that A can be converted to B by a sequence of row operations. Then $B = UA$ for some matrix U that can be expressed as a product of elementary matrices.

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Proof.
The assumption is that there is a sequence of matrices $A_0, A_1, \ldots, A_r$ starting with $A_0 = A$ and ending with $A_r = B$ such that A_i is obtained from A_{i-1} by a single row operation.

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$$A_1 = U_1 A_0 = U_1 A$$

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$$A_1 = U_1 A_0 = U_1 A$$
$$A_2 = U_2 A_1$$

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A_2 &= U_2 A_1 = U_2 U_1 A \\
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and so on. Eventually we get $B = A_r = U_r U_{r-1} \cdots U_1 A$.

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and so on. Eventually we get $B = A_r = U_r U_{r-1} \cdots U_1 A$. We can thus take $U = U_r U_{r-1} \cdots U_1$ and we have $B = UA$ as required. $\square$

Invertibility

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(a) A can be row-reduced to I_n.

(b) The columns of A are linearly independent.

(c) The columns of A span $\mathbb{R}^n$.

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Moreover, if these statements are all true then there is a unique matrix U that satisfies $UA = I_n$

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(i) There is a matrix U such that $UA = I_n$.

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Moreover, if these statements are all true then there is a unique matrix U that satisfies $UA = I_n$, and this is also the unique matrix that satisfies $AU = I_n$ (so the matrix V in (j) is necessarily the same as the matrix U in (i)).

Invertibility — what we already know

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(d) The columns of A form a basis for $\mathbb{R}^n$.

(e),(f),(g),(h): same for A^T

(i) There is a matrix U such that $UA = I_n$.

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Statements (a) to (d) are equivalent to each other by the "numerical criteria" (Proposition 10.12).

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Similarly statements (e) to (h) are equivalent to each other.

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Similarly statements (e) to (h) are equivalent to each other.

Moreover, (a) to (d) are equivalent to (e) to (h) by "duality for bases" (Proposition 10.11).

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Similarly statements (e) to (h) are equivalent to each other.

Moreover, (a) to (d) are equivalent to (e) to (h) by "duality for bases" (Proposition 10.11).

The real issue is to prove that (a) to (h) are equivalent to (i) and (j).

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(i) There is a matrix U such that $UA = I_n$.
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- If (a) holds then each row operation corresponds to an elementary matrix, and the product of those is a matrix U with $UA = I_n$; so (i) holds.

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- Conversely, suppose that (i) holds.

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- Similarly, if (e) holds then there exists W with $WA^T = I_n$, so $AW^T = I_n$, so can take $V = W^T$ to see that (j) holds.
- Conversely, suppose that (i) holds. Let $v_1, \ldots, v_r$ be the columns of A.

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We say that A is *invertible* if (any one of) the conditions (a) to (j) hold.

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Remark 11.7: It is clear that A is invertible if and only if A^T is invertible.

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The proof is similar.

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- Thus, U is invertible.

□

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Consider the matrix $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$. We have the following row-reduction:

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & b-ac & 1 & -a & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

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We conclude that

$$A^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -5/2 & 4 & -3/2 \\ 1/2 & -1 & 1/2 \end{bmatrix}.$$

Lecture 7

Definition : For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is defined as

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Determinants

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$$\det(A) = a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

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We will now discuss determinants for square matrices of any size.

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We will now discuss determinants for square matrices of any size. There are more details in an appendix to the printed notes, which will not be examined.

Definition 12.1: Let A be an $n \times n$ matrix,
and let a_{ij} denote the entry in the i'th row of the j'th column.

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One can check that this agrees with the standard formulae on the previous slide, if $n = 2$ or $n = 3$.

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$$\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix} = 1 \times 5 \times 8 \times 10 = 400.$$

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(c) In particular, if A is a diagonal matrix (so all entries off the diagonal are zero) then both (a) and (b) apply and we have $\det(A) = \prod_{i=1}^{n} a_{ii}$.

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(b) Similarly, if all the entries above the diagonal are zero, then the determinant is just the product of the diagonal entries.

(c) In particular, if A is a diagonal matrix (so all entries off the diagonal are zero) then both (a) and (b) apply and we have $\det(A) = \prod_{i=1}^{n} a_{ii}$.

(d) In particular, we have $\det(I_n) = 1$.

Basic facts about determinants

Example 12.5: If any row or column of A is zero, then $\det(A) = 0$.

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Proposition 12.6: The determinants of elementary matrices are
$\det(D_p(\lambda)) = \lambda$

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Proposition 12.6: The determinants of elementary matrices are $\det(D_p(\lambda)) = \lambda$ and $\det(E_{pq}(\mu)) = 1$

$$D_2(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_{24}(\mu) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \mu \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Theorem 12.8: If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

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Determinants and row operations

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- If $B \neq I_n$ then B must have a row of zeros so $\det(B) = 0$ and $\det(A) = 0$.

It will often be more efficient to stop the row-reduction at an earlier stage.

Example determinant by row-reduction

$$A = \begin{bmatrix} 3 & 5 & 5 & 5 \\ 1 & 3 & 5 & 5 \\ 1 & 1 & 3 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

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$$A = \begin{bmatrix} 3 & 5 & 5 & 5 \\ 1 & 3 & 5 & 5 \\ 1 & 1 & 3 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 2 & -4 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

- Add multiples of row 4 to the other rows: no factor.

Example determinant by row-reduction

$$A = \begin{bmatrix} 3 & 5 & 5 & 5 \\ 1 & 3 & 5 & 5 \\ 1 & 1 & 3 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 2 & -4 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{\frac{1}{8}} \begin{bmatrix} 0 & 1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

- Add multiples of row 4 to the other rows: no factor.
- Multiply each of the first three rows by $\frac{1}{2}$: overall factor of $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

Example determinant by row-reduction

$$A = \begin{bmatrix} 3 & 5 & 5 & 5 \\ 1 & 3 & 5 & 5 \\ 1 & 1 & 3 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 2 & -4 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{\frac{1}{8}} \begin{bmatrix} 0 & 1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \to$$

$$\begin{bmatrix} 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

- Add multiples of row 4 to the other rows: no factor.
- Multiply each of the first three rows by $\frac{1}{2}$: overall factor of $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.
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- Add multiples of row 4 to the other rows: no factor.
- Multiply each of the first three rows by $\frac{1}{2}$: overall factor of $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.
- Subtract row 1 from row 2: no factor.
- Subtract row 3 from row 2: no factor.
- Exchange rows 2 and 4: factor of -1.

Example determinant by row-reduction

$$A = \begin{bmatrix} 3 & 5 & 5 & 5 \\ 1 & 3 & 5 & 5 \\ 1 & 1 & 3 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 2 & -4 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{\color{red}\frac{1}{8}} \begin{bmatrix} 0 & 1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \to$$

$$\begin{bmatrix} 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{\color{red}-1} \begin{bmatrix} 0 & 1 & 1 & -2 \\ 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{\color{red}-1} \begin{bmatrix} \color{blue}1 & 1 & 1 & 3 \\ 0 & \color{blue}1 & 1 & -2 \\ 0 & 0 & \color{blue}1 & 1 \\ 0 & 0 & 0 & \color{blue}2 \end{bmatrix}$$

- Add multiples of row 4 to the other rows: no factor.
- Multiply each of the first three rows by $\frac{1}{2}$: overall factor of $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.
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The final matrix B is upper-triangular

Example determinant by row-reduction

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- Add multiples of row 4 to the other rows: no factor.
- Multiply each of the first three rows by $\frac{1}{2}$: overall factor of $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.
- Subtract row 1 from row 2: no factor.
- Subtract row 3 from row 2: no factor.
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The final matrix B is upper-triangular, so the determinant is just the product of the diagonal entries

Example determinant by row-reduction

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$$\begin{bmatrix} 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 0 & 1 & 1 & -2 \\ 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

- Add multiples of row 4 to the other rows: no factor.
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- Subtract row 3 from row 2: no factor.
- Exchange rows 2 and 4: factor of -1.
- Exchange rows 1 and 2: another factor of -1.

The final matrix B is upper-triangular, so the determinant is just the product of the diagonal entries, which is $\det(B) = 2$.

Example determinant by row-reduction

$$A = \begin{bmatrix} 3 & 5 & 5 & 5 \\ 1 & 3 & 5 & 5 \\ 1 & 1 & 3 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 2 & -4 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{\frac{1}{8}} \begin{bmatrix} 0 & 1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \to$$

$$\begin{bmatrix} 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 0 & 1 & 1 & -2 \\ 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

- Add multiples of row 4 to the other rows: no factor.
- Multiply each of the first three rows by $\frac{1}{2}$: overall factor of $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.
- Subtract row 1 from row 2: no factor.
- Subtract row 3 from row 2: no factor.
- Exchange rows 2 and 4: factor of -1.
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The final matrix B is upper-triangular, so the determinant is just the product of the diagonal entries, which is $\det(B) = 2$. The product of the factors is $\mu = 1/8$

Example determinant by row-reduction

$$A = \begin{bmatrix} 3 & 5 & 5 & 5 \\ 1 & 3 & 5 & 5 \\ 1 & 1 & 3 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 2 & -4 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{\frac{1}{8}} \begin{bmatrix} 0 & 1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \to$$

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- Add multiples of row 4 to the other rows: no factor.
- Multiply each of the first three rows by $\frac{1}{2}$: overall factor of $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.
- Subtract row 1 from row 2: no factor.
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- Exchange rows 2 and 4: factor of -1.
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The final matrix B is upper-triangular, so the determinant is just the product of the diagonal entries, which is $\det(B) = 2$. The product of the factors is $\mu = 1/8$, so $\det(A) = \det(B)/\mu = 16$.

Example determinant by row-reduction

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$$

Example determinant by row-reduction

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}$$

- Subtract row 1 from each of the other rows: no factor.

Example determinant by row-reduction

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Subtract row 1 from each of the other rows: no factor.
- Subtract multiples of row 2 from rows 3 and 4: no factor.

Example determinant by row-reduction

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

- Subtract row 1 from each of the other rows: no factor.
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Example determinant by row-reduction

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

- Subtract row 1 from each of the other rows: no factor.
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As B has two rows of zeros, we see that $\det(B) = 0$.

Example determinant by row-reduction

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

- Subtract row 1 from each of the other rows: no factor.
- Subtract multiples of row 2 from rows 3 and 4: no factor.

As B has two rows of zeros, we see that $\det(B) = 0$.
The method therefore tells us that $\det(A) = \det(B)/\mu = 0$ as well.

Definition 12.12: Let A be an $n \times n$ matrix, and let p and q be integers with $1 \leq p, q \leq n$.

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(a) We let M_{pq} be the matrix obtained by deleting the p'th row and the q'th column from A.

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(b) We put $m_{pq} = \det(M_{pq})$.

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The matrices M_{pq} the *minor matrices* for A, and the numbers m_{pq} the *minor determinants*.

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Minors and the adjugate

Definition 12.12: Let A be an $n \times n$ matrix, and let p and q be integers with $1 \leq p, q \leq n$.

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$\operatorname{adj}(A)_{qp} = (-1)^{p+q} m_{pq}$.

The matrices M_{pq} the *minor matrices* for A, and the numbers m_{pq} the *minor determinants*. The matrix $\operatorname{adj}(A)$ is the *adjugate* (or *classical adjoint*) of A.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad M_{11} = \begin{bmatrix} e & f \\ h & i \end{bmatrix} \qquad m_{11} = ei - fh$$

$$\operatorname{adj}(A) = \begin{bmatrix} m_{11} & -m_{21} & m_{31} \\ -m_{12} & m_{22} & -m_{32} \\ m_{13} & -m_{23} & m_{33} \end{bmatrix} = \begin{bmatrix} ei - fh & & \\ & & \\ & & \end{bmatrix}$$

Definition 12.12: Let A be an $n \times n$ matrix, and let p and q be integers with $1 \leq p, q \leq n$.

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The matrices M_{pq} the *minor matrices* for A, and the numbers m_{pq} the *minor determinants*. The matrix $\operatorname{adj}(A)$ is the *adjugate* (or *classical adjoint*) of A.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad M_{12} = \begin{bmatrix} d & f \\ g & i \end{bmatrix} \qquad m_{12} = di - fg$$

$$\operatorname{adj}(A) = \begin{bmatrix} m_{11} & -m_{21} & m_{31} \\ -m_{12} & m_{22} & -m_{32} \\ m_{13} & -m_{23} & m_{33} \end{bmatrix} = \begin{bmatrix} ei - fh & & \\ -di + fg & & \\ & & \end{bmatrix}$$

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$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad M_{13} = \begin{bmatrix} d & e \\ g & h \end{bmatrix} \qquad m_{13} = dh - eg$$

$$\mathrm{adj}(A) = \begin{bmatrix} m_{11} & -m_{21} & m_{31} \\ -m_{12} & m_{22} & -m_{32} \\ m_{13} & -m_{23} & m_{33} \end{bmatrix} = \begin{bmatrix} ei - fh & & \\ -di + fg & & \\ dh - eg & & \end{bmatrix}$$

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$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad M_{21} = \begin{bmatrix} b & c \\ h & i \end{bmatrix} \qquad m_{21} = bi - ch$$

$$\operatorname{adj}(A) = \begin{bmatrix} m_{11} & -m_{21} & m_{31} \\ -m_{12} & m_{22} & -m_{32} \\ m_{13} & -m_{23} & m_{33} \end{bmatrix} = \begin{bmatrix} ei - fh & -bi + ch & \\ -di + fg & & \\ dh - eg & & \end{bmatrix}$$

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$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad M_{22} = \begin{bmatrix} a & c \\ g & i \end{bmatrix} \qquad m_{22} = ai - cg$$

$$\operatorname{adj}(A) = \begin{bmatrix} m_{11} & -m_{21} & m_{31} \\ -m_{12} & m_{22} & -m_{32} \\ m_{13} & -m_{23} & m_{33} \end{bmatrix} = \begin{bmatrix} ei - fh & -bi + ch & \\ -di + fg & ai - cg & \\ dh - eg & & \end{bmatrix}$$

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$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad M_{23} = \begin{bmatrix} a & b \\ g & h \end{bmatrix} \qquad m_{23} = ah - bg$$

$$\operatorname{adj}(A) = \begin{bmatrix} m_{11} & -m_{21} & m_{31} \\ -m_{12} & m_{22} & -m_{32} \\ m_{13} & -m_{23} & m_{33} \end{bmatrix} = \begin{bmatrix} ei - fh & -bi + ch & \\ -di + fg & ai - cg & \\ dh - eg & -ah + bg & \end{bmatrix}$$

Definition 12.12: Let A be an $n \times n$ matrix, and let p and q be integers with $1 \leq p, q \leq n$.

(a) We let M_{pq} be the matrix obtained by deleting the p'th row and the q'th column from A. This is a square matrix of shape $(n-1) \times (n-1)$.

(b) We put $m_{pq} = \det(M_{pq})$.

(c) We let $\operatorname{adj}(A)$ denote the $n \times n$ matrix with entries $\operatorname{adj}(A)_{qp} = (-1)^{p+q} m_{pq}$.

The matrices M_{pq} the *minor matrices* for A, and the numbers m_{pq} the *minor determinants*. The matrix $\operatorname{adj}(A)$ is the *adjugate* (or *classical adjoint*) of A.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad M_{31} = \begin{bmatrix} b & c \\ e & f \end{bmatrix} \qquad m_{31} = bf - ce$$

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$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad M_{32} = \begin{bmatrix} a & c \\ d & f \end{bmatrix} \qquad m_{32} = af - cd$$

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Expanding the determinant

Proposition 12.13: $\det(A)$ can be "expanded along the first row":

$$\det(A) = a_{11}m_{11} - a_{12}m_{12} + \cdots \pm a_{1n}m_{1n} = \sum_{j=1}^{n}(-1)^{1+j}a_{1j}m_{1j}.$$

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$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = +a\det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b\det\begin{bmatrix} d & f \\ g & i \end{bmatrix}$$

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Similarly, it can be expanded down the q'th column for any q, in the sense that

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Example of expanding a determinant

Consider $\det(A)$, where

$$A = \begin{bmatrix} a & 0 & b & c \\ 0 & 0 & 0 & d \\ e & f & g & h \\ i & 0 & j & k \end{bmatrix}$$

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- Expand $\det(B)$ down the middle column

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- Expand $\det(A)$ along the second row to get $\det(A) = (-1)^{2+4} d \det(B) = d \det(B)$.
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- Expand $\det(B)$ down the middle column to get $\det(B) = (-1)^{2+2} f \det(C) = f \det(C)$
- $\det(C) = aj - bi$
- So $\det(A) = df(aj - bi) = adfi - bdfj$.

Example of expanding a determinant

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Expanding along the top row gives

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where

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Consider the matrix $U = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

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$$\det(U) = \det(V_1) - 0 \times \det(V_2) + \det(V_3) - 0 \times \det(V_4)$$

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In V_1 the first and last rows are the same, so after a single row operation we have a row of zeros, which means that $\det(V_1) = 0$.

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Consider the matrix $U = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Expanding along the top row gives

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In V_1 the first and last rows are the same, so after a single row operation we have a row of zeros, which means that $\det(V_1) = 0$. We need not work out $\det(V_2)$ and $\det(V_4)$ because they will be multiplied by zero anyway.

Example of expanding a determinant

Consider the matrix $U = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$
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In V_1 the first and last rows are the same, so after a single row operation we have a row of zeros, which means that $\det(V_1) = 0$. We need not work out $\det(V_2)$ and $\det(V_4)$ because they will be multiplied by zero anyway. This just leaves $\det(U) = \det(V_3)$, which we can expand along the top row again:

$$\det(V_3) = 0 \times \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0 - 1 + 0 = -1.$$

Example of expanding a determinant

Consider the matrix $U = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Expanding along the top row gives

$$\det(U) = \det(V_1) - 0 \times \det(V_2) + \det(V_3) - 0 \times \det(V_4)$$

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In V_1 the first and last rows are the same, so after a single row operation we have a row of zeros, which means that $\det(V_1) = 0$. We need not work out $\det(V_2)$ and $\det(V_4)$ because they will be multiplied by zero anyway. This just leaves $\det(U) = \det(V_3)$, which we can expand along the top row again:

$$\det(V_3) = 0 \times \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0 - 1 + 0 = -1.$$

We conclude that $\det(U) = -1$.

Theorem 12.16: Let A be an $n \times n$ matrix.

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(b) If $\det(A) = 0$ then A has no inverse; also, there exists a vector $v \neq 0$ with $Av = 0$.

Example 12.17: For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the minor matrices are 1×1 matrices or in other words just numbers, so $m_{ij} = \det(M_{ij}) = M_{ij}$.

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also, the only $v \in \mathbb{R}^n$ with $Av = 0$ is $v = 0$.

(b) If $\det(A) = 0$ then A has no inverse;
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Example 12.17: For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the minor matrices are 1×1 matrices or in other words just numbers, so $m_{ij} = \det(M_{ij}) = M_{ij}$.

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so we get the well-known formula

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$$A^{-1} = \operatorname{adj}(A)/\det(A) = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The inverse of an upper-triangular matrix

Consider an upper triangular matrix $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$.

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$$\mathrm{adj}(A) = \begin{bmatrix} +m_{11} & -m_{21} & +m_{31} \\ -m_{12} & +m_{22} & -m_{32} \\ +m_{13} & -m_{23} & +m_{33} \end{bmatrix} = \begin{bmatrix} 1 & -a & \\ 0 & & \\ 0 & & \end{bmatrix}.$$

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$$m_{21} = \det \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = a \qquad m_{22} = \det \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = 1 \qquad m_{23} = \det \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} = 0$$

$$m_{31} = \det \begin{bmatrix} a & b \\ 1 & c \end{bmatrix} = ac - b$$

$$\operatorname{adj}(A) = \begin{bmatrix} +m_{11} & -m_{21} & +m_{31} \\ -m_{12} & +m_{22} & -m_{32} \\ +m_{13} & -m_{23} & +m_{33} \end{bmatrix} = \begin{bmatrix} 1 & -a & ac - b \\ 0 & 1 & \\ 0 & 0 & \end{bmatrix}.$$

The inverse of an upper-triangular matrix

Consider an upper triangular matrix $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$.

This has $\det(A) = 1$ by Example 12.4. The minor determinants are

$$m_{11} = \det\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} = 1 \qquad m_{12} = \det\begin{bmatrix} 0 & c \\ 0 & 1 \end{bmatrix} = 0 \qquad m_{13} = \det\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0$$

$$m_{21} = \det\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = a \qquad m_{22} = \det\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = 1 \qquad m_{23} = \det\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} = 0$$

$$m_{31} = \det\begin{bmatrix} a & b \\ 1 & c \end{bmatrix} = ac - b \qquad m_{32} = \det\begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix} = c$$

$$\operatorname{adj}(A) = \begin{bmatrix} +m_{11} & -m_{21} & +m_{31} \\ -m_{12} & +m_{22} & -m_{32} \\ +m_{13} & -m_{23} & +m_{33} \end{bmatrix} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & \end{bmatrix}.$$

The inverse of an upper-triangular matrix

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This has $\det(A) = 1$ by Example 12.4. The minor determinants are

$$
\begin{aligned}
m_{11} &= \det \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} = 1 & m_{12} &= \det \begin{bmatrix} 0 & c \\ 0 & 1 \end{bmatrix} = 0 & m_{13} &= \det \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0 \\
m_{21} &= \det \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = a & m_{22} &= \det \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = 1 & m_{23} &= \det \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} = 0 \\
m_{31} &= \det \begin{bmatrix} a & b \\ 1 & c \end{bmatrix} = ac - b & m_{32} &= \det \begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix} = c & m_{33} &= \det \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = 1
\end{aligned}
$$

$$
\operatorname{adj}(A) = \begin{bmatrix} +m_{11} & -m_{21} & +m_{31} \\ -m_{12} & +m_{22} & -m_{32} \\ +m_{13} & -m_{23} & +m_{33} \end{bmatrix} = \begin{bmatrix} 1 & -a & ac - b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}.
$$

We also have $A^{-1} = \operatorname{adj}(A)/\det(A)$

The inverse of an upper-triangular matrix

Consider an upper triangular matrix $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$.

This has $\det(A) = 1$ by Example 12.4. The minor determinants are

$$m_{11} = \det\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} = 1 \qquad m_{12} = \det\begin{bmatrix} 0 & c \\ 0 & 1 \end{bmatrix} = 0 \qquad m_{13} = \det\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0$$

$$m_{21} = \det\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = a \qquad m_{22} = \det\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = 1 \qquad m_{23} = \det\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} = 0$$

$$m_{31} = \det\begin{bmatrix} a & b \\ 1 & c \end{bmatrix} = ac - b \qquad m_{32} = \det\begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix} = c \qquad m_{33} = \det\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = 1$$

$$\operatorname{adj}(A) = \begin{bmatrix} +m_{11} & -m_{21} & +m_{31} \\ -m_{12} & +m_{22} & -m_{32} \\ +m_{13} & -m_{23} & +m_{33} \end{bmatrix} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}.$$

We also have $A^{-1} = \operatorname{adj}(A)/\det(A)$ but $\det(A) = 1$

The inverse of an upper-triangular matrix

Consider an upper triangular matrix $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$.

This has $\det(A) = 1$ by Example 12.4. The minor determinants are

$$m_{11} = \det \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} = 1 \qquad m_{12} = \det \begin{bmatrix} 0 & c \\ 0 & 1 \end{bmatrix} = 0 \qquad m_{13} = \det \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0$$

$$m_{21} = \det \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = a \qquad m_{22} = \det \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = 1 \qquad m_{23} = \det \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} = 0$$

$$m_{31} = \det \begin{bmatrix} a & b \\ 1 & c \end{bmatrix} = ac - b \qquad m_{32} = \det \begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix} = c \qquad m_{33} = \det \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = 1$$

$$\operatorname{adj}(A) = \begin{bmatrix} +m_{11} & -m_{21} & +m_{31} \\ -m_{12} & +m_{22} & -m_{32} \\ +m_{13} & -m_{23} & +m_{33} \end{bmatrix} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}.$$

We also have $A^{-1} = \operatorname{adj}(A)/\det(A)$ but $\det(A) = 1$ so $A^{-1} = \operatorname{adj}(A)$.

The inverse of an upper-triangular matrix

Consider an upper triangular matrix $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$.

This has $\det(A) = 1$ by Example 12.4. The minor determinants are

$$
\begin{aligned}
m_{11} &= \det \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} = 1 & m_{12} &= \det \begin{bmatrix} 0 & c \\ 0 & 1 \end{bmatrix} = 0 & m_{13} &= \det \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0 \\
m_{21} &= \det \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = a & m_{22} &= \det \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = 1 & m_{23} &= \det \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} = 0 \\
m_{31} &= \det \begin{bmatrix} a & b \\ 1 & c \end{bmatrix} = ac - b & m_{32} &= \det \begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix} = c & m_{33} &= \det \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = 1
\end{aligned}
$$

$$
\operatorname{adj}(A) = \begin{bmatrix} +m_{11} & -m_{21} & +m_{31} \\ -m_{12} & +m_{22} & -m_{32} \\ +m_{13} & -m_{23} & +m_{33} \end{bmatrix} = \begin{bmatrix} 1 & -a & ac - b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}.
$$

We also have $A^{-1} = \operatorname{adj}(A)/\det(A)$ but $\det(A) = 1$ so $A^{-1} = \operatorname{adj}(A)$. Note that this is the same answer as we obtained in Example 11.12.

Inverse of a Jordan block

Consider the matrix $P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Inverse of a Jordan block

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$$M_{11} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{13} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{14} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{22} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{23} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{24} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{33} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{34} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{41} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad M_{42} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad M_{43} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{44} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse of a Jordan block

Consider the matrix $P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The minor matrices are:

$$M_{11} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{13} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{14} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{22} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{23} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{24} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{33} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{34} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{41} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad M_{42} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad M_{43} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{44} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Each of these matrices is either upper triangular or lower triangular, so the determinant is the product of the diagonal entries.

Inverse of a Jordan block

Consider the matrix

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The corresponding minor determinants are as follows:

$$\begin{aligned} m_{11} &= 1 & m_{12} &= 0 & m_{13} &= 0 & m_{14} &= 0 \\ m_{21} &= 1 & m_{22} &= 1 & m_{23} &= 0 & m_{24} &= 0 \\ m_{31} &= 1 & m_{32} &= 1 & m_{33} &= 1 & m_{34} &= 0 \\ m_{41} &= 1 & m_{42} &= 1 & m_{43} &= 1 & m_{44} &= 1 \end{aligned}$$

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and thus

$$\operatorname{adj}(P) = \begin{bmatrix} +m_{11} & -m_{21} & +m_{31} & -m_{41} \\ -m_{12} & +m_{22} & -m_{32} & +m_{42} \\ +m_{13} & -m_{23} & +m_{33} & -m_{43} \\ -m_{14} & +m_{24} & -m_{34} & +m_{44} \end{bmatrix}$$

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As P is upper triangular it is easy to see that $\det(P) = 1$

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As P is upper triangular it is easy to see that $\det(P) = 1$ and so P^{-1} is the same as $\operatorname{adj}(P)$.

Lecture 8

Definition 13.1: Let A be an $n \times n$ matrix, and let λ be a real number.

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- This is for *square* matrices only.

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Eigenvalues and eigenvectors

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Eigenvector example

Consider the case

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Eigenvector example

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$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We have

$$Aa = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

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so a is a 2-eigenvector and b is a 0-eigenvector

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Subtract $1 - \lambda$ times row 2 from row 1 to get $\begin{bmatrix} 0 & 1-(1-\lambda)^2 \\ 1 & 1-\lambda \end{bmatrix}$.

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so a is a 2-eigenvector and b is a 0-eigenvector, so 2 and 0 are eigenvalues. We claim that these are the only eigenvalues, or equivalently that when $\lambda \not\in \{0,2\}$ the only solution to $(A-\lambda I_2)v = 0$ is $v = 0$, or equivalently that the matrix $A - \lambda I_2 = \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$ row-reduces to I_2.

Subtract $1-\lambda$ times row 2 from row 1 to get $\begin{bmatrix} 0 & 1-(1-\lambda)^2 \\ 1 & 1-\lambda \end{bmatrix}$.

Here $1-(1-\lambda)^2 = 2\lambda - \lambda^2 = \lambda(2-\lambda)$

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$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We have

$$Aa = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2a \qquad Ab = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0b$$

so a is a 2-eigenvector and b is a 0-eigenvector, so 2 and 0 are eigenvalues. We claim that these are the only eigenvalues, or equivalently that when $\lambda \notin \{0, 2\}$ the only solution to $(A - \lambda I_2)v = 0$ is $v = 0$, or equivalently that the matrix $A - \lambda I_2 = \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$ row-reduces to I_2.

Subtract $1 - \lambda$ times row 2 from row 1 to get $\begin{bmatrix} 0 & 1-(1-\lambda)^2 \\ 1 & 1-\lambda \end{bmatrix}$.

Here $1 - (1-\lambda)^2 = 2\lambda - \lambda^2 = \lambda(2 - \lambda)$, which is nonzero because $\lambda \notin \{0, 2\}$.

Eigenvector example

Consider the case

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We have

$$Aa = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2a \qquad Ab = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0b$$

so a is a 2-eigenvector and b is a 0-eigenvector, so 2 and 0 are eigenvalues. We claim that these are the only eigenvalues, or equivalently that when $\lambda \notin \{0, 2\}$ the only solution to $(A - \lambda I_2)v = 0$ is $v = 0$, or equivalently that the matrix $A - \lambda I_2 = \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$ row-reduces to I_2.

Subtract $1 - \lambda$ times row 2 from row 1 to get $\begin{bmatrix} 0 & 1-(1-\lambda)^2 \\ 1 & 1-\lambda \end{bmatrix}$.

Here $1 - (1-\lambda)^2 = 2\lambda - \lambda^2 = \lambda(2-\lambda)$, which is nonzero because $\lambda \notin \{0, 2\}$.

Divide the row 1 by this to get $\begin{bmatrix} 0 & 1 \\ 1 & 1-\lambda \end{bmatrix}$

Eigenvector example

Consider the case

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We have

$$Aa = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2a \qquad Ab = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0b$$

so a is a 2-eigenvector and b is a 0-eigenvector, so 2 and 0 are eigenvalues. We claim that these are the only eigenvalues, or equivalently that when $\lambda \not\in \{0, 2\}$ the only solution to $(A - \lambda I_2)v = 0$ is $v = 0$, or equivalently that the matrix $A - \lambda I_2 = \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$ row-reduces to I_2.

Subtract $1 - \lambda$ times row 2 from row 1 to get $\begin{bmatrix} 0 & 1-(1-\lambda)^2 \\ 1 & 1-\lambda \end{bmatrix}$.

Here $1 - (1-\lambda)^2 = 2\lambda - \lambda^2 = \lambda(2 - \lambda)$, which is nonzero because $\lambda \not\in \{0, 2\}$.

Divide the row 1 by this to get $\begin{bmatrix} 0 & 1 \\ 1 & 1-\lambda \end{bmatrix}$; more steps then give $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$.

Eigenvector example

Consider

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Eigenvector example

Consider

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 0 \end{bmatrix} \qquad d = \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix}.$$

Eigenvector example

Consider

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 0 \end{bmatrix} \qquad d = \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix}.$$

We have

$$Ad = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix}$$

Eigenvector example

Consider

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 0 \end{bmatrix} \qquad d = \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix}.$$

We have

$$Ad = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 32 \\ 48 \\ 36 \\ 12 \end{bmatrix}$$

Eigenvector example

Consider

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 0 \end{bmatrix} \qquad d = \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix}.$$

We have

$$Ad = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 32 \\ 48 \\ 36 \\ 12 \end{bmatrix} = 4d$$

Eigenvector example

Consider

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 0 \end{bmatrix} \qquad d = \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix}.$$

We have

$$Ad = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 32 \\ 48 \\ 36 \\ 12 \end{bmatrix} = 4d,$$

which means that d is a 4-eigenvector for A

Eigenvector example

Consider

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 0 \end{bmatrix} \qquad d = \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix}.$$

We have

$$Ad = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 32 \\ 48 \\ 36 \\ 12 \end{bmatrix} = 4d,$$

which means that d is a 4-eigenvector for A, and 4 is an eigenvalue of A.

Eigenvector example

Consider

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 0 \end{bmatrix} \qquad d = \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix}.$$

We have

$$Ad = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 32 \\ 48 \\ 36 \\ 12 \end{bmatrix} = 4d,$$

which means that d is a 4-eigenvector for A, and 4 is an eigenvalue of A. Equally direct calculation shows that $Aa = a$ and $Ab = 2b$ and $Ac = 3c$

Eigenvector example

Consider

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 0 \end{bmatrix} \qquad d = \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix}.$$

We have

$$Ad = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 32 \\ 48 \\ 36 \\ 12 \end{bmatrix} = 4d,$$

which means that d is a 4-eigenvector for A, and 4 is an eigenvalue of A. Equally direct calculation shows that $Aa = a$ and $Ab = 2b$ and $Ac = 3c$, so a, b and c are also eigenvectors

Eigenvector example

Consider

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 0 \end{bmatrix} \qquad d = \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix}.$$

We have

$$Ad = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 32 \\ 48 \\ 36 \\ 12 \end{bmatrix} = 4d,$$

which means that d is a 4-eigenvector for A, and 4 is an eigenvalue of A. Equally direct calculation shows that $Aa = a$ and $Ab = 2b$ and $Ac = 3c$, so a, b and c are also eigenvectors, and 1, 2 and 3 are also eigenvalues of A.

Eigenvector example

Consider

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 0 \end{bmatrix} \qquad d = \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix}.$$

We have

$$Ad = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 32 \\ 48 \\ 36 \\ 12 \end{bmatrix} = 4d,$$

which means that d is a 4-eigenvector for A, and 4 is an eigenvalue of A. Equally direct calculation shows that $Aa = a$ and $Ab = 2b$ and $Ac = 3c$, so a, b and c are also eigenvectors, and 1, 2 and 3 are also eigenvalues of A. Using the general theory that we will discuss below, we can show that

(a) The only 1-eigenvectors are the nonzero multiples of a.

Eigenvector example

Consider

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 0 \end{bmatrix} \qquad d = \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix}.$$

We have

$$Ad = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 32 \\ 48 \\ 36 \\ 12 \end{bmatrix} = 4d,$$

which means that d is a 4-eigenvector for A, and 4 is an eigenvalue of A. Equally direct calculation shows that $Aa = a$ and $Ab = 2b$ and $Ac = 3c$, so a, b and c are also eigenvectors, and 1, 2 and 3 are also eigenvalues of A. Using the general theory that we will discuss below, we can show that

(a) The only 1-eigenvectors are the nonzero multiples of a.

(b) The only 2-eigenvectors are the nonzero multiples of b.

Eigenvector example

Consider

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 0 \end{bmatrix} \qquad d = \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix}.$$

We have

$$Ad = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 32 \\ 48 \\ 36 \\ 12 \end{bmatrix} = 4d,$$

which means that d is a 4-eigenvector for A, and 4 is an eigenvalue of A. Equally direct calculation shows that $Aa = a$ and $Ab = 2b$ and $Ac = 3c$, so a, b and c are also eigenvectors, and 1, 2 and 3 are also eigenvalues of A. Using the general theory that we will discuss below, we can show that

(a) The only 1-eigenvectors are the nonzero multiples of a.
(b) The only 2-eigenvectors are the nonzero multiples of b.
(c) The only 3-eigenvectors are the nonzero multiples of c.

Eigenvector example

Consider

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 0 \end{bmatrix} \qquad d = \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix}.$$

We have

$$Ad = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 32 \\ 48 \\ 36 \\ 12 \end{bmatrix} = 4d,$$

which means that d is a 4-eigenvector for A, and 4 is an eigenvalue of A. Equally direct calculation shows that $Aa = a$ and $Ab = 2b$ and $Ac = 3c$, so a, b and c are also eigenvectors, and 1, 2 and 3 are also eigenvalues of A. Using the general theory that we will discuss below, we can show that

(a) The only 1-eigenvectors are the nonzero multiples of a.
(b) The only 2-eigenvectors are the nonzero multiples of b.
(c) The only 3-eigenvectors are the nonzero multiples of c.
(d) The only 4-eigenvectors are the nonzero multiples of d.

Eigenvector example

Consider

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 0 \end{bmatrix} \qquad d = \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix}.$$

We have

$$Ad = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 32 \\ 48 \\ 36 \\ 12 \end{bmatrix} = 4d,$$

which means that d is a 4-eigenvector for A, and 4 is an eigenvalue of A. Equally direct calculation shows that $Aa = a$ and $Ab = 2b$ and $Ac = 3c$, so a, b and c are also eigenvectors, and 1, 2 and 3 are also eigenvalues of A. Using the general theory that we will discuss below, we can show that

(a) The only 1-eigenvectors are the nonzero multiples of a.
(b) The only 2-eigenvectors are the nonzero multiples of b.
(c) The only 3-eigenvectors are the nonzero multiples of c.
(d) The only 4-eigenvectors are the nonzero multiples of d.
(e) There are no more eigenvalues

Eigenvector example

Consider

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 0 \end{bmatrix} \qquad d = \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix}.$$

We have

$$Ad = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 32 \\ 48 \\ 36 \\ 12 \end{bmatrix} = 4d,$$

which means that d is a 4-eigenvector for A, and 4 is an eigenvalue of A. Equally direct calculation shows that $Aa = a$ and $Ab = 2b$ and $Ac = 3c$, so a, b and c are also eigenvectors, and 1, 2 and 3 are also eigenvalues of A. Using the general theory that we will discuss below, we can show that

(a) The only 1-eigenvectors are the nonzero multiples of a.
(b) The only 2-eigenvectors are the nonzero multiples of b.
(c) The only 3-eigenvectors are the nonzero multiples of c.
(d) The only 4-eigenvectors are the nonzero multiples of d.
(e) There are no more eigenvalues: if λ is a real number other than 1, 2, 3 and 4, then the equation $Av = \lambda v$ has $v = 0$ as the only solution, so there are no λ-eigenvectors.

Definition 13.8: Let A be an $n \times n$ matrix. We define $\chi_A(t) = \det(A - t\, I_n)$ (where I_n is the identity matrix). This is the *characteristic polynomial* of A.

The characteristic polynomial

Definition 13.8: Let A be an $n \times n$ matrix. We define $\chi_A(t) = \det(A - t\,I_n)$ (where I_n is the identity matrix). This is the *characteristic polynomial* of A.

Example 13.9: For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Definition 13.8: Let A be an $n \times n$ matrix. We define $\chi_A(t) = \det(A - t\, I_n)$ (where I_n is the identity matrix). This is the *characteristic polynomial* of A.

Example 13.9: For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have $A - tI_2 = \begin{bmatrix} a - t & b \\ c & d - t \end{bmatrix}$

Definition 13.8: Let A be an $n \times n$ matrix. We define $\chi_A(t) = \det(A - t\, I_n)$ (where I_n is the identity matrix). This is the *characteristic polynomial* of A.

Example 13.9: For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have $A - tI_2 = \begin{bmatrix} a-t & b \\ c & d-t \end{bmatrix}$ so

$$\chi_A(t) = \det \begin{bmatrix} a-t & b \\ c & d-t \end{bmatrix}$$

Definition 13.8: Let A be an $n \times n$ matrix. We define $\chi_A(t) = \det(A - t\, I_n)$ (where I_n is the identity matrix). This is the *characteristic polynomial* of A.

Example 13.9: For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have $A - tI_2 = \begin{bmatrix} a-t & b \\ c & d-t \end{bmatrix}$ so

$$\chi_A(t) = \det \begin{bmatrix} a-t & b \\ c & d-t \end{bmatrix} = (a-t)(d-t) - bc$$

The characteristic polynomial

Definition 13.8: Let A be an $n \times n$ matrix. We define $\chi_A(t) = \det(A - t\, I_n)$ (where I_n is the identity matrix). This is the *characteristic polynomial* of A.

Example 13.9: For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have $A - tI_2 = \begin{bmatrix} a-t & b \\ c & d-t \end{bmatrix}$ so

$$\chi_A(t) = \det \begin{bmatrix} a-t & b \\ c & d-t \end{bmatrix} = (a-t)(d-t) - bc = t^2 - (a+d)t + (ad - bc).$$

Definition 13.8: Let A be an $n \times n$ matrix. We define $\chi_A(t) = \det(A - t\, I_n)$ (where I_n is the identity matrix). This is the *characteristic polynomial* of A.

Example 13.9: For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have $A - tI_2 = \begin{bmatrix} a-t & b \\ c & d-t \end{bmatrix}$ so

$$\chi_A(t) = \det \begin{bmatrix} a-t & b \\ c & d-t \end{bmatrix} = (a-t)(d-t) - bc = t^2 - (a+d)t + (ad - bc).$$

When $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ we have

$$\chi_A(t) = t^2 - (1+4)t + (1 \times 4 - 2 \times 3) = t^2 - 5t - 2.$$

Definition 13.8: Let A be an $n \times n$ matrix. We define $\chi_A(t) = \det(A - t\, I_n)$ (where I_n is the identity matrix). This is the *characteristic polynomial* of A.

Example 13.9: For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have $A - tI_2 = \begin{bmatrix} a-t & b \\ c & d-t \end{bmatrix}$ so

$$\chi_A(t) = \det \begin{bmatrix} a-t & b \\ c & d-t \end{bmatrix} = (a-t)(d-t) - bc = t^2 - (a+d)t + (ad - bc).$$

When $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ we have

$$\chi_A(t) = t^2 - (1+4)t + (1 \times 4 - 2 \times 3) = t^2 - 5t - 2.$$

Theorem 13.11: The eigenvalues of A are the roots of the characteristic polynomial.

Characteristic polynomial example

Consider $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 3 & -1 \\ 2 & -1 & 2 \end{bmatrix}$

Characteristic polynomial example

Consider $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 3 & -1 \\ 2 & -1 & 2 \end{bmatrix}$, so $\chi_A(t) = \det \begin{bmatrix} 2-t & -1 & 2 \\ -1 & 3-t & -1 \\ 2 & -1 & 2-t \end{bmatrix}$

Characteristic polynomial example

Consider $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 3 & -1 \\ 2 & -1 & 2 \end{bmatrix}$, so $\chi_A(t) = \det \begin{bmatrix} 2-t & -1 & 2 \\ -1 & 3-t & -1 \\ 2 & -1 & 2-t \end{bmatrix}$

$$= (2-t)\det\begin{bmatrix} 3-t & -1 \\ -1 & 2-t \end{bmatrix} - (-1)\det\begin{bmatrix} -1 & -1 \\ 2 & 2-t \end{bmatrix} + 2\det\begin{bmatrix} -1 & 3-t \\ 2 & -1 \end{bmatrix}$$

Characteristic polynomial example

Consider $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 3 & -1 \\ 2 & -1 & 2 \end{bmatrix}$, so $\chi_A(t) = \det \begin{bmatrix} 2-t & -1 & 2 \\ -1 & 3-t & -1 \\ 2 & -1 & 2-t \end{bmatrix}$

$$= (2-t)\det \begin{bmatrix} 3-t & -1 \\ -1 & 2-t \end{bmatrix} - (-1)\det \begin{bmatrix} -1 & -1 \\ 2 & 2-t \end{bmatrix} + 2\det \begin{bmatrix} -1 & 3-t \\ 2 & -1 \end{bmatrix}$$

$$\det \begin{bmatrix} 3-t & -1 \\ -1 & 2-t \end{bmatrix} = (3-t)(2-t) - (-1)(-1) = t^2 - 5t + 5$$

Characteristic polynomial example

Consider $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 3 & -1 \\ 2 & -1 & 2 \end{bmatrix}$, so $\chi_A(t) = \det \begin{bmatrix} 2-t & -1 & 2 \\ -1 & 3-t & -1 \\ 2 & -1 & 2-t \end{bmatrix}$

$$= (2-t)\det\begin{bmatrix} 3-t & -1 \\ -1 & 2-t \end{bmatrix} - (-1)\det\begin{bmatrix} -1 & -1 \\ 2 & 2-t \end{bmatrix} + 2\det\begin{bmatrix} -1 & 3-t \\ 2 & -1 \end{bmatrix}$$

$$\det\begin{bmatrix} 3-t & -1 \\ -1 & 2-t \end{bmatrix} = (3-t)(2-t) - (-1)(-1) = t^2 - 5t + 5$$

$$\det\begin{bmatrix} -1 & -1 \\ 2 & 2-t \end{bmatrix} = (-1)(2-t) - (-1)(2) = t$$

Characteristic polynomial example

Consider $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 3 & -1 \\ 2 & -1 & 2 \end{bmatrix}$, so $\chi_A(t) = \det \begin{bmatrix} 2-t & -1 & 2 \\ -1 & 3-t & -1 \\ 2 & -1 & 2-t \end{bmatrix}$

$$= (2-t)\det \begin{bmatrix} 3-t & -1 \\ -1 & 2-t \end{bmatrix} - (-1)\det \begin{bmatrix} -1 & -1 \\ 2 & 2-t \end{bmatrix} + 2\det \begin{bmatrix} -1 & 3-t \\ 2 & -1 \end{bmatrix}$$

$$\det \begin{bmatrix} 3-t & -1 \\ -1 & 2-t \end{bmatrix} = (3-t)(2-t) - (-1)(-1) = t^2 - 5t + 5$$

$$\det \begin{bmatrix} -1 & -1 \\ 2 & 2-t \end{bmatrix} = (-1)(2-t) - (-1)(2) = t$$

$$\det \begin{bmatrix} -1 & 3-t \\ 2 & -1 \end{bmatrix} = (-1)(-1) - (3-t)(2) = 2t - 5$$

Characteristic polynomial example

Consider $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 3 & -1 \\ 2 & -1 & 2 \end{bmatrix}$, so $\chi_A(t) = \det \begin{bmatrix} 2-t & -1 & 2 \\ -1 & 3-t & -1 \\ 2 & -1 & 2-t \end{bmatrix}$

$$= (2-t)\det \begin{bmatrix} 3-t & -1 \\ -1 & 2-t \end{bmatrix} - (-1)\det \begin{bmatrix} -1 & -1 \\ 2 & 2-t \end{bmatrix} + 2\det \begin{bmatrix} -1 & 3-t \\ 2 & -1 \end{bmatrix}$$

$$\begin{aligned}
\det \begin{bmatrix} 3-t & -1 \\ -1 & 2-t \end{bmatrix} &= (3-t)(2-t) - (-1)(-1) = t^2 - 5t + 5 \\
\det \begin{bmatrix} -1 & -1 \\ 2 & 2-t \end{bmatrix} &= (-1)(2-t) - (-1)(2) = t \\
\det \begin{bmatrix} -1 & 3-t \\ 2 & -1 \end{bmatrix} &= (-1)(-1) - (3-t)(2) = 2t - 5 \\
\chi_A(t) &= (2-t)(t^2 - 5t + 5) + t + 2(2t - 5)
\end{aligned}$$

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&= -t(t-2)(t-5).
\end{aligned}$$

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\end{aligned}$$

The eigenvalues of A are the roots of $\chi_A(t)$, namely 0, 2 and 5.

Eigenvalue example

Consider $A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$

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or ($y=0$ and $-x+y+z=0$ and $-x+z=0$). These equations reduce to $x=z$ with $y=0$, so $\begin{bmatrix} x & y & z \end{bmatrix} = z\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$. This means that the (-1)-eigenvectors are just the nonzero multiples of the vector $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$.

General method for eigenvectors

Method 13.14: Suppose we have an $n \times n$ matrix A, and we want to find the eigenvalues and eigenvectors.

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(d) Read off solutions to the equation $Bu = 0$ (which is easy because B is in RREF). These are the λ_i-eigenvectors of the matrix A.

Eigenvector example

Consider the matrix
$$A = \begin{bmatrix} 16 & 2 & 1 & 1 \\ 2 & 16 & 1 & 1 \\ 1 & 1 & 16 & 2 \\ 1 & 1 & 2 & 16 \end{bmatrix}$$

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$$\begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -3 & -3 \\ 0 & 0 & -3 & -3 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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If we write $u = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$, then the equation $Bu = 0$ just gives $a + b = c + d = 0$

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$$u = \begin{bmatrix} -b & b & -d & d \end{bmatrix}^T$$

for some $b, d \in \mathbb{R}$. The eigenvectors of eigenvalue 14 are precisely the nonzero vectors of the above form.

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$$A = \begin{bmatrix} 16 & 2 & 1 & 1 \\ 2 & 16 & 1 & 1 \\ 1 & 1 & 16 & 2 \\ 1 & 1 & 2 & 16 \end{bmatrix}$$

We will take it as given here that $\chi_A(t) = (t-14)^2(t-16)(t-20)$. Thus, the eigenvalues of A are 14, 16 and 20. To find the eigenvectors of eigenvalue 14, we write down the matrix $A - 14I_4$ and row-reduce it to get a matrix B as follows:

$$\begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -3 & -3 \\ 0 & 0 & -3 & -3 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If we write $u = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$, then the equation $Bu = 0$ just gives $a + b = c + d = 0$, so $a = -b$ and $c = -d$ (with b and d arbitrary), so

$$u = \begin{bmatrix} -b & b & -d & d \end{bmatrix}^T$$

for some $b, d \in \mathbb{R}$. The eigenvectors of eigenvalue 14 are precisely the nonzero vectors of the above form. (Recall that eigenvectors are nonzero, by definition.)

Nasty eigenvalues

Using Maple, we find that one eigenvalue of the matrix

$$A = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

Nasty eigenvalues

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$$A = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix} \qquad \text{is}$$

$$-1/2+1/12\,\sqrt{6}\sqrt{\frac{10\,\sqrt[3]{892+36\,\sqrt{597}}+\left(892+36\,\sqrt{597}\right)^{2/3}+28}{\sqrt[3]{892+36\,\sqrt{597}}}}+$$

$$1/12\,\sqrt{6}\sqrt{\frac{20\,\sqrt[3]{892+36\,\sqrt{597}}\sqrt{\frac{10\,\sqrt[3]{892+36\,\sqrt{597}}+\left(892+36\,\sqrt{597}\right)^{2/3}+28}{\sqrt[3]{892+36\,\sqrt{597}}}}-\sqrt{\frac{10\,\sqrt[3]{892+36\,\sqrt{597}}+\left(892+36\,\sqrt{597}\right)^{2/3}+28}{\sqrt[3]{892+36\,\sqrt{597}}}}\left(892+36\,\sqrt{}\right.}{\sqrt[3]{892+36\,\sqrt{597}}\sqrt{\frac{10\,\sqrt[3]{892+36\,\sqrt{597}}+\left(892+36\,\sqrt{59\right.}{\sqrt[3]{892+36\,\sqrt{597}}}}}}$$

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$$1/12\,\sqrt{6}\sqrt{\frac{20\,\sqrt[3]{892+36\,\sqrt{597}}\sqrt{\frac{10\,\sqrt[3]{892+36\,\sqrt{597}}+\left(892+36\,\sqrt{597}\right)^{2/3}+28}{\sqrt[3]{892+36\,\sqrt{597}}}}-\sqrt{\frac{10\,\sqrt[3]{892+36\,\sqrt{597}}+\left(892+36\,\sqrt{597}\right)^{2/3}+28}{\sqrt[3]{892+36\,\sqrt{597}}}}\left(892+36\,\sqrt{\cdots}\right.}{\sqrt[3]{892+36\,\sqrt{597}}\sqrt{\frac{10\,\sqrt[3]{892+36\,\sqrt{597}}+\left(892+36\,\sqrt{59\cdots}\right.}{\sqrt[3]{892+36\,\sqrt{597}}}}}}$$

This level of complexity is quite normal, even for matrices whose entries are all 0 or ± 1.

Nasty eigenvalues

Using Maple, we find that one eigenvalue of the matrix

$$A = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix} \qquad \text{is}$$

$$-1/2 + 1/12\,\sqrt{6}\sqrt{\frac{10\,\sqrt[3]{892+36\,\sqrt{597}} + \left(892+36\,\sqrt{597}\right)^{2/3} + 28}{\sqrt[3]{892+36\,\sqrt{597}}}} +$$

$$1/12\,\sqrt{6}\sqrt{\frac{20\,\sqrt[3]{892+36\,\sqrt{597}}\sqrt{\frac{10\,\sqrt[3]{892+36\,\sqrt{597}} + \left(892+36\,\sqrt{597}\right)^{2/3} + 28}{\sqrt[3]{892+36\,\sqrt{597}}}} - \sqrt{\frac{10\,\sqrt[3]{892+36\,\sqrt{597}} + \left(892+36\,\sqrt{597}\right)^{2/3} + 28}{\sqrt[3]{892+36\,\sqrt{597}}}}\left(892+36\,\sqrt{\right.}{\sqrt[3]{892+36\,\sqrt{597}}\sqrt{\frac{10\,\sqrt[3]{892+36\,\sqrt{597}} + \left(892+36\,\sqrt{59\right.}{\sqrt[3]{892+36\,\sqrt{597}}}}}}$$

This level of complexity is quite normal, even for matrices whose entries are all 0 or ± 1. Most examples in this course are carefully constructed to have simple eigenvalues and eigenvectors, but you should be aware that this is not typical.

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Using Maple, we find that one eigenvalue of the matrix

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Nasty eigenvalues

Using Maple, we find that one eigenvalue of the matrix

$$A = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix} \qquad \text{is}$$

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$$1/12\sqrt{6}\sqrt{\frac{20\sqrt[3]{892+36\sqrt{597}}\sqrt{\frac{10\sqrt[3]{892+36\sqrt{597}}+\left(892+36\sqrt{597}\right)^{2/3}+28}{\sqrt[3]{892+36\sqrt{597}}}}-\sqrt{\frac{10\sqrt[3]{892+36\sqrt{597}}+\left(892+36\sqrt{597}\right)^{2/3}+28}{\sqrt[3]{892+36\sqrt{597}}}}\left(892+36\sqrt{\right.}{\sqrt[3]{892+36\sqrt{597}}\sqrt{\frac{10\sqrt[3]{892+36\sqrt{597}}+\left(892+36\sqrt{59\right.}{\sqrt[3]{892+36\sqrt{597}}}}}}$$

This level of complexity is quite normal, even for matrices whose entries are all 0 or ± 1. Most examples in this course are carefully constructed to have simple eigenvalues and eigenvectors, but you should be aware that this is not typical. The methods that we discuss will work perfectly well for all matrices, but in practice we need to use computers to do the calculations. Also, it is rarely useful to work with exact expressions for the eigenvalues when they are as complicated as those above. Instead we should use the numerical approximation $\lambda \simeq 1.496698205$.

Eigenvector example

Consider $A = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$.

Eigenvector example

Consider $A = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$. We will take it as given that $\chi_A(t) = (t+1)(t+2)(t-2)(t-4)$

Eigenvector example

Consider $A = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$. We will take it as given that $\chi_A(t) = (t+1)(t+2)(t-2)(t-4)$, so the eigenvalues are -1, -2, 2 and 4.

Eigenvector example

Consider $A = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$. We will take it as given that $\chi_A(t) = (t+1)(t+2)(t-2)(t-4)$, so the eigenvalues are -1, -2, 2 and 4. To find the eigenvectors of eigenvalue 2, we write down the matrix $A - 2I_4$ and row-reduce it to get a matrix B in RREF:

Eigenvector example

Consider $A = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$. We will take it as given that $\chi_A(t) = (t+1)(t+2)(t-2)(t-4)$, so the eigenvalues are -1, -2, 2 and 4.
To find the eigenvectors of eigenvalue 2, we write down the matrix $A - 2I_4$ and row-reduce it to get a matrix B in RREF:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 2 & 0 & 0 & -2 \end{bmatrix}$$

Eigenvector example

Consider $A = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$. We will take it as given that $\chi_A(t) = (t+1)(t+2)(t-2)(t-4)$, so the eigenvalues are -1, -2, 2 and 4. To find the eigenvectors of eigenvalue 2, we write down the matrix $A - 2I_4$ and row-reduce it to get a matrix B in RREF:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 2 & 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

Eigenvector example

Consider $A = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$. We will take it as given that $\chi_A(t) = (t+1)(t+2)(t-2)(t-4)$, so the eigenvalues are -1, -2, 2 and 4. To find the eigenvectors of eigenvalue 2, we write down the matrix $A - 2I_4$ and row-reduce it to get a matrix B in RREF:

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Consider $A = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$. We will take it as given that $\chi_A(t) = (t+1)(t+2)(t-2)(t-4)$, so the eigenvalues are -1, -2, 2 and 4. To find the eigenvectors of eigenvalue 2, we write down the matrix $A - 2I_4$ and row-reduce it to get a matrix B in RREF:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 2 & 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

Eigenvector example

Consider $A = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$. We will take it as given that
$\chi_A(t) = (t+1)(t+2)(t-2)(t-4)$, so the eigenvalues are -1, -2, 2 and 4.
To find the eigenvectors of eigenvalue 2, we write down the matrix $A - 2I_4$ and row-reduce it to get a matrix B in RREF:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 2 & 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

If we write $u = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$, then the equation $Bu = 0$ just gives
$a = b - c = d = 0$

Eigenvector example

Consider $A = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$. We will take it as given that $\chi_A(t) = (t+1)(t+2)(t-2)(t-4)$, so the eigenvalues are -1, -2, 2 and 4. To find the eigenvectors of eigenvalue 2, we write down the matrix $A - 2I_4$ and row-reduce it to get a matrix B in RREF:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 2 & 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

If we write $u = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$, then the equation $Bu = 0$ just gives $a = b - c = d = 0$, so

$$u = \begin{bmatrix} 0 & c & c & 0 \end{bmatrix}^T = c \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T.$$

for some $c \in \mathbb{R}$.

Eigenvector example

Consider $A = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$. We will take it as given that $\chi_A(t) = (t+1)(t+2)(t-2)(t-4)$, so the eigenvalues are -1, -2, 2 and 4. To find the eigenvectors of eigenvalue 2, we write down the matrix $A - 2I_4$ and row-reduce it to get a matrix B in RREF:

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If we write $u = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$, then the equation $Bu = 0$ just gives $a = b - c = d = 0$, so

$$u = \begin{bmatrix} 0 & c & c & 0 \end{bmatrix}^T = c \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T.$$

for some $c \in \mathbb{R}$. The eigenvectors of eigenvalue 2 are precisely the nonzero vectors of the above form.

Eigenvector example

Consider $A = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$. We will take it as given that $\chi_A(t) = (t+1)(t+2)(t-2)(t-4)$, so the eigenvalues are -1, -2, 2 and 4. To find the eigenvectors of eigenvalue 2, we write down the matrix $A - 2I_4$ and row-reduce it to get a matrix B in RREF:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 2 & 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

If we write $u = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$, then the equation $Bu = 0$ just gives $a = b - c = d = 0$, so

$$u = \begin{bmatrix} 0 & c & c & 0 \end{bmatrix}^T = c \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T.$$

for some $c \in \mathbb{R}$. The eigenvectors of eigenvalue 2 are precisely the nonzero vectors of the above form. In particular, the vector $\begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$ is an eigenvector of eigenvalue 2.

Eigenvector example

We will find the eigenvalues and eigenvectors for $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 4 & 0 & 0 \end{bmatrix}$.

Eigenvector example

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$$\chi_A(t) = \det \begin{bmatrix} -t & 0 & 1 \\ 0 & 3-t & 0 \\ 4 & 0 & -t \end{bmatrix}$$

Eigenvector example

We will find the eigenvalues and eigenvectors for $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 4 & 0 & 0 \end{bmatrix}$.

$$\chi_A(t) = \det \begin{bmatrix} -t & 0 & 1 \\ 0 & 3-t & 0 \\ 4 & 0 & -t \end{bmatrix} = -t \det \begin{bmatrix} 3-t & 0 \\ 0 & -t \end{bmatrix} + \det \begin{bmatrix} 0 & 3-t \\ 4 & 0 \end{bmatrix}$$

Eigenvector example

We will find the eigenvalues and eigenvectors for $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 4 & 0 & 0 \end{bmatrix}$.

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$$= -t^3 + 3t^2 + 4t - 12$$

Eigenvector example

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$$= -t^3 + 3t^2 + 4t - 12 = (4 - t^2)(t-3)$$

Eigenvector example

We will find the eigenvalues and eigenvectors for $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 4 & 0 & 0 \end{bmatrix}$.

$$\chi_A(t) = \det \begin{bmatrix} -t & 0 & 1 \\ 0 & 3-t & 0 \\ 4 & 0 & -t \end{bmatrix} = -t \det \begin{bmatrix} 3-t & 0 \\ 0 & -t \end{bmatrix} + \det \begin{bmatrix} 0 & 3-t \\ 4 & 0 \end{bmatrix}$$

$$= -t^3 + 3t^2 + 4t - 12 = (4-t^2)(t-3) = -(t-2)(t+2)(t-3)$$

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$$= -t^3 + 3t^2 + 4t - 12 = (4-t^2)(t-3) = -(t-2)(t+2)(t-3)$$

Thus, the eigenvalues are -2, 2 and 3.

Eigenvector example

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$$= -t^3 + 3t^2 + 4t - 12 = (4-t^2)(t-3) = -(t-2)(t+2)(t-3)$$

Thus, the eigenvalues are -2, 2 and 3.
For the eigenvectors $\begin{bmatrix} a & b & c \end{bmatrix}^T$ of eigenvalue -2:

$$A + 2I_3 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvector example

We will find the eigenvalues and eigenvectors for $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 4 & 0 & 0 \end{bmatrix}$.

$$\chi_A(t) = \det \begin{bmatrix} -t & 0 & 1 \\ 0 & 3-t & 0 \\ 4 & 0 & -t \end{bmatrix} = -t \det \begin{bmatrix} 3-t & 0 \\ 0 & -t \end{bmatrix} + \det \begin{bmatrix} 0 & 3-t \\ 4 & 0 \end{bmatrix}$$
$$= -t^3 + 3t^2 + 4t - 12 = (4-t^2)(t-3) = -(t-2)(t+2)(t-3)$$

Thus, the eigenvalues are -2, 2 and 3.
For the eigenvectors $\begin{bmatrix} a & b & c \end{bmatrix}^T$ of eigenvalue -2:

$$A + 2I_3 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 4 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenvector example

We will find the eigenvalues and eigenvectors for $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 4 & 0 & 0 \end{bmatrix}$.

$$\chi_A(t) = \det \begin{bmatrix} -t & 0 & 1 \\ 0 & 3-t & 0 \\ 4 & 0 & -t \end{bmatrix} = -t \det \begin{bmatrix} 3-t & 0 \\ 0 & -t \end{bmatrix} + \det \begin{bmatrix} 0 & 3-t \\ 4 & 0 \end{bmatrix}$$
$$= -t^3 + 3t^2 + 4t - 12 = (4-t^2)(t-3) = -(t-2)(t+2)(t-3)$$

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For the eigenvectors $\begin{bmatrix} a & b & c \end{bmatrix}^T$ of eigenvalue -2:

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The eigenvectors of eigenvalue -2 are solutions to the equation

$$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvector example

We will find the eigenvalues and eigenvectors for $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 4 & 0 & 0 \end{bmatrix}$.

$$\chi_A(t) = \det \begin{bmatrix} -t & 0 & 1 \\ 0 & 3-t & 0 \\ 4 & 0 & -t \end{bmatrix} = -t \det \begin{bmatrix} 3-t & 0 \\ 0 & -t \end{bmatrix} + \det \begin{bmatrix} 0 & 3-t \\ 4 & 0 \end{bmatrix}$$
$$= -t^3 + 3t^2 + 4t - 12 = (4 - t^2)(t-3) = -(t-2)(t+2)(t-3)$$

Thus, the eigenvalues are -2, 2 and 3.
For the eigenvectors $\begin{bmatrix} a & b & c \end{bmatrix}^T$ of eigenvalue -2:

$$A + 2I_3 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 4 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The eigenvectors of eigenvalue -2 are solutions to the equation

$$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

or $a + c/2 = 0$ and $b = 0$.

Eigenvector example

We will find the eigenvalues and eigenvectors for $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 4 & 0 & 0 \end{bmatrix}$.

$$\chi_A(t) = \det \begin{bmatrix} -t & 0 & 1 \\ 0 & 3-t & 0 \\ 4 & 0 & -t \end{bmatrix} = -t \det \begin{bmatrix} 3-t & 0 \\ 0 & -t \end{bmatrix} + \det \begin{bmatrix} 0 & 3-t \\ 4 & 0 \end{bmatrix}$$
$$= -t^3 + 3t^2 + 4t - 12 = (4 - t^2)(t-3) = -(t-2)(t+2)(t-3)$$

Thus, the eigenvalues are -2, 2 and 3.
For the eigenvectors $\begin{bmatrix} a & b & c \end{bmatrix}^T$ of eigenvalue -2:

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The eigenvectors of eigenvalue -2 are solutions to the equation

$$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

or $a + c/2 = 0$ and $b = 0$. Take $c = 2$ to get the eigenvector $\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}^T$.

Eigenvector example

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 4 & 0 & 0 \end{bmatrix}$$

eigenvalues -2, 2 and 3

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This gives $a = c = 0$. Take $b = 1$ to get the eigenvector $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$.

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As the eigenvalues are all different $(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2) \neq 0$. As v_1 is an eigenvector it is nonzero. It follows that $\alpha_1 = 0$. Similarly, multiplying (P) by $(A - \lambda_1 I)(A - \lambda_3 I)$ makes the first and third terms go away leaving $\alpha_2(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)v_2 = 0$ and so $\alpha_2 = 0$. Similarly, multiplying (P) by $(A - \lambda_1 I)(A - \lambda_2 I)$ gives $\alpha_3(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)v_3 = 0$ and $\alpha_3 = 0$.

$\square$

Independence of eigenvectors

Proposition 13.19: Let A be a $d \times d$ matrix, and let $v_1, \ldots, v_n$ be eigenvectors of A. Suppose that the corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ are all different. Then the list $v_1, \ldots, v_n$ is linearly independent.

Proof for $n = 3$.

Suppose we have a linear relation $\qquad \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0. \qquad$ (P)

Multiply both sides by $A - \lambda_3 I$, then by $A - \lambda_2 I$

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We now see that $\alpha_1 = \alpha_2 = \alpha_3 = 0$, so relation (P) is the trivial relation.

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We can then combine the lists $\mathcal{V}_1, \dots, \mathcal{V}_r$ into a single list

$$\mathcal{W} = (v_{1,1}, \cdots, v_{1,h_1}, v_{2,1}, \cdots, v_{2,h_2}, \cdots, v_{r,1}, \cdots, v_{r,h_r}).$$

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The problem sheet asks you to prove this.

Lecture 9

Eigenvector bases

Let A be an $n \times n$ matrix.

Eigenvector bases

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(a) If $u_1, \ldots, u_k$ are eigenvectors, with eigenvalues $\lambda_1, \ldots, \lambda_k$, and these eigenvalues are all different, then the vectors $u_1, \ldots, u_k$ are independent.

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Eigenvector basis example

Consider $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

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Consider $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, so $\chi_A(t) = \det(A - tI) = (1-t)(2-t)(3-t)$

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$$
\begin{aligned}
A - I &= \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} & u_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
A - 2I &= \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
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$$

Eigenvector basis example

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Consider $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, so $\chi_A(t) = \det \begin{bmatrix} -t & 1 \\ -1 & -t \end{bmatrix} = t^2 + 1$.

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This example and the previous one are typical. If we pick an $n \times n$ matrix at random, it will usually have n different eigenvalues (some of which will usually be complex), and so the corresponding eigenvectors will form a basis for $\mathbb{C}^n$. However, there are some exceptions, as we will see soon. Such exceptions usually arise because of some symmetry or other interesting feature of the problem that gives rise to the matrix.

Eigenvector basis example

Consider $A = \begin{bmatrix} 5 & 5 & 0 \\ 0 & 5 & 5 \\ 0 & 0 & 5 \end{bmatrix}$

Eigenvector basis example

Consider $A = \begin{bmatrix} 5 & 5 & 0 \\ 0 & 5 & 5 \\ 0 & 0 & 5 \end{bmatrix}$, so $\chi_A(t) = (5-t)^3$

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Consider $A = \begin{bmatrix} 5 & 5 & 0 \\ 0 & 5 & 5 \\ 0 & 0 & 5 \end{bmatrix}$, so $\chi_A(t) = (5 - t)^3$, so the only eigenvalue is 5.

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The eigenvectors are the solutions of $\begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

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This means that any two eigenvectors are multiples of each other, and so are linearly dependent. Thus, we cannot find a basis consisting of eigenvectors.

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Eigenvector basis example

Consider $A = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 0 & 0 \end{bmatrix}$. The characteristic polynomial is

$$\chi_A(t) = \det \begin{bmatrix} -t & 0 & 5 \\ 0 & 5-t & 0 \\ 5 & 0 & -t \end{bmatrix}$$

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$$= t^2(5-t) - 25(5-t)$$

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The only eigenvalues (even in $\mathbb{C}$) are 5 and -5.

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$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (x+z)/2 \\ 0 \\ (x+z)/2 \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} (x-z)/2 \\ 0 \\ (z-x)/2 \end{bmatrix} = \frac{x+z}{2} u_1 + y u_2 + \frac{x-z}{2} u_3.$$

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- We say that A is *diagonalisable* if there exist matrices U and D with these properties.

Diagonalisation and eigenvectors

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The proof will be given after a lemma.

Lemma 14.5: Let A and U be $n \times n$ matrices, let $\lambda_1, \ldots, \lambda_n$ be real numbers, and put $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

A matrix multiplication lemma

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as claimed.

Proposition 14.4: Suppose we have a basis $u_1, \dots, u_n$ for $\mathbb{R}^n$ such that each vector u_i is an eigenvector for A, with eigenvalue λ_i say. Put $U = [u_1 | \cdots | u_n]$ and $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Then $U^{-1}AU = D$, so we have a diagonalisation of A. Moreover, every diagonalisation arises in this way.

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- It follows that $U^{-1}AU = U^{-1}UD$

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Conversely: suppose we have an invertible matrix U and a diagonal matrix $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ such that $U^{-1}AU = D$.

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$\square$

Diagonalisation and eigenvectors

Proposition 14.4: Suppose we have a basis $u_1, \ldots, u_n$ for $\mathbb{R}^n$ such that each vector u_i is an eigenvector for A, with eigenvalue λ_i say. Put $U = [u_1|\cdots|u_n]$ and $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Then $U^{-1}AU = D$, so we have a diagonalisation of A. Moreover, every diagonalisation arises in this way.

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Conversely: suppose we have an invertible matrix U and a diagonal matrix $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ such that $U^{-1}AU = D$. Let $u_1, \ldots, u_n$ be the columns of U. By reversing the above steps: u_i is an eigenvector of eigenvalue λ_i, and $u_1, \ldots, u_n$ is a basis for $\mathbb{R}^n$. $\square$

Diagonalisation example

Example 13.23: the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ and $\lambda_3 = 3$; and eigenvectors

$$u_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \qquad u_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T \qquad u_3 = \begin{bmatrix} 3/2 & 2 & 1 \end{bmatrix}^T.$$

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Diagonalisation example

In Example 13.24 we showed that the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ does not have any real eigenvalues or eigenvectors, but that over the complex numbers we have eigenvectors $u_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ and $u_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ with eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$.

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$$U = [u_1 | u_2]$$

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As expected, this is the same as A.

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Powers and eigenvectors

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This is a key point in many applications of eigenvalues and eigenvectors.

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We will diagonalise the matrix $A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -3 \end{bmatrix}$ and thus find A^k.

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- Multiply rows 3 and 4 by $1/t$ (factor $1/t^2$)

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$$B = A - tI_4 = \begin{bmatrix} 2-t & 2 & 2 & 2 \\ 2 & 5-t & 5 & 2 \\ 2 & 5 & 5-t & 2 \\ 2 & 2 & 2 & 2-t \end{bmatrix}.$$

Method 12.9: row-reduce B and keep track of row operation factors.

$$\begin{bmatrix} 2-t & 2 & 2 & 2 \\ 2 & 5-t & 5 & 2 \\ 2 & 5 & 5-t & 2 \\ 2 & 2 & 2 & 2-t \end{bmatrix} \to \begin{bmatrix} 2-t & 2 & 2 & 2 \\ 2 & 5-t & 5 & 2 \\ 0 & t & -t & 0 \\ t & 0 & 0 & -t \end{bmatrix} \xrightarrow{1/t^2} \begin{bmatrix} 2-t & 2 & 2 & 2 \\ 2 & 5-t & 5 & 2 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \to$$

$$\begin{bmatrix} 0 & 0 & 4 & 4-t \\ 0 & 0 & 10-t & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

- Subtract row 1 from row 4, and row 2 from row 3.
- Multiply rows 3 and 4 by $1/t$ (factor $1/t^2$)
- Subtract multiples of rows 3 and 4 from rows 1 and 2.

Diagonalisation example

We will diagonalise the matrix $A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}$. Recall $\chi_A(t) = \det(B)$,

where

$$B = A - tI_4 = \begin{bmatrix} 2-t & 2 & 2 & 2 \\ 2 & 5-t & 5 & 2 \\ 2 & 5 & 5-t & 2 \\ 2 & 2 & 2 & 2-t \end{bmatrix}.$$

Method 12.9: row-reduce B and keep track of row operation factors.

$$\begin{bmatrix} 2-t & 2 & 2 & 2 \\ 2 & 5-t & 5 & 2 \\ 2 & 5 & 5-t & 2 \\ 2 & 2 & 2 & 2-t \end{bmatrix} \to \begin{bmatrix} 2-t & 2 & 2 & 2 \\ 2 & 5-t & 5 & 2 \\ 0 & t & -t & 0 \\ t & 0 & 0 & -t \end{bmatrix} \xrightarrow{1/t^2} \begin{bmatrix} 2-t & 2 & 2 & 2 \\ 2 & 5-t & 5 & 2 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \to$$

$$\begin{bmatrix} 0 & 0 & 4 & 4-t \\ 0 & 0 & 10-t & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 10-t & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 4 & 4-t \end{bmatrix}$$

- Subtract row 1 from row 4, and row 2 from row 3.
- Multiply rows 3 and 4 by $1/t$ (factor $1/t^2$)
- Subtract multiples of rows 3 and 4 from rows 1 and 2.
- Swap rows 1 and 4 (factor -1);

Diagonalisation example

We will diagonalise the matrix $A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}$. Recall $\chi_A(t) = \det(B)$,

where

$$B = A - tI_4 = \begin{bmatrix} 2-t & 2 & 2 & 2 \\ 2 & 5-t & 5 & 2 \\ 2 & 5 & 5-t & 2 \\ 2 & 2 & 2 & 2-t \end{bmatrix}.$$

Method 12.9: row-reduce B and keep track of row operation factors.

$$\begin{bmatrix} 2-t & 2 & 2 & 2 \\ 2 & 5-t & 5 & 2 \\ 2 & 5 & 5-t & 2 \\ 2 & 2 & 2 & 2-t \end{bmatrix} \to \begin{bmatrix} 2-t & 2 & 2 & 2 \\ 2 & 5-t & 5 & 2 \\ 0 & t & -t & 0 \\ t & 0 & 0 & -t \end{bmatrix} \xrightarrow{1/t^2} \begin{bmatrix} 2-t & 2 & 2 & 2 \\ 2 & 5-t & 5 & 2 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \to$$

$$\begin{bmatrix} 0 & 0 & 4 & 4-t \\ 0 & 0 & 10-t & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 10-t & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 4 & 4-t \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}$$

- Subtract row 1 from row 4, and row 2 from row 3.
- Multiply rows 3 and 4 by $1/t$ (factor $1/t^2$)
- Subtract multiples of rows 3 and 4 from rows 1 and 2.
- Swap rows 1 and 4 (factor -1); Swap rows 2 and 3 (factor -1).

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix} \text{ ; product of factors } \mu = 1/t^2$$

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}; \text{ product of factors } \mu = 1/t^2$$

Expand down the columns to get

$$\det(C) = \det \begin{bmatrix} 10-t & 4 \\ 4 & 4-t \end{bmatrix}$$

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix} ;\ \text{product of factors } \mu = 1/t^2$$

Expand down the columns to get

$$\det(C) = \det \begin{bmatrix} 10-t & 4 \\ 4 & 4-t \end{bmatrix} = (10-t)(4-t)-16$$

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix};\ \text{product of factors } \mu = 1/t^2$$

Expand down the columns to get

$$\det(C) = \det \begin{bmatrix} 10-t & 4 \\ 4 & 4-t \end{bmatrix} = (10-t)(4-t)-16 = t^2-14t+24$$

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix} ; \text{ product of factors } \mu = 1/t^2$$

Expand down the columns to get

$$\det(C) = \det \begin{bmatrix} 10-t & 4 \\ 4 & 4-t \end{bmatrix} = (10-t)(4-t)-16 = t^2-14t+24 = (t-2)(t-12).$$

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}; \text{ product of factors } \mu = 1/t^2$$

Expand down the columns to get

$$\det(C) = \det \begin{bmatrix} 10-t & 4 \\ 4 & 4-t \end{bmatrix} = (10-t)(4-t)-16 = t^2-14t+24 = (t-2)(t-12).$$

Thus $\chi_A(t) = \det(B) = \det(C)/\mu = (t-2)(t-12)t^2$.

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix};\ \text{product of factors } \mu = 1/t^2$$

Expand down the columns to get

$$\det(C) = \det \begin{bmatrix} 10-t & 4 \\ 4 & 4-t \end{bmatrix} = (10-t)(4-t)-16 = t^2-14t+24 = (t-2)(t-12).$$

Thus $\chi_A(t) = \det(B) = \det(C)/\mu = (t-2)(t-12)t^2$.
This means that the eigenvalues of A are 2, 12 and 0.

Diagonalisation example

$$A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}$$

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix} ; \text{ product of factors } \mu = 1/t^2$$

$$\chi_A(t) = \det(B) = \det(C)/\mu = (t-2)(t-12)t^2$$

Eigenvalues are 0, 2 and 12.

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}; \text{ eigenvalues } 2, 12, 0.$$

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}; \text{ eigenvalues } 2, 12, 0.$$

To find an eigenvector of eigenvalue 2 we need to row-reduce the matrix $A - 2I_4$

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix};\ \text{eigenvalues } 2, 12, 0.$$

To find an eigenvector of eigenvalue 2 we need to row-reduce the matrix $A - 2I_4$, which is just the matrix B with $t = 2$.

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}; \text{ eigenvalues } 2, 12, 0.$$

To find an eigenvector of eigenvalue 2 we need to row-reduce the matrix $A - 2I_4$, which is just the matrix B with $t = 2$. We can therefore substitute $t = 2$ in C

$$A - 2I_4 \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 8 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix} \to$$

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}; \text{ eigenvalues } 2, 12, 0.$$

To find an eigenvector of eigenvalue 2 we need to row-reduce the matrix $A - 2I_4$, which is just the matrix B with $t = 2$. We can therefore substitute $t = 2$ in C and then perform a few more steps to complete the row-reduction.

$$A - 2I_4 \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 8 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix} ; \text{ eigenvalues } 2, 12, 0.$$

To find an eigenvector of eigenvalue 2 we need to row-reduce the matrix $A - 2I_4$, which is just the matrix B with $t = 2$. We can therefore substitute $t = 2$ in C and then perform a few more steps to complete the row-reduction.

$$A - 2I_4 \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 8 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}; \text{ eigenvalues } 2, 12, 0.$$

To find an eigenvector of eigenvalue 2 we need to row-reduce the matrix $A - 2I_4$, which is just the matrix B with $t = 2$. We can therefore substitute $t = 2$ in C and then perform a few more steps to complete the row-reduction.

$$A - 2I_4 \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 8 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvector $u_1 = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ of eigenvalue 2 must therefore satisfy $w - z = x + z/2 = y + z/2 = 0$

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}; \text{ eigenvalues } 2, 12, 0.$$

To find an eigenvector of eigenvalue 2 we need to row-reduce the matrix $A - 2I_4$, which is just the matrix B with $t = 2$. We can therefore substitute $t = 2$ in C and then perform a few more steps to complete the row-reduction.

$$A - 2I_4 \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 8 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvector $u_1 = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ of eigenvalue 2 must therefore satisfy $w - z = x + z/2 = y + z/2 = 0$, so $u_1 = z \begin{bmatrix} 1 & -1/2 & -1/2 & 1 \end{bmatrix}^T$, with z arbitrary.

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix};\ \text{eigenvalues } 2, 12, 0.$$

To find an eigenvector of eigenvalue 2 we need to row-reduce the matrix $A - 2I_4$, which is just the matrix B with $t = 2$. We can therefore substitute $t = 2$ in C and then perform a few more steps to complete the row-reduction.

$$A - 2I_4 \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 8 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvector $u_1 = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ of eigenvalue 2 must therefore satisfy $w - z = x + z/2 = y + z/2 = 0$, so $u_1 = z \begin{bmatrix} 1 & -1/2 & -1/2 & 1 \end{bmatrix}^T$, with z arbitrary. It will be convenient to take $z = 2$ so $u_1 = \begin{bmatrix} 2 & -1 & -1 & 2 \end{bmatrix}^T$.

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix};\ \text{eigenvalues } 2, 12, 0.$$

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}; \text{ eigenvalues } 2, 12, 0.$$

To find an eigenvector of eigenvalue 12 we need to row-reduce the matrix $A - 12I_4$

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}; \text{ eigenvalues } 2, 12, 0.$$

To find an eigenvector of eigenvalue 12 we need to row-reduce the matrix $A - 12I_4$, which is just the matrix B with $t = 12$.

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix};\ \text{eigenvalues } 2, 12, 0.$$

To find an eigenvector of eigenvalue 12 we need to row-reduce the matrix $A - 12I_4$, which is just the matrix B with $t = 12$. We can therefore substitute $t = 12$ in C

$$A - 12I_4 \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 4 & -8 \end{bmatrix}$$

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}; \text{ eigenvalues } 2, 12, 0.$$

To find an eigenvector of eigenvalue 12 we need to row-reduce the matrix $A - 12I_4$, which is just the matrix B with $t = 12$. We can therefore substitute $t = 12$ in C and then perform a few more steps to complete the row-reduction.

$$A - 12I_4 \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 4 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}; \text{ eigenvalues } 2, 12, 0.$$

To find an eigenvector of eigenvalue 12 we need to row-reduce the matrix $A - 12I_4$, which is just the matrix B with $t = 12$. We can therefore substitute $t = 12$ in C and then perform a few more steps to complete the row-reduction.

$$A - 12I_4 \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 4 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix} \text{; eigenvalues } 2, 12, 0.$$

To find an eigenvector of eigenvalue 12 we need to row-reduce the matrix $A - 12I_4$, which is just the matrix B with $t = 12$. We can therefore substitute $t = 12$ in C and then perform a few more steps to complete the row-reduction.

$$A - 12I_4 \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 4 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The eigenvector $u_2 = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ of eigenvalue 12 must therefore satisfy $w - z = x - 2z = y - 2z = 0$

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix};\ \text{eigenvalues } 2, 12, 0.$$

To find an eigenvector of eigenvalue 12 we need to row-reduce the matrix $A - 12I_4$, which is just the matrix B with $t = 12$. We can therefore substitute $t = 12$ in C and then perform a few more steps to complete the row-reduction.

$$A - 12I_4 \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 4 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The eigenvector $u_2 = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ of eigenvalue 12 must therefore satisfy $w - z = x - 2z = y - 2z = 0$, so $u_2 = z \begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix}^T$, with z arbitrary.

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}; \text{ eigenvalues } 2, 12, 0.$$

To find an eigenvector of eigenvalue 12 we need to row-reduce the matrix $A - 12I_4$, which is just the matrix B with $t = 12$. We can therefore substitute $t = 12$ in C and then perform a few more steps to complete the row-reduction.

$$A - 12I_4 \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 4 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The eigenvector $u_2 = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ of eigenvalue 12 must therefore satisfy $w - z = x - 2z = y - 2z = 0$, so $u_2 = z \begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix}^T$, with z arbitrary. It will be convenient to take $z = 1$ so $u_2 = \begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix}^T$.

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}; \text{ eigenvalues } 2, 12, 0.$$

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}; \text{ eigenvalues } 2, 12, 0.$$

Finally, we need to find the eigenvectors of eigenvalue 0.

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}; \text{ eigenvalues } 2, 12, 0.$$

Finally, we need to find the eigenvectors of eigenvalue 0. Our reduction $B \to C$ involved division by t, so it is not valid in this case where $t = 0$.

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}; \text{ eigenvalues } 2, 12, 0.$$

Finally, we need to find the eigenvectors of eigenvalue 0. Our reduction $B \to C$ involved division by t, so it is not valid in this case where $t = 0$. We must therefore start again and row-reduce the matrix $A - 0I_4 = A$ directly, but that is easy:

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}; \text{ eigenvalues } 2, 12, 0.$$

Finally, we need to find the eigenvectors of eigenvalue 0. Our reduction $B \to C$ involved division by t, so it is not valid in this case where $t = 0$. We must therefore start again and row-reduce the matrix $A - 0I_4 = A$ directly, but that is easy:

$$\begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}$$

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix} \text{ ; eigenvalues } 2, 12, 0.$$

Finally, we need to find the eigenvectors of eigenvalue 0. Our reduction $B \to C$ involved division by t, so it is not valid in this case where $t = 0$. We must therefore start again and row-reduce the matrix $A - 0I_4 = A$ directly, but that is easy:

$$\begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 2 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix};\ \text{eigenvalues } 2, 12, 0.$$

Finally, we need to find the eigenvectors of eigenvalue 0. Our reduction $B \to C$ involved division by t, so it is not valid in this case where $t = 0$. We must therefore start again and row-reduce the matrix $A - 0I_4 = A$ directly, but that is easy:

$$\begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 2 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}; \text{ eigenvalues } 2, 12, 0.$$

Finally, we need to find the eigenvectors of eigenvalue 0. Our reduction $B \to C$ involved division by t, so it is not valid in this case where $t = 0$. We must therefore start again and row-reduce the matrix $A - 0I_4 = A$ directly, but that is easy:

$$\begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 2 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We conclude that the eigenvectors of eigenvalue 0 are the vectors $\begin{bmatrix} w & x & y & z \end{bmatrix}^T$ with $w + z = x + y = 0$.

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We conclude that the eigenvectors of eigenvalue 0 are the vectors $\begin{bmatrix} w & x & y & z \end{bmatrix}^T$ with $w + z = x + y = 0$. These form a two-dimensional space, and the vectors

$$u_3 = \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^T \qquad u_4 = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T$$

form a basis.

Diagonalisation example

$$\begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= 12 \\ \lambda_3 &= 0 \\ \lambda_4 &= 0 \end{aligned} \qquad u_1 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

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Row reduce $[U|I_4] \to [I_4|U^{-1}]$.

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$$U^{-1} = \frac{1}{10}\begin{bmatrix} 2 & -1 & -1 & 2\\ 1 & 2 & 2 & 1\\ 5 & 0 & 0 & -5\\ 0 & 5 & -5 & 0\end{bmatrix}.$$

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We now have a diagonalisation $A = UDU^{-1}$.

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$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ \dot{x}_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned} \qquad \text{so } \dot{x} = Ax \text{ where } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

- To solve this, diagonalise $A = UDU^{-1}$ with $D = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ say, so $\dot{x} = UDU^{-1}x$. Put $y = U^{-1}x$ and $d = U^{-1}c$ so $\dot{y} = U^{-1}\dot{x} = DU^{-1}x = Dy$, with $y = d$ at $t = 0$. This gives $y = e^{Dt}d$, where

$$e^{Dt} = \mathrm{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t});$$

  so $x = Uy = Ue^{Dt}d = Ue^{Dt}U^{-1}c$.

Differential equations example

If $\dot{x} = Ax$, $x = c$ at $t = 0$, $A = UDU^{-1}$, then $x = Ue^{Dt}U^{-1}c$ where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and $e^{Dt} = \mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$.

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If $\dot{x} = Ax$, $x = c$ at $t = 0$, $A = UDU^{-1}$, then $x = Ue^{Dt}U^{-1}c$ where $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and $e^{Dt} = \text{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$.

Example 15.1: Suppose $\dot{x}_1 = x_1 + x_2 + x_3; \quad \dot{x}_2 = 2x_2 + 2x_3; \quad \dot{x}_3 = 3x_3$

Differential equations example

If $\dot{x} = Ax$, $x = c$ at $t = 0$, $A = UDU^{-1}$, then $x = Ue^{Dt}U^{-1}c$ where $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and $e^{Dt} = \text{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$.

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$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

Differential equations example

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$$U = \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad U^{-1} = \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

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So $x = Ue^{Dt}U^{-1}c$, where $c =$ initial value $= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$.

Differential equations example

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$$= \begin{bmatrix} e^t & e^{2t} & \frac{3}{2}e^{3t} \\ 0 & e^{2t} & 2e^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1/2 \\ -2 \\ 1 \end{bmatrix}$$

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$$= \begin{bmatrix} e^t & e^{2t} & \frac{3}{2}e^{3t} \\ 0 & e^{2t} & 2e^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1/2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^t - 2e^{2t} + \frac{3}{2}e^{3t} \\ -2e^{2t} + 2e^{3t} \\ e^{3t} \end{bmatrix}.$$

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Suppose $\dot{x} = \dot{y} = \dot{z} = x + y + z$ with $x = z = 0$ and $y = 1$ at $t = 0$.

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Thus $\dot{v} = Av$, where $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

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The characteristic polynomial is

$$\chi_A(t) = \det \begin{bmatrix} 1-t & 1 & 1 \\ 1 & 1-t & 1 \\ 1 & 1 & 1-t \end{bmatrix}$$

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$$u_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

we find that $Au_1 = Au_2 = 0$ and $Au_3 = 3u_3$.

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we find that $Au_1 = Au_2 = 0$ and $Au_3 = 3u_3$. Thus, the vectors u_i form a basis for $\mathbb{R}^3$ consisting of eigenvectors for A.

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we find that $Au_1 = Au_2 = 0$ and $Au_3 = 3u_3$. Thus, the vectors u_i form a basis for $\mathbb{R}^3$ consisting of eigenvectors for A. This means that we have a diagonalisation $A = UDU^{-1}$, where

$$U = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Differential equations example

$\dot{v} = UDU^{-1}v$ and $v = c$ at $t = 0$ where

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We can find U^{-1} by the following row-reduction:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \end{array}\right]$$

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$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & -1/3 & -1/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \\ 0 & -1 & 0 & -1/3 & -1/3 & 2/3 \end{array}\right]$$

Differential equations example

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The conclusion is that

$$U^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Differential equations example

$\dot{v} = UDU^{-1}v$ and $v = c$ at $t = 0$ where

$$U = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad U^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} \quad c = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

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The solution to our differential equation is now $v = Ue^{Dt}U^{-1}c$:

Differential equations example

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$$U = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad U^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} \quad c = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The solution to our differential equation is now $v = Ue^{Dt}U^{-1}c$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Differential equations example

$\dot{v} = UDU^{-1}v$ and $v = c$ at $t = 0$ where

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Solving difference equations

Problem: find a formula for the sequence where $a_0 = -1$, $a_1 = 0$, and $a_{i+2} = 6a_{i+1} - 8a_i$ for all $i \geq 0$.

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We can be more explicit by finding the eigenvalues and eigenvectors of A.

Solving difference equations

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The characteristic polynomial is

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$$\chi_A(t) = \det \begin{bmatrix} -t & 1 \\ -8 & 6-t \end{bmatrix} = t^2 - 6t + 8$$

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so the eigenvectors are 2 and 4.

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$$v_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = A^n v_0 \qquad A = \begin{bmatrix} 0 & 1 \\ -8 & 6 \end{bmatrix} \qquad v_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

The characteristic polynomial is

$$\chi_A(t) = \det \begin{bmatrix} -t & 1 \\ -8 & 6-t \end{bmatrix} = t^2 - 6t + 8 = (t-2)(t-4),$$

so the eigenvectors are 2 and 4. Using the row-reductions

$$A - 2I = \begin{bmatrix} -2 & 1 \\ -8 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \qquad A - 4I = \begin{bmatrix} -4 & 1 \\ -8 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1/4 \\ 0 & 0 \end{bmatrix}$$

we see that $u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ are eigenvectors of eigenvalues 2 and 4 (forming a basis for $\mathbb{R}^2$). Recall that $A^n u_1 = 2^n u_1$ and $A^n u_2 = 4^n u_2$. We can express $v_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ in terms of this basis by row-reducing $[u_1|u_2|v_0]$:

$$\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 2 & 4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 2 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 1 \end{array}\right].$$

Solving difference equations

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By reading off the last column, we deduce that $v_0 = -2u_1 + u_2$

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By reading off the last column, we deduce that $v_0 = -2u_1 + u_2$ (which could also have been obtained by inspection).

Solving difference equations

$$v_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = A^n v_0 \qquad A = \begin{bmatrix} 0 & 1 \\ -8 & 6 \end{bmatrix} \qquad v_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \qquad u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \qquad u_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

$$v_0 = -2u_1 + u_2 \qquad v_n = A^n v_0 \qquad A^n u_1 = 2^n u_1, \quad A^n u_2 = 4^n u_2$$

Solving difference equations

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It follows that

$$v_n = A^n v_0$$

Solving difference equations

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$$v_n = A^n v_0 = A^n u_2 - 2A^n u_1$$

Solving difference equations

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It follows that

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Solving difference equations

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Solving difference equations

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We will check that this formula does indeed give the required properties:

$$v_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = A^n v_0 \qquad A = \begin{bmatrix} 0 & 1 \\ -8 & 6 \end{bmatrix} \qquad v_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \qquad u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \qquad u_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

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$$a_0 = 2^0 - 2^1 = 1 - 2 = -1$$

Solving difference equations

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$$6a_{i+1} - 8a_i = 6(2^{2i+2} - 2^{i+2}) - 8(2^{2i} - 2^{i+1})$$

$$v_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = A^n v_0 \qquad A = \begin{bmatrix} 0 & 1 \\ -8 & 6 \end{bmatrix} \qquad v_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \qquad u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \qquad u_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

$$v_0 = -2u_1 + u_2 \qquad v_n = A^n v_0 \qquad A^n u_1 = 2^n u_1, \quad A^n u_2 = 4^n u_2$$

It follows that

$$v_n = A^n v_0 = A^n u_2 - 2A^n u_1 = 4^n u_2 - 2 \times 2^n u_1$$

$$= 2^{2n} \begin{bmatrix} 1 \\ 4 \end{bmatrix} - 2^{n+1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2^{2n} - 2^{n+1} \\ 2^{2n+2} - 2^{n+2} \end{bmatrix}.$$

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$$6a_{i+1} - 8a_i = 6(2^{2i+2} - 2^{i+2}) - 8(2^{2i} - 2^{i+1}) = 24 \times 2^{2i} - 24 \times 2^i - 8 \times 2^{2i} + 16 \times 2^i$$

Solving difference equations

$$v_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = A^n v_0 \qquad A = \begin{bmatrix} 0 & 1 \\ -8 & 6 \end{bmatrix} \qquad v_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \qquad u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \qquad u_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

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It follows that

$$\begin{aligned} v_n &= A^n v_0 = A^n u_2 - 2A^n u_1 = 4^n u_2 - 2 \times 2^n u_1 \\ &= 2^{2n} \begin{bmatrix} 1 \\ 4 \end{bmatrix} - 2^{n+1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2^{2n} - 2^{n+1} \\ 2^{2n+2} - 2^{n+2} \end{bmatrix}. \end{aligned}$$

Moreover, v_n was defined to be $\begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$, so a_n is the top entry in v_n, so we conclude that

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We will check that this formula does indeed give the required properties:

$$\begin{aligned} a_0 &= 2^0 - 2^1 = 1 - 2 = -1 \\ a_1 &= 2^2 - 2^2 = 0 \\ 6a_{i+1} - 8a_i &= 6(2^{2i+2} - 2^{i+2}) - 8(2^{2i} - 2^{i+1}) = 24 \times 2^{2i} - 24 \times 2^i - 8 \times 2^{2i} + 16 \times 2^i \\ &= 16 \times 2^{2i} - 8 \times 2^i \end{aligned}$$

Solving difference equations

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$$= 16 \times 2^{2i} - 8 \times 2^i = 2^{2i+4} - 2^{i+3}$$

Solving difference equations

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Solving difference equations

$$v_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = A^n v_0 \qquad A = \begin{bmatrix} 0 & 1 \\ -8 & 6 \end{bmatrix} \qquad v_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \qquad u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \qquad u_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

$$v_0 = -2u_1 + u_2 \qquad v_n = A^n v_0 \qquad A^n u_1 = 2^n u_1, \quad A^n u_2 = 4^n u_2$$

It follows that

$$\begin{aligned} v_n &= A^n v_0 = A^n u_2 - 2A^n u_1 = 4^n u_2 - 2 \times 2^n u_1 \\ &= 2^{2n} \begin{bmatrix} 1 \\ 4 \end{bmatrix} - 2^{n+1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2^{2n} - 2^{n+1} \\ 2^{2n+2} - 2^{n+2} \end{bmatrix}. \end{aligned}$$

Moreover, v_n was defined to be $\begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$, so a_n is the top entry in v_n, so we conclude that

$$a_n = 2^{2n} - 2^{n+1}.$$

We will check that this formula does indeed give the required properties:

$$\begin{aligned} a_0 &= 2^0 - 2^1 = 1 - 2 = -1 \\ a_1 &= 2^2 - 2^2 = 0 \\ 6a_{i+1} - 8a_i &= 6(2^{2i+2} - 2^{i+2}) - 8(2^{2i} - 2^{i+1}) = 24 \times 2^{2i} - 24 \times 2^i - 8 \times 2^{2i} + 16 \times 2^i \\ &= 16 \times 2^{2i} - 8 \times 2^i = 2^{2i+4} - 2^{i+3} = 2^{2(i+2)} - 2^{(i+2)+1} = a_{i+2}. \end{aligned}$$

Another difference equation

We will find the sequence satisfying $b_0 = 3$ and $b_1 = 6$ and $b_2 = 14$ and

$$b_{i+3} = 6b_i - 11b_{i+1} + 6b_{i+2}.$$

Another difference equation

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The vectors $v_i = \begin{bmatrix} b_i & b_{i+1} & b_{i+2} \end{bmatrix}^T$

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Another difference equation

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It follows that $v_n = B^n v_0$ for all n

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It follows that $v_n = B^n v_0$ for all n, and b_n is the top entry in the vector v_n.

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It follows that $v_n = B^n v_0$ for all n, and b_n is the top entry in the vector v_n. Now write v_0 in terms of the eigenvectors of B.

Another difference equation

We will find the sequence satisfying $b_0 = 3$ and $b_1 = 6$ and $b_2 = 14$ and

$$b_{i+3} = 6b_i - 11b_{i+1} + 6b_{i+2}.$$

The vectors $v_i = \begin{bmatrix} b_i & b_{i+1} & b_{i+2} \end{bmatrix}^T$ satisfy $v_0 = \begin{bmatrix} 3 & 6 & 14 \end{bmatrix}^T$ and

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It follows that $v_n = B^n v_0$ for all n, and b_n is the top entry in the vector v_n. Now write v_0 in terms of the eigenvectors of B. The characteristic polynomial is

$$\chi_B(t) = \det \begin{bmatrix} -t & 1 & 0 \\ 0 & -t & 1 \\ 6 & -11 & 6-t \end{bmatrix}$$

Another difference equation

We will find the sequence satisfying $b_0 = 3$ and $b_1 = 6$ and $b_2 = 14$ and

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Another difference equation

We will find the sequence satisfying $b_0 = 3$ and $b_1 = 6$ and $b_2 = 14$ and

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$$= -t(t^2 - 6t + 11) - (-6)$$

Another difference equation

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The vectors $v_i = \begin{bmatrix} b_i & b_{i+1} & b_{i+2} \end{bmatrix}^T$ satisfy $v_0 = \begin{bmatrix} 3 & 6 & 14 \end{bmatrix}^T$ and

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$$\chi_B(t) = \det \begin{bmatrix} -t & 1 & 0 \\ 0 & -t & 1 \\ 6 & -11 & 6-t \end{bmatrix} = -t \det \begin{bmatrix} -t & 1 \\ -11 & 6-t \end{bmatrix} - \det \begin{bmatrix} 0 & 1 \\ 6 & 6-t \end{bmatrix}$$

$$= -t(t^2 - 6t + 11) - (-6) = 6 - 11t + 6t^2 - t^3 = (1-t)(2-t)(3-t)$$

Another difference equation

We will find the sequence satisfying $b_0 = 3$ and $b_1 = 6$ and $b_2 = 14$ and

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The vectors $v_i = \begin{bmatrix} b_i & b_{i+1} & b_{i+2} \end{bmatrix}^T$ satisfy $v_0 = \begin{bmatrix} 3 & 6 & 14 \end{bmatrix}^T$ and

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It follows that $v_n = B^n v_0$ for all n, and b_n is the top entry in the vector v_n. Now write v_0 in terms of the eigenvectors of B. The characteristic polynomial is

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$$= -t(t^2 - 6t + 11) - (-6) = 6 - 11t + 6t^2 - t^3 = (1-t)(2-t)(3-t),$$

so the eigenvalues are 1, 2 and 3.

Another difference equation

$$v_n = \begin{bmatrix} b_n \\ b_{n+1} \\ b_{n+2} \end{bmatrix} = B^n \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \text{ has eigenvalues } 1, 2, 3.$$

Another difference equation

$$v_n = \begin{bmatrix} b_n \\ b_{n+1} \\ b_{n+2} \end{bmatrix} = B^n \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \text{ has eigenvalues } 1, 2, 3.$$

Now find the eigenvectors:

Another difference equation

$$v_n = \begin{bmatrix} b_n \\ b_{n+1} \\ b_{n+2} \end{bmatrix} = B^n \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$$ has eigenvalues $1, 2, 3$.

Now find the eigenvectors:

$$B - I \quad = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 6 & -11 & 5 \end{bmatrix}$$

Another difference equation

$$v_n = \begin{bmatrix} b_n \\ b_{n+1} \\ b_{n+2} \end{bmatrix} = B^n \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \text{ has eigenvalues } 1, 2, 3.$$

Now find the eigenvectors:

$$B - I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 6 & -11 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Another difference equation

$$v_n = \begin{bmatrix} b_n \\ b_{n+1} \\ b_{n+2} \end{bmatrix} = B^n \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \text{ has eigenvalues } 1, 2, 3.$$

Now find the eigenvectors:

$$B - I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 6 & -11 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Another difference equation

$$v_n = \begin{bmatrix} b_n \\ b_{n+1} \\ b_{n+2} \end{bmatrix} = B^n \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$$ has eigenvalues $1, 2, 3$.

Now find the eigenvectors:

$$B - I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 6 & -11 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$B - 2I = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 6 & -11 & 4 \end{bmatrix}$$

Another difference equation

$$v_n = \begin{bmatrix} b_n \\ b_{n+1} \\ b_{n+2} \end{bmatrix} = B^n \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$$ has eigenvalues $1, 2, 3$.

Now find the eigenvectors:

$$B - I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 6 & -11 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$B - 2I = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 6 & -11 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Another difference equation

$$v_n = \begin{bmatrix} b_n \\ b_{n+1} \\ b_{n+2} \end{bmatrix} = B^n \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \text{ has eigenvalues } 1, 2, 3.$$

Now find the eigenvectors:

$$\begin{aligned} B - I &= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 6 & -11 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} & u_1 &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ B - 2I &= \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 6 & -11 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} & u_2 &= \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \end{aligned}$$

Another difference equation

$$v_n = \begin{bmatrix} b_n \\ b_{n+1} \\ b_{n+2} \end{bmatrix} = B^n \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \text{ has eigenvalues } 1, 2, 3.$$

Now find the eigenvectors:

$$B - I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 6 & -11 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$B - 2I = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 6 & -11 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$B - 3I = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 6 & -11 & 3 \end{bmatrix}$$

Another difference equation

$$v_n = \begin{bmatrix} b_n \\ b_{n+1} \\ b_{n+2} \end{bmatrix} = B^n \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \text{ has eigenvalues } 1, 2, 3.$$

Now find the eigenvectors:

$$\begin{aligned}
B - I &= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 6 & -11 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} & u_1 &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
B - 2I &= \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 6 & -11 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} & u_2 &= \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \\
B - 3I &= \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 6 & -11 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/9 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

Another difference equation

$$v_n = \begin{bmatrix} b_n \\ b_{n+1} \\ b_{n+2} \end{bmatrix} = B^n \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \text{ has eigenvalues } 1, 2, 3.$$

Now find the eigenvectors:

$$\begin{aligned}
B - I &= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 6 & -11 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} & u_1 &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
B - 2I &= \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 6 & -11 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} & u_2 &= \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \\
B - 3I &= \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 6 & -11 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1/9 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix} & u_3 &= \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}.
\end{aligned}$$

Another difference equation

$$v_n = B^n \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad u_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \quad u_3 = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} \qquad Bu_k = ku_k.$$

Another difference equation

$$v_n = B^n \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad u_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \quad u_3 = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} \qquad Bu_k = ku_k.$$

By inspection: $v_0 = \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} = u_1 + u_2 + u_3.$

Another difference equation

$$v_n = B^n \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad u_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \quad u_3 = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} \qquad Bu_k = ku_k.$$

By inspection: $v_0 = \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} = u_1 + u_2 + u_3.$

This could also have been obtained by row-reducing $[u_1|u_2|u_3|v_0]$:

Another difference equation

$$v_n = B^n \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad u_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \quad u_3 = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} \qquad Bu_k = ku_k.$$

By inspection: $v_0 = \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} = u_1 + u_2 + u_3.$

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Moreover, b_n is the top entry in v_n, so we conclude that

$$b_n = 1 + 2^n + 3^n.$$

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Take $x = 1$ to get an eigenvector $u_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$ of eigenvalue λ_1.
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$$\alpha \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{or} \qquad \begin{aligned} \beta &= -\alpha \\ \alpha(\lambda_1 - \lambda_2) &= 1. \end{aligned}$$

Fibonacci numbers

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Now $\lambda_1 - \lambda_2 = \sqrt{5}$

Fibonacci numbers

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$$v_n = A^n v_0$$

Fibonacci numbers

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$$v_n = A^n v_0 = \frac{A^n u_1 - A^n u_2}{\sqrt{5}}$$

Fibonacci numbers

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Moreover, F_n is the top entry in v_n, so we obtain the formula

$$F_n = \frac{\lambda_1^n - \lambda_2^n}{\sqrt{5}} = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

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It is also useful to note here that $\lambda_1 \simeq 1.618033988$ and $\lambda_2 \simeq -0.6180339880$.

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Fibonacci numbers

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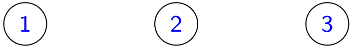
It is also useful to note here that $\lambda_1 \simeq 1.618033988$ and $\lambda_2 \simeq -0.6180339880$. As $|\lambda_1| > 1$ and $|\lambda_2| < 1$ we see that $|\lambda_1^n| \to \infty$ and $|\lambda_2^n| \to 0$ as $n \to \infty$. When n is large we can neglect the λ_2 term and we have $F_n \simeq \lambda_1^n/\sqrt{5}$.

Lecture 12

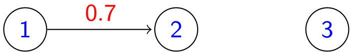
Markov chains

Consider a system that can be in three different states.



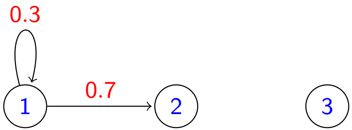
Once per second, it can change state in a random way.

Consider a system that can be in three different states.



Once per second, it can change state in a random way. If it is in state 1, it jumps to state 2 with probability 0.7

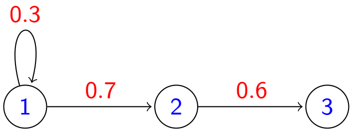
Consider a system that can be in three different states.



Once per second, it can change state in a random way. If it is in state 1, it jumps to state 2 with probability 0.7 and stays in state 1 with probability 0.3.

Markov chains

Consider a system that can be in three different states.



Once per second, it can change state in a random way. If it is in state 1, it jumps to state 2 with probability 0.7 and stays in state 1 with probability 0.3. If it is in state 2, it jumps to state 3 with probability 0.6

Markov chains

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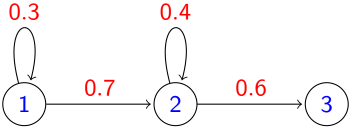

Once per second, it can change state in a random way. If it is in state 1, it jumps to state 2 with probability 0.7 and stays in state 1 with probability 0.3. If it is in state 2, it jumps to state 3 with probability 0.6 and stays in state 1 with probability 0.4.

Markov chains

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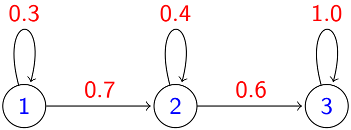

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Markov chains

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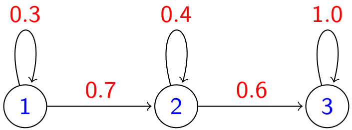

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This is an example of a *Markov chain*.

Markov chains

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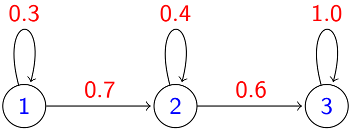

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This is an example of a *Markov chain*. These are widely used to model (pseudo)-random processes in economics, population biology, information technology and other areas.

Markov chains

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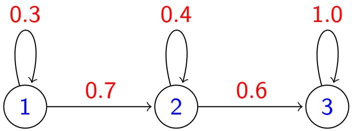

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Markov chains

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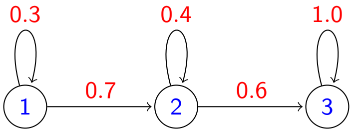

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- How much time to we spend in state i, on average?

Markov chains

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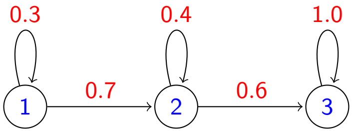

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- How much time to we spend in state i, on average?
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Markov chains

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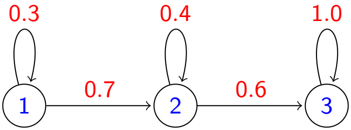

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- If we start in state i, what is the probability of reaching j before k?

Markov chains

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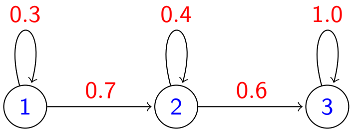

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- How much time to we spend in state i, on average?
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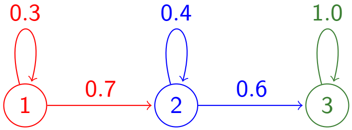
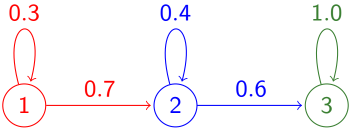
- If we start in state i, what is the probability of reaching j before k?

We will take the first steps towards answering such questions.

Markov chains

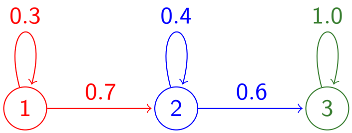


Markov chains



Notation: $p_{j \leftarrow i}$ is the probability of jumping from state i to state j.

Markov chains



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The *transition matrix* has $p_{j\leftarrow i}$ in the i'th column of the j'th row.

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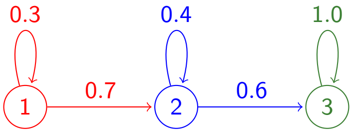
Markov chains



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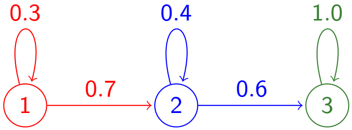

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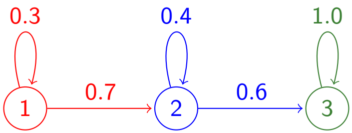

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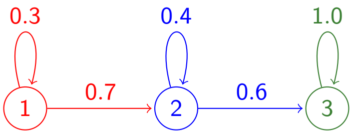

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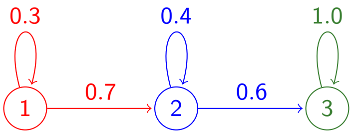

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Distribution vectors

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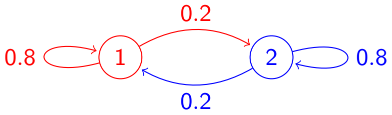
Markov chain example

Consider a two-state Markov chain which stays in the same state with probability 0.8, and flips to the other state with probability 0.2.



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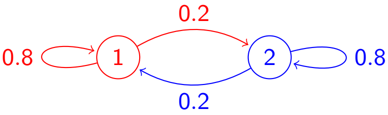

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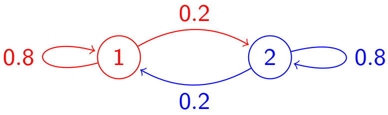

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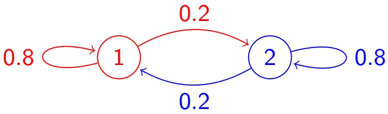

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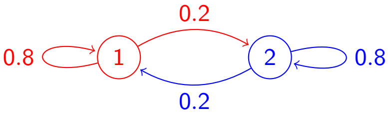

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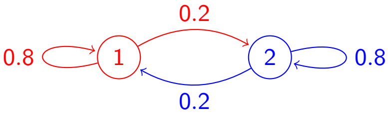

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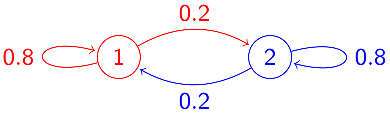

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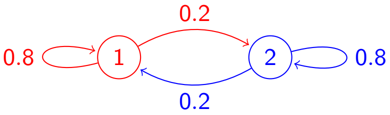

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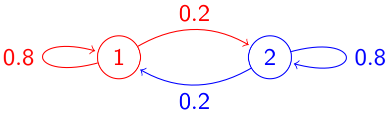

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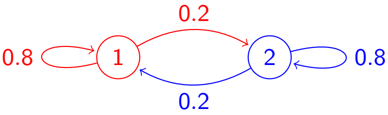

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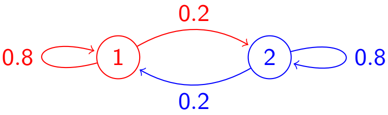

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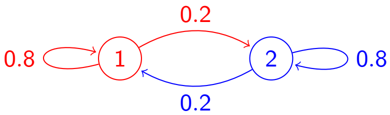

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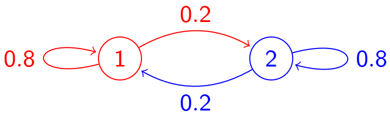

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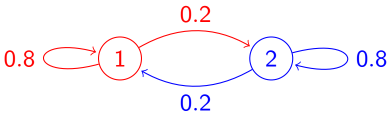
Suppose we are given that the system starts at $t = 0$ in state 1, so $r_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Markov chain example



$$P^n = \begin{bmatrix} 0.5(1+0.6^n) & 0.5(1-0.6^n) \\ 0.5(1-0.6^n) & 0.5(1-0.6^n) \end{bmatrix}$$

Suppose we are given that the system starts at $t = 0$ in state 1, so $r_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. It follows that

$$r_n = P^n r_0$$

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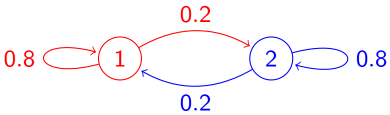

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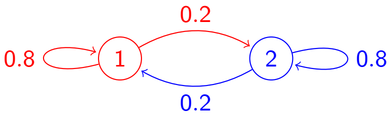
Markov chain example



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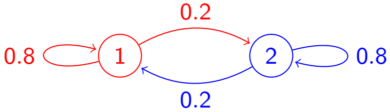
Markov chain example



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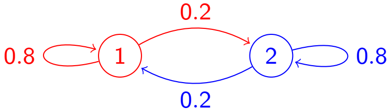
Thus, at time n the probability of being in state 1 is $0.5(1+0.6^n)$, and the probability of being in state 2 is $0.5(1-0.6^n)$.

Markov chain example



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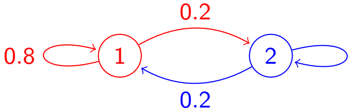
When n is large, we observe that $(0.6)^n$ will be very small, so $r_n \simeq \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, so it is almost equally probable that X will be in either of the two states.

Markov chain example



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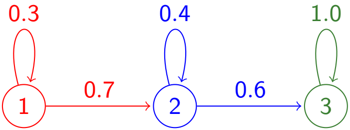
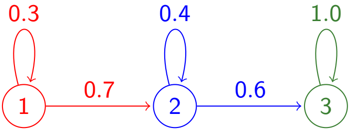
When n is large, we observe that $(0.6)^n$ will be very small, so $r_n \simeq \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, so it is almost equally probable that X will be in either of the two states. This should be intuitively plausible, given the symmetry of the situation.

Markov chain example



$$P = \begin{bmatrix} 0.3 & 0.0 & 0.0 \\ 0.7 & 0.4 & 0.0 \\ 0.0 & 0.6 & 1.0 \end{bmatrix}.$$

Markov chain example



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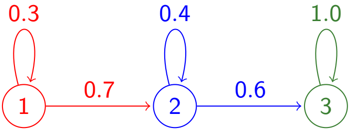
We start in state 1 at $t = 0$. What is the probability that we are in state 3 at $t = 5$?

Markov chain example



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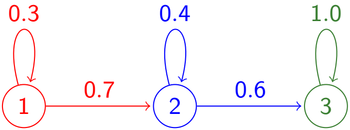
Markov chain example



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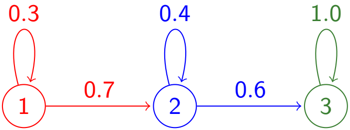
$$\chi_P(t) = \det \begin{bmatrix} 0.3 - t & 0.0 & 0.0 \\ 0.7 & 0.4 - t & 0.0 \\ 0.0 & 0.6 & 1.0 - t \end{bmatrix}$$

Markov chain example



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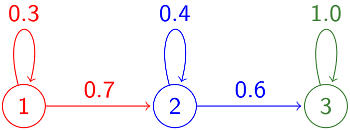

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so the eigenvalues are 0.3, 0.4 and 1.

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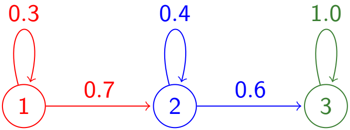

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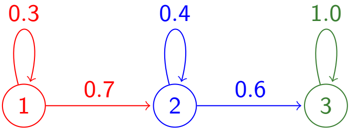
To find an eigenvector of eigenvalue 0.3, we row-reduce the matrix $P - 0.3I$:

Markov chain example



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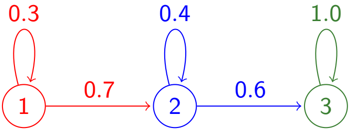
Markov chain example



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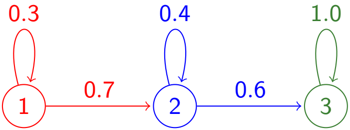
Markov chain example



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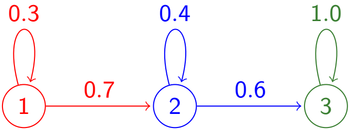
Markov chain example



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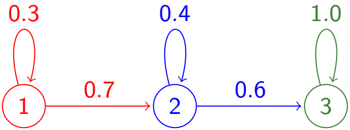
$$\begin{bmatrix} 0 & 0 & 0 \\ 7/10 & 1/10 & 0 \\ 0 & 6/10 & 7/10 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1/7 & 0 \\ 0 & 1 & 7/6 \end{bmatrix} \to \begin{bmatrix} 1 & 1/7 & 0 \\ 0 & 1 & 7/6 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/6 \\ 0 & 1 & 7/6 \\ 0 & 0 & 0 \end{bmatrix}.$$

Markov chain example



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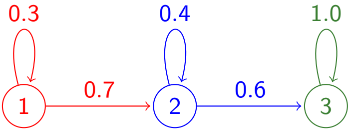
Thus take $u_1 = \begin{bmatrix} 1 & -7 & 6 \end{bmatrix}^T$ as an eigenvector of eigenvalue 0.3.

Markov chain example



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so the eigenvalues are 0.3, 0.4 and 1.

To find an eigenvector of eigenvalue 0.3, we row-reduce the matrix $P - 0.3I$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 7/10 & 1/10 & 0 \\ 0 & 6/10 & 7/10 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1/7 & 0 \\ 0 & 1 & 7/6 \end{bmatrix} \to \begin{bmatrix} 1 & 1/7 & 0 \\ 0 & 1 & 7/6 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/6 \\ 0 & 1 & 7/6 \\ 0 & 0 & 0 \end{bmatrix}.$$

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Eigenvectors u_2 and u_3 can be found similarly.

Markov chain example

$$P = \begin{bmatrix} 0.3 & 0.0 & 0.0 \\ 0.7 & 0.4 & 0.0 \\ 0.0 & 0.6 & 1.0 \end{bmatrix} \quad u_1 = \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix} \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_1 = 0.3 \qquad \lambda_2 = 0.4 \qquad \lambda_3 = 1$$

Markov chain example

$$P = \begin{bmatrix} 0.3 & 0.0 & 0.0 \\ 0.7 & 0.4 & 0.0 \\ 0.0 & 0.6 & 1.0 \end{bmatrix} \quad \underset{\lambda_1 = 0.3}{u_1 = \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix}} \quad \underset{\lambda_2 = 0.4}{u_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}} \quad \underset{\lambda_3 = 1}{u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

We have $P = UDU^{-1}$ where

$D = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$

Markov chain example

$$P = \begin{bmatrix} 0.3 & 0.0 & 0.0 \\ 0.7 & 0.4 & 0.0 \\ 0.0 & 0.6 & 1.0 \end{bmatrix} \quad \begin{array}{c} u_1 = \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix} \\ \lambda_1 = 0.3 \end{array} \quad \begin{array}{c} u_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \\ \lambda_2 = 0.4 \end{array} \quad \begin{array}{c} u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ \lambda_3 = 1 \end{array}$$

We have $P = UDU^{-1}$ where

$$D = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 1.0 \end{bmatrix}$$

Markov chain example

$$P = \begin{bmatrix} 0.3 & 0.0 & 0.0 \\ 0.7 & 0.4 & 0.0 \\ 0.0 & 0.6 & 1.0 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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For the probability p that X is in state 3 at time $t = 5$, we need to take $k = 5$ and look at the third component

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Stochastic matrices have eigenvalue 1

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Proposition 17.10: If P is an $n \times n$ stochastic matrix, then 1 is an eigenvalue of P.

We will prove this after two lemmas.

A and A^T have the same eigenvalues

Lemma: Let B be an $n \times n$ matrix.
Then 0 is an eigenvalue of B iff 0 is an eigenvalue of B^T.

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Corollary: For any $n \times n$ matrix A, the eigenvalues of A are the same as the eigenvalues of A^T.

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Let λ be an eigenvalue of A, so there is a nonzero vector u with $Au = \lambda u$.

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Let λ be an eigenvalue of A, so there is a nonzero vector u with $Au = \lambda u$. This means that $(A - \lambda I_n)u = 0$

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Let λ be an eigenvalue of A, so there is a nonzero vector u with $Au = \lambda u$. This means that $(A - \lambda I_n)u = 0$, so 0 is an eigenvalue of $A - \lambda I_n$.

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The lemma then tells us that 0 is also an eigenvalue of $(A - \lambda I_n)^T$

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This means that there is a nonzero vector v with $(A^T - \lambda I_n)v = 0$

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This means that $(A - \lambda I_n)u = 0$, so 0 is an eigenvalue of $A - \lambda I_n$.
The lemma then tells us that 0 is also an eigenvalue of $(A - \lambda I_n)^T = A^T - \lambda I_n$.
This means that there is a nonzero vector v with $(A^T - \lambda I_n)v = 0$, or equivalently $A^T v = \lambda v$.

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A and A^T have the same eigenvalues

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The whole argument can be reversed to prove the converse as well: if λ is an eigenvalue of A^T, then it is also an eigenvalue of A. $\square$

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Let the columns of P be $v_1, \ldots, v_n$.

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Thus, d is an eigenvector of P^T with eigenvalue 1.
It follows by the Corollary that 1 is also an eigenvalue of P, as required. $\square$

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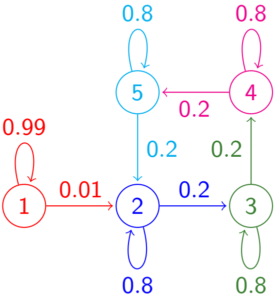
so r_∞ is a stationary distribution. Moreover, it often happens that there is only one stationary distribution. There are many theorems about this sort of thing, but we will not explore them in this course.

Stationary distribution example



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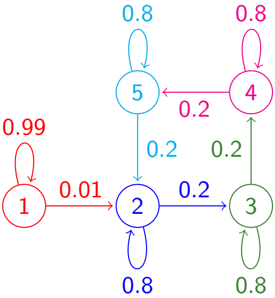

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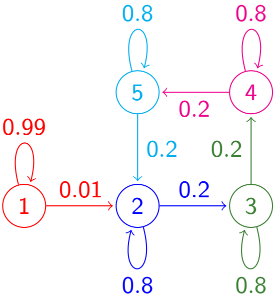

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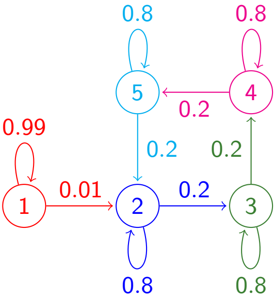

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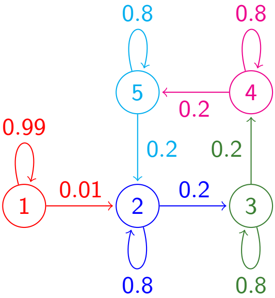

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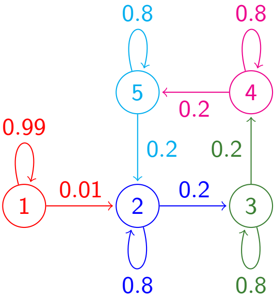

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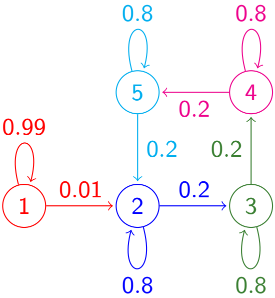

Once we have entered the square things are symmetric so we spend $\frac{1}{4}$ of the time in each of states $2, \ldots, 5$.

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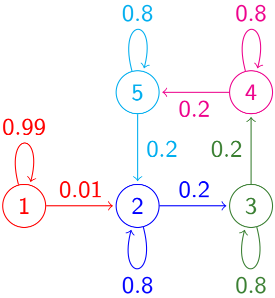

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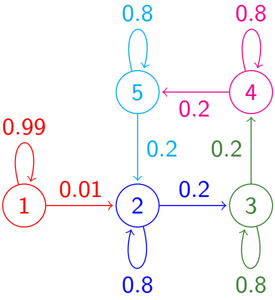

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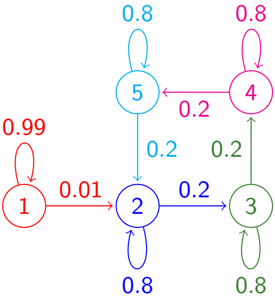

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$$Pq = \begin{bmatrix} 0.99 & 0 & 0 & 0 & 0 \\ 0.01 & 0.8 & 0 & 0 & 0.2 \\ 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{bmatrix}$$

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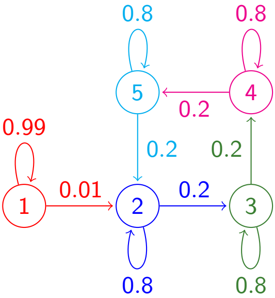

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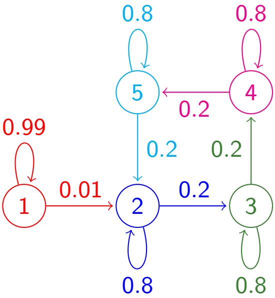

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Lecture 13

Page rank

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- A link from S_j to S_i is a vote by S_j that S_i is important.

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- Say S_j links to N_j different pages, and assume $N_j > 0$.
- We want rankings $r_i \geq 0$, normalised so that $\sum_i r_i = 1$; so r is a probability vector in $\mathbb{R}^n$.
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- Links from important pages should count for more; links from pages with many links should count for less.
- We use this rule: a link from S_j to S_i contributes r_j/N_j to r_i.
- Thus, the following consistency condition should be satisfied:

$$r_i = \sum_{\text{pages } S_j \text{ that link to } S_i} r_j/N_j.$$

PageRank as an eigenvector

Pages $S_1, \ldots, S_n$; rankings $r_i \geq 0$ with $\sum_i r_i = 1$; S_j links to N_j pages;
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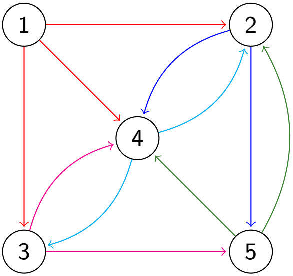

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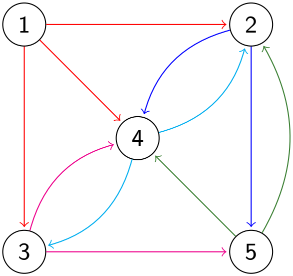

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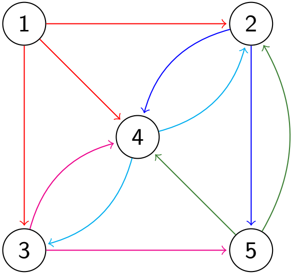
PageRank as a Markov chain



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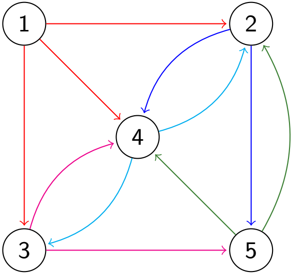

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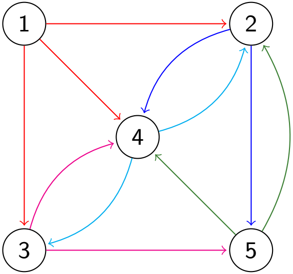

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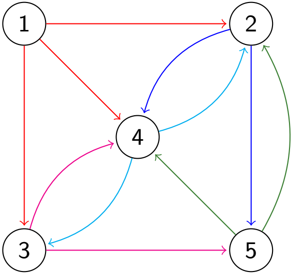

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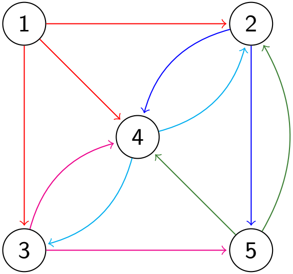

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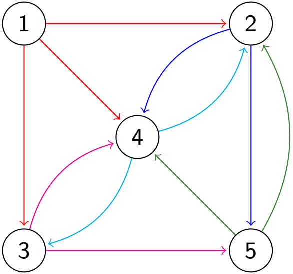

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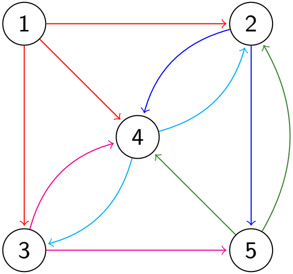

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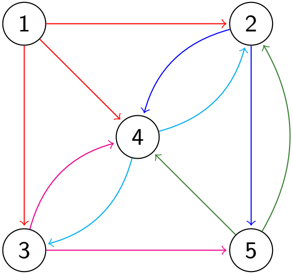

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Conceptually: r_i is the long run average proportion of time that a random surfer spends on page i.

Calculating PageRank in Maple

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```
with(LinearAlgebra):
```

Load the linear algebra package.

Calculating PageRank in Maple

```
with(LinearAlgebra):
n := 5;
```

Declare the number of web pages.

Calculating PageRank in Maple

```
with(LinearAlgebra):
n := 5;
P := << 0 | 0 | 0 | 0 | 0 >,
 <1/3 | 0 | 0 | 1/2 | 1/2 >,
 <1/3 | 0 | 0 | 1/2 | 0 >,
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Enter the transition matrix.

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NS := NullSpace(P - IdentityMatrix(n));
```

Find solutions to $(P - I_n)v = 0$. Maple returns a set of solutions enclosed in curly brackets; usually, there will only be one element in the set.

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NS := NullSpace(P - IdentityMatrix(n));
r := NS[1];
```

Define r to be the first element in the set (which is usually the only element).

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 <1/3 | 1/2 | 1/2 | 0 | 1/2 >,
 < 0 | 1/2 | 1/2 | 0 | 0 >>;
NS := NullSpace(P - IdentityMatrix(n));
r := NS[1];
r := r / add(r[i],i=1..n);
```

The solution found by Maple is not usually a probability vector. To fix this, we just divide by $\sum_i r_i$.

Calculating PageRank in Maple

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NS := NullSpace(P - IdentityMatrix(n));
r := NS[1];
r := r / add(r[i],i=1..n);
r := evalf(r);
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It is more convenient to have the answer in decimals rather than fractions, so we use `evalf()`.

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Result: $r = \begin{bmatrix} 0.0 \\ 0.2777777778 \\ 0.1666666667 \\ 0.3333333333 \\ 0.2222222222 \end{bmatrix}$

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Result: $r = \begin{bmatrix} 0.0 \\ 0.2777777778 \\ 0.1666666667 \\ 0.3333333333 \\ 0.2222222222 \end{bmatrix}$; so

page 1 has rank 0.0
page 2 has rank 0.2777777778
page 3 has rank 0.1666666667
page 4 has rank 0.3333333333
page 5 has rank 0.2222222222

Calculating PageRank as a limit

```
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q := Vector(n,[1/n $ n]);
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q is a vector of length n, whose entries are $1/n$, repeated n times.

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We have seen that $r = \lim_{k\to\infty} P^k q$, so $r = P^{10} q$ should be approximately right.

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Result: $r = \begin{bmatrix} 0.0 \\ 0.2783203125 \\ 0.1667317708 \\ 0.3332682292 \\ 0.2216796875 \end{bmatrix}$, close to the exact value of $\begin{bmatrix} 0.0 \\ 0.2777777778 \\ 0.1666666667 \\ 0.3333333333 \\ 0.2222222222 \end{bmatrix}$

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This gives a new transition matrix:

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Q := d * P + (1-d) * R;
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r := r / add(r[i],i=1..n);
```
or
```
r := Q^10 .  q;
```

Lecture 14

Subspaces

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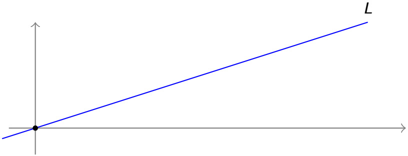

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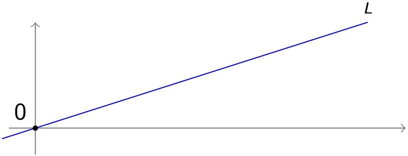

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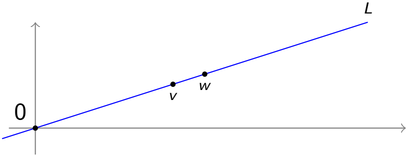

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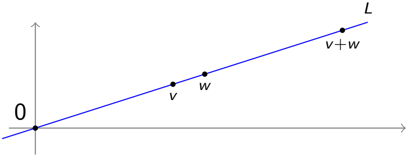

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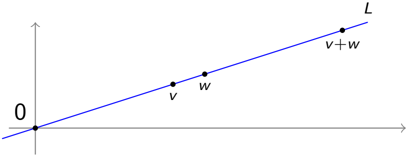

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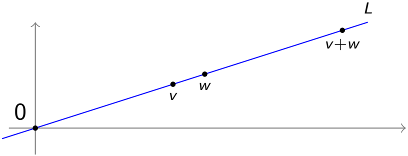

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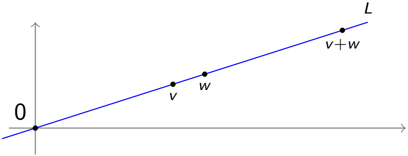

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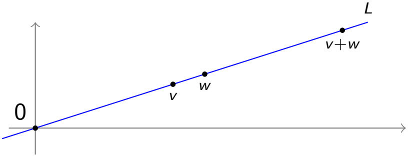

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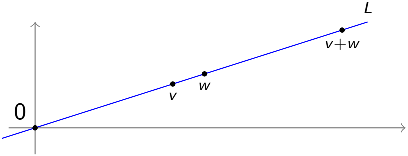

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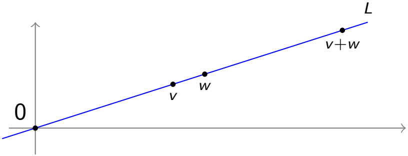

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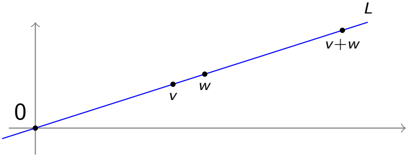

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So L is a subspace.

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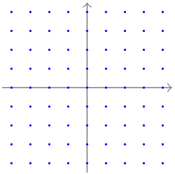

V_1

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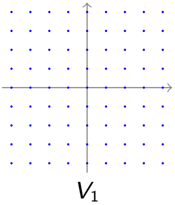

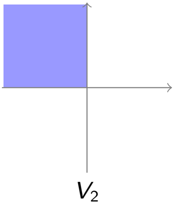

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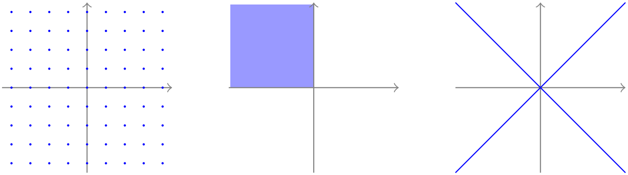

V_1 V_2 V_3

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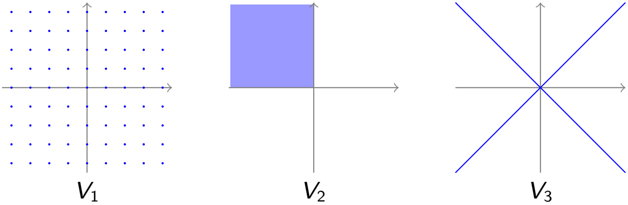

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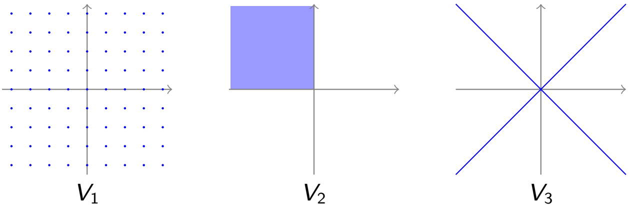
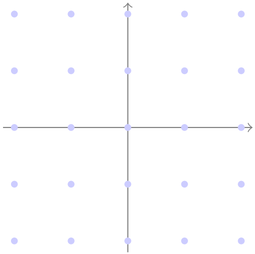


None of these are subspaces.

V_1 is not a subspace



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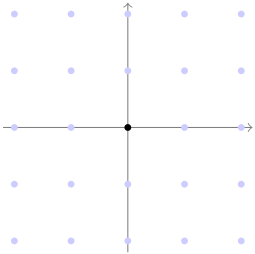

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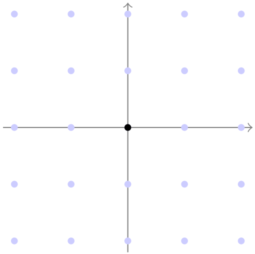
It is clear that the zero vector has integer coordinates and so lies in V_1.

V_1 is not a subspace



$$V_1 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \text{ and } y \text{ are integers} \right\}$$

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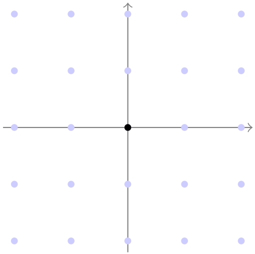

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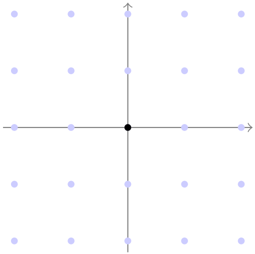

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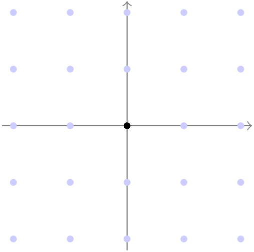

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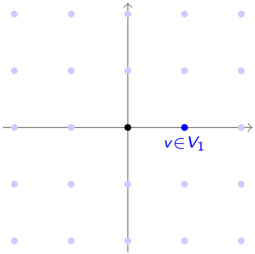
V_1 is not a subspace



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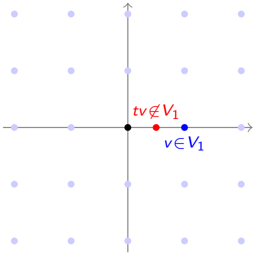
then $v \in V_1$ and $t \in \mathbb{R}$

V_1 is not a subspace



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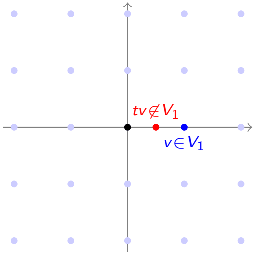
V_1 is not a subspace



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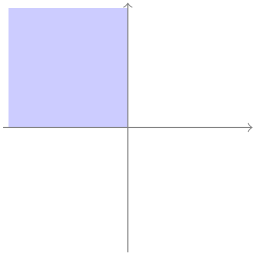
(This is generally the best way to prove that a set is not a subspace: provide a completely specific and explicit example where one of the conditions is not satisfied.)

V_2 is not a subspace



$$V_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \leq 0 \leq y \right\}$$

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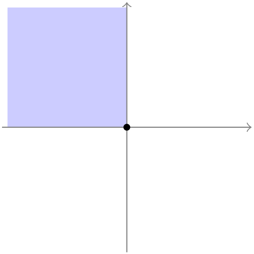

$$V_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \leq 0 \leq y \right\}$$

As $0 \leq 0 \leq 0$ we see that $0 \in V_2$.

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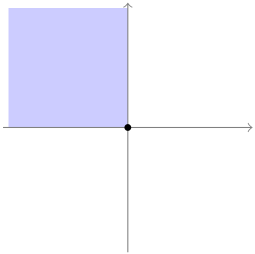

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Suppose we have vectors $v = \begin{bmatrix} x & y \end{bmatrix}^T$ and $v' = \begin{bmatrix} x' & y' \end{bmatrix}^T$ in V_2

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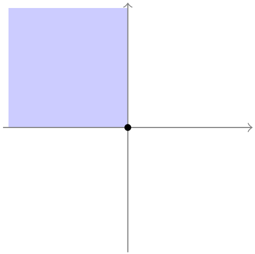

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so $x \leq 0 \leq y$ and $x' \leq 0 \leq y'$.

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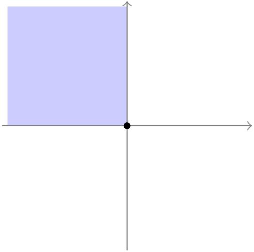

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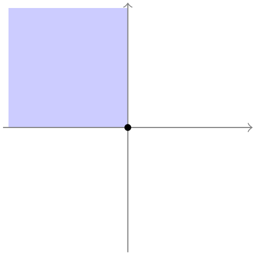

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As $y, y' \geq 0$ it follows that $y + y' \geq 0$.

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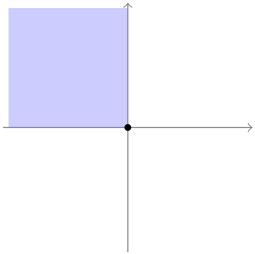

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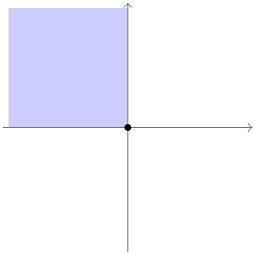
V_2 is not a subspace



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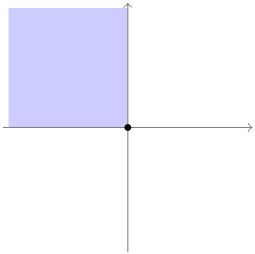
V_2 is not a subspace



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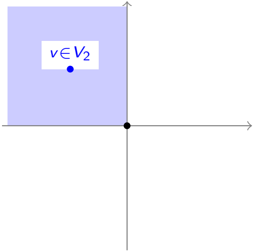

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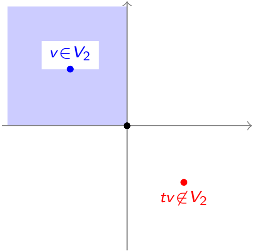
Indeed, if we take $v = \begin{bmatrix} -1 & 1 \end{bmatrix}^T$ and $t = -1$ then $v \in V_2$ and $t \in \mathbb{R}$

V_2 is not a subspace



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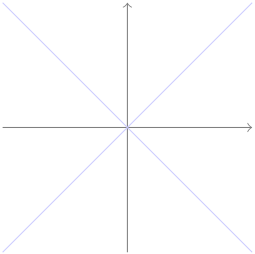
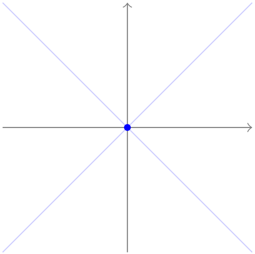
V_3 is not a subspace



$$V_3 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x^2 = y^2 \right\}$$
$$= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x = \pm y \right\}.$$

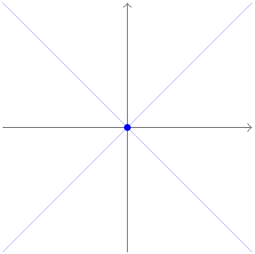
V_3 is not a subspace



It is again clear that $0 \in V_3$.

$$
\begin{aligned}
V_3 &= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x^2 = y^2 \right\} \\
&= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x = \pm y \right\}.
\end{aligned}
$$

V_3 is not a subspace



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It is again clear that $0 \in V_3$.
Now suppose we have $v = \begin{bmatrix} x & y \end{bmatrix}^T \in V_3$ (so $x^2 = y^2$) and $t \in \mathbb{R}$.

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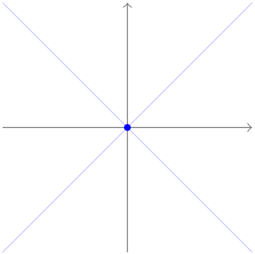

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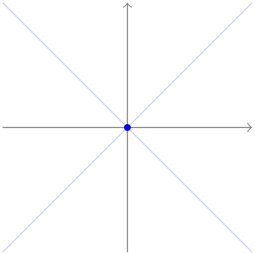
It follows that $(tx)^2 = t^2x^2 = t^2y^2 = (ty)^2$

V_3 is not a subspace



$$V_3 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x^2 = y^2 \right\} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x = \pm y \right\}.$$

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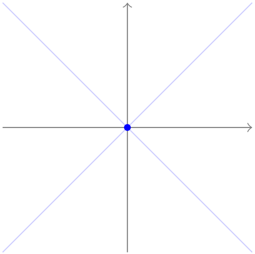
It follows that $(tx)^2 = t^2x^2 = t^2y^2 = (ty)^2$,
so the vector $tv = \begin{bmatrix} tx & ty \end{bmatrix}^T$ again lies in V_3.

V_3 is not a subspace



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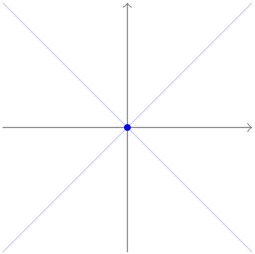
It follows that $(tx)^2 = t^2x^2 = t^2y^2 = (ty)^2$,
so the vector $tv = \begin{bmatrix} tx & ty \end{bmatrix}^T$ again lies in V_3.
This means that V_3 is closed under scalar multiplication.

V_3 is not a subspace



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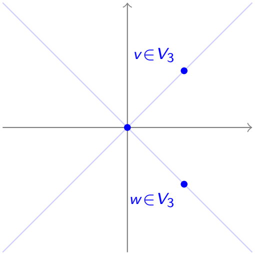
Now suppose we have $v = \begin{bmatrix} x & y \end{bmatrix}^T \in V_3$ (so $x^2 = y^2$) and $t \in \mathbb{R}$.
It follows that $(tx)^2 = t^2x^2 = t^2y^2 = (ty)^2$,
so the vector $tv = \begin{bmatrix} tx & ty \end{bmatrix}^T$ again lies in V_3.
This means that V_3 is closed under scalar multiplication.
However, it is not closed under addition

V_3 is not a subspace



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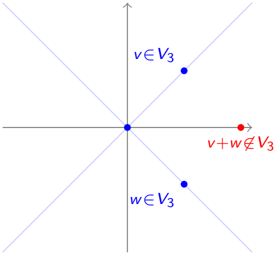
It follows that $(tx)^2 = t^2x^2 = t^2y^2 = (ty)^2$,
so the vector $tv = \begin{bmatrix} tx & ty \end{bmatrix}^T$ again lies in V_3.
This means that V_3 is closed under scalar multiplication.
However, it is not closed under addition,
because the vectors $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ lie in V_3

V_3 is not a subspace



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It is again clear that $0 \in V_3$.
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It follows that $(tx)^2 = t^2x^2 = t^2y^2 = (ty)^2$,
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This means that V_3 is closed under scalar multiplication.
However, it is not closed under addition,
because the vectors $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ lie in V_3,
but $v + w$ does not.

Two extreme cases

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(a) The set $\{0\}$ (just consisting of the zero vector) is a subspace of $\mathbb{R}^n$.

Two extreme cases

(a) The set $\{0\}$ (just consisting of the zero vector) is a subspace of $\mathbb{R}^n$.

(b) The whole set $\mathbb{R}^n$ is a subspace of itself.

Linear combinations in subspaces

Proposition 19.6: Let V be a subspace of $\mathbb{R}^n$. Then any linear combination of elements of V is again in V.

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Proof.

$\square$

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Linear combinations in subspaces

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Spans and annihilators

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Remark 19.9: The terminology in (a) is related in an obvious way to the terminology used earlier: the list $\mathcal{W}$ spans $\mathbb{R}^n$ if and only if every vector in $\mathbb{R}^n$ is a linear combination of $\mathcal{W}$, or in other words $\operatorname{span}(\mathcal{W}) = \mathbb{R}^n$.

Span and annihilator example

$\text{span}(w_1, \ldots, w_r) = \{$ linear combinations of $w_1, \ldots, w_r\}$;
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If we put $v = \begin{bmatrix} x & y & z \end{bmatrix}^T$ and $t = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, then we have $v.t = x + y + z$.

Span and annihilator example

$\mathrm{span}(w_1, \ldots, w_r) = \{$ linear combinations of $w_1, \ldots, w_r\}$;
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Span and annihilator example

Put

$$V = \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w + 2x + 3y + 4z = 4w + 3x + 2y + z = 0\}.$$

Span and annihilator example

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$$V = \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w + 2x + 3y + 4z = 4w + 3x + 2y + z = 0\}.$$

If we put $a = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^T$ and $b = \begin{bmatrix} 4 & 3 & 2 & 1 \end{bmatrix}^T$ then

$$w+2x+3y+4z = a.\begin{bmatrix} w & x & y & z \end{bmatrix}^T \qquad 4w+3x+2y+z = b.\begin{bmatrix} w & x & y & z \end{bmatrix}^T$$

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so we can describe V as $\mathrm{ann}(a, b)$.

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On the other hand, suppose we have a vector $v = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ in V, so that

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$$4w+3x+2y+z = 0 \tag{B}$$

Span and annihilator example

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$$w + 2x + 3y + 4z = 0 \qquad \text{(A)}$$

$$4w + 3x + 2y + z = 0 \qquad \text{(B)}$$

If we subtract 4 times (A) from (B) and then divide by -15 we get equation (C) below.

$$\tfrac{1}{3}x + \tfrac{2}{3}y + z = 0 \qquad \text{(C)}$$

$$\text{(D)}$$

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$$4w + 3x + 2y + z = 0 \tag{B}$$

If we subtract 4 times (A) from (B) and then divide by -15 we get equation (C) below. Similarly, if we subtract 4 times (B) from (A) and divide by -15 we get (D).

$$\tfrac{1}{3}x + \tfrac{2}{3}y + z = 0 \tag{C}$$

$$w + \tfrac{2}{3}x + \tfrac{1}{3}y = 0 \tag{D}$$

Span and annihilator example

$$V = \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w + 2x + 3y + 4z = 4w + 3x + 2y + z = 0\}$$
$$= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \mid w = -\tfrac{2}{3}x - \tfrac{1}{3}y, \qquad z = -\tfrac{1}{3}x - \tfrac{2}{3}y\}$$

Span and annihilator example

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\begin{aligned}
V &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w + 2x + 3y + 4z = 4w + 3x + 2y + z = 0\} \\
&= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \mid w = -\tfrac{2}{3}x - \tfrac{1}{3}y, \qquad z = -\tfrac{1}{3}x - \tfrac{2}{3}y\} \\
&= \left\{ \begin{bmatrix} -\frac{2}{3}x - \frac{1}{3}y \\ x \\ y \\ -\frac{1}{3}x - \frac{2}{3}y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}
\end{aligned}
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$$

Thus, if we put

$$
c = \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \\ -\frac{1}{3} \end{bmatrix} \qquad d = \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \\ -\frac{2}{3} \end{bmatrix}
$$

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$$
\begin{aligned}
V &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w+2x+3y+4z = 4w+3x+2y+z = 0\} \\
&= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \mid w = -\tfrac{2}{3}x - \tfrac{1}{3}y, \qquad z = -\tfrac{1}{3}x - \tfrac{2}{3}y\} \\
&= \left\{ \begin{bmatrix} -\frac{2}{3}x - \frac{1}{3}y \\ x \\ y \\ -\frac{1}{3}x - \frac{2}{3}y \end{bmatrix} \mid x, y \in \mathbb{R} \right\} = \left\{ x \begin{bmatrix} -2/3 \\ 1 \\ 0 \\ -1/3 \end{bmatrix} + y \begin{bmatrix} -1/3 \\ 0 \\ 1 \\ -2/3 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}
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Span and annihilator example

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Annihilators are subspaces

A subspace must contain 0, and be closed under addition and scalar multiplication.

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Proposition 19.23: For any list $\mathcal{W} = (w_1, \ldots, w_r)$ of vectors in $\mathbb{R}^n$, the set

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Proof.

(a) The zero vector clearly has $0.w_i = 0$ for all i, so $0 \in \mathrm{ann}(\mathcal{W})$.

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In particular, V has a basis.

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Let u be an arbitrary vector in V. Consider the list $\mathcal{V}' = (v_1, \dots, v_d, u)$. This is a list in V of length $d + 1$, but d is the maximum possible length for any linearly independent list in V, so the list $\mathcal{V}'$ must be dependent.

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Proof.
By assumption the list $\mathcal{V}$ is linearly independent and $\operatorname{span}(\mathcal{V}) = V$.

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By assumption the list $\mathcal{V}$ is linearly independent and $\operatorname{span}(\mathcal{V}) = V$. Consider an arbitrary vector $u \in V$.

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Corollary: Let V be a d-dimensional subspace of $\mathbb{R}^n$.

(a) Any linearly independent list in V has at most d elements.
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This means that the original relation is the trivial one, as required.
As $\mathcal{W}$ is linearly independent and spans V, it is a basis for V.

Lecture 16

Canonical bases

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Let $\mathcal{U} = (u_1, \ldots, u_d)$ be any basis for V, and let A be the matrix with rows $u_1^T, \ldots, u_d^T$.

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Let B be the row-reduction of A, let $v_1^T, \dots, v_d^T$ be the rows of B, and put $\mathcal{V} = (v_1, \dots, v_d) =$ the list of columns of B^T.

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$w = \begin{bmatrix} 0 & b & -b & d & e & -d-e \end{bmatrix}^T$, where b, d and e are arbitrary.

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$w = \begin{bmatrix} 0 & b & -b & d & e & -d-e \end{bmatrix}^T$, where b, d and e are arbitrary.
If $b \neq 0$ then w starts in slot 2.

Examples of jumps

Example: Consider $V = \{\begin{bmatrix} s & -s & t+s & t-s \end{bmatrix}^T \mid s,t \in \mathbb{R}\} \subseteq \mathbb{R}^4$.
If $s \neq 0$ then the vector $x = \begin{bmatrix} s & -s & t+s & t-s \end{bmatrix}^T$ starts in slot 1.
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Lemma: Let B be an RREF matrix, and suppose that the columns of B^T form a basis for a subspace $V \subseteq \mathbb{R}^n$. Then $J(V) = \{$cols of B that contain pivots$\}$.

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Jumps and pivots

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- If $\lambda_1 \neq 0$ then $x = \begin{bmatrix} 0 & \boldsymbol{\lambda_1} & * & * & * & * & * \end{bmatrix}^T$

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- If $\lambda_1 \neq 0$ then $x = \begin{bmatrix} 0 & \boldsymbol{\lambda_1} & * & * & * & * & * \end{bmatrix}^T$ and so x starts in slot 2 (the first pivot column).

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- If $\lambda_1 = 0 \neq \lambda_2$ then $x = \begin{bmatrix} 0 & 0 & 0 & \boldsymbol{\lambda_2} & * & * & * \end{bmatrix}^T$

Jumps and pivots

Lemma: Let B be an RREF matrix, and suppose that the columns of B^T form a basis for a subspace $V \subseteq \mathbb{R}^n$. Then $J(V) = \{\text{cols of } B \text{ that contain pivots}\}$.

Example proof: Consider $B = \left[\begin{array}{c} v_1^T \\ \hline v_2^T \\ \hline v_3^T \end{array}\right] = \begin{bmatrix} 0 & \mathbf{1} & \alpha & 0 & \beta & 0 & \gamma \\ 0 & 0 & 0 & \mathbf{1} & \delta & 0 & \epsilon \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \zeta \end{bmatrix}$.

Put $V = \text{span}(v_1, v_2, v_3) \subseteq \mathbb{R}^7$, so the v_i ($=$ cols of B^T) form a basis for V. There are pivots in columns 2, 4 and 6, so we must show that $J(V) = \{2, 4, 6\}$. Any $x \in V$ has the form $x = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$

$$= \begin{bmatrix} 0 & \boldsymbol{\lambda_1} & \lambda_1\alpha_1 & \boldsymbol{\lambda_2} & \lambda_1\beta + \lambda_2\delta & \boldsymbol{\lambda_3} & \lambda_1\gamma + \lambda_2\epsilon + \lambda_3\zeta \end{bmatrix}^T.$$

Note that $\boldsymbol{\lambda_k}$ occurs on its own in the k'th pivot column, and all entries to the left of that involve only $\lambda_1, \dots, \lambda_{k-1}$. Thus, if $\lambda_1, \dots, \lambda_{k-1}$ are all zero but $\lambda_k \neq 0$ then x starts in the k'th pivot column. In more detail:

- If $\lambda_1 \neq 0$ then $x = \begin{bmatrix} 0 & \boldsymbol{\lambda_1} & * & * & * & * & * \end{bmatrix}^T$ and so x starts in slot 2 (the first pivot column).
- If $\lambda_1 = 0 \neq \lambda_2$ then $x = \begin{bmatrix} 0 & 0 & 0 & \boldsymbol{\lambda_2} & * & * & * \end{bmatrix}^T$ and so x starts in slot 4

Jumps and pivots

Lemma: Let B be an RREF matrix, and suppose that the columns of B^T form a basis for a subspace $V \subseteq \mathbb{R}^n$. Then $J(V) = \{\text{cols of } B \text{ that contain pivots}\}$.

Example proof: Consider $B = \left[\begin{array}{c} v_1^T \\ \hline v_2^T \\ \hline v_3^T \end{array}\right] = \begin{bmatrix} 0 & \mathbf{1} & \alpha & 0 & \beta & 0 & \gamma \\ 0 & 0 & 0 & \mathbf{1} & \delta & 0 & \epsilon \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \zeta \end{bmatrix}$.

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Note that $\boldsymbol{\lambda_k}$ occurs on its own in the k'th pivot column, and all entries to the left of that involve only $\lambda_1, \dots, \lambda_{k-1}$. Thus, if $\lambda_1, \dots, \lambda_{k-1}$ are all zero but $\lambda_k \neq 0$ then x starts in the k'th pivot column. In more detail:

- If $\lambda_1 \neq 0$ then $x = \begin{bmatrix} 0 & \boldsymbol{\lambda_1} & * & * & * & * & * \end{bmatrix}^T$ and so x starts in slot 2 (the first pivot column).
- If $\lambda_1 = 0 \neq \lambda_2$ then $x = \begin{bmatrix} 0 & 0 & 0 & \boldsymbol{\lambda_2} & * & * & * \end{bmatrix}^T$ and so x starts in slot 4 (the second pivot column).

Jumps and pivots

Lemma: Let B be an RREF matrix, and suppose that the columns of B^T form a basis for a subspace $V \subseteq \mathbb{R}^n$. Then $J(V) = \{\text{cols of } B \text{ that contain pivots}\}$.

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Put $V = \text{span}(v_1, v_2, v_3) \subseteq \mathbb{R}^7$, so the v_i ($=$ cols of B^T) form a basis for V. There are pivots in columns 2, 4 and 6, so we must show that $J(V) = \{2, 4, 6\}$. Any $x \in V$ has the form $x = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$

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Note that $\boldsymbol{\lambda_k}$ occurs on its own in the k'th pivot column, and all entries to the left of that involve only $\lambda_1, \dots, \lambda_{k-1}$. Thus, if $\lambda_1, \dots, \lambda_{k-1}$ are all zero but $\lambda_k \neq 0$ then x starts in the k'th pivot column. In more detail:

- If $\lambda_1 \neq 0$ then $x = \begin{bmatrix} 0 & \boldsymbol{\lambda_1} & * & * & * & * & * \end{bmatrix}^T$ and so x starts in slot 2 (the first pivot column).
- If $\lambda_1 = 0 \neq \lambda_2$ then $x = \begin{bmatrix} 0 & 0 & 0 & \boldsymbol{\lambda_2} & * & * & * \end{bmatrix}^T$ and so x starts in slot 4 (the second pivot column).
- If $\lambda_1 = \lambda_2 = 0 \neq \lambda_3$ then $x = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \boldsymbol{\lambda_3} & * \end{bmatrix}^T$

Jumps and pivots

Lemma: Let B be an RREF matrix, and suppose that the columns of B^T form a basis for a subspace $V \subseteq \mathbb{R}^n$. Then $J(V) = \{\text{cols of } B \text{ that contain pivots}\}$.

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Note that $\boldsymbol{\lambda_k}$ occurs on its own in the k'th pivot column, and all entries to the left of that involve only $\lambda_1, \dots, \lambda_{k-1}$. Thus, if $\lambda_1, \dots, \lambda_{k-1}$ are all zero but $\lambda_k \neq 0$ then x starts in the k'th pivot column. In more detail:

- If $\lambda_1 \neq 0$ then $x = \begin{bmatrix} 0 & \boldsymbol{\lambda_1} & * & * & * & * & * \end{bmatrix}^T$ and so x starts in slot 2 (the first pivot column).
- If $\lambda_1 = 0 \neq \lambda_2$ then $x = \begin{bmatrix} 0 & 0 & 0 & \boldsymbol{\lambda_2} & * & * & * \end{bmatrix}^T$ and so x starts in slot 4 (the second pivot column).
- If $\lambda_1 = \lambda_2 = 0 \neq \lambda_3$ then $x = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \boldsymbol{\lambda_3} & * \end{bmatrix}^T$ and so x starts in slot 6

Jumps and pivots

Lemma: Let B be an RREF matrix, and suppose that the columns of B^T form a basis for a subspace $V \subseteq \mathbb{R}^n$. Then $J(V) = \{$cols of B that contain pivots$\}$.

Example proof: Consider $B = \left[\begin{array}{c} v_1^T \\ \hline v_2^T \\ \hline v_3^T \end{array}\right] = \begin{bmatrix} 0 & \mathbf{1} & \alpha & 0 & \beta & 0 & \gamma \\ 0 & 0 & 0 & \mathbf{1} & \delta & 0 & \epsilon \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \zeta \end{bmatrix}$.

Put $V = \text{span}(v_1, v_2, v_3) \subseteq \mathbb{R}^7$, so the v_i ($=$ cols of B^T) form a basis for V. There are pivots in columns 2, 4 and 6, so we must show that $J(V) = \{2, 4, 6\}$. Any $x \in V$ has the form $x = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$

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Note that $\boldsymbol{\lambda_k}$ occurs on its own in the k'th pivot column, and all entries to the left of that involve only $\lambda_1, \dots, \lambda_{k-1}$. Thus, if $\lambda_1, \dots, \lambda_{k-1}$ are all zero but $\lambda_k \neq 0$ then x starts in the k'th pivot column. In more detail:

- If $\lambda_1 \neq 0$ then $x = \begin{bmatrix} 0 & \boldsymbol{\lambda_1} & * & * & * & * & * \end{bmatrix}^T$ and so x starts in slot 2 (the first pivot column).
- If $\lambda_1 = 0 \neq \lambda_2$ then $x = \begin{bmatrix} 0 & 0 & 0 & \boldsymbol{\lambda_2} & * & * & * \end{bmatrix}^T$ and so x starts in slot 4 (the second pivot column).
- If $\lambda_1 = \lambda_2 = 0 \neq \lambda_3$ then $x = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \boldsymbol{\lambda_3} & * \end{bmatrix}^T$ and so x starts in slot 6 (the third pivot column).

Jumps and pivots

Lemma: Let B be an RREF matrix, and suppose that the columns of B^T form a basis for a subspace $V \subseteq \mathbb{R}^n$. Then $J(V) = \{$cols of B that contain pivots$\}$.

Example proof: Consider $B = \begin{bmatrix} v_1^T \\ \hline v_2^T \\ \hline v_3^T \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{1} & \alpha & 0 & \beta & 0 & \gamma \\ 0 & 0 & 0 & \mathbf{1} & \delta & 0 & \epsilon \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \zeta \end{bmatrix}$.

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- If $\lambda_1 \neq 0$ then $x = \begin{bmatrix} 0 & \boldsymbol{\lambda_1} & * & * & * & * & * \end{bmatrix}^T$ and so x starts in slot 2 (the first pivot column).
- If $\lambda_1 = 0 \neq \lambda_2$ then $x = \begin{bmatrix} 0 & 0 & 0 & \boldsymbol{\lambda_2} & * & * & * \end{bmatrix}^T$ and so x starts in slot 4 (the second pivot column).
- If $\lambda_1 = \lambda_2 = 0 \neq \lambda_3$ then $x = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \boldsymbol{\lambda_3} & * \end{bmatrix}^T$ and so x starts in slot 6 (the third pivot column).
- If $\lambda_1 = \lambda_2 = \lambda_3 = 0$ then $x = 0$ and so x does not start anywhere.

Canonical bases — proof of uniqueness

Proposition 20.6: Let V be a subspace of $\mathbb{R}^n$. Then there is a **unique** RREF matrix B such that the columns of B^T form a basis for V.

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Suppose we have a subspace $V \subseteq \mathbb{R}^n$ and two RREF matrices B and C such that the columns of B^T form a basis for V, and the columns of C^T also form a basis for V.

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Suppose we have a subspace $V \subseteq \mathbb{R}^n$ and two RREF matrices B and C such that the columns of B^T form a basis for V, and the columns of C^T also form a basis for V. Both B and C must be $d \times n$ matrices, where $d = \dim(V)$.

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Now consider one of the vectors v_i.

$\square$

Canonical bases — proof of uniqueness

Proposition 20.6: Let V be a subspace of $\mathbb{R}^n$. Then there is a **unique** RREF matrix B such that the columns of B^T form a basis for V.

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Now consider one of the vectors v_i. As $v_i \in V$ and $V = \mathrm{span}(w_1, \dots, w_d)$ we can write v_i as a linear combination of the vectors w_j

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Suppose we have a subspace $V \subseteq \mathbb{R}^n$ and two RREF matrices B and C such that the columns of B^T form a basis for V, and the columns of C^T also form a basis for V. Both B and C must be $d \times n$ matrices, where $d = \dim(V)$. Let $v_1, \dots, v_d$ be the columns of B and let $w_1, \dots, w_d$ be the columns of C. Both B and C have all rows nonzero, and so have d pivots each. The pivot columns are the jumps for V and so are the same for B and C: say columns $p_1, \dots, p_d$.

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$\square$

Canonical bases — proof of uniqueness

Proposition 20.6: Let V be a subspace of $\mathbb{R}^n$. Then there is a **unique** RREF matrix B such that the columns of B^T form a basis for V.

Sketch proof of uniqueness.

Suppose we have a subspace $V \subseteq \mathbb{R}^n$ and two RREF matrices B and C such that the columns of B^T form a basis for V, and the columns of C^T also form a basis for V. Both B and C must be $d \times n$ matrices, where $d = \dim(V)$. Let $v_1, \dots, v_d$ be the columns of B and let $w_1, \dots, w_d$ be the columns of C. Both B and C have all rows nonzero, and so have d pivots each. The pivot columns are the jumps for V and so are the same for B and C: say columns $p_1, \dots, p_d$.

Now consider one of the vectors v_i. As $v_i \in V$ and $V = \operatorname{span}(w_1, \dots, w_d)$ we can write v_i as a linear combination of the vectors w_j, say $v_i = \lambda_1 w_1 + \cdots + \lambda_d w_d$. By looking in slot p_i we see that $1 = \lambda_i$. By looking in slot p_j (where $j \neq i$) we see that $\lambda_j = 0$. Thus, the sum on the right is just w_i □

Proposition 20.6: Let V be a subspace of $\mathbb{R}^n$. Then there is a **unique** RREF matrix B such that the columns of B^T form a basis for V.

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Suppose we have a subspace $V \subseteq \mathbb{R}^n$ and two RREF matrices B and C such that the columns of B^T form a basis for V, and the columns of C^T also form a basis for V. Both B and C must be $d \times n$ matrices, where $d = \dim(V)$. Let $v_1, \dots, v_d$ be the columns of B and let $w_1, \dots, w_d$ be the columns of C. Both B and C have all rows nonzero, and so have d pivots each. The pivot columns are the jumps for V and so are the same for B and C: say columns $p_1, \dots, p_d$.

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Canonical bases — proof of uniqueness

Proposition 20.6: Let V be a subspace of $\mathbb{R}^n$. Then there is a **unique** RREF matrix B such that the columns of B^T form a basis for V.

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Suppose we have a subspace $V \subseteq \mathbb{R}^n$ and two RREF matrices B and C such that the columns of B^T form a basis for V, and the columns of C^T also form a basis for V. Both B and C must be $d \times n$ matrices, where $d = \dim(V)$. Let $v_1, \dots, v_d$ be the columns of B and let $w_1, \dots, w_d$ be the columns of C. Both B and C have all rows nonzero, and so have d pivots each. The pivot columns are the jumps for V and so are the same for B and C: say columns $p_1, \dots, p_d$.

Now consider one of the vectors v_i. As $v_i \in V$ and $V = \mathrm{span}(w_1, \dots, w_d)$ we can write v_i as a linear combination of the vectors w_j, say $v_i = \lambda_1 w_1 + \cdots + \lambda_d w_d$. By looking in slot p_i we see that $1 = \lambda_i$. By looking in slot p_j (where $j \neq i$) we see that $\lambda_j = 0$. Thus, the sum on the right is just w_i and we get $v_i = w_i$. This holds for all i, so we have $B = C$ as claimed. $\square$

Finding the canonical basis for a span

Method: To find the canonical basis for a subspace $V = \text{span}(v_1, \ldots, v_r)$

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Then row-reduce to get an RREF matrix B, and discard any rows of zeros to get another RREF matrix C.

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Consider again the plane

$$P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid x + y + z = 0 \right\}.$$

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We conclude that the vectors $u_1 = \begin{bmatrix} 1 & 0 & -3 & 2 \end{bmatrix}^T$ and $u_2 = \begin{bmatrix} 0 & 1 & -2 & 1 \end{bmatrix}^T$ form the canonical basis for V.

Finding the canonical basis for an annihilator

Method: Suppose $V = \mathrm{ann}(u_1, \ldots, u_r) = \{x \in \mathbb{R}^n \mid x.u_1 = \cdots = x.u_r = 0\}$.

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- These constant vectors form the canonical basis for V.

Finding the canonical basis for an annihilator

Example: Put $V = \operatorname{ann}(u_1, u_2, u_3)$, where

$$u_1 = \begin{bmatrix} 9 & 13 & 5 & 3 \end{bmatrix}^T \qquad u_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \qquad u_3 = \begin{bmatrix} 7 & 11 & 3 & 1 \end{bmatrix}^T.$$

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The equations $x.u_3 = x.u_2 = x.u_1 = 0$ can be written as follows:

$$x_4 + 3x_3 + 11x_2 + 7x_1 = 0 \qquad x_4 + x_3 + x_2 + x_1 = 0 \qquad 3x_4 + 5x_3 + 13x_2 + 9x_1 = 0$$

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$$\begin{bmatrix} 1 & 3 & 11 & 7 \\ 1 & 1 & 1 & 1 \\ 3 & 5 & 13 & 9 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 10 & 6 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 10 & 6 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 5 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & -2 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This gives $x_4 - 4x_2 - 2x_1 = x_3 + 5x_2 + 3x_1 = 0$

Finding the canonical basis for an annihilator

Example: Put $V = \mathrm{ann}(u_1, u_2, u_3)$, where

$$u_1 = \begin{bmatrix} 9 & 13 & 5 & 3 \end{bmatrix}^T \qquad u_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \qquad u_3 = \begin{bmatrix} 7 & 11 & 3 & 1 \end{bmatrix}^T.$$

The equations $x.u_3 = x.u_2 = x.u_1 = 0$ can be written as follows:

$$x_4+3x_3+11x_2+7x_1 = 0 \qquad x_4+x_3+x_2+x_1 = 0 \qquad 3x_4+5x_3+13x_2+9x_1 = 0$$

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so $x_4 = 4x_2 + 2x_1$ and $x_3 = -5x_2 - 3x_1$.

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$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Finding the canonical basis for an annihilator

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Finding the canonical basis for an annihilator

Example: Put $V = \mathrm{ann}(u_1, u_2, u_3)$, where

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$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -5x_2 - 3x_1 \\ 4x_2 + 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -5 \\ 4 \end{bmatrix}.$$

Finding the canonical basis for an annihilator

Example: Put $V = \mathrm{ann}(u_1, u_2, u_3)$, where

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$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -5x_2 - 3x_1 \\ 4x_2 + 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -5 \\ 4 \end{bmatrix}.$$

so $\begin{bmatrix} 1 & 0 & -3 & 2 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 1 & -5 & 4 \end{bmatrix}^T$ form the canonical basis for V.

Finding the canonical basis for an annihilator

Example: Put $V = \operatorname{ann}(u_1, u_2, u_3)$, where

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Finding the canonical basis for an annihilator

Example: Put $V = \operatorname{ann}(u_1, u_2, u_3)$, where

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To find the canonical basis, write the equations $x.u_3 = x.u_2 = x.u_1 = 0$ as:

$$\begin{aligned} x_5 + x_4 + x_3 + x_2 + x_1 &= 0 \\ 3x_5 + 3x_4 + 3x_3 + 2x_2 + x_1 &= 0 \\ 5x_5 + 4x_4 + 3x_3 + 2x_2 + x_1 &= 0. \end{aligned}$$

Finding the canonical basis for an annihilator

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We now row-reduce the matrix of coefficients:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 2 & 1 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}$$

Finding the canonical basis for an annihilator

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We now row-reduce the matrix of coefficients:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 2 & 1 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -2 \\ 0 & -1 & -2 & -3 & -4 \end{bmatrix}$$

Finding the canonical basis for an annihilator

Example: Put $V = \operatorname{ann}(u_1, u_2, u_3)$, where

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$$\to \begin{bmatrix} 1 & 1 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Finding the canonical basis for an annihilator

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Finding the canonical basis for an annihilator

Example: Put $V = \operatorname{ann}(u_1, u_2, u_3)$, where

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x_5 + x_4 + x_3 + x_2 + x_1 &= 0\\
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\end{aligned}$$

We now row-reduce the matrix of coefficients:

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This gives $x_5 - x_3 + x_1 = 0$

Finding the canonical basis for an annihilator

Example: Put $V = \operatorname{ann}(u_1, u_2, u_3)$, where

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$$\to \begin{bmatrix}1 & 1 & 1 & 0 & -1\\ 0 & 1 & 2 & 0 & -2\\ 0 & 0 & 0 & 1 & 2\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1 & 0 & 1\\ 0 & 1 & 2 & 0 & -2\\ 0 & 0 & 0 & 1 & 2\end{bmatrix}$$

This gives $x_5 - x_3 + x_1 = 0$ and $x_4 + 2x_3 - 2x_1 = 0$

Finding the canonical basis for an annihilator

Example: Put $V = \mathrm{ann}(u_1, u_2, u_3)$, where

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This gives $x_5 - x_3 + x_1 = 0$ and $x_4 + 2x_3 - 2x_1 = 0$ and $x_2 + 2x_1 = 0$

Finding the canonical basis for an annihilator

Example: Put $V = \mathrm{ann}(u_1, u_2, u_3)$, where

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This gives $x_5 - x_3 + x_1 = 0$ and $x_4 + 2x_3 - 2x_1 = 0$ and $x_2 + 2x_1 = 0$
so $x_5 = x_3 - x_1$

Finding the canonical basis for an annihilator

Example: Put $V = \mathrm{ann}(u_1, u_2, u_3)$, where

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This gives $x_5 - x_3 + x_1 = 0$ and $x_4 + 2x_3 - 2x_1 = 0$ and $x_2 + 2x_1 = 0$ so $x_5 = x_3 - x_1$ and $x_4 = -2x_3 + 2x_1$ and $x_2 = -2x_1$

Finding the canonical basis for an annihilator

Example: Put $V = \text{ann}(u_1, u_2, u_3)$, where

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$$\to \begin{bmatrix} 1 & 1 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

This gives $x_5 - x_3 + x_1 = 0$ and $x_4 + 2x_3 - 2x_1 = 0$ and $x_2 + 2x_1 = 0$
so $x_5 = x_3 - x_1$ and $x_4 = -2x_3 + 2x_1$ and $x_2 = -2x_1$, (x_1, x_3 independent)

Finding the canonical basis for an annihilator

$V = \mathrm{ann}(u_1, u_2, u_3)$, where

$$u_1 = \begin{bmatrix}1 & 2 & 3 & 4 & 5\end{bmatrix}^T \qquad u_2 = \begin{bmatrix}1 & 2 & 3 & 3 & 3\end{bmatrix}^T \qquad u_3 = \begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}^T.$$

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$x_5 = x_3 - x_1$ and $x_4 = -2x_3 + 2x_1$ and $x_2 = -2x_1$
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Finding the canonical basis for an annihilator

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Equations $x.u_3 = x.u_2 = x.u_1 = 0$ give
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Thus

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1 \\ x_3 \\ -2x_3 + 2x_1 \\ x_3 - x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} = x_1 v_1 + x_3 v_2 \text{ say.}$$

It follows that the vectors

$$v_1 = \begin{bmatrix}1 & -2 & 0 & 2 & 1\end{bmatrix}^T \qquad \text{and} \qquad v_2 = \begin{bmatrix}0 & 0 & 1 & -2 & 1\end{bmatrix}^T$$

form the canonical basis for V.

Pure matrix method for annihilators

Method: Let A be a $k \times n$ matrix, and let $V \subseteq \mathbb{R}^n$ be the annihilator of the columns of A^T.

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(a) Rotate A through 180° to get a matrix A^*.

(b) Row-reduce A^* and discard any rows of zeros to obtain a matrix B^* in RREF. This will have shape $m \times n$ for some m with $m \leq \min(k, n)$.

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(c) The matrix B^* will have m pivots (one in each row). Let columns $p_1, \dots, p_m$ be the ones with pivots, and let columns $q_1, \dots, q_{n-m}$ be the ones without pivots.

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(d) Delete the pivot columns from B^* to leave an $m \times (n - m)$ matrix, which we call C^*. Let the i'th row of C^* be c_i^T (so $c_i \in \mathbb{R}^{n-m}$ for $1 \leq i \leq m$).

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(e) Now construct a new matrix D^* of shape $(n-m) \times n$ as follows: the p_i'th column is $-c_i$, and the q_j'th column is the standard basis vector e_j.

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Method: Let A be a $k \times n$ matrix, and let $V \subseteq \mathbb{R}^n$ be the annihilator of the columns of A^T. We can find the canonical basis for V as follows:

(a) Rotate A through 180° to get a matrix A^*.

(b) Row-reduce A^* and discard any rows of zeros to obtain a matrix B^* in RREF. This will have shape $m \times n$ for some m with $m \leq \min(k, n)$.

(c) The matrix B^* will have m pivots (one in each row). Let columns $p_1, \dots, p_m$ be the ones with pivots, and let columns $q_1, \dots, q_{n-m}$ be the ones without pivots.

(d) Delete the pivot columns from B^* to leave an $m \times (n-m)$ matrix, which we call C^*. Let the i'th row of C^* be c_i^T (so $c_i \in \mathbb{R}^{n-m}$ for $1 \leq i \leq m$).

(e) Now construct a new matrix D^* of shape $(n-m) \times n$ as follows: the p_i'th column is $-c_i$, and the q_j'th column is the standard basis vector e_j.

(f) Rotate D through 180° to get a matrix D.

Pure matrix method for annihilators

Method: Let A be a $k \times n$ matrix, and let $V \subseteq \mathbb{R}^n$ be the annihilator of the columns of A^T. We can find the canonical basis for V as follows:

(a) Rotate A through 180° to get a matrix A^*.

(b) Row-reduce A^* and discard any rows of zeros to obtain a matrix B^* in RREF. This will have shape $m \times n$ for some m with $m \leq \min(k, n)$.

(c) The matrix B^* will have m pivots (one in each row). Let columns $p_1, \dots, p_m$ be the ones with pivots, and let columns $q_1, \dots, q_{n-m}$ be the ones without pivots.

(d) Delete the pivot columns from B^* to leave an $m \times (n-m)$ matrix, which we call C^*. Let the i'th row of C^* be c_i^T (so $c_i \in \mathbb{R}^{n-m}$ for $1 \leq i \leq m$).

(e) Now construct a new matrix D^* of shape $(n-m) \times n$ as follows: the p_i'th column is $-c_i$, and the q_j'th column is the standard basis vector e_j.

(f) Rotate D through 180° to get a matrix D.

(g) The columns of D^T then form the canonical basis for V.

Example: Again consider $V = \operatorname{ann}(u_1, u_2, u_3)$, where

$$u_1 = \begin{bmatrix} 9 & 13 & 5 & 3 \end{bmatrix}^T \qquad u_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \qquad u_3 = \begin{bmatrix} 7 & 11 & 3 & 1 \end{bmatrix}^T.$$

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The pivot columns are $p_1 = 1$ and $p_2 = 2$, whereas the non-pivot columns are $q_1 = 3$ and $q_2 = 4$.

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The pivot columns are $p_1 = 1$ and $p_2 = 2$, whereas the non-pivot columns are $q_1 = 3$ and $q_2 = 4$. We now delete the pivot columns to get

$$C^* = \left[\begin{array}{c} c_1^T \\ \hline c_2^T \end{array}\right] = \begin{bmatrix} -4 & -2 \\ 5 & 3 \end{bmatrix}.$$

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$$D^* = \left[\begin{array}{c|c|c|c} -c_1 & -c_2 & e_1 & e_2 \end{array}\right]$$

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Example: Again consider $V = \operatorname{ann}(u_1, u_2, u_3)$, where

$$u_1 = \begin{bmatrix} 9 & 13 & 5 & 3 \end{bmatrix}^T \qquad u_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \qquad u_3 = \begin{bmatrix} 7 & 11 & 3 & 1 \end{bmatrix}^T.$$

$$A = \left[\begin{array}{c} u_1^T \\ \hline u_2^T \\ \hline u_3^T \end{array}\right] = \begin{bmatrix} 9 & 13 & 5 & 3 \\ 1 & 1 & 1 & 1 \\ 7 & 11 & 3 & 1 \end{bmatrix} \qquad A^* = \begin{bmatrix} 1 & 3 & 11 & 7 \\ 1 & 1 & 1 & 1 \\ 3 & 5 & 13 & 9 \end{bmatrix}.$$

The matrix A^* is the the matrix of coefficients appearing in our previous approach; as we saw we can row-reduce and delete zeros as follows:

$$A^* = \begin{bmatrix} 1 & 3 & 11 & 7 \\ 1 & 1 & 1 & 1 \\ 3 & 5 & 13 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & -2 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & -2 \\ 0 & 1 & 5 & 3 \end{bmatrix} = B^*.$$

The pivot columns are $p_1 = 1$ and $p_2 = 2$, whereas the non-pivot columns are $q_1 = 3$ and $q_2 = 4$. We now delete the pivot columns to get

$$C^* = \left[\begin{array}{c} c_1^T \\ \hline c_2^T \end{array}\right] = \begin{bmatrix} -4 & -2 \\ 5 & 3 \end{bmatrix}.$$

$$D^* = \left[\begin{array}{c|c|c|c} -c_1 & -c_2 & e_1 & e_2 \end{array}\right] = \begin{bmatrix} 4 & -5 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{bmatrix}; \; D = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \end{bmatrix}.$$

Pure matrix method for annihilators

Example: Again consider $V = \operatorname{ann}(u_1, u_2, u_3)$, where

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Canonical basis for V: $\begin{bmatrix} 1 & 0 & -3 & 2 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 1 & -5 & 4 \end{bmatrix}^T$.

Pure matrix method for annihilators

Example: Again consider $V = \mathrm{ann}(u_1, u_2, u_3)$, where

$$u_1 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}^T \qquad u_2 = \begin{bmatrix} 1 & 2 & 3 & 3 & 3 \end{bmatrix}^T \qquad u_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T.$$

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Pivot cols $p_1 = 1$, $p_2 = 2$ and $p_3 = 4$; non-pivot cols $q_1 = 3$ and $q_2 = 5$.

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Pivot cols $p_1 = 1$, $p_2 = 2$ and $p_3 = 4$; non-pivot cols $q_1 = 3$ and $q_2 = 5$.

Deleting pivot columns leaves $C^* = \left[\begin{array}{c} c_1^T \\ \hline c_2^T \\ \hline c_3^T \end{array}\right] = \begin{bmatrix}-1 & -1\\ 2 & -2\\ 0 & 2\end{bmatrix}$

Pure matrix method for annihilators

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$$D^* = \left[\begin{array}{c|c|c|c|c} -c_1 & -c_2 & e_1 & -c_3 & e_2 \end{array}\right]$$

Pure matrix method for annihilators

Example: Again consider $V = \mathrm{ann}(u_1, u_2, u_3)$, where

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$$A = \begin{bmatrix}1 & 2 & 3 & 4 & 5\\ 1 & 2 & 3 & 3 & 3\\ 1 & 1 & 1 & 1 & 1\end{bmatrix} \qquad A^* = \begin{bmatrix}1 & 1 & 1 & 1 & 1\\ 3 & 3 & 3 & 2 & 1\\ 5 & 4 & 3 & 2 & 1\end{bmatrix}$$

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Pure matrix method for annihilators

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Rotate: $D = \begin{bmatrix}1 & -2 & 0 & -2 & -1\\0 & 0 & 1 & 2 & -1\end{bmatrix}$.

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Rotate: $D = \begin{bmatrix}1 & -2 & 0 & -2 & -1\\ 0 & 0 & 1 & 2 & -1\end{bmatrix}$. Rows of D give canonical basis for V.

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In more detail: given $v_1, \ldots, v_r$ find $u_1, \ldots, u_s$ such that $\mathrm{span}(v_1, \ldots, v_r) = \mathrm{ann}(u_1, \ldots, u_s)$.

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(d) Call these constant vectors $u_1, \dots, u_s$. Then $V = \mathrm{ann}(u_1, \dots, u_s)$.

Lecture 17

Sums and intersections of subspaces

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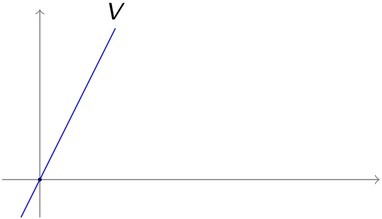
Example: Put $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid y = 2x \right\}$



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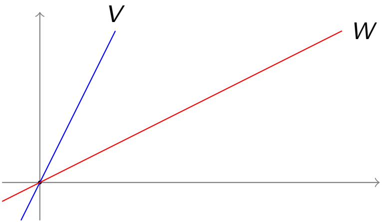

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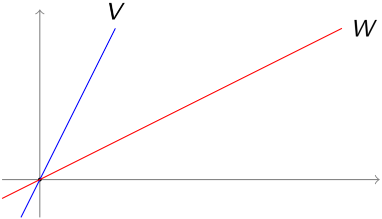

Then $V \cap W$ is the set of points lying on both lines, but the lines only meet at the origin

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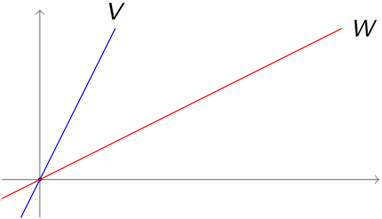

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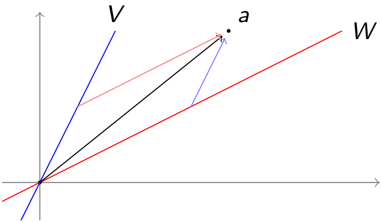

Then $V \cap W$ is the set of points lying on both lines, but the lines only meet at the origin, so $V \cap W = \{0\}$.
Every point $a \in \mathbb{R}^2$ can be expressed as the sum of a point on V with a point on W

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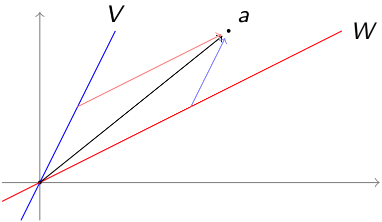
Example: Put $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid y = 2x \right\}$ $\quad W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid 2y = x \right\}$



Then $V \cap W$ is the set of points lying on both lines, but the lines only meet at the origin, so $V \cap W = \{0\}$.
Every point $a \in \mathbb{R}^2$ can be expressed as the sum of a point on V with a point on W, so $V + W = \mathbb{R}^2$.

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Sum and intersection example

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$$V = \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w = y \text{ and } x = z\}$$
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- Suppose $u \in V + W$. Then $u = v + w$ for some $v \in V$ and $w \in W$, say $v = \begin{bmatrix} p & q & p & q\end{bmatrix}$ and $w = \begin{bmatrix} -r & -s & s & r\end{bmatrix}^T$. This gives $u = v + w = \begin{bmatrix} p - r & q - s & p + s & q + r\end{bmatrix}$

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- Suppose $u \in U$

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Proposition: For lists $v_1, \ldots, v_r$ and $w_1, \ldots, w_s$ of vectors in $\mathbb{R}^n$, we have

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This is also the general form for an element of $\mathrm{span}(v_1, \ldots, v_r, w_1, \ldots, w_s)$.

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Sum of spans, intersection of annihilators

Proposition: For lists $v_1, \dots, v_r$ and $w_1, \dots, w_s$ of vectors in $\mathbb{R}^n$, we have

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Finding sums and intersections

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Dimensions of V, W, $V \cap W$ and $V + W$ are linked by the following formula:

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- If we know three of the numbers $\dim(V \cap W)$, $\dim(V + W)$, $\dim(V)$ and $\dim(W)$, we can rearrange the formula to find the fourth.
- If you believe that you have found bases for V, W, $V \cap W$ and $V + W$, you can use the formula as a check that your bases are correct.

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V + W &= \operatorname{span}(e_1, \ldots, e_{p+q+r}) \\
\dim(V \cap W) + \dim(V + W) &= p + (p + q + r) = 2p + q + r \\
\dim(V) + \dim(W) &= (p + q) + (p + r) = 2p + q + r.
\end{aligned}$$

- If we know three of the numbers $\dim(V \cap W)$, $\dim(V + W)$, $\dim(V)$ and $\dim(W)$, we can rearrange the formula to find the fourth.
- If you believe that you have found bases for V, W, $V \cap W$ and $V + W$, you can use the formula as a check that your bases are correct.

We will not prove the formula, except to say that one can choose bases to make the proof like the above example.

The dimension formula

Dimensions of V, W, $V \cap W$ and $V + W$ are linked by the following formula:

$$\dim(V \cap W) + \dim(V + W) = \dim(V) + \dim(W).$$

Example:

$$\begin{aligned}
V &= \operatorname{span}(e_1, \dots, e_p, e_{p+1}, \dots, e_{p+q}) \\
W &= \operatorname{span}(e_1, \dots, e_p, e_{p+q+1}, \dots, e_{p+q+r}) \\
V \cap W &= \operatorname{span}(e_1, \dots, e_p) \\
V + W &= \operatorname{span}(e_1, \dots, e_{p+q+r}) \\
\dim(V \cap W) + \dim(V + W) &= p + (p + q + r) = 2p + q + r \\
\dim(V) + \dim(W) &= (p + q) + (p + r) = 2p + q + r.
\end{aligned}$$

- If we know three of the numbers $\dim(V \cap W)$, $\dim(V + W)$, $\dim(V)$ and $\dim(W)$, we can rearrange the formula to find the fourth.
- If you believe that you have found bases for V, W, $V \cap W$ and $V + W$, you can use the formula as a check that your bases are correct.

We will not prove the formula, except to say that one can choose bases to make the proof like the above example. Details would be a digression.

Sum and intersection example

Put $V = \text{span}(v_1, v_2, v_3)$ and $W = \text{span}(w_1, w_2, w_3)$ where

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \qquad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \qquad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Sum and intersection example

Put $V = \text{span}(v_1, v_2, v_3)$ and $W = \text{span}(w_1, w_2, w_3)$ where

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Claim: $V + W = \mathbb{R}^4$.

Sum and intersection example

Put $V = \text{span}(v_1, v_2, v_3)$ and $W = \text{span}(w_1, w_2, w_3)$ where

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Claim: $V + W = \mathbb{R}^4$. Systematic proof: recall
$V + W = \text{span}(v_1, v_2, v_3, w_1, w_2, w_3))$ and row-reduce:

Sum and intersection example

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Claim: $V + W = \mathbb{R}^4$. Systematic proof: recall
$V + W = \mathrm{span}(v_1, v_2, v_3, w_1, w_2, w_3))$ and row-reduce:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \\ \hline v_3^T \\ \hline w_1^T \\ \hline w_2^T \\ \hline w_3^T \end{array}\right]$$

Sum and intersection example

Put $V = \mathrm{span}(v_1, v_2, v_3)$ and $W = \mathrm{span}(w_1, w_2, w_3)$ where

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \quad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \quad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

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$V + W = \mathrm{span}(v_1, v_2, v_3, w_1, w_2, w_3))$ and row-reduce:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \\ \hline v_3^T \\ \hline w_1^T \\ \hline w_2^T \\ \hline w_3^T \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Sum and intersection example

Put $V = \text{span}(v_1, v_2, v_3)$ and $W = \text{span}(w_1, w_2, w_3)$ where

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \qquad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \qquad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Claim: $V + W = \mathbb{R}^4$. Systematic proof: recall
$V + W = \text{span}(v_1, v_2, v_3, w_1, w_2, w_3))$ and row-reduce:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \\ \hline v_3^T \\ \hline w_1^T \\ \hline w_2^T \\ \hline w_3^T \end{array}\right] = \begin{bmatrix}1&0&0&0\\1&1&1&0\\1&1&1&1\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{bmatrix} \to \left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\ \hline 0&0&0&0\\0&0&0&0\end{array}\right]$$

Sum and intersection example

Put $V = \operatorname{span}(v_1, v_2, v_3)$ and $W = \operatorname{span}(w_1, w_2, w_3)$ where

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \qquad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \qquad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Claim: $V + W = \mathbb{R}^4$. Systematic proof: recall $V + W = \operatorname{span}(v_1, v_2, v_3, w_1, w_2, w_3))$ and row-reduce:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \\ \hline v_3^T \\ \hline w_1^T \\ \hline w_2^T \\ \hline w_3^T \end{array}\right] = \begin{bmatrix}1&0&0&0\\1&1&1&0\\1&1&1&1\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{bmatrix} \to \left[\begin{array}{cccc} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\ \hline 0&0&0&0\\0&0&0&0\end{array}\right] = \left[\begin{array}{c} e_1^T \\ \hline e_2^T \\ \hline e_3^T \\ \hline e_4^T \\ \hline 0 \\ \hline 0 \end{array}\right]$$

Sum and intersection example

Put $V = \text{span}(v_1, v_2, v_3)$ and $W = \text{span}(w_1, w_2, w_3)$ where

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \qquad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \qquad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Claim: $V + W = \mathbb{R}^4$. Systematic proof: recall
$V + W = \text{span}(v_1, v_2, v_3, w_1, w_2, w_3))$ and row-reduce:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \\ \hline v_3^T \\ \hline w_1^T \\ \hline w_2^T \\ \hline w_3^T \end{array}\right] = \begin{bmatrix}1&0&0&0\\1&1&1&0\\1&1&1&1\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{bmatrix} \to \left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\ \hline 0&0&0&0\\0&0&0&0\end{array}\right] = \left[\begin{array}{c} e_1^T \\ \hline e_2^T \\ \hline e_3^T \\ \hline e_4^T \\ \hline 0 \\ \hline 0 \end{array}\right]$$

Conclusion: (e_1, e_2, e_3, e_4) is the canonical basis for $V + W$

Sum and intersection example

Put $V = \mathrm{span}(v_1, v_2, v_3)$ and $W = \mathrm{span}(w_1, w_2, w_3)$ where

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \qquad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \qquad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Claim: $V + W = \mathbb{R}^4$. Systematic proof: recall
$V + W = \mathrm{span}(v_1, v_2, v_3, w_1, w_2, w_3))$ and row-reduce:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \\ \hline v_3^T \\ \hline w_1^T \\ \hline w_2^T \\ \hline w_3^T \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c} e_1^T \\ \hline e_2^T \\ \hline e_3^T \\ \hline e_4^T \\ \hline 0 \\ \hline 0 \end{array}\right]$$

Conclusion: (e_1, e_2, e_3, e_4) is the canonical basis for $V + W$, so $V + W = \mathbb{R}^4$.

Sum and intersection example

Put $V = \text{span}(v_1, v_2, v_3)$ and $W = \text{span}(w_1, w_2, w_3)$ where

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \quad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \quad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Claim: $V + W = \mathbb{R}^4$. Systematic proof: recall
$V + W = \text{span}(v_1, v_2, v_3, w_1, w_2, w_3)$ and row-reduce:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \\ \hline v_3^T \\ \hline w_1^T \\ \hline w_2^T \\ \hline w_3^T \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c} e_1^T \\ \hline e_2^T \\ \hline e_3^T \\ \hline e_4^T \\ \hline 0 \\ \hline 0 \end{array}\right]$$

Conclusion: (e_1, e_2, e_3, e_4) is the canonical basis for $V + W$, so $V + W = \mathbb{R}^4$.
More efficiently:

$$e_1 = v_1$$

Sum and intersection example

Put $V = \text{span}(v_1, v_2, v_3)$ and $W = \text{span}(w_1, w_2, w_3)$ where

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Claim: $V + W = \mathbb{R}^4$. Systematic proof: recall
$V + W = \text{span}(v_1, v_2, v_3, w_1, w_2, w_3))$ and row-reduce:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \\ \hline v_3^T \\ \hline w_1^T \\ \hline w_2^T \\ \hline w_3^T \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c} e_1^T \\ \hline e_2^T \\ \hline e_3^T \\ \hline e_4^T \\ \hline 0 \\ \hline 0 \end{array}\right]$$

Conclusion: (e_1, e_2, e_3, e_4) is the canonical basis for $V + W$, so $V + W = \mathbb{R}^4$.
More efficiently:

$$e_1 = v_1 \qquad e_2 = w_1 - v_1$$

Sum and intersection example

Put $V = \text{span}(v_1, v_2, v_3)$ and $W = \text{span}(w_1, w_2, w_3)$ where

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \qquad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \qquad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Claim: $V + W = \mathbb{R}^4$. Systematic proof: recall
$V + W = \text{span}(v_1, v_2, v_3, w_1, w_2, w_3))$ and row-reduce:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \\ \hline v_3^T \\ \hline w_1^T \\ \hline w_2^T \\ \hline w_3^T \end{array}\right] = \begin{bmatrix}1&0&0&0\\1&1&1&0\\1&1&1&1\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{bmatrix} \to \left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\ \hline 0&0&0&0\\0&0&0&0\end{array}\right] = \left[\begin{array}{c} e_1^T \\ \hline e_2^T \\ \hline e_3^T \\ \hline e_4^T \\ \hline 0 \\ \hline 0 \end{array}\right]$$

Conclusion: (e_1, e_2, e_3, e_4) is the canonical basis for $V + W$, so $V + W = \mathbb{R}^4$.
More efficiently:

$$e_1 = v_1 \qquad e_2 = w_1 - v_1 \qquad e_3 = v_2 - w_1$$

Sum and intersection example

Put $V = \operatorname{span}(v_1, v_2, v_3)$ and $W = \operatorname{span}(w_1, w_2, w_3)$ where

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \quad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \quad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Claim: $V + W = \mathbb{R}^4$. Systematic proof: recall $V + W = \operatorname{span}(v_1, v_2, v_3, w_1, w_2, w_3))$ and row-reduce:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \\ \hline v_3^T \\ \hline w_1^T \\ \hline w_2^T \\ \hline w_3^T \end{array}\right] = \begin{bmatrix}1&0&0&0\\1&1&1&0\\1&1&1&1\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{bmatrix} \to \left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\ \hline 0&0&0&0\\0&0&0&0\end{array}\right] = \left[\begin{array}{c} e_1^T \\ \hline e_2^T \\ \hline e_3^T \\ \hline e_4^T \\ \hline 0 \\ \hline 0 \end{array}\right]$$

Conclusion: (e_1, e_2, e_3, e_4) is the canonical basis for $V + W$, so $V + W = \mathbb{R}^4$.
More efficiently:

$$e_1 = v_1 \qquad e_2 = w_1 - v_1 \qquad e_3 = v_2 - w_1 \qquad e_4 = v_3 - v_2.$$

Sum and intersection example

Put $V = \text{span}(v_1, v_2, v_3)$ and $W = \text{span}(w_1, w_2, w_3)$ where

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \quad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \quad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Claim: $V + W = \mathbb{R}^4$. Systematic proof: recall
$V + W = \text{span}(v_1, v_2, v_3, w_1, w_2, w_3))$ and row-reduce:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \\ \hline v_3^T \\ \hline w_1^T \\ \hline w_2^T \\ \hline w_3^T \end{array}\right] = \begin{bmatrix}1&0&0&0\\1&1&1&0\\1&1&1&1\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{bmatrix} \to \left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\ \hline 0&0&0&0\\0&0&0&0\end{array}\right] = \left[\begin{array}{c} e_1^T \\ \hline e_2^T \\ \hline e_3^T \\ \hline e_4^T \\ \hline 0 \\ \hline 0 \end{array}\right]$$

Conclusion: (e_1, e_2, e_3, e_4) is the canonical basis for $V + W$, so $V + W = \mathbb{R}^4$.
More efficiently:

$$e_1 = v_1 \qquad e_2 = w_1 - v_1 \qquad e_3 = v_2 - w_1 \qquad e_4 = v_3 - v_2.$$

It follows that e_1, e_2, e_3 and e_4 are all in $V + W$

Sum and intersection example

Put $V = \text{span}(v_1, v_2, v_3)$ and $W = \text{span}(w_1, w_2, w_3)$ where

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \quad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \quad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Claim: $V + W = \mathbb{R}^4$. Systematic proof: recall
$V + W = \text{span}(v_1, v_2, v_3, w_1, w_2, w_3))$ and row-reduce:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \\ \hline v_3^T \\ \hline w_1^T \\ \hline w_2^T \\ \hline w_3^T \end{array}\right] = \begin{bmatrix}1&0&0&0\\1&1&1&0\\1&1&1&1\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{bmatrix} \to \left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\ \hline 0&0&0&0\\0&0&0&0\end{array}\right] = \left[\begin{array}{c} e_1^T \\ \hline e_2^T \\ \hline e_3^T \\ \hline e_4^T \\ \hline 0 \\ \hline 0 \end{array}\right]$$

Conclusion: (e_1, e_2, e_3, e_4) is the canonical basis for $V + W$, so $V + W = \mathbb{R}^4$.
More efficiently:

$$e_1 = v_1 \qquad e_2 = w_1 - v_1 \qquad e_3 = v_2 - w_1 \qquad e_4 = v_3 - v_2.$$

It follows that e_1, e_2, e_3 and e_4 are all in $V + W$, so $V + W = \mathbb{R}^4$.

Sum and intersection example

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Sum and intersection example

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Now find $V \cap W$.

Sum and intersection example

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Now find $V \cap W$. First step: describe V as an annihilator.

Sum and intersection example

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \qquad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \qquad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Now find $V \cap W$. First step: describe V as an annihilator. Write equations $x.v_3 = 0$, $x.v_2 = 0$ and $x.v_1 = 0$, with the variables x_i in descending order:

$$\begin{aligned} x_4 + x_3 + x_2 + x_1 &= 0 \\ x_3 + x_2 + x_1 &= 0 \\ x_1 &= 0. \end{aligned}$$

Sum and intersection example

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \quad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \quad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

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$$\begin{aligned} x_4 + x_3 + x_2 + x_1 &= 0 \\ x_3 + x_2 + x_1 &= 0 \\ x_1 &= 0. \end{aligned}$$

Clearly $x_1 = x_4 = 0$ and $x_3 = -x_2$, with x_2 arbitrary.

Sum and intersection example

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \quad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \quad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

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$$\begin{aligned} x_4 + x_3 + x_2 + x_1 &= 0 \\ x_3 + x_2 + x_1 &= 0 \\ x_1 &= 0. \end{aligned}$$

Clearly $x_1 = x_4 = 0$ and $x_3 = -x_2$, with x_2 arbitrary. Thus:

$$x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$

Sum and intersection example

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \quad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \quad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Now find $V \cap W$. First step: describe V as an annihilator. Write equations $x.v_3 = 0$, $x.v_2 = 0$ and $x.v_1 = 0$, with the variables x_i in descending order:

$$\begin{aligned} x_4 + x_3 + x_2 + x_1 &= 0 \\ x_3 + x_2 + x_1 &= 0 \\ x_1 &= 0. \end{aligned}$$

Clearly $x_1 = x_4 = 0$ and $x_3 = -x_2$, with x_2 arbitrary. Thus:

$$x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\x_2\\-x_2\\0\end{bmatrix}$$

Sum and intersection example

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \quad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \quad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Now find $V \cap W$. First step: describe V as an annihilator. Write equations $x.v_3 = 0$, $x.v_2 = 0$ and $x.v_1 = 0$, with the variables x_i in descending order:

$$\begin{aligned} x_4 + x_3 + x_2 + x_1 &= 0 \\ x_3 + x_2 + x_1 &= 0 \\ x_1 &= 0. \end{aligned}$$

Clearly $x_1 = x_4 = 0$ and $x_3 = -x_2$, with x_2 arbitrary. Thus:

$$x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\x_2\\-x_2\\0\end{bmatrix} = x_2 \begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$$

Sum and intersection example

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \quad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \quad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Now find $V \cap W$. First step: describe V as an annihilator. Write equations $x.v_3 = 0$, $x.v_2 = 0$ and $x.v_1 = 0$, with the variables x_i in descending order:

$$\begin{aligned} x_4 + x_3 + x_2 + x_1 &= 0 \\ x_3 + x_2 + x_1 &= 0 \\ x_1 &= 0. \end{aligned}$$

Clearly $x_1 = x_4 = 0$ and $x_3 = -x_2$, with x_2 arbitrary. Thus:

$$x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\x_2\\-x_2\\0\end{bmatrix} = x_2 \begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$$

We conclude that $V = \operatorname{ann}(a)$, where $a = \begin{bmatrix}0 & 1 & -1 & 0\end{bmatrix}^T$.

Sum and intersection example

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \qquad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \qquad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Sum and intersection example

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Next step: describe W as an annihilator.

Sum and intersection example

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \qquad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \qquad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Next step: describe W as an annihilator. Write down the equations $x.w_3 = 0$, $x.w_2 = 0$ and $x.w_1 = 0$, with the variables x_i in descending order:

$$x_4 + x_3 = 0$$
$$x_3 + x_2 = 0$$
$$x_2 + x_1 = 0.$$

Sum and intersection example

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \qquad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \qquad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Next step: describe W as an annihilator. Write down the equations $x.w_3 = 0$, $x.w_2 = 0$ and $x.w_1 = 0$, with the variables x_i in descending order:

$$\begin{aligned} x_4 + x_3 &= 0 \\ x_3 + x_2 &= 0 \\ x_2 + x_1 &= 0. \end{aligned}$$

This easily gives $x_4 = -x_3 = x_2 = -x_1$

Sum and intersection example

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \quad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \quad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Next step: describe W as an annihilator. Write down the equations $x.w_3 = 0$, $x.w_2 = 0$ and $x.w_1 = 0$, with the variables x_i in descending order:

$$\begin{aligned} x_4 + x_3 &= 0 \\ x_3 + x_2 &= 0 \\ x_2 + x_1 &= 0. \end{aligned}$$

This easily gives $x_4 = -x_3 = x_2 = -x_1$, so

$$x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$

Sum and intersection example

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \quad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \quad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Next step: describe W as an annihilator. Write down the equations $x.w_3 = 0$, $x.w_2 = 0$ and $x.w_1 = 0$, with the variables x_i in descending order:

$$\begin{aligned} x_4 + x_3 &= 0 \\ x_3 + x_2 &= 0 \\ x_2 + x_1 &= 0. \end{aligned}$$

This easily gives $x_4 = -x_3 = x_2 = -x_1$, so

$$x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}x_1\\-x_1\\x_1\\-x_1\end{bmatrix}$$

Sum and intersection example

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \qquad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \qquad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Next step: describe W as an annihilator. Write down the equations $x.w_3 = 0$, $x.w_2 = 0$ and $x.w_1 = 0$, with the variables x_i in descending order:

$$\begin{aligned} x_4 + x_3 &= 0 \\ x_3 + x_2 &= 0 \\ x_2 + x_1 &= 0. \end{aligned}$$

This easily gives $x_4 = -x_3 = x_2 = -x_1$, so

$$x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}x_1\\-x_1\\x_1\\-x_1\end{bmatrix} = x_1 \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$$

Sum and intersection example

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \qquad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \qquad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Next step: describe W as an annihilator. Write down the equations $x.w_3 = 0$, $x.w_2 = 0$ and $x.w_1 = 0$, with the variables x_i in descending order:

$$\begin{aligned} x_4 + x_3 &= 0 \\ x_3 + x_2 &= 0 \\ x_2 + x_1 &= 0. \end{aligned}$$

This easily gives $x_4 = -x_3 = x_2 = -x_1$, so

$$x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}x_1\\-x_1\\x_1\\-x_1\end{bmatrix} = x_1 \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$$

We conclude that $W = \operatorname{ann}(b)$, where $b = \begin{bmatrix}1 & -1 & 1 & -1\end{bmatrix}^T$.

Sum and intersection example

$$a = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T \qquad b = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$$

Sum and intersection example

$$a = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T \qquad\qquad b = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$$

We now have $V = \operatorname{ann}(a)$ and $W = \operatorname{ann}(b)$ so $V \cap W = \operatorname{ann}(a, b)$.

Sum and intersection example

$$a = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T \qquad b = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$$

We now have $V = \text{ann}(a)$ and $W = \text{ann}(b)$ so $V \cap W = \text{ann}(a, b)$. To find the canonical basis for this, we write the equations $x.b = 0$ and $x.a = 0$, again with the variables in decreasing order:

$$\begin{aligned} -x_4 + x_3 - x_2 + x_1 &= 0 \\ -x_3 + x_2 &= 0 \end{aligned}$$

Sum and intersection example

$$a = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T \qquad\qquad b = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$$

We now have $V = \text{ann}(a)$ and $W = \text{ann}(b)$ so $V \cap W = \text{ann}(a, b)$. To find the canonical basis for this, we write the equations $x.b = 0$ and $x.a = 0$, again with the variables in decreasing order:

$$\begin{aligned} -x_4 + x_3 - x_2 + x_1 &= 0 \\ -x_3 + x_2 &= 0 \end{aligned}$$

After row-reduction we get $x_4 = x_1$ and $x_3 = x_2$ with x_1 and x_2 arbitrary.

Sum and intersection example

$$a = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T \qquad b = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$$

We now have $V = \mathrm{ann}(a)$ and $W = \mathrm{ann}(b)$ so $V \cap W = \mathrm{ann}(a, b)$. To find the canonical basis for this, we write the equations $x.b = 0$ and $x.a = 0$, again with the variables in decreasing order:

$$\begin{aligned} -x_4 + x_3 - x_2 + x_1 &= 0 \\ -x_3 + x_2 &= 0 \end{aligned}$$

After row-reduction we get $x_4 = x_1$ and $x_3 = x_2$ with x_1 and x_2 arbitrary. This gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Sum and intersection example

$$a = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T \qquad\qquad b = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$$

We now have $V = \mathrm{ann}(a)$ and $W = \mathrm{ann}(b)$ so $V \cap W = \mathrm{ann}(a, b)$. To find the canonical basis for this, we write the equations $x.b = 0$ and $x.a = 0$, again with the variables in decreasing order:

$$\begin{aligned} -x_4 + x_3 - x_2 + x_1 &= 0 \\ -x_3 + x_2 &= 0 \end{aligned}$$

After row-reduction we get $x_4 = x_1$ and $x_3 = x_2$ with x_1 and x_2 arbitrary. This gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_1 \end{bmatrix}$$

Sum and intersection example

$$a = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T \qquad b = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$$

We now have $V = \mathrm{ann}(a)$ and $W = \mathrm{ann}(b)$ so $V \cap W = \mathrm{ann}(a, b)$. To find the canonical basis for this, we write the equations $x.b = 0$ and $x.a = 0$, again with the variables in decreasing order:

$$\begin{aligned} -x_4 + x_3 - x_2 + x_1 &= 0 \\ -x_3 + x_2 &= 0 \end{aligned}$$

After row-reduction we get $x_4 = x_1$ and $x_3 = x_2$ with x_1 and x_2 arbitrary. This gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Sum and intersection example

$$a = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T \qquad b = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$$

We now have $V = \mathrm{ann}(a)$ and $W = \mathrm{ann}(b)$ so $V \cap W = \mathrm{ann}(a, b)$. To find the canonical basis for this, we write the equations $x.b = 0$ and $x.a = 0$, again with the variables in decreasing order:

$$\begin{aligned} -x_4 + x_3 - x_2 + x_1 &= 0 \\ -x_3 + x_2 &= 0 \end{aligned}$$

After row-reduction we get $x_4 = x_1$ and $x_3 = x_2$ with x_1 and x_2 arbitrary. This gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

We conclude that the vectors $u_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$ and $u_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$ form the canonical basis for $V \cap W$.

Sum and intersection example

$$a = \begin{bmatrix}0 & 1 & -1 & 0\end{bmatrix}^T \qquad b = \begin{bmatrix}1 & -1 & 1 & -1\end{bmatrix}^T$$

We now have $V = \text{ann}(a)$ and $W = \text{ann}(b)$ so $V \cap W = \text{ann}(a, b)$. To find the canonical basis for this, we write the equations $x.b = 0$ and $x.a = 0$, again with the variables in decreasing order:

$$\begin{aligned} -x_4 + x_3 - x_2 + x_1 &= 0 \\ -x_3 + x_2 &= 0 \end{aligned}$$

After row-reduction we get $x_4 = x_1$ and $x_3 = x_2$ with x_1 and x_2 arbitrary. This gives

$$x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_2\\x_1\end{bmatrix} = x_1\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + x_2\begin{bmatrix}0\\1\\1\\0\end{bmatrix}.$$

We conclude that the vectors $u_1 = \begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix}^T$ and $u_2 = \begin{bmatrix}0 & 1 & 1 & 0\end{bmatrix}^T$ form the canonical basis for $V \cap W$. As a sanity check we can note that

$$u_1 = v_1 - v_2 + v_3$$

$$a = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T \qquad b = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$$

We now have $V = \mathrm{ann}(a)$ and $W = \mathrm{ann}(b)$ so $V \cap W = \mathrm{ann}(a, b)$. To find the canonical basis for this, we write the equations $x.b = 0$ and $x.a = 0$, again with the variables in decreasing order:

$$\begin{aligned} -x_4 + x_3 - x_2 + x_1 &= 0 \\ -x_3 + x_2 &= 0 \end{aligned}$$

After row-reduction we get $x_4 = x_1$ and $x_3 = x_2$ with x_1 and x_2 arbitrary. This gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

We conclude that the vectors $u_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$ and $u_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$ form the canonical basis for $V \cap W$. As a sanity check we can note that

$$u_1 = v_1 - v_2 + v_3 \in V$$

Sum and intersection example

$$a = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T \qquad b = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$$

We now have $V = \operatorname{ann}(a)$ and $W = \operatorname{ann}(b)$ so $V \cap W = \operatorname{ann}(a, b)$. To find the canonical basis for this, we write the equations $x.b = 0$ and $x.a = 0$, again with the variables in decreasing order:

$$\begin{aligned} -x_4 + x_3 - x_2 + x_1 &= 0 \\ -x_3 + x_2 &= 0 \end{aligned}$$

After row-reduction we get $x_4 = x_1$ and $x_3 = x_2$ with x_1 and x_2 arbitrary. This gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

We conclude that the vectors $u_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$ and $u_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$ form the canonical basis for $V \cap W$. As a sanity check we can note that

$$u_1 = v_1 - v_2 + v_3 \in V \qquad u_2 = v_2 - v_1$$

Sum and intersection example

$$a = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T \qquad b = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$$

We now have $V = \text{ann}(a)$ and $W = \text{ann}(b)$ so $V \cap W = \text{ann}(a, b)$. To find the canonical basis for this, we write the equations $x.b = 0$ and $x.a = 0$, again with the variables in decreasing order:

$$\begin{aligned} -x_4 + x_3 - x_2 + x_1 &= 0 \\ -x_3 + x_2 &= 0 \end{aligned}$$

After row-reduction we get $x_4 = x_1$ and $x_3 = x_2$ with x_1 and x_2 arbitrary. This gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

We conclude that the vectors $u_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$ and $u_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$ form the canonical basis for $V \cap W$. As a sanity check we can note that

$$u_1 = v_1 - v_2 + v_3 \in V \qquad u_2 = v_2 - v_1 \in V$$

Sum and intersection example

$$a = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T \qquad b = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$$

We now have $V = \mathrm{ann}(a)$ and $W = \mathrm{ann}(b)$ so $V \cap W = \mathrm{ann}(a, b)$. To find the canonical basis for this, we write the equations $x.b = 0$ and $x.a = 0$, again with the variables in decreasing order:

$$\begin{aligned} -x_4 + x_3 - x_2 + x_1 &= 0 \\ -x_3 + x_2 &= 0 \end{aligned}$$

After row-reduction we get $x_4 = x_1$ and $x_3 = x_2$ with x_1 and x_2 arbitrary. This gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

We conclude that the vectors $u_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$ and $u_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$ form the canonical basis for $V \cap W$. As a sanity check we can note that

$$u_1 = v_1 - v_2 + v_3 \in V \qquad u_2 = v_2 - v_1 \in V$$

$$u_1 = w_1 - w_2 + w_3$$

Sum and intersection example

$$a = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T \qquad b = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$$

We now have $V = \mathrm{ann}(a)$ and $W = \mathrm{ann}(b)$ so $V \cap W = \mathrm{ann}(a, b)$. To find the canonical basis for this, we write the equations $x.b = 0$ and $x.a = 0$, again with the variables in decreasing order:

$$\begin{aligned} -x_4 + x_3 - x_2 + x_1 &= 0 \\ -x_3 + x_2 &= 0 \end{aligned}$$

After row-reduction we get $x_4 = x_1$ and $x_3 = x_2$ with x_1 and x_2 arbitrary. This gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

We conclude that the vectors $u_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$ and $u_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$ form the canonical basis for $V \cap W$. As a sanity check we can note that

$$\begin{aligned} u_1 &= v_1 - v_2 + v_3 \in V \qquad & u_2 &= v_2 - v_1 \in V \\ u_1 &= w_1 - w_2 + w_3 \in W \end{aligned}$$

Sum and intersection example

$$a = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T \qquad b = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$$

We now have $V = \operatorname{ann}(a)$ and $W = \operatorname{ann}(b)$ so $V \cap W = \operatorname{ann}(a, b)$. To find the canonical basis for this, we write the equations $x.b = 0$ and $x.a = 0$, again with the variables in decreasing order:

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We conclude that the vectors $u_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$ and $u_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$ form the canonical basis for $V \cap W$. As a sanity check we can note that

$$\begin{aligned} u_1 &= v_1 - v_2 + v_3 \in V & \qquad u_2 &= v_2 - v_1 \in V \\ u_1 &= w_1 - w_2 + w_3 \in W & u_2 &= w_2 \in W. \end{aligned}$$

These equations show directly that u_1 and u_2 lie in $V \cap W$.

Dimension check

We will use the dimension formula to check our calculation.

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- We showed that $V + W = \mathbb{R}^4$ so $\dim(V + W) = 4$.

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- We showed that $V + W = \mathbb{R}^4$ so $\dim(V + W) = 4$.
- We showed that u_1, u_2 is a basis for $V \cap W$ so $\dim(V \cap W) = 2$.

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- We showed that $V + W = \mathbb{R}^4$ so $\dim(V + W) = 4$.
- We showed that u_1, u_2 is a basis for $V \cap W$ so $\dim(V \cap W) = 2$.
- Now $\dim(V + W) + \dim(V \cap W) = 4 + 2 = 6$

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- We showed that $V + W = \mathbb{R}^4$ so $\dim(V + W) = 4$.
- We showed that u_1, u_2 is a basis for $V \cap W$ so $\dim(V \cap W) = 2$.
- Now $\dim(V + W) + \dim(V \cap W) = 4 + 2 = 6$ and $\dim(V) + \dim(W) = 3 + 3 = 6$.

Dimension check

We will use the dimension formula to check our calculation.

- $V = \text{span}(v_1, v_2, v_3)$ and one can check that this list is independent so $\dim(V) = 3$.
- $W = \text{span}(w_1, w_2, w_3)$ and one can check that this list is independent so $\dim(W) = 3$.
- We showed that $V + W = \mathbb{R}^4$ so $\dim(V + W) = 4$.
- We showed that u_1, u_2 is a basis for $V \cap W$ so $\dim(V \cap W) = 2$.
- Now $\dim(V + W) + \dim(V \cap W) = 4 + 2 = 6$ and $\dim(V) + \dim(W) = 3 + 3 = 6$. As expected, these are the same.

Sum and intersection example

Put $V = \operatorname{span}(v_1, v_2)$ and $W = \operatorname{span}(w_1, w_2)$ where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad w_1 = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

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We will find the canonical bases for V, W, $V + W$ and $V \cap W$.

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We will find the canonical bases for V, W, $V + W$ and $V \cap W$. For V:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \end{array}\right]$$

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Put $V = \mathrm{span}(v_1, v_2)$ and $W = \mathrm{span}(w_1, w_2)$ where

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We will find the canonical bases for V, W, $V + W$ and $V \cap W$. For V:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \end{array} \right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Sum and intersection example

Put $V = \text{span}(v_1, v_2)$ and $W = \text{span}(w_1, w_2)$ where

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We will find the canonical bases for V, W, $V + W$ and $V \cap W$. For V:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \end{array}\right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Sum and intersection example

Put $V = \mathrm{span}(v_1, v_2)$ and $W = \mathrm{span}(w_1, w_2)$ where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad w_1 = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

We will find the canonical bases for V, W, $V + W$ and $V \cap W$. For V:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \end{array}\right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Sum and intersection example

Put $V = \mathrm{span}(v_1, v_2)$ and $W = \mathrm{span}(w_1, w_2)$ where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad w_1 = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

We will find the canonical bases for V, W, $V + W$ and $V \cap W$. For V:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \end{array}\right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Thus: the vectors $v_1' = \begin{bmatrix} 1 & 0 & -1 & -2 \end{bmatrix}^T$ and $v_2' = \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}^T$ form the canonical basis for V.

Sum and intersection example

Put $V = \operatorname{span}(v_1, v_2)$ and $W = \operatorname{span}(w_1, w_2)$ where

$$v_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix} \qquad w_1 = \begin{bmatrix}-3\\-1\\1\\3\end{bmatrix} \qquad w_2 = \begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$$

We will find the canonical bases for V, W, $V + W$ and $V \cap W$. For V:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \end{array}\right] = \begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 2 & 3 & 4\end{bmatrix} \to \begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 3\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1 & -2\\ 0 & 1 & 2 & 3\end{bmatrix}$$

Thus: the vectors $v_1' = \begin{bmatrix}1 & 0 & -1 & -2\end{bmatrix}^T$ and $v_2' = \begin{bmatrix}0 & 1 & 2 & 3\end{bmatrix}^T$ form the canonical basis for V.
Similarly, the row-reduction

$$\left[\begin{array}{c} w_1^T \\ \hline w_2^T \end{array}\right]$$

Sum and intersection example

Put $V = \text{span}(v_1, v_2)$ and $W = \text{span}(w_1, w_2)$ where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad w_1 = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

We will find the canonical bases for V, W, $V + W$ and $V \cap W$. For V:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \end{array}\right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Thus: the vectors $v_1' = \begin{bmatrix} 1 & 0 & -1 & -2 \end{bmatrix}^T$ and $v_2' = \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}^T$ form the canonical basis for V.
Similarly, the row-reduction

$$\left[\begin{array}{c} w_1^T \\ \hline w_2^T \end{array}\right] = \begin{bmatrix} -3 & -1 & 1 & 3 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

Sum and intersection example

Put $V = \mathrm{span}(v_1, v_2)$ and $W = \mathrm{span}(w_1, w_2)$ where

$$v_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix} \qquad w_1 = \begin{bmatrix}-3\\-1\\1\\3\end{bmatrix} \qquad w_2 = \begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$$

We will find the canonical bases for V, W, $V + W$ and $V \cap W$. For V:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \end{array}\right] = \begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 2 & 3 & 4\end{bmatrix} \to \begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 3\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1 & -2\\ 0 & 1 & 2 & 3\end{bmatrix}$$

Thus: the vectors $v_1' = \begin{bmatrix}1 & 0 & -1 & -2\end{bmatrix}^T$ and $v_2' = \begin{bmatrix}0 & 1 & 2 & 3\end{bmatrix}^T$ form the canonical basis for V.
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$$\left[\begin{array}{c} w_1^T \\ \hline w_2^T \end{array}\right] = \begin{bmatrix}-3 & -1 & 1 & 3\\ 0 & 1 & -1 & 0\end{bmatrix} \to \begin{bmatrix}-3 & 0 & 0 & 3\\ 0 & 1 & -1 & 0\end{bmatrix}$$

Sum and intersection example

Put $V = \text{span}(v_1, v_2)$ and $W = \text{span}(w_1, w_2)$ where

$$v_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix} \qquad w_1 = \begin{bmatrix}-3\\-1\\1\\3\end{bmatrix} \qquad w_2 = \begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$$

We will find the canonical bases for V, W, $V + W$ and $V \cap W$. For V:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \end{array}\right] = \begin{bmatrix}1 & 1 & 1 & 1\\1 & 2 & 3 & 4\end{bmatrix} \to \begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 2 & 3\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1 & -2\\0 & 1 & 2 & 3\end{bmatrix}$$

Thus: the vectors $v_1' = \begin{bmatrix}1 & 0 & -1 & -2\end{bmatrix}^T$ and $v_2' = \begin{bmatrix}0 & 1 & 2 & 3\end{bmatrix}^T$ form the canonical basis for V.
Similarly, the row-reduction

$$\left[\begin{array}{c} w_1^T \\ \hline w_2^T \end{array}\right] = \begin{bmatrix}-3 & -1 & 1 & 3\\0 & 1 & -1 & 0\end{bmatrix} \to \begin{bmatrix}-3 & 0 & 0 & 3\\0 & 1 & -1 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & -1\\0 & 1 & -1 & 0\end{bmatrix}$$

Sum and intersection example

Put $V = \operatorname{span}(v_1, v_2)$ and $W = \operatorname{span}(w_1, w_2)$ where

$$v_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix} \qquad w_1 = \begin{bmatrix}-3\\-1\\1\\3\end{bmatrix} \qquad w_2 = \begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$$

We will find the canonical bases for V, W, $V + W$ and $V \cap W$. For V:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \end{array}\right] = \begin{bmatrix}1 & 1 & 1 & 1\\1 & 2 & 3 & 4\end{bmatrix} \to \begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 2 & 3\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1 & -2\\0 & 1 & 2 & 3\end{bmatrix}$$

Thus: the vectors $v_1' = \begin{bmatrix}1 & 0 & -1 & -2\end{bmatrix}^T$ and $v_2' = \begin{bmatrix}0 & 1 & 2 & 3\end{bmatrix}^T$ form the canonical basis for V.

Similarly, the row-reduction

$$\left[\begin{array}{c} w_1^T \\ \hline w_2^T \end{array}\right] = \begin{bmatrix}-3 & -1 & 1 & 3\\0 & 1 & -1 & 0\end{bmatrix} \to \begin{bmatrix}-3 & 0 & 0 & 3\\0 & 1 & -1 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & -1\\0 & 1 & -1 & 0\end{bmatrix}$$

shows that the vectors $w_1' = \begin{bmatrix}1 & 0 & 0 & -1\end{bmatrix}^T$ and $w_2' = \begin{bmatrix}0 & 1 & -1 & 0\end{bmatrix}^T$ form the canonical basis for W.

Sum and intersection example

$V = \text{span}(v_1', v_2')$ and $W = \text{span}(w_1', w_2')$ where

$$v_1' = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} \qquad v_2' = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad w_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad w_2' = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Sum and intersection example

$V = \text{span}(v_1', v_2')$ and $W = \text{span}(w_1', w_2')$ where

$$v_1' = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} \qquad v_2' = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad w_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad w_2' = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Next find the canonical basis for $V + W = \text{span}(v_1', v_2', w_1', w_2')$

Sum and intersection example

$V = \text{span}(v_1', v_2')$ and $W = \text{span}(w_1', w_2')$ where

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Next find the canonical basis for $V + W = \text{span}(v_1', v_2', w_1', w_2')$, by row-reducing either the matrix $[v_1'|v_2'|w_1'|w_2']^T$:

Sum and intersection example

$V = \text{span}(v_1', v_2')$ and $W = \text{span}(w_1', w_2')$ where

$$v_1' = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} \qquad v_2' = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad w_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad w_2' = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Next find the canonical basis for $V + W = \text{span}(v_1', v_2', w_1', w_2')$, by row-reducing either the matrix $[v_1'|v_2'|w_1'|w_2']^T$:

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

Sum and intersection example

$V = \mathrm{span}(v_1', v_2')$ and $W = \mathrm{span}(w_1', w_2')$ where

$$v_1' = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} \qquad v_2' = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad w_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad w_2' = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Next find the canonical basis for $V + W = \mathrm{span}(v_1', v_2', w_1', w_2')$, by row-reducing either the matrix $[v_1'|v_2'|w_1'|w_2']^T$:

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & -3 \end{bmatrix}$$

Sum and intersection example

$V = \text{span}(v_1', v_2')$ and $W = \text{span}(w_1', w_2')$ where

$$v_1' = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} \qquad v_2' = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad w_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad w_2' = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Next find the canonical basis for $V + W = \text{span}(v_1', v_2', w_1', w_2')$, by row-reducing either the matrix $[v_1'|v_2'|w_1'|w_2']^T$:

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Sum and intersection example

$V = \text{span}(v_1', v_2')$ and $W = \text{span}(w_1', w_2')$ where

$$v_1' = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} \qquad v_2' = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad w_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad w_2' = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Next find the canonical basis for $V + W = \text{span}(v_1', v_2', w_1', w_2')$, by row-reducing either the matrix $[v_1'|v_2'|w_1'|w_2']^T$:

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We conclude that the following vectors form the canonical basis for $V + W$:

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

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In particular, we have $\dim(V + W) = 3$.

Sum and intersection example

$$V = \operatorname{span}(v_1', v_2') \qquad v_1' = \begin{bmatrix}1 & 0 & -1 & -2\end{bmatrix}^T \quad v_2' = \begin{bmatrix}0 & 1 & 2 & 3\end{bmatrix}^T$$

$$W = \operatorname{span}(w_1', w_2') \quad w_1' = \begin{bmatrix}1 & 0 & 0 & -1\end{bmatrix}^T \quad w_2' = \begin{bmatrix}0 & 1 & -1 & 0\end{bmatrix}^T$$

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Next, to understand $V \cap W$, we need to write V and W as annihilators.

Sum and intersection example

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For W: put $b_1 = \begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix}^T$ and $b_2 = \begin{bmatrix}0 & 1 & 1 & 0\end{bmatrix}^T$.

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After considering the form of the vectors w_1' and w_2' we see that

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ -x_2 \\ -x_1 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R} \right\}$$

Sum and intersection example

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Sum and intersection example

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For V: the equations $x.v_1' = 0$ and $x.v_2' = 0$ are
$-2x_4 - x_3 + x_1 = 0$ and $3x_4 + 2x_3 + x_2 = 0$.

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$$x_3 = -2x_2 - 3x_1 \qquad x_4 = x_2 + 2x_1 \qquad (\ x_2 \text{ and } x_1 \text{ arbitrary})$$

Sum and intersection example

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$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Sum and intersection example

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For W: put $b_1 = \begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix}^T$ and $b_2 = \begin{bmatrix}0 & 1 & 1 & 0\end{bmatrix}^T$.
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Sum and intersection example

$$V = \operatorname{span}(v_1', v_2') \qquad v_1' = \begin{bmatrix} 1 & 0 & -1 & -2 \end{bmatrix}^T \quad v_2' = \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}^T$$

$$W = \operatorname{span}(w_1', w_2') \quad w_1' = \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^T \quad w_2' = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T$$

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$$V = \operatorname{span}(v_1', v_2') \qquad v_1' = \begin{bmatrix}1 & 0 & -1 & -2\end{bmatrix}^T \quad v_2' = \begin{bmatrix}0 & 1 & 2 & 3\end{bmatrix}^T$$

$$W = \operatorname{span}(w_1', w_2') \quad w_1' = \begin{bmatrix}1 & 0 & 0 & -1\end{bmatrix}^T \quad w_2' = \begin{bmatrix}0 & 1 & -1 & 0\end{bmatrix}^T$$

Next, to understand $V \cap W$, we need to write V and W as annihilators.
For W: put $b_1 = \begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix}^T$ and $b_2 = \begin{bmatrix}0 & 1 & 1 & 0\end{bmatrix}^T$.
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Thus $V = \operatorname{ann}(a_1, a_2)$, where $a_1 = \begin{bmatrix}1 & 0 & -3 & 2\end{bmatrix}^T$, $a_2 = \begin{bmatrix}0 & 1 & -2 & 1\end{bmatrix}^T$.

Sum and intersection example

$$a_1 = \begin{bmatrix} 1 & 0 & -3 & 2 \end{bmatrix}^T \qquad a_2 = \begin{bmatrix} 0 & 1 & -2 & 1 \end{bmatrix}^T$$

$$b_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T \qquad b_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$$

Sum and intersection example

$$a_1 = \begin{bmatrix} 1 & 0 & -3 & 2 \end{bmatrix}^T \qquad a_2 = \begin{bmatrix} 0 & 1 & -2 & 1 \end{bmatrix}^T$$
$$b_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T \qquad b_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$$

We now have

$$V \cap W = \operatorname{ann}(a_1, a_2) \cap \operatorname{ann}(b_1, b_2)$$

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We now have

$$V \cap W = \operatorname{ann}(a_1, a_2) \cap \operatorname{ann}(b_1, b_2) = \operatorname{ann}(a_1, a_2, b_1, b_2).$$

Sum and intersection example

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To find the canonical basis, solve $x.b_2 = x.b_1 = x.a_2 = x.a_1 = 0$:

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$$\begin{aligned} x_3 + x_2 &= 0 & x_4 + x_1 &= 0 \\ x_4 - 2x_3 + x_2 &= 0 & 2x_4 - 3x_3 + x_1 &= 0. \end{aligned}$$

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$$x_3 + x_2 = 0 \qquad x_4 + x_1 = 0$$
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The first two equations give $x_3 = -x_2$ and $x_4 = -x_1$

Sum and intersection example

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The first two equations give $x_3 = -x_2$ and $x_4 = -x_1$, which we can substitute into the remaining equations to get $x_2 = x_1/3$.

Sum and intersection example

$$a_1 = \begin{bmatrix} 1 & 0 & -3 & 2 \end{bmatrix}^T \qquad a_2 = \begin{bmatrix} 0 & 1 & -2 & 1 \end{bmatrix}^T$$
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We now have

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The first two equations give $x_3 = -x_2$ and $x_4 = -x_1$, which we can substitute into the remaining equations to get $x_2 = x_1/3$. This leads to
$x = x_1 \begin{bmatrix} 1 & 1/3 & -1/3 & -1 \end{bmatrix}^T$

Sum and intersection example

$$a_1 = \begin{bmatrix} 1 & 0 & -3 & 2 \end{bmatrix}^T \qquad a_2 = \begin{bmatrix} 0 & 1 & -2 & 1 \end{bmatrix}^T$$
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We now have

$$V \cap W = \operatorname{ann}(a_1, a_2) \cap \operatorname{ann}(b_1, b_2) = \operatorname{ann}(a_1, a_2, b_1, b_2).$$

To find the canonical basis, solve $x.b_2 = x.b_1 = x.a_2 = x.a_1 = 0$:

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As a check, we note that

$$\dim(V + W) + \dim(V \cap W) = 3 + 1 = 2 + 2 = \dim(V) + \dim(W),$$

as expected.

Lecture 18

Rank

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$$\begin{aligned}\operatorname{rank}(A) &= \dim(\text{ span of the columns of } A)\\ &= \dim(\text{ span of the rows of } A^T).\end{aligned}$$

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Rank of an RCEF matrix

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This proves that the list $u_1, \dots, u_r$ is linearly independent, so it forms a basis for U, so $\dim(U) = r$. We thus have $\mathrm{rank}(A) = r$ as claimed. $\square$

Basic facts about column operations

Proposition 22.5: Any matrix A can be converted to RCEF by a sequence of elementary column operations.

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Proposition 22.7: Suppose that A can be converted to B by a sequence of elementary column operations.

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We thus have $B = B^{TT} = (UA^T)^T = A^{TT}U^T = AU^T$.
Here U^T is also invertible, so we can take $V = U^T$. $\square$

Proposition 22.7: Suppose that A can be converted to B by a sequence of elementary column operations. Then the span of the columns of A is the same as the span of the columns of B

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Proposition 22.7: Suppose that A can be converted to B by a sequence of elementary column operations. Then the span of the columns of A is the same as the span of the columns of B (and so $\mathrm{rank}(A) = \mathrm{rank}(B)$).

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Proof: Analogous to Corollary 9.16 for row operations. $\square$

Invariance under row operations

Proposition 22.8: Suppose that A can be converted to B by a sequence of elementary *row* operations. Then $\text{rank}(A) = \text{rank}(B)$.

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Proof: Let the columns of A be $v_1, \dots, v_n$ and put $V = \mathrm{span}(v_1, \dots, v_n)$ so $\mathrm{rank}(A) = \dim(V)$. There is an invertible matrix P such that

$$B = PA$$

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We need to check that it is also linearly independent.

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We have now shown that $Pa_1, \ldots, Pa_r$ is a basis for W, so $\dim(W) = r$. In conclusion, we have $\mathrm{rank}(A) = r = \mathrm{rank}(B)$ as required. $\square$

Normal form

Definition 22.9: An $n \times m$ matrix A is in *normal form* if it has the form

$$A = \left[\begin{array}{c|c} I_r & 0_{r\times(m-r)} \\ \hline 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{array}\right]$$

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Now just move all columns of zeros to the right hand end, which leaves a matrix C in normal form.

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As each pivot column contains nothing but the pivot, the only effect of these column operations is to set everything to the right of a pivot equal to zero.
However, every nonzero entry in B is either a pivot or to the right of a pivot, so after these ops we just have the pivots from B and everything else is zero.
Now just move all columns of zeros to the right hand end, which leaves a matrix C in normal form. As C was obtained from B by a sequence of elementary column operations, we have $C = BV$ for some invertible $m \times m$ matrix V. As $B = UA$, it follows that $C = UAV$. Propositions 22.7 and 22.8 tell us that neither row nor column operations affect the rank, so $\text{rank}(A) = \text{rank}(C)$

Reduction to normal form

Proposition 22.11: Any $n \times m$ matrix A can be converted to a matrix C in normal form by a sequence of row and column operations. Moreover:

(a) There is an invertible $n \times n$ matrix U and an invertible $m \times m$ matrix V such that $C = UAV$.

(b) $\text{rank}(A) = \text{rank}(C) =$ the number of ones in C.

Proof: Perform row operations to get a matrix B in RREF.
By Corollary 11.10 there is an invertible matrix U such that $B = UA$.
(This has to be an $n \times n$ matrix for the product UA to make sense.)
Now subtract multiples of pivot columns from columns further to the right. As each pivot column contains nothing but the pivot, the only effect of these column operations is to set everything to the right of a pivot equal to zero. However, every nonzero entry in B is either a pivot or to the right of a pivot, so after these ops we just have the pivots from B and everything else is zero. Now just move all columns of zeros to the right hand end, which leaves a matrix C in normal form. As C was obtained from B by a sequence of elementary column operations, we have $C = BV$ for some invertible $m \times m$ matrix V. As $B = UA$, it follows that $C = UAV$. Propositions 22.7 and 22.8 tell us that neither row nor column operations affect the rank, so $\text{rank}(A) = \text{rank}(C)$, and because C is in normal form, $\text{rank}(C)$ is just the number of ones in C.

Example of reduction to normal form

Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 6 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 2 & 9 \end{bmatrix}.$$

Example of reduction to normal form

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$$A = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 6 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 2 & 9 \end{bmatrix}.$$

This can be row-reduced as follows:

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 6 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 2 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 8 \end{bmatrix}$$

Example of reduction to normal form

Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 6 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 2 & 9 \end{bmatrix}.$$

This can be row-reduced as follows:

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 6 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 2 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example of reduction to normal form

Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 6 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 2 & 9 \end{bmatrix}.$$

This can be row-reduced as follows:

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 6 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 2 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example of reduction to normal form

Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 6 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 2 & 9 \end{bmatrix}.$$

This can be row-reduced as follows:

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 6 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 2 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We now perform column operations:

Example of reduction to normal form

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(Subtract column 1 from column 4, and 3 times column 1 from column 2;
)

Example of reduction to normal form

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$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 6 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 2 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We now perform column operations:

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(Subtract column 1 from column 4, and 3 times column 1 from column 2; subtract 4 times column 3 from column 4;)

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Consider the matrix

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$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(Subtract column 1 from column 4, and 3 times column 1 from column 2; subtract 4 times column 3 from column 4; exchange columns 2 and 3.)

Example of reduction to normal form

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$$A = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 6 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 2 & 9 \end{bmatrix}.$$

This can be row-reduced as follows:

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 6 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 2 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We now perform column operations:

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(Subtract column 1 from column 4, and 3 times column 1 from column 2; subtract 4 times column 3 from column 4; exchange columns 2 and 3.)
We are left with a matrix of rank 2 in normal form, so $\operatorname{rank}(A) = 2$.

Example of reduction to normal form

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

This can be reduced to normal form as follows: $A \to$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \\ 0 & -3 & -6 \end{bmatrix}$$

- Subtract multiples of row 1 from the other rows

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$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

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- Subtract multiples of row 1 from the other rows
- Multiply row 2 by -1

Example of reduction to normal form

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

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- Subtract multiples of row 1 from the other rows
- Multiply row 2 by -1
- Subtract multiples of row 2 from the other rows

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- Subtract multiples of row 1 from the other rows
- Multiply row 2 by -1
- Subtract multiples of row 2 from the other rows
- Add column 1 to column 3

Example of reduction to normal form

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

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- Subtract multiples of row 1 from the other rows
- Multiply row 2 by -1
- Subtract multiples of row 2 from the other rows
- Add column 1 to column 3
- Subtract 2 times column 2 from column 3.

Example of reduction to normal form

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

This can be reduced to normal form as follows: $A \to$

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- Subtract multiples of row 1 from the other rows
- Multiply row 2 by -1
- Subtract multiples of row 2 from the other rows
- Add column 1 to column 3
- Subtract 2 times column 2 from column 3.

The final matrix has rank 2, so we must also have $\mathrm{rank}(A) = 2$.

Proposition 22.14: For any matrix A we have $\operatorname{rank}(A) = \operatorname{rank}(A^T)$.

Proposition 22.14: For any matrix A we have $\operatorname{rank}(A) = \operatorname{rank}(A^T)$.

Proof: We can convert A by row and column ops to a matrix C in normal form, and $\operatorname{rank}(A)$ is the number of ones in C.

$\mathrm{rank}(A) = \mathrm{rank}(A^T)$

Proposition 22.14: For any matrix A we have $\mathrm{rank}(A) = \mathrm{rank}(A^T)$.

Proof: We can convert A by row and column ops to a matrix C in normal form, and $\mathrm{rank}(A)$ is the number of ones in C. If we transpose everything then the row ops become column ops and *vice-versa*, so A^T can be converted to C^T by row and column ops

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rank(A) = rank(A^T)

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Alternative terminology:

column rank of $A = \dim(\operatorname{span}(\text{ columns of } A))$

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With this terminology, the proposition says row rank=column rank.

Proposition 22.14: For any matrix A we have $\text{rank}(A) = \text{rank}(A^T)$.

Proof: We can convert A by row and column ops to a matrix C in normal form, and $\text{rank}(A)$ is the number of ones in C. If we transpose everything then the row ops become column ops and *vice-versa*, so A^T can be converted to C^T by row and column ops, and C^T is also in normal form, so $\text{rank}(A^T) =$ the number of ones in $C^T =$ the number of ones in $C = \text{rank}(A)$. $\square$

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Corollary 22.16: If A is an $n \times m$ matrix. Then $\text{rank}(A) \leq \min(n, m)$.

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Proof: We can convert A by row and column ops to a matrix C in normal form, and $\operatorname{rank}(A)$ is the number of ones in C. If we transpose everything then the row ops become column ops and *vice-versa*, so A^T can be converted to C^T by row and column ops, and C^T is also in normal form, so $\operatorname{rank}(A^T) =$ the number of ones in C^T = the number of ones in $C = \operatorname{rank}(A)$. $\square$

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Corollary 22.16: If A is an $n \times m$ matrix. Then $\operatorname{rank}(A) \leq \min(n, m)$.

Proof: Let V be the span of the columns of A, and let W be the span of the columns of A^T.

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With this terminology, the proposition says row rank=column rank.

Corollary 22.16: If A is an $n \times m$ matrix. Then $\text{rank}(A) \leq \min(n, m)$.

Proof: Let V be the span of the columns of A, and let W be the span of the columns of A^T. Now V is a subspace of $\mathbb{R}^n$, so $\dim(V) \leq n$

$\mathrm{rank}(A) = \mathrm{rank}(A^T)$

Proposition 22.14: For any matrix A we have $\mathrm{rank}(A) = \mathrm{rank}(A^T)$.

Proof: We can convert A by row and column ops to a matrix C in normal form, and $\mathrm{rank}(A)$ is the number of ones in C. If we transpose everything then the row ops become column ops and *vice-versa*, so A^T can be converted to C^T by row and column ops, and C^T is also in normal form, so $\mathrm{rank}(A^T) =$ the number of ones in C^T = the number of ones in $C = \mathrm{rank}(A)$. $\square$

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With this terminology, the proposition says row rank=column rank.

Corollary 22.16: If A is an $n \times m$ matrix. Then $\mathrm{rank}(A) \leq \min(n, m)$.

Proof: Let V be the span of the columns of A, and let W be the span of the columns of A^T. Now V is a subspace of $\mathbb{R}^n$, so $\dim(V) \leq n$, but W is a subspace of $\mathbb{R}^m$, so $\dim(W) \leq m$.

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Proposition 22.14: For any matrix A we have $\mathrm{rank}(A) = \mathrm{rank}(A^T)$.

Proof: We can convert A by row and column ops to a matrix C in normal form, and $\mathrm{rank}(A)$ is the number of ones in C. If we transpose everything then the row ops become column ops and *vice-versa*, so A^T can be converted to C^T by row and column ops, and C^T is also in normal form, so $\mathrm{rank}(A^T) =$ the number of ones in C^T = the number of ones in $C = \mathrm{rank}(A)$. $\square$

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A linearly independent list of n vectors in $\mathbb{R}^n$ is automatically a basis by
Proposition 10.12. $\square$

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Symmetric matrices

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$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 10 & 1000 \\ 1 & 10 & 1000 \\ 1 & 10 & 1000 \end{bmatrix}$$

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Put $p = A^T u$ and $q = Av$, so the claim is that $u.q = p.v$.

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By the definition of matrix multiplication, we have $q_i = \sum_j A_{ij} v_j$

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By assumption $u \neq 0$ so ($v \neq 0$ or $w \neq 0$) so $\|v\|^2 + \|w\|^2 > 0$.

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Let $v, w \in \mathbb{R}^n$ be the real and imaginary parts of u, so $u = v + iw$.

$$Av + iAw = A(v + iw) = Au = \lambda u = (\alpha + i\beta)(v + iw) = (\alpha v - \beta w) + i(\beta v + \alpha w).$$

As the entries in A are real, we see that the vectors Av and Aw are real.
Compare real and imaginary parts to get

$$\begin{aligned} Av &= \alpha v - \beta w & \qquad Aw &= \beta v + \alpha w \\ (Av).w &= \alpha v.w - \beta w.w & \qquad v.(Aw) &= \beta v.v + \alpha v.w. \end{aligned}$$

However, A is symmetric, so $(Av).w = v.(Aw)$ by Lemma 23.9.
Rearrange to get $\beta(v.v + w.w) = 0$ or $\beta(\|v\|^2 + \|w\|^2) = 0$.
By assumption $u \neq 0$ so ($v \neq 0$ or $w \neq 0$) so $\|v\|^2 + \|w\|^2 > 0$.
Divide by this to get $\beta = 0$ and $\lambda = \alpha \in \mathbb{R}$ as claimed.

Proposition 23.10: Let A be an $n \times n$ symmetric matrix (with real entries).

(a) All eigenvalues of A are real numbers.

(b) If u and v are (real) eigenvectors for A with distinct eigenvalues, then u and v are orthogonal.

Proof of (b):

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Rearrange to get $(\lambda - \mu)u.v = 0$.

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which means that u and v are orthogonal. $\square$

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This is the sum of two squares, so it is nonnegative, so the square root is real, so the two eigenvalues are both real.

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Proposition 13.22: any n eigenvectors in $\mathbb{R}^n$ with distinct eigenvalues form a basis.
Either of these results implies that $v_1, \dots, v_n$ is a basis. $\square$

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Some of them might be the same, but that would be a concidence which could only happen if the matrix A was very simple or had some kind of hidden symmetry.

Thus, our proof of Proposition 23.12 covers almost all cases
(but some of the cases that are not covered are the most interesting ones).

Orthonormal eigenvector example

Consider the symmetric matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

(which appeared on one of the problem sheets)

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$$u_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad u_5 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

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so the v_i are eigenvectors and are orthogonal to each other.

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However, the list $v_1, \dots, v_5$ is not orthonormal, because

$$v_1.v_1 = 2$$

Orthonormal eigenvector example

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$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \\ 0 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -4 \end{bmatrix} \qquad v_5 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

These satisfy $Av_1 = Av_2 = Av_3 = Av_4 = 0$ and $Av_5 = 5v_5$, and they are orthogonal to each other.

However, the list $v_1, \dots, v_5$ is not orthonormal, because

$$v_1.v_1 = 2 \qquad v_2.v_2 = 6 \qquad v_3.v_3 = 12 \qquad v_4.v_4 = 20 \qquad v_5.v_5 = 5.$$

This is easily fixed: if we put

$$w_1 = \frac{v_1}{\sqrt{2}} \qquad w_2 = \frac{v_2}{\sqrt{6}} \qquad w_3 = \frac{v_3}{\sqrt{12}} \qquad w_4 = \frac{v_4}{\sqrt{20}} \qquad w_5 = \frac{v_5}{\sqrt{5}}$$

then $w_1, \dots, w_5$ is an orthonormal basis for $\mathbb{R}^5$ consisting of eigenvectors for A.

Lecture 20

Corollary 23.15: Let A be an $n \times n$ symmetric matrix.
Then there is an orthogonal matrix U and a diagonal matrix D such that $A = UDU^T = UDU^{-1}$.

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Proof.
Choose an orthonormal basis of eigenvectors $u_1, \ldots, u_n$, and let λ_i be the eigenvalue of u_i.

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Proof.

Choose an orthonormal basis of eigenvectors $u_1, \dots, u_n$, and let λ_i be the eigenvalue of u_i.

Put $U = [u_1|\cdots|u_n]$ and $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$.

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Put $U = [u_1 | \cdots | u_n]$ and $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$.

Proposition 14.4 tells us that $U^{-1}AU = D$ and so $A = UDU^{-1}$.

$\square$

Orthogonal diagonalisation of symmetric matrices

Corollary 23.15: Let A be an $n \times n$ symmetric matrix.
Then there is an orthogonal matrix U and a diagonal matrix D such that $A = UDU^T = UDU^{-1}$.

Proof.
Choose an orthonormal basis of eigenvectors $u_1, \dots, u_n$, and let λ_i be the eigenvalue of u_i.

Put $U = [u_1 | \cdots | u_n]$ and $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$.

Proposition 14.4 tells us that $U^{-1}AU = D$ and so $A = UDU^{-1}$.

Proposition 23.5 tells us that U is an orthogonal matrix, so $U^{-1} = U^T$. $\square$

Example of orthogonal diagonalisation

Let A be the 5×5 matrix in which every entry is one, as in Example 23.14.

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Example of orthogonal diagonalisation

Let A be the 5×5 matrix in which every entry is one, as in Example 23.14. Following the prescription in the above proof, we put

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Example of orthogonal diagonalisation

Write $\rho = \sqrt{3}$ for brevity (so $\rho^2 = 3$), and consider the symmetric matrix

$$A = \begin{bmatrix} 0 & 1 & \rho \\ 1 & 0 & -\rho \\ \rho & -\rho & 0 \end{bmatrix}.$$

The characteristic polynomial is

$$\begin{aligned}
\chi_A(t) &= \det \begin{bmatrix} -t & 1 & \rho \\ 1 & -t & -\rho \\ \rho & -\rho & -t \end{bmatrix} \\
&= -t \det \begin{bmatrix} -t & -\rho \\ -\rho & -t \end{bmatrix} - \det \begin{bmatrix} 1 & -\rho \\ \rho & -t \end{bmatrix} + \rho \det \begin{bmatrix} 1 & -t \\ \rho & -\rho \end{bmatrix} \\
&= -t(t^2 - \rho^2) - (-t + \rho^2) + \rho(-\rho + t\rho) = -t^3 + 3t + t - 3 - 3 + 3t \\
&= -t^3 + 7t - 6 = -(t-1)(t-2)(t+3).
\end{aligned}$$

It follows that the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = -3$.

Example of orthogonal diagonalisation

$$\rho = \sqrt{3} \qquad A = \begin{bmatrix} 0 & 1 & \rho \\ 1 & 0 & -\rho \\ \rho & -\rho & 0 \end{bmatrix} \qquad \begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 2 \\ \lambda_3 &= -3. \end{aligned}$$

Eigenvectors can be found by row-reduction:

$$A - I = \begin{bmatrix} -1 & 1 & \rho \\ 1 & -1 & -\rho \\ \rho & -\rho & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -\rho \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} -2 & 1 & \rho \\ 1 & -2 & -\rho \\ \rho & -\rho & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -\rho \\ 0 & -3 & -\rho \\ 0 & \rho & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\rho/3 \\ 0 & 1 & \rho/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A + 3I = \begin{bmatrix} 3 & 1 & \rho \\ 1 & 3 & -\rho \\ \rho & -\rho & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -\rho \\ 0 & -8 & 4\rho \\ 0 & -4\rho & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \rho/2 \\ 0 & 1 & -\rho/2 \\ 0 & 0 & 0 \end{bmatrix}$$

From this we can read off the following eigenvectors:

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} \rho/3 \\ -\rho/3 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -\rho/2 \\ \rho/2 \\ 1 \end{bmatrix}.$$

Example of orthogonal diagonalisation

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 2 \\ \lambda_3 &= -3 \end{aligned} \qquad u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} \rho/3 \\ -\rho/3 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -\rho/2 \\ \rho/2 \\ 1 \end{bmatrix}$$

Because the matrix A is symmetric and the eigenvalues are distinct, it is automatic that the eigenvectors u_i are orthogonal to each other. However, they are not normalised: instead we have

$$\begin{aligned} u_1.u_1 &= 1^2 + 1^2 = 2 \\ u_2.u_2 &= (\rho/3)^2 + (-\rho/3)^2 + 1^2 = 1/3 + 1/3 + 1 = 5/3 \\ u_3.u_3 &= (-\rho/2)^2 + (\rho/2)^2 + 1^2 = 3/4 + 3/4 + 1 = 5/2. \end{aligned}$$

The vectors $v_i = u_i/\sqrt{u_i.u_i}$ form an orthonormal basis of eigenvectors. Explicitly, this works out as follows:

$$v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1/\sqrt{5} \\ -1/\sqrt{5} \\ \sqrt{3/5} \end{bmatrix} \qquad v_3 = \begin{bmatrix} -\sqrt{3/10} \\ \sqrt{3/10} \\ \sqrt{2/5} \end{bmatrix}.$$

Example of orthogonal diagonalisation

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= 2\\ \lambda_3 &= -3\end{aligned} \qquad v_1 = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix} \qquad v_2 = \begin{bmatrix}1/\sqrt{5}\\-1/\sqrt{5}\\\sqrt{3/5}\end{bmatrix} \qquad v_3 = \begin{bmatrix}-\sqrt{3/10}\\\sqrt{3/10}\\\sqrt{2/5}\end{bmatrix}.$$

The eigenvectors v_i form orthonormal basis for $\mathbb{R}^3$.

It follows that if we put

$$U = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{5} & -\sqrt{3/10}\\ 1/\sqrt{2} & -1/\sqrt{5} & \sqrt{3/10}\\ 0 & \sqrt{3/5} & \sqrt{2/5}\end{bmatrix} \qquad D = \begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & -3\end{bmatrix}$$

then U is an orthogonal matrix and $A = UDU^T$.

Corollary 23.18: Let A be an $n \times n$ real symmetric matrix, and suppose that all the eigenvalues of A are positive.

Square roots of positive matrices

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Proof.
Choose an orthonormal basis of eigenvectors $u_1, \dots, u_n$, and let λ_i be the eigenvalue of u_i.

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Choose an orthonormal basis of eigenvectors $u_1, \dots, u_n$, and let λ_i be the eigenvalue of u_i.
Put $U = [u_1 | \cdots | u_n]$ and $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$.

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We saw in Corollary 23.15 that U is orthogonal (so $U^T U = I = UU^T$)

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We saw in Corollary 23.15 that U is orthogonal (so $U^T U = I = UU^T$) and that $A = UDU^T$.

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We saw in Corollary 23.15 that U is orthogonal (so $U^T U = I = UU^T$) and that $A = UDU^T$.

By assumption the eigenvectors λ_i are positive, so we have a real diagonal matrix $E = \mathrm{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$.

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We saw in Corollary 23.15 that U is orthogonal (so $U^T U = I = UU^T$) and that $A = UDU^T$.

By assumption the eigenvectors λ_i are positive, so we have a real diagonal matrix $E = \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$. Put $B = UEU^T$.

$\square$

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Proof.
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We saw in Corollary 23.15 that U is orthogonal (so $U^T U = I = UU^T$) and that $A = UDU^T$.

By assumption the eigenvectors λ_i are positive, so we have a real diagonal matrix $E = \mathrm{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$. Put $B = UEU^T$. It is clear that $E^T = E$

□

Corollary 23.18: Let A be an $n \times n$ real symmetric matrix, and suppose that all the eigenvalues of A are positive.
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Given a linear form $L(x) = \sum_i a_i x_i$, we can form the vector $a = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}^T$, and clearly $L(x) = a.x = a^T x$.

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Symmetric expressions for quadratic forms

For example, we considered above the quadratic form

$$Q(x) = 10x_1x_2 + 12x_3x_4 - 14x_1x_4 - 16x_2x_3.$$

This can be rewritten symmetrically as

$$Q(x) = 5x_1x_2 + 5x_2x_1 + 6x_3x_4 + 6x_4x_3 - 7x_1x_4 - 7x_4x_1 - 8x_2x_3 - 8x_3x_2,$$

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Then there are integers $r, s \geq 0$ and nonzero vectors $v_1, \dots, v_r, w_1, \dots, w_s$ such that all the v's and w's are orthogonal to each other, and

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$$U = [u_1|\cdots|u_n] \qquad D = \mathrm{diag}(\lambda_1,\dots,\lambda_n) \qquad Q(x) = \lambda_1(u_1.x)^2 + \cdots + \lambda_n(u_n.x)^2.$$

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We thus have

$$Q(x) = (x.v_1)^2 + \cdots + (x.v_r)^2 - (x.w_1)^2 - \cdots - (x.w_s)^2$$

as required.

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$$v_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ -1/2 \end{bmatrix} \qquad w_1 = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \\ 0 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

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and it is easy to see that the v's and w's are all orthogonal.

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$$Q(x) = x^T Bx \qquad B = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \qquad \chi_B(t) = (t-1)(t+1)(t-4)(t+4)$$

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Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = -1$ and $\lambda_4 = -4$.

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Row-reduce $B - \lambda_i I$ to find the eigenvectors:

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In each case we see that $t_i.t_i = 10$

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Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = -1$ and $\lambda_4 = -4$.
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$$t_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix} \qquad t_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \qquad t_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix} \qquad t_4 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix}$$

In each case we see that $t_i.t_i = 10$ so the corresponding orthonormal basis consists of the vectors $u_i = t_i/\sqrt{10}$. Following Proposition 23.23:

$$v_1 = \sqrt{\lambda_1} u_1$$

Example of diagonalising a quadratic form

$$Q(x) = x^T Bx \qquad B = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \qquad \chi_B(t) = (t-1)(t+1)(t-4)(t+4)$$

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = -1$ and $\lambda_4 = -4$.
Row-reduce $B - \lambda_i I$ to find the eigenvectors:

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In each case we see that $t_i.t_i = 10$ so the corresponding orthonormal basis consists of the vectors $u_i = t_i/\sqrt{10}$. Following Proposition 23.23:

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$$v_1 \quad = \sqrt{\lambda_1} u_1 \quad = t_1/\sqrt{10} \quad = \sqrt{1/10} \begin{bmatrix} 2 & 1 & -1 & -2 \end{bmatrix}^T$$

Example of diagonalising a quadratic form

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Example of diagonalising a quadratic form

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Example of diagonalising a quadratic form

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Example of diagonalising a quadratic form

$$Q(x) = x^T B x \qquad B = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \qquad \chi_B(t) = (t-1)(t+1)(t-4)(t+4)$$

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$$t_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix} \qquad t_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \qquad t_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix} \qquad t_4 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix}$$

In each case we see that $t_i . t_i = 10$ so the corresponding orthonormal basis consists of the vectors $u_i = t_i / \sqrt{10}$. Following Proposition 23.23:

$$\begin{aligned} v_1 &= \sqrt{\lambda_1} u_1 && = t_1/\sqrt{10} && = \sqrt{1/10} \begin{bmatrix} 2 & 1 & -1 & -2 \end{bmatrix}^T \\ v_2 &= \sqrt{\lambda_2} u_2 && = \sqrt{4} t_2/\sqrt{10} && = \sqrt{2/5} \begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix}^T \\ w_1 &= \sqrt{|\lambda_3|} u_3 && = t_3/\sqrt{10} && \end{aligned}$$

Example of diagonalising a quadratic form

$$Q(x) = x^T B x \qquad B = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \qquad \chi_B(t) = (t-1)(t+1)(t-4)(t+4)$$

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = -1$ and $\lambda_4 = -4$.
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Example of diagonalising a quadratic form

$$Q(x) = x^T B x \qquad B = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \qquad \chi_B(t) = (t-1)(t+1)(t-4)(t+4)$$

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = -1$ and $\lambda_4 = -4$.
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$$t_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix} \qquad t_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \qquad t_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix} \qquad t_4 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix}$$

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$$\begin{aligned} v_1 &= \sqrt{\lambda_1}u_1 && = t_1/\sqrt{10} && = \sqrt{1/10}\begin{bmatrix} 2 & 1 & -1 & -2 \end{bmatrix}^T \\ v_2 &= \sqrt{\lambda_2}u_2 && = \sqrt{4}t_2/\sqrt{10} && = \sqrt{2/5}\begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix}^T \\ w_1 &= \sqrt{|\lambda_3|}u_3 && = t_3/\sqrt{10} && = \sqrt{1/10}\begin{bmatrix} 2 & -1 & -1 & 2 \end{bmatrix}^T \\ w_2 &= \sqrt{|\lambda_4|}u_4 \end{aligned}$$

Example of diagonalising a quadratic form

$$Q(x) = x^T B x \qquad B = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \qquad \chi_B(t) = (t-1)(t+1)(t-4)(t+4)$$

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = -1$ and $\lambda_4 = -4$.
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In each case we see that $t_i.t_i = 10$ so the corresponding orthonormal basis consists of the vectors $u_i = t_i/\sqrt{10}$. Following Proposition 23.23:

$$\begin{aligned} v_1 &= \sqrt{\lambda_1}u_1 && = t_1/\sqrt{10} && = \sqrt{1/10}\begin{bmatrix} 2 & 1 & -1 & -2 \end{bmatrix}^T \\ v_2 &= \sqrt{\lambda_2}u_2 && = \sqrt{4}t_2/\sqrt{10} && = \sqrt{2/5}\begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix}^T \\ w_1 &= \sqrt{|\lambda_3|}u_3 && = t_3/\sqrt{10} && = \sqrt{1/10}\begin{bmatrix} 2 & -1 & -1 & 2 \end{bmatrix}^T \\ w_2 &= \sqrt{|\lambda_4|}u_4 && = \sqrt{4}t_4/\sqrt{10} && \end{aligned}$$

Example of diagonalising a quadratic form

$$Q(x) = x^T B x \qquad B = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \qquad \chi_B(t) = (t-1)(t+1)(t-4)(t+4)$$

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In each case we see that $t_i.t_i = 10$ so the corresponding orthonormal basis consists of the vectors $u_i = t_i/\sqrt{10}$. Following Proposition 23.23:

$$\begin{aligned}
v_1 &= \sqrt{\lambda_1} u_1 &&= t_1/\sqrt{10} &&= \sqrt{1/10}\begin{bmatrix} 2 & 1 & -1 & -2 \end{bmatrix}^T \\
v_2 &= \sqrt{\lambda_2} u_2 &&= \sqrt{4} t_2/\sqrt{10} &&= \sqrt{2/5}\begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix}^T \\
w_1 &= \sqrt{|\lambda_3|} u_3 &&= t_3/\sqrt{10} &&= \sqrt{1/10}\begin{bmatrix} 2 & -1 & -1 & 2 \end{bmatrix}^T \\
w_2 &= \sqrt{|\lambda_4|} u_4 &&= \sqrt{4} t_4/\sqrt{10} &&= \sqrt{2/5}\begin{bmatrix} 1 & -2 & 2 & -1 \end{bmatrix}^T
\end{aligned}$$

Example of diagonalising a quadratic form

$$Q(x) = 4x_1x_2 + 6x_2x_3 + 4x_3x_4$$

$$v_1 = \sqrt{\frac{1}{10}} \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix} \qquad v_2 = \sqrt{\frac{2}{5}} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \qquad w_1 = \sqrt{\frac{1}{10}} \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix} \qquad w_2 = \sqrt{\frac{2}{5}} \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix}$$

Conclusion: $Q(x) = (x.v_1)^2 + (x.v_2)^2 - (x.w_1)^2 - (x.w_2)^2$.