Mathematics IV (Electrical) MAS243

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We need to understand the various different manifestations of the differential operator ∇. Moreover, these equations involve quantities like $\mathbf{E}$ and $\mathbf{v}$, the electric field and the fluid velocity at a single point. To calculate the total energy of the electric field in a region, or the total fluid flow through a pipe, we need to integrate (with respect to several variables).

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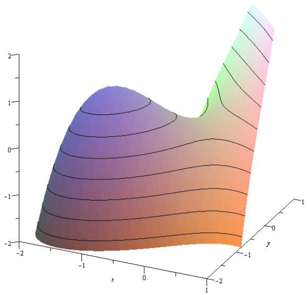
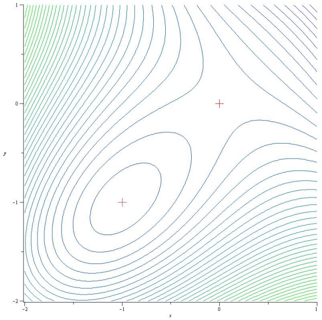
- Stokes's theorem and the Divergence theorem. Examples and applications.

3D Diagrams

There will be many three-dimensional diagrams for this course. It is often helpful to have a version that you can rotate with your mouse to inspect from different angles. Unfortunately I cannot embed such versions in these slides, but they will be available on the course web page.





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When calculating u_z, we treat z as a variable and x and y as constants, and we differentiate with respect to z.

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- These partial derivatives measure the sensitivity of u to small changes in x, y and z. If these variables change by small amounts δx, δy and δz, then the resulting change δu in u is approximately

$$\delta u \simeq u_x\,\delta x + u_y\,\delta y + u_z\,\delta z.$$

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- We have the usual product rule and quotient rule:

$$(uv)_x = u_x v + u v_x \qquad (u/v)_x = \frac{u_x v - u v_x}{v^2}$$

and similarly for the partial derivatives with respect to y or z.

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$$(uv)_x = u_xv + uv_x \qquad (u/v)_x = \frac{u_xv - uv_x}{v^2}$$

  and similarly for the partial derivatives with respect to y or z.
- We also have a chain rule: if v is a function of x, y and z, and $u = f(v)$, then

$$u_x = f'(v)v_x \qquad u_y = f'(v)v_y \qquad u_z = f'(v)v_z.$$

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u_{zx} = 0 & u_{zy} = 3y^2 & u_{zz} = 0
\end{array}
$$

Note that $u_{xy} = u_{yx}$, $u_{xz} = u_{zx}$ and $u_{yz} = u_{zy}$. This is a general principle: if we take partial derivatives with respect to two different variables, then the order does not matter.

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$$\begin{array}{lll} u_x = 2xy & u_y = x^2 + 3y^2z & u_z = y^3 \\ u_{xx} = 2y & u_{xy} = 2x & u_{xz} = 0 \\ u_{yx} = 2x & u_{yy} = 6yz & u_{yz} = 3y^2 \\ u_{zx} = 0 & u_{zy} = 3y^2 & u_{zz} = 0 \end{array}$$

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We can write all the second-order partial derivatives as a symmetric square matrix, called the *Hessian matrix*:

$$H = \begin{bmatrix} u_{xx} & u_{xy} & u_{xz} \\ u_{yx} & u_{yy} & u_{yz} \\ u_{zx} & u_{zy} & u_{zz} \end{bmatrix} = \begin{bmatrix} 2y & 2x & 0 \\ 2x & 6yz & 3y^2 \\ 0 & 3y^2 & 0 \end{bmatrix}.$$

Partial derivatives example

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$$P_R = -V^2/R^2$$
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P_{RR} &= 2V^2/R^3
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$$u_{aa} = 0$$

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The Hessian is therefore

$$H = \begin{bmatrix} u_{aa} & u_{ab} & u_{ac} \\ u_{ba} & u_{bb} & u_{bc} \\ u_{ca} & u_{cb} & u_{cc} \end{bmatrix}$$

Partial derivatives example

Consider the function $u = a + ab^2 + ab^2c^3$.

$$\begin{aligned} u_a &= 1 + b^2 + b^2c^3 \\ u_b &= 2ab + 2abc^3 \\ u_c &= 3ab^2c^2 \\ u_{aa} &= 0 \\ u_{ab} = u_{ba} &= 2b + 2bc^3 \\ u_{ac} = u_{ca} &= 3b^2c^2 \\ u_{bb} &= 2a + 2ac^3 \\ u_{bc} = u_{cb} &= 6abc^2 \\ u_{cc} &= 6ab^2c. \end{aligned}$$

The Hessian is therefore

$$H = \begin{bmatrix} u_{aa} & u_{ab} & u_{ac} \\ u_{ba} & u_{bb} & u_{bc} \\ u_{ca} & u_{cb} & u_{cc} \end{bmatrix} = \begin{bmatrix} 0 & 2b + 2bc^3 & 3b^2c^2 \\ 2b + 2bc^3 & 2a + 2ac^3 & 6abc^2 \\ 3b^2c^2 & 6abc^2 & 6ab^2c \end{bmatrix}.$$

Partial derivatives example

Consider $f(x, y, z) = \ln(ax + by + cz)$ (a, b, c constant).

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This means that the Hessian matrix is

$$H = \frac{-1}{(ax + by + cz)^2}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}.$$

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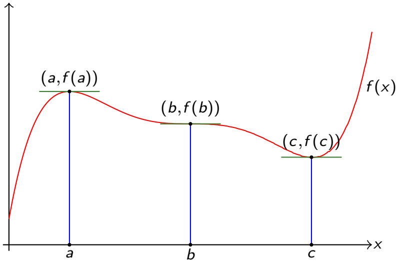
- For a function of several variables, look for points where *all* partial derivatives vanish.
- To distinguish between maxima, minima and saddles, consider the Hessian matrix of all second-order partial derivatives.

Optimisation with one variable

- The *critical points* of $f(x)$ are the values of x where $f'(x) = 0$.



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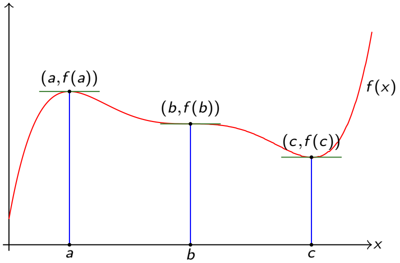

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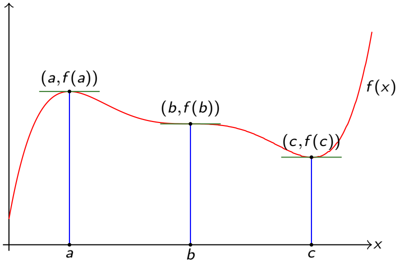

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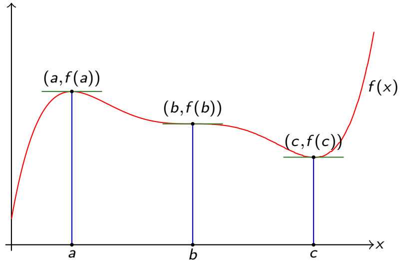

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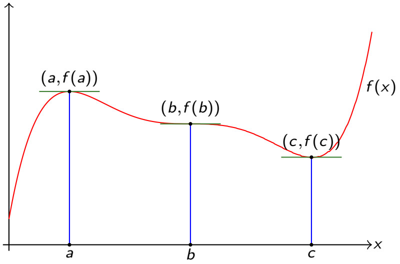

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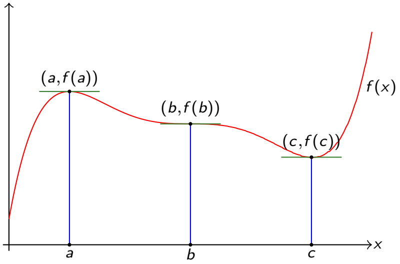

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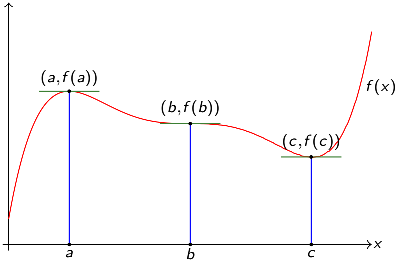

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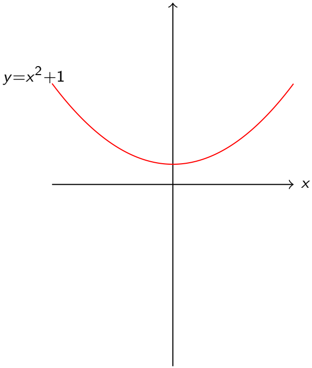

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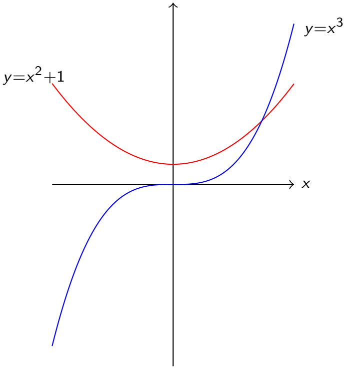

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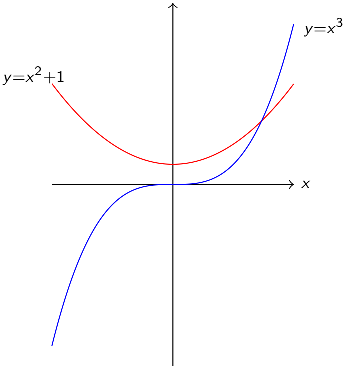
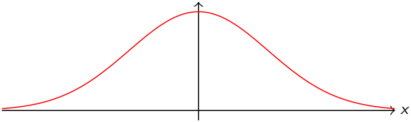
- When trying to find maxima and minima, you should remember the possibility that they might not exist.



Limiting maxima/minima that are not attained

- Here is the graph of $y = e^{-x^2}$:



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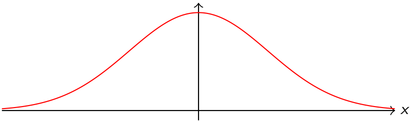

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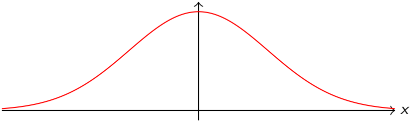

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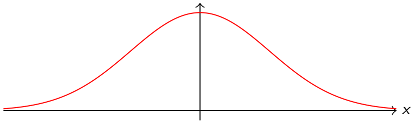

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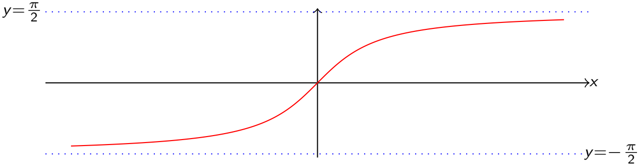

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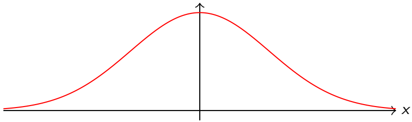

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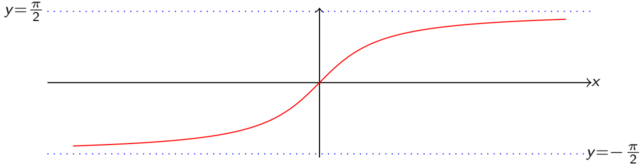

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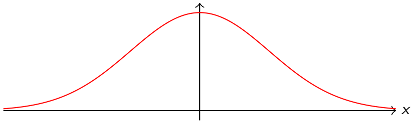

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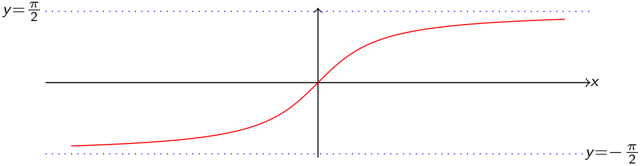

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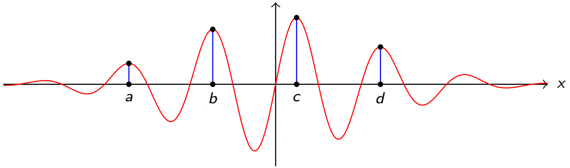
- In these cases we cannot find the maximum and minimum values by looking for critical points.

Other complications

- The function $y = e^{-x^2}\sin(10x)$ has local maxima at a, b, c and d. Only the one at $x = c$ is a global maximum.



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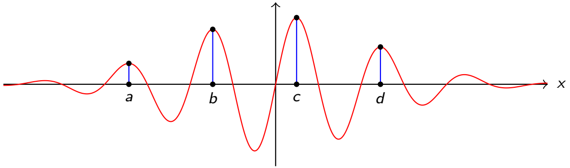

To find the global maximum, we need to check *all* the critical points.

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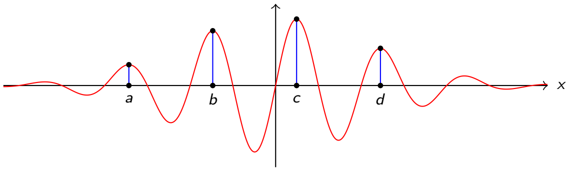
- The function $y = e^{-x^2}\sin(10x)$ has local maxima at a, b, c and d. Only the one at $x = c$ is a global maximum.



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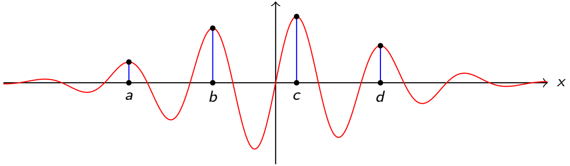

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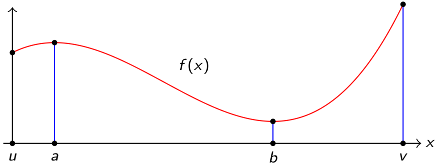

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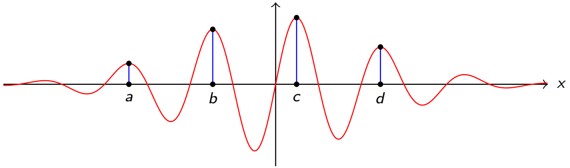

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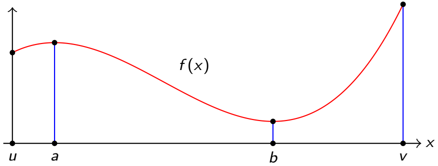

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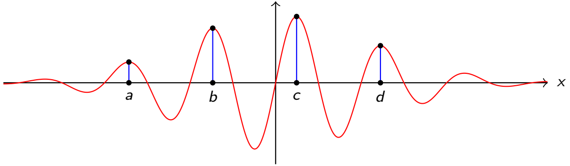

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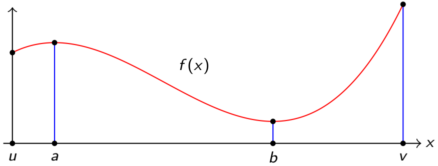

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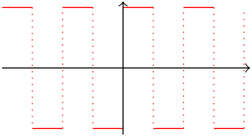
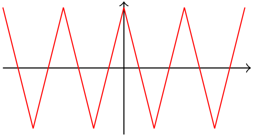
Bad functions

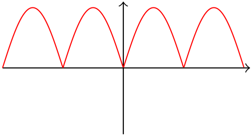
These methods do not work well if the graph of $f(x)$ has jumps or kinks.



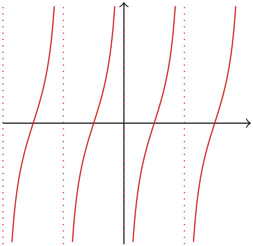
square wave



sawtooth

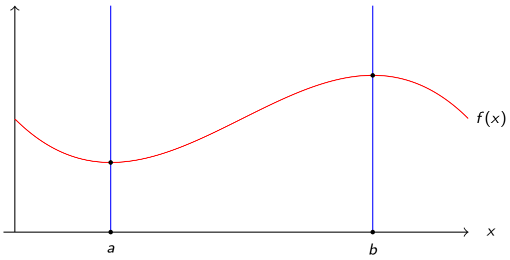


rectified sin wave



cotangent

Classifying critical points



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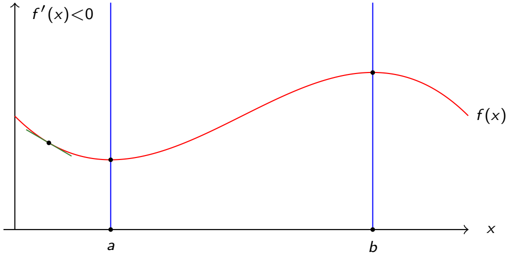

- The slope $f'(x)$ is negative to the left of a

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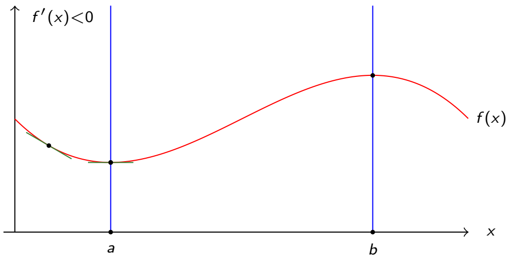

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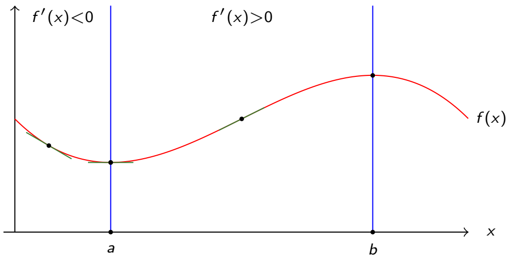

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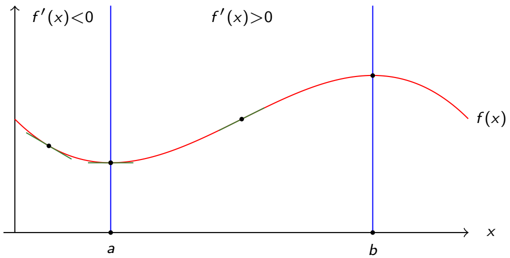
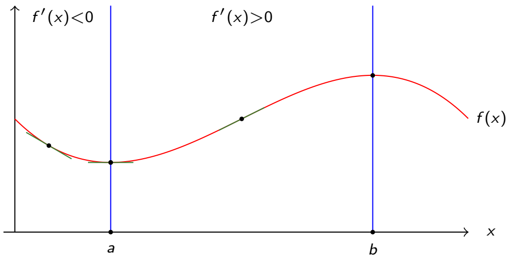


- The slope $f'(x)$ is negative to the left of a, and zero at a, and positive to the right of a; so $f''(a) > 0$.

Classifying critical points



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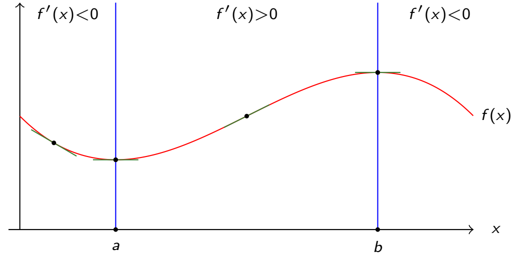

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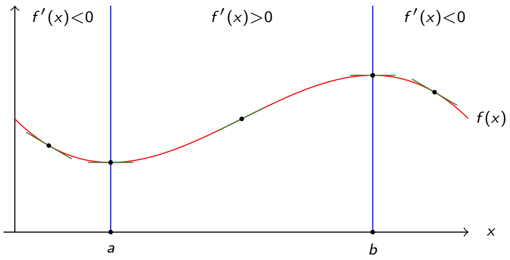

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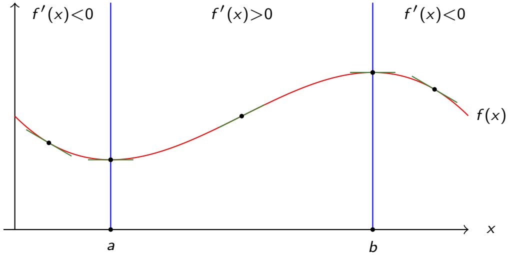

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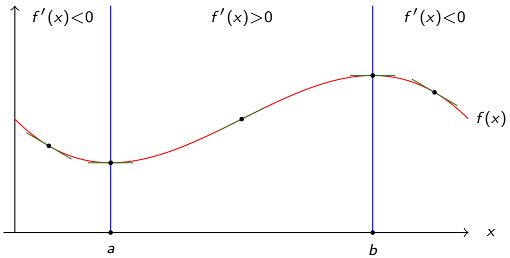

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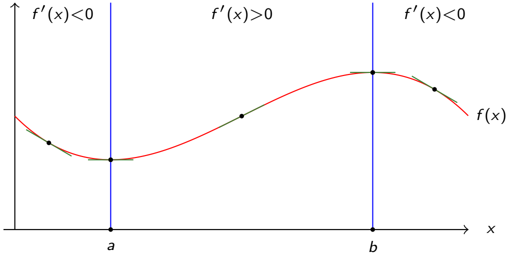

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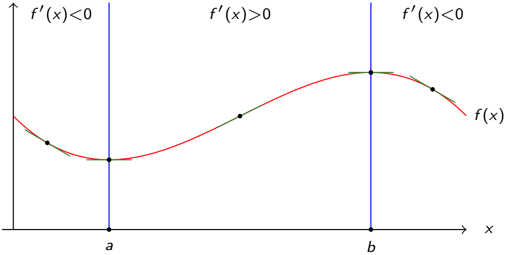

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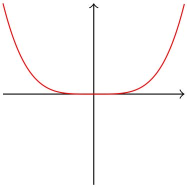
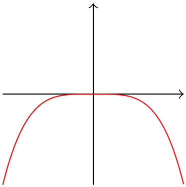
- In general, a critical point with $f'' > 0$ is a local minimum, and a critical point with $f'' < 0$ is a local maximum.
- It can happen that there is a critical point where $f'' = 0$. Such points are rare. They may be local maxima, local minima or neither.

What if $f''(x) = 0$ at a critical point?



$f(x) = x^4$



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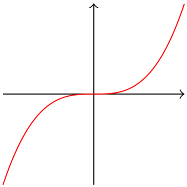

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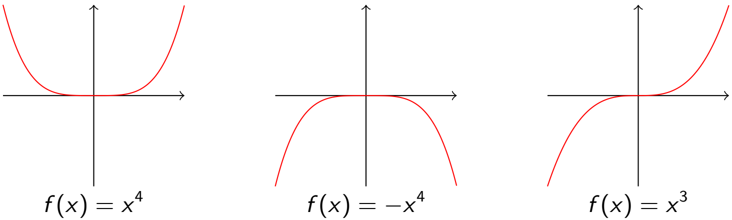

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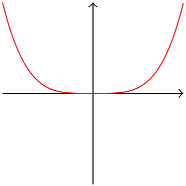

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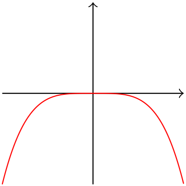

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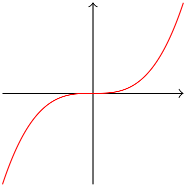

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- These three functions each have a critical point at $x = 0$ with $f''(0) = f'(0) = 0$.
- The first has a local minimum at $x = 0$, the second has a local maximum, and the third has an *inflection point* (a critical point that is neither a local maximum nor a local minimum).

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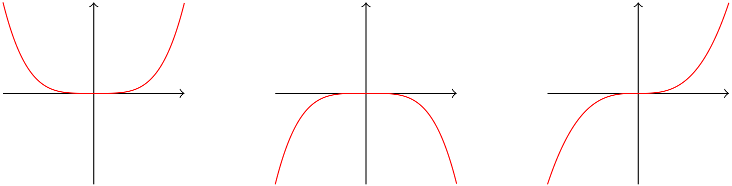

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- We will not take the time to explain general rules for this situation, as it does not occur very often.

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- For $0 < x < 4$ we see that all three factors (x^3, $4 - x$ and e^{-x}) are positive so $f'(x) > 0$ and the function increases from zero.

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- The minimum is zero, and the maximum is $f(4) = 256e^{-4} \simeq 4.69$.

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- Consider $f(x) = x^4 e^{-x}$ for $x \geq 0$.
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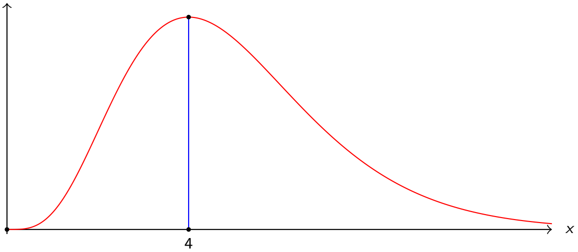

Critical points in two variables

- For a function $f(x, y)$, a *critical point* is a point where f_x and f_y both vanish.

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- At a critical point we must have (A) $4x + 2y = 6$ and (B) $2x + 2y = 4$

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- If we make small changes δx and δy to x and y, the resulting change in f is $\delta f \simeq f_x.\delta x + f_y.\delta y$.
- By taking δx and δy to have the same sign as f_x and f_y, we can make $\delta f > 0$ unless $f_x = f_y = 0$.

Critical points in two variables

- For a function $f(x, y)$, a *critical point* is a point where f_x and f_y both vanish.
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- By taking δx and δy to have the same sign as f_x and f_y, we can make $\delta f > 0$ unless $f_x = f_y = 0$.
- If we are at a local maximum it is impossible to have $\delta f > 0$, so we must have $f_x = f_y = 0$, so we have a critical point.

Critical points in two variables

- For a function $f(x, y)$, a *critical point* is a point where f_x and f_y both vanish.
- Consider $f(x, y) = 2x^2 + 2xy - 6x + y^2 - 4y + 5$, so $f_x(x, y) = 4x + 2y - 6$ and $f_y(x, y) = 2x + 2y - 4$.
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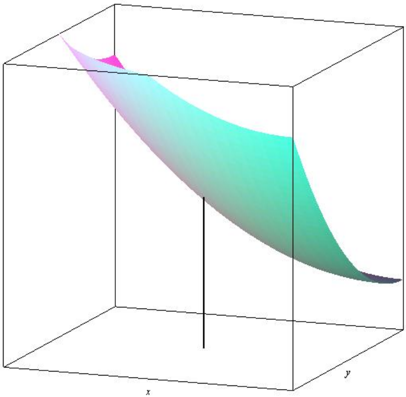
- Similarly, local minima are always critical points.

Geometry of partial derivatives



Here is a surface $z = f(x, y)$ with a marked point. The bottom of the black line is $(a, b, 0)$ for some a and b, and the top is $(a, b, f(a, b))$.

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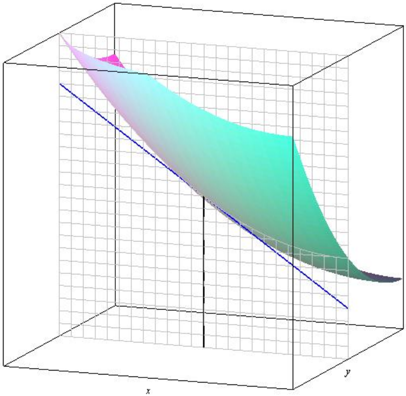

Here is a surface $z = f(x, y)$ with a marked point. We have added the plane $y = b$, shown in grey. The blue line lies in that plane and is tangent to the surface. Along this line x varies and y is held constant. The slope is the partial derivative $f_x(a, b)$.

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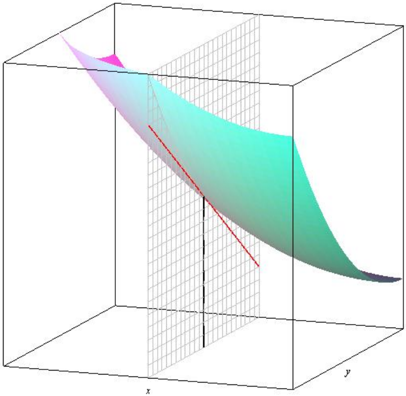

Here is a surface $z = f(x, y)$ with a marked point. We have added the plane $x = a$, shown in grey. The red line lies in that plane and is tangent to the surface. Along this line y varies and x is held constant. The slope is the partial derivative $f_y(a, b)$.

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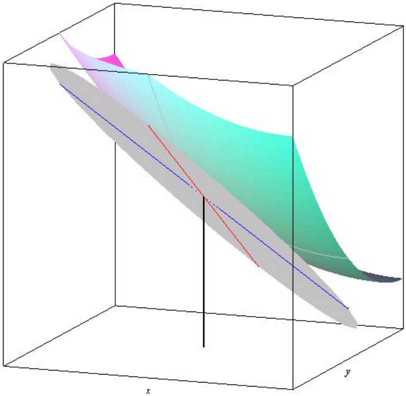
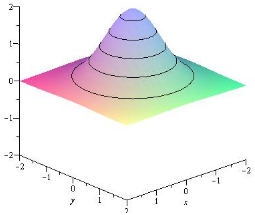
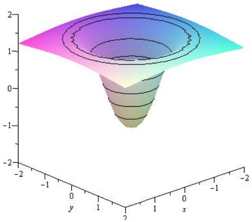


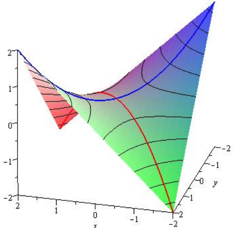
Here is a surface $z = f(x, y)$ with a marked point. The grey disk is the tangent plane to the surface at the point $(a, b, f(a, b))$. This will be horizontal if (a, b) is a critical point (which is clearly not the case for this example.)

Types of critical points

There are three main types of critical point for a function of two variables: local maxima, local minima and saddle points.







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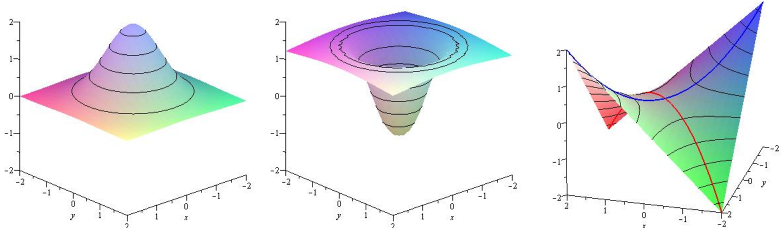

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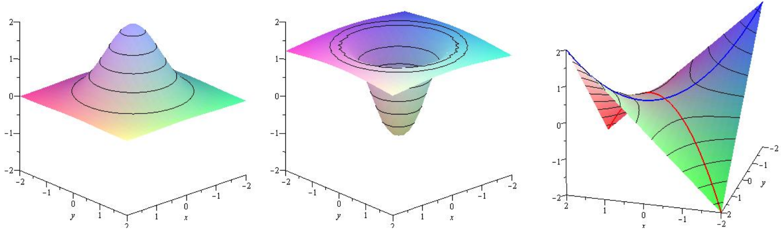

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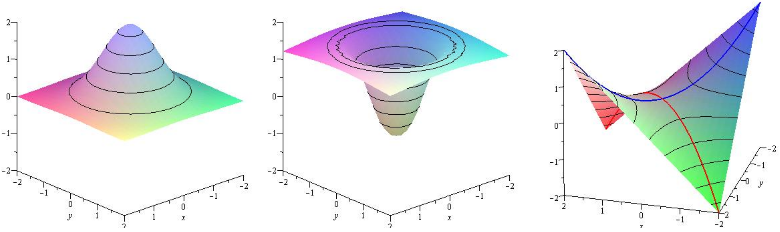

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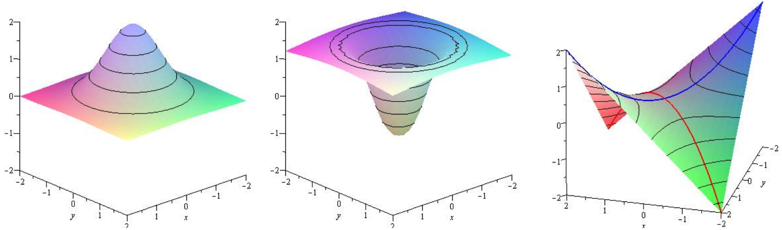

If we walk along the blue curve, the saddle looks like a local minimum. If we walk along the red curve, it looks like a local maximum instead. Saddle points are common, not like inflection points. If we have found a critical point and we want to know whether it is a local maximum, a local minimum or a saddle, we need to look at the Hessian matrix.

Classifying critical points

- Suppose $f(x, y)$ has a critical point at (a, b) (so $f_x(a, b) = f_y(a, b) = 0$).

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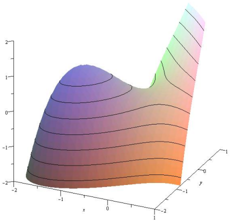

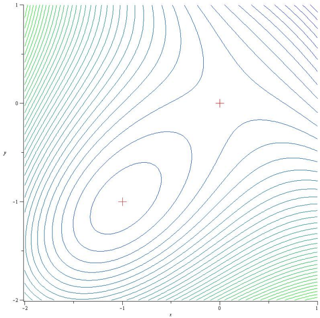

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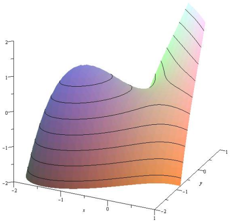

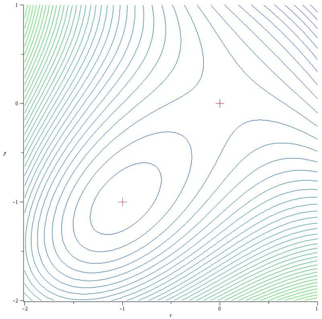

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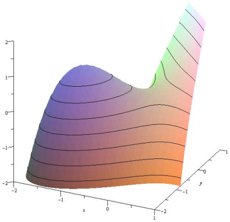

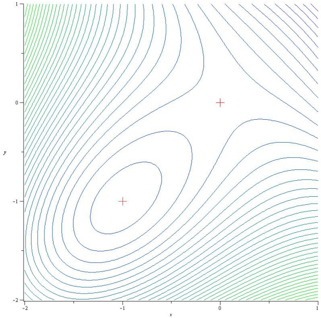

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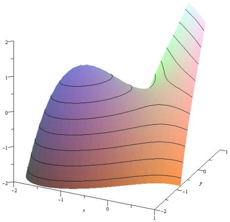

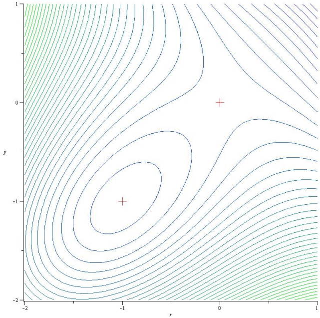

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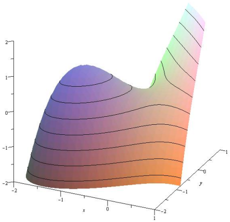

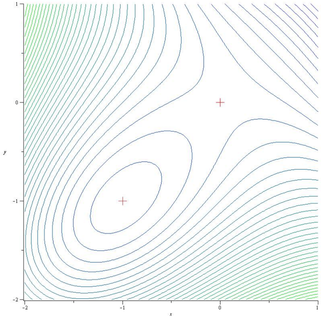

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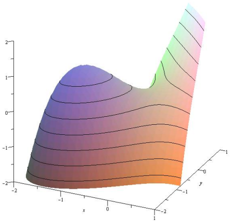

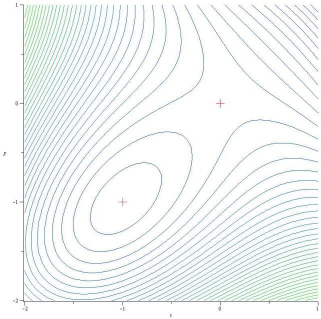

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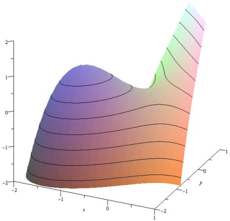

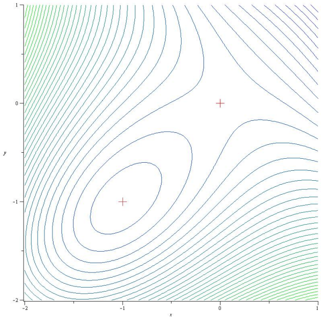

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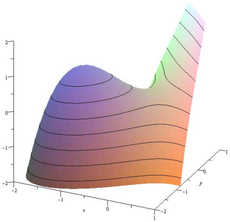

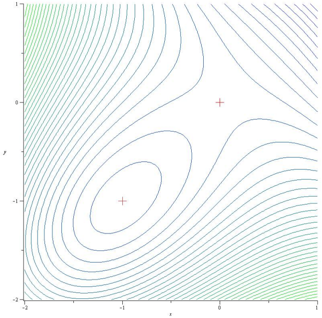

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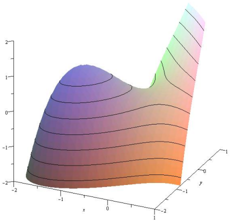

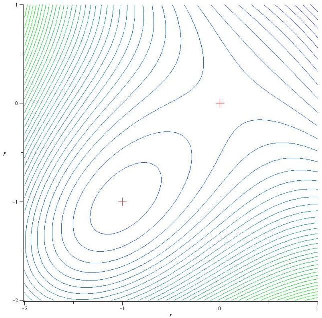

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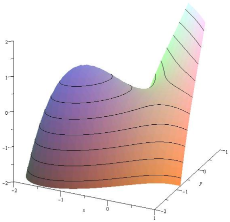

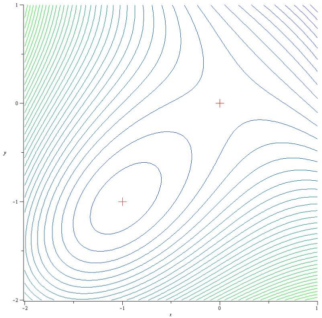

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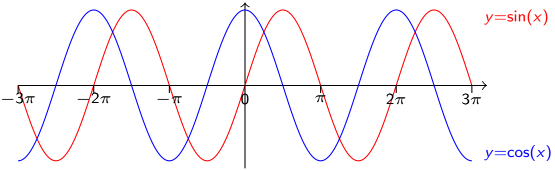

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- $H = \begin{bmatrix} -\sin(x)\sin(y) & \cos(x)\cos(y) \\ \cos(x)\cos(y) & -\sin(x)\sin(y) \end{bmatrix}$

The function $\sin(x)\sin(y)$

$f(x, y) = \sin(x)\sin(y)$; $f_x(x, y) = \cos(x)\sin(y)$; $f_y(x, y) = \sin(x)\cos(y)$.
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- $A_1 = -\sin(x)\sin(y)$

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- Note that $\cos((k + \frac{1}{2})\pi) = 0$, $\sin((k + \frac{1}{2})\pi) = (-1)^k$.

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- At $((n + \frac{1}{2})\pi, (m + \frac{1}{2})\pi)$: $A_1 = (-1)^{n+m+1}$, $A_2 = 1 > 0$.

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- If $n + m$ is even: $A_1 = -1 < 0$ and $A_2 = 1 > 0$ so we have a local maximum.

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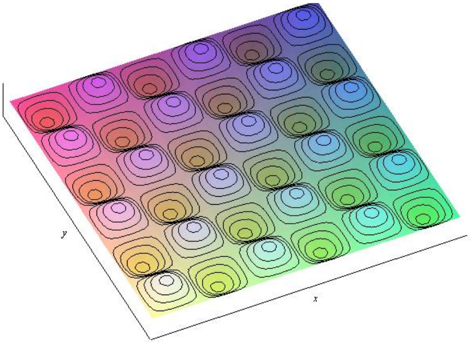
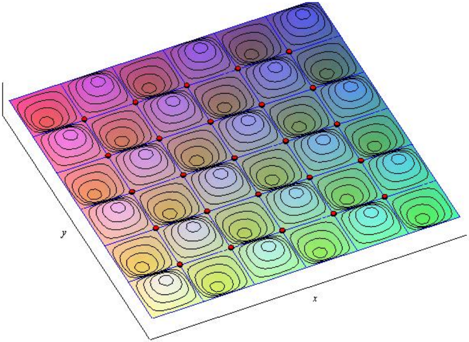
- If $n + m$ is odd: $A_1 = 1 > 0$ and $A_2 = 1 > 0$ so we have a local minimum.

The function $\sin(x)\sin(y)$

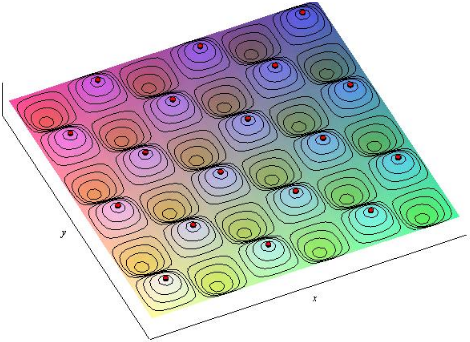


The function $\sin(x)\sin(y)$



- There are saddle points at $(n\pi, m\pi)$ for all integers n and m.

The function $\sin(x)\sin(y)$



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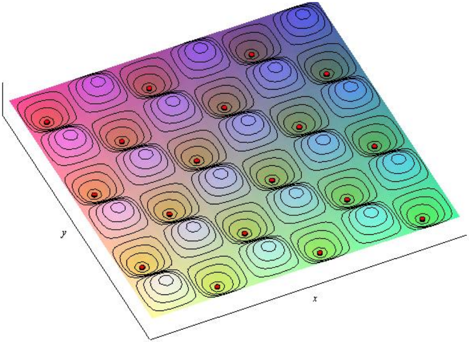

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The function $e^{-x^2-y^2-2y}$

- Take $f(x, y) = e^{-x^2-y^2-2y}$.

The function $e^{-x^2-y^2-2y}$

- Take $f(x,y) = e^{-x^2-y^2-2y}$.
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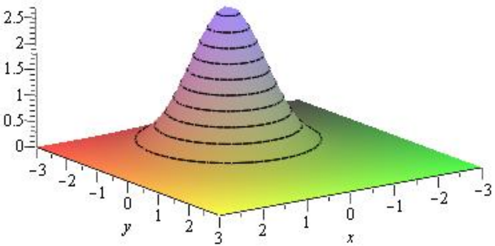
- Now $A_1 = -2e < 0$ and $A_2 = (-2e)^2 - 0^2 = 4e^2 > 0$ so we have a local maximum at $(0,-1)$.

The function $e^{-x^2-y^2-2y}$



There is a local maximum at $(0, -1)$ and no other critical points.

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- These phenomena can be important, but we will not discuss them further here.

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- At $(0, 0, 3)$ we have $A_3 = -2048 \neq 0$ and the signs do not alternate so we have some kind of saddle.

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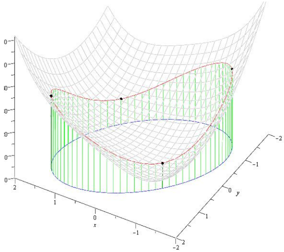

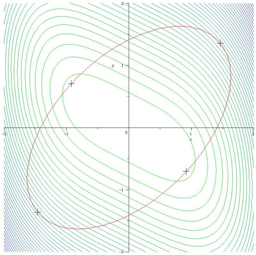

Constrained optimisation - applications

- Suppose we want to build a 5kW motor that is as light as possible. We have come up with a design with parameters a, b and c that we can adjust. The weight is $W(a, b, c)$ and the power (in kW) is $P(a, b, c)$. We want to minimise $W(a, b, c)$ subject to the constraint $P(a, b, c) - 5 = 0$.

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- More generally, whenever we design a device, there will be some requirements that are not negotiable; these will be expressed by constraint equations. There will be other functions that measure the effectiveness of the device. We want to maximise these, but we have to do so subject to the constraints.

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Geometric interpretation

$f(x, y) = x^2 + y^2 =$ squared distance from (x, y) to $(0, 0)$
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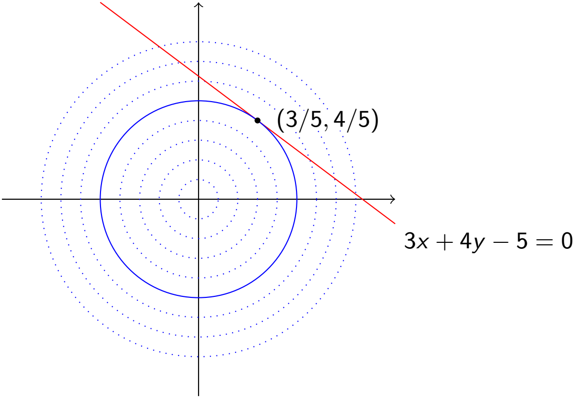
Geometrically, we have found the closest point to the origin on the line $3x + 4y = 5$.



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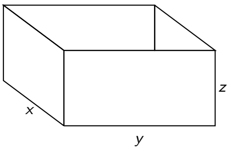
- Geometrically, the vector $\mathbf{u} = \begin{bmatrix} g_x \\ g_y \end{bmatrix}$ is normal to the constraint curve, and $\mathbf{v} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$ is normal to the contour of f. The equations $f_x = \lambda g_x$ and $f_y = \lambda g_y$ say that $\mathbf{v}$ is a multiple of $\mathbf{u}$, so the constraint curve is running parallel to the contour.

A constrained optimization example

Consider a metal tank, open at the top. The volume is $V = xyz$, and the area is $S = xy + 2xz + 2yz$. We want the volume to be $4m^3$, and we want to minimise S, to use as little metal as possible.



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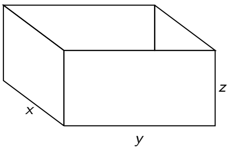

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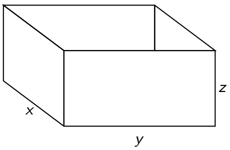

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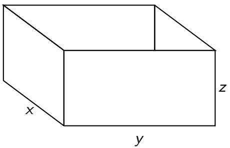

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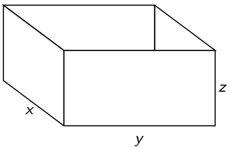

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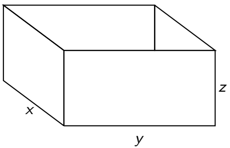

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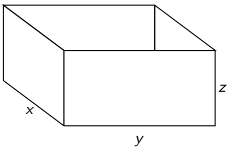

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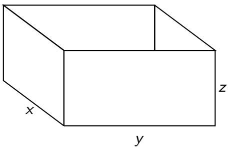

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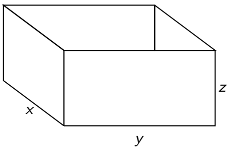

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- Subtract (B) and (C) to get $x^{-1} = y^{-1}$ so $x = y$. Substitute in (D) to get $4x^{-1} = 4y^{-1} = \lambda$, so $x = y = 4/\lambda$.

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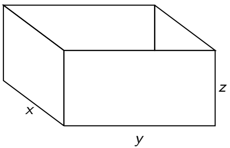

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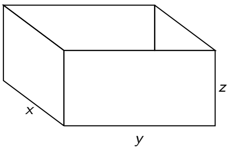

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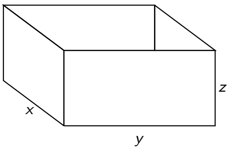

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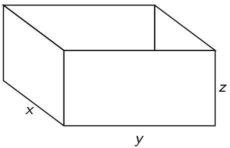

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- For these values, we have $S = 12$. Thus, the minimum possible area of metal sheet that we need is $12m^2$.

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This means that the maximum possible value of f (subject to the constraint) is $\sqrt{a+b}$, and the minimum is $-\sqrt{a+b}$.

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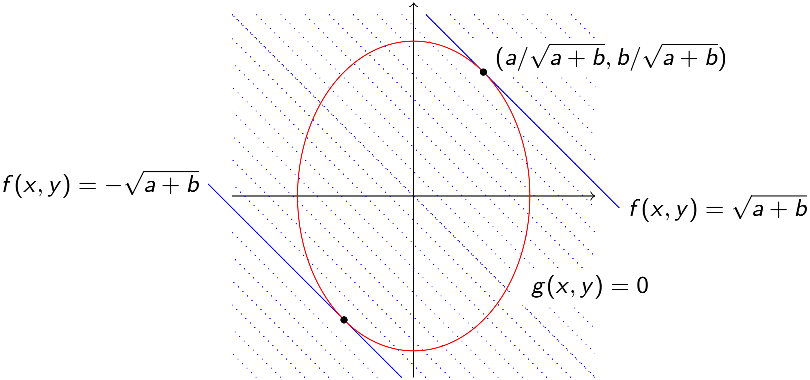

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- Thus, the minimum value of z is $-6/7$ at $(9/7, 18/7, -6/7)$, and the maximum is 2 at $(-1, 2, 2)$.

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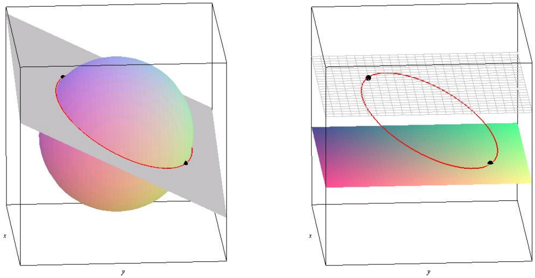

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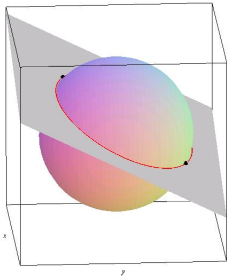

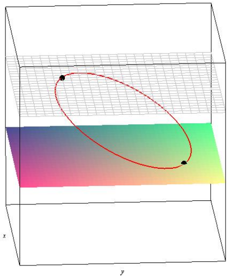

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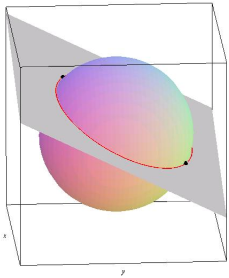

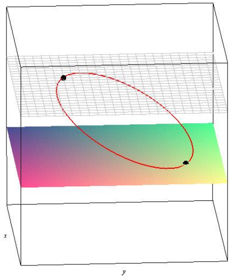

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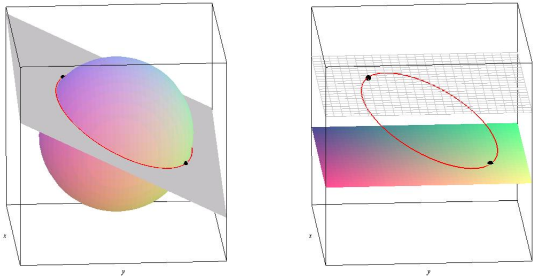

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Integrals over plane regions

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To get the exact value, we divide D into a larger and larger number of smaller and smaller pieces, and then pass to the limit.

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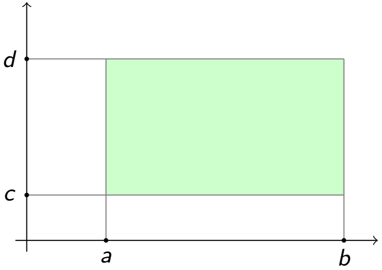
(d) The total area of a region D is just $\iint_D 1\, dA$.

Rectangular regions

In the simplest case, the region D is a rectangle aligned with the axes, given by $a \leq x \leq b$ and $c \leq y \leq d$ say.



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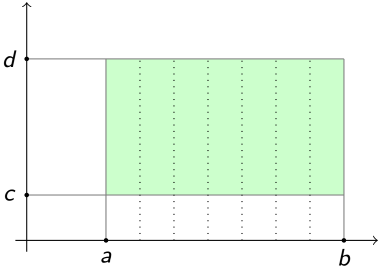

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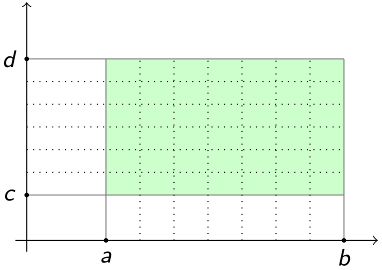

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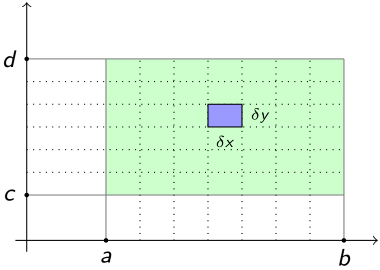

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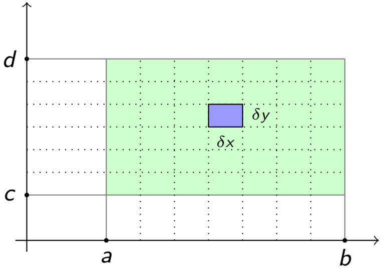

Using this kind of subdivision, we see that the area integral is just obtained by integrating with respect to both variables x and y:

$$\iint_D f(x,y)\, dA = \int_{x=a}^{b} \left(\int_{y=c}^{d} f(x,y)\, dy \right) dx.$$

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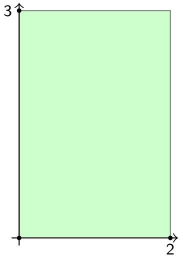
$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.



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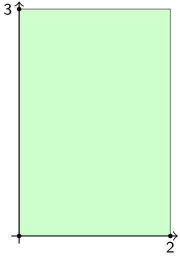

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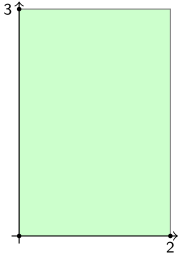

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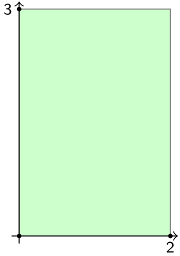

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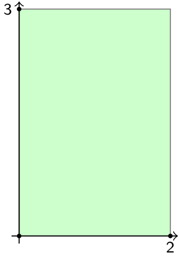

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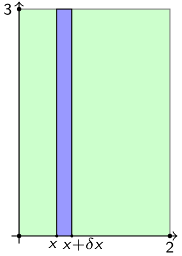

Meaning: if we take a thin strip running horizontally from x to $x + \delta x$, and vertically all the way from 0 to 3, then the sum of the corresponding contributions is approximately $(3x^3 + 9)\delta x$

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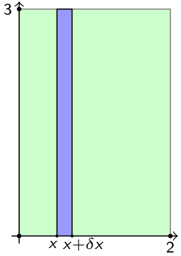

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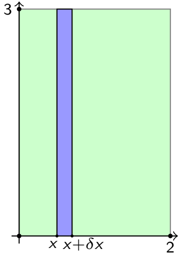

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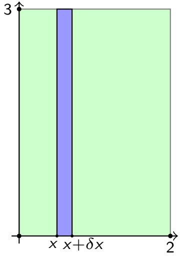

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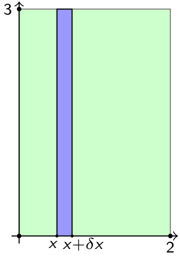

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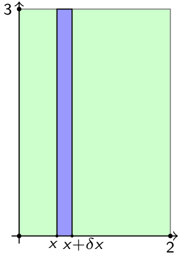

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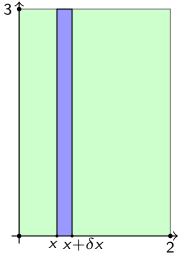

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The conclusion is that $\iint_D x^3 + y^2 \, dA = 30$.

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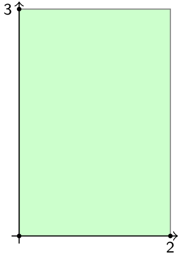

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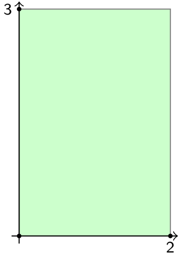

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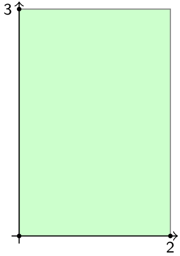

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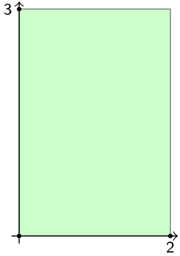

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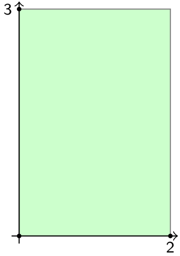

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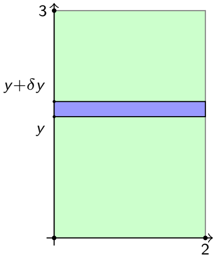

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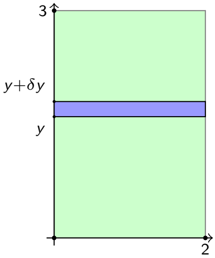

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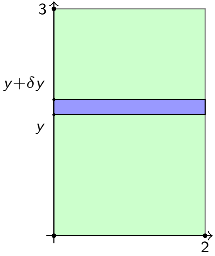

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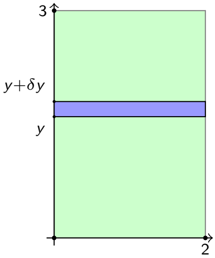

Meaning: if we take a thin strip running vertically from y to $y + \delta y$, and horizontally all the way from 0 to 2, then the sum of the corresponding contributions is approximately $(4 + 2y^2)\delta y$ (and the approximation becomes exact in the limit as $\delta y \to 0$). Outer integral: add up the contributions from all such horizontal strips.

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Rectangular example — horizontal strips

$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.

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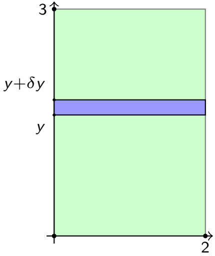

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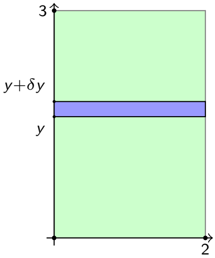

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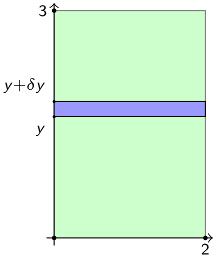

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Square region example

Let E be the square where $0 \leq x \leq \pi$ and $-\pi/2 \leq y \leq \pi/2$.

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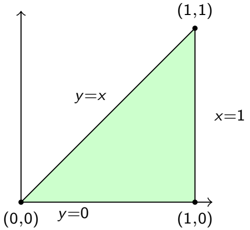
Triangular example

$D =$ triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.



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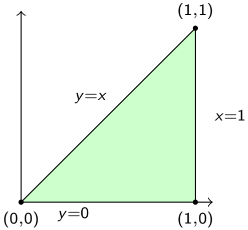

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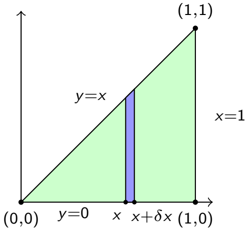

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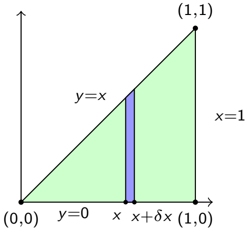

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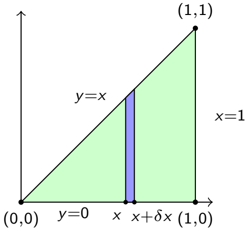

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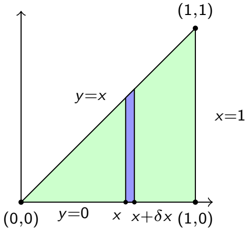

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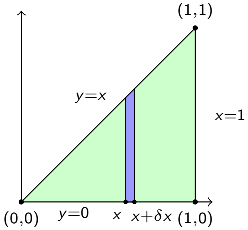

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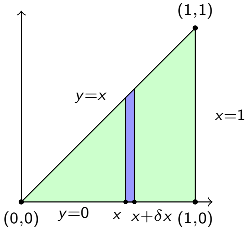

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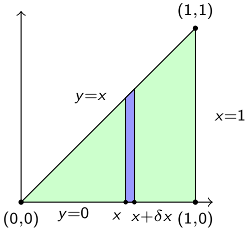

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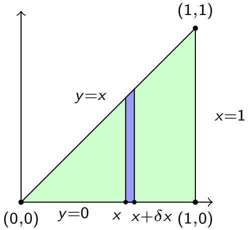

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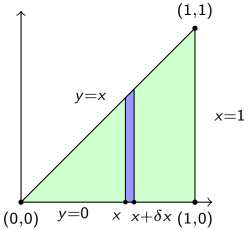

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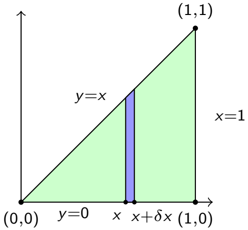

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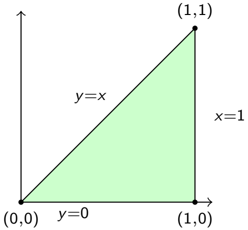

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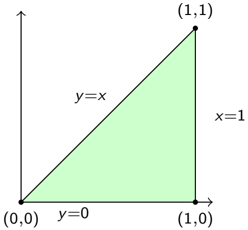

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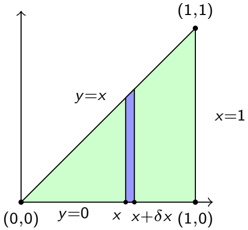

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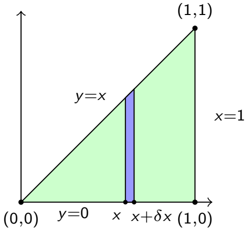

Triangular example

$D =$ triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

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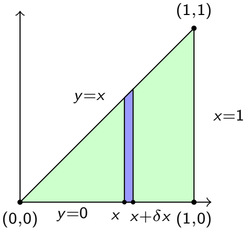

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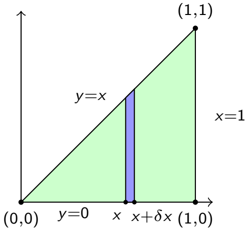

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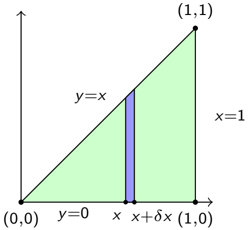

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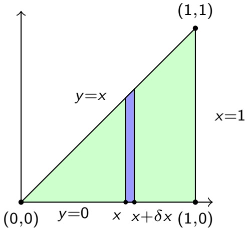

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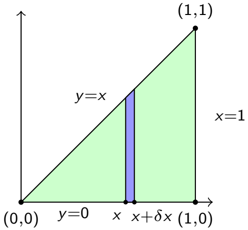

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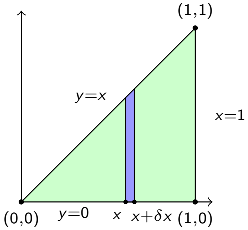

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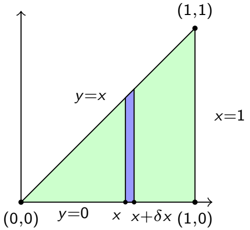

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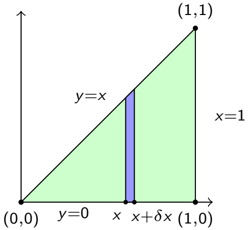

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$$= (e^2 - 3)/4.$$

Triangular example — horizontal strips

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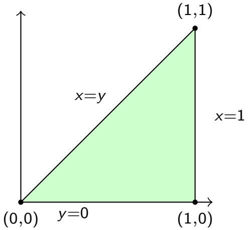

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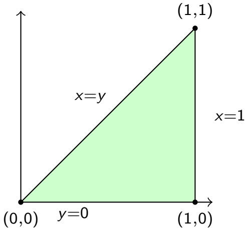

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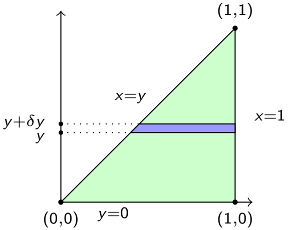

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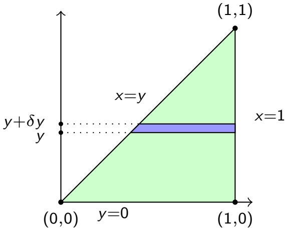

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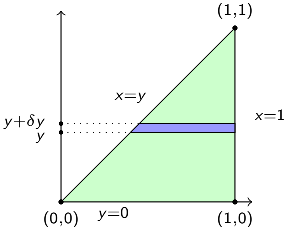

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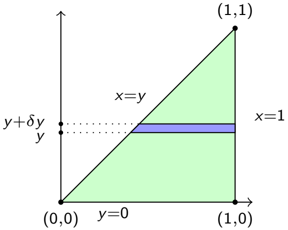

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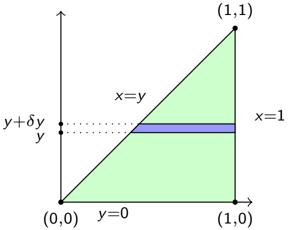

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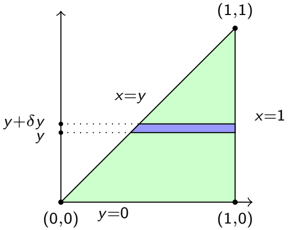

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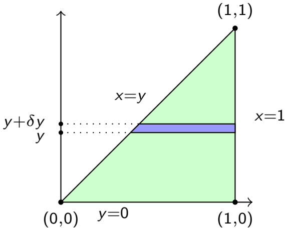

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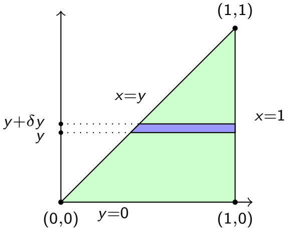

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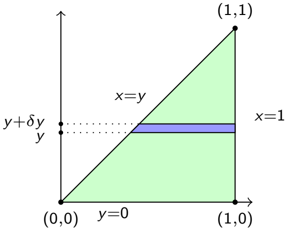

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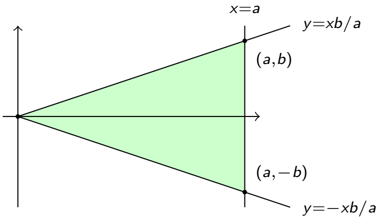
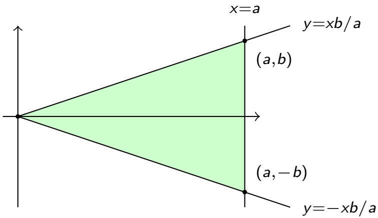
Moment of inertia



Moment of inertia

For the moment of inertia, we need $\iint_D x^2 + y^2 \, dA$.



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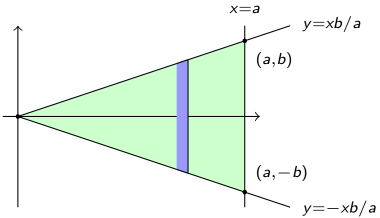
If we fix x with $0 \leq x \leq a$, then y will run from $-xb/a$ to $+xb/a$.



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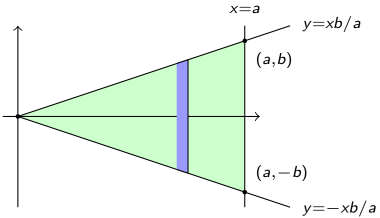

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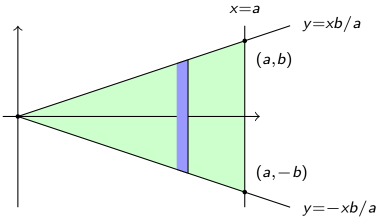

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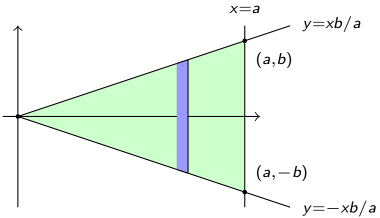

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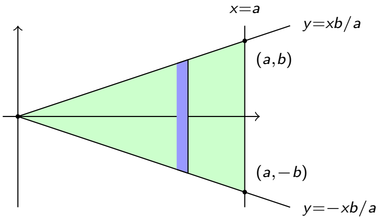

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$$\int_{y=-xb/a}^{xb/a} x^2 + y^2\, dy = \left[x^2 y + \tfrac{1}{3} y^3\right]_{y=-xb/a}^{xb/a} = \frac{2x^3 b}{a} + \frac{2x^3 b^3}{3a^3}$$

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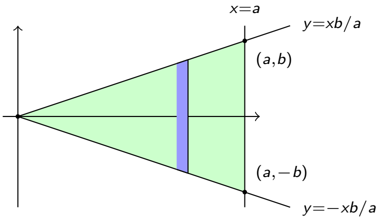

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Moment of inertia

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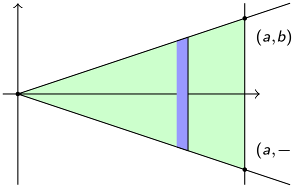

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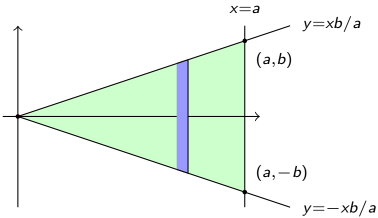

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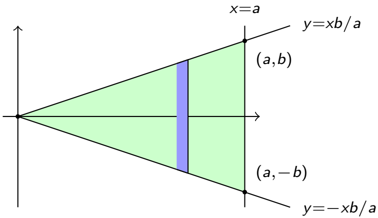

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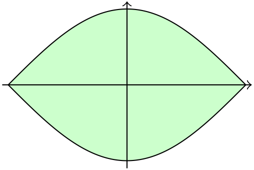
Area of a curved region

Let D be the region where $-\pi/2 \leq x \leq \pi/2$ and $-\cos(x) \leq y \leq \cos(x)$.



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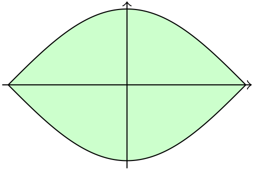

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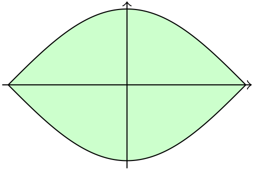

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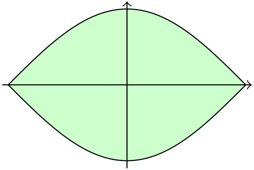

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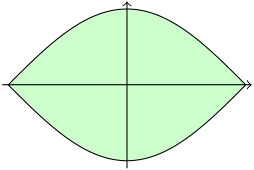

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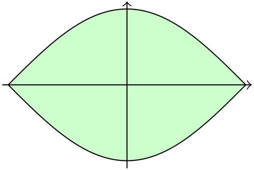

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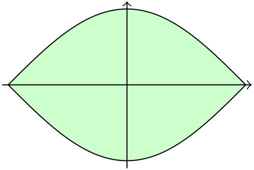

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$$= 2 - (-2) = 4.$$

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Consider $I = \int_{y=0}^{1} \int_{x=y^2}^{y} x^{-1} y e^{x} \, dx \, dy.$

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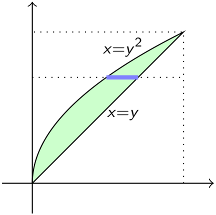

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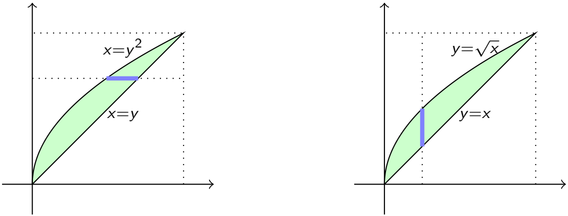

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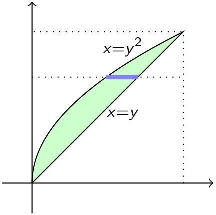

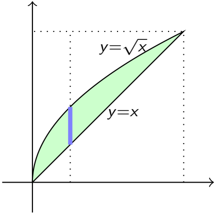

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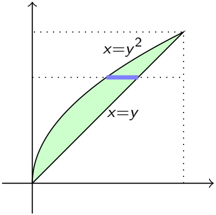

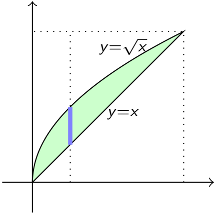

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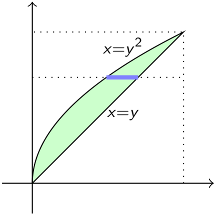

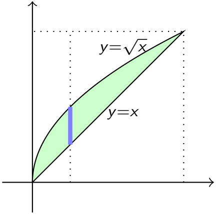

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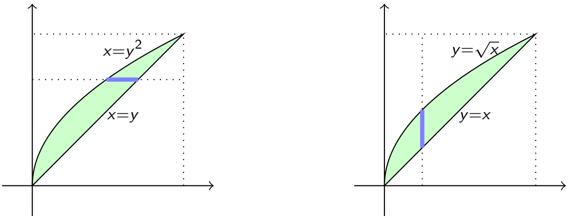

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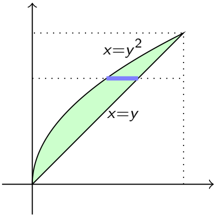

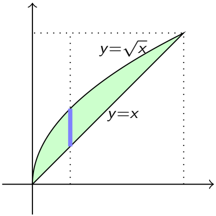

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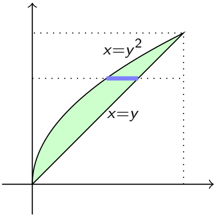

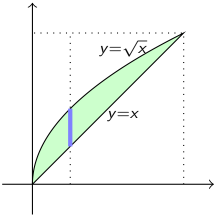

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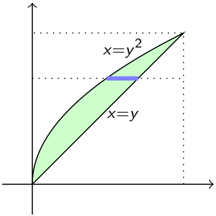

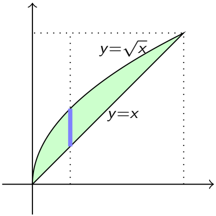

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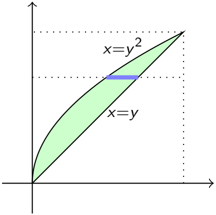

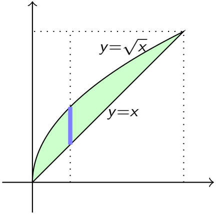

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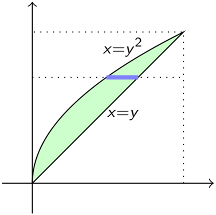

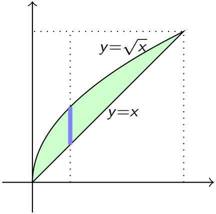

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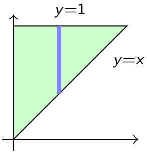

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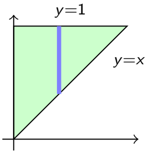

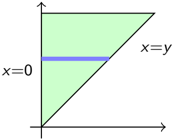

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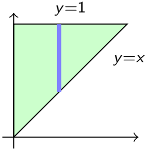
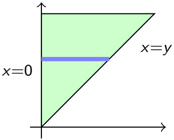


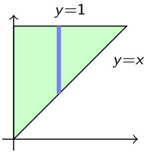


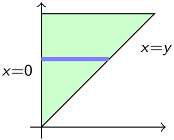
Rewrite in the opposite order:

$$I = \int_{y=0}^{1} \int_{x=0}^{y} \frac{xy}{\sqrt{1+y^4}} \, dx \, dy$$

Reversing the order of integration

Consider $I = \int_{x=0}^{1} \int_{y=x}^{1} \frac{xy}{\sqrt{1+y^4}}\, dy\, dx.$





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$$I = \int_{y=0}^{1} \int_{x=0}^{y} \frac{xy}{\sqrt{1+y^4}}\, dx\, dy = \int_{y=0}^{1} \left[\frac{x^2 y}{2\sqrt{1+y^4}} \right]_{x=0}^{y} dy$$

Reversing the order of integration

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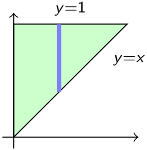

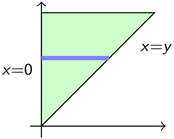

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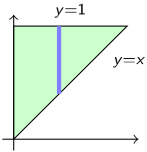
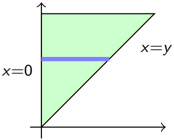
Reversing the order of integration

Consider $I = \int_{x=0}^{1} \int_{y=x}^{1} \frac{xy}{\sqrt{1+y^4}} \, dy \, dx$.





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We now substitute $u = 1 + y^4$

Reversing the order of integration

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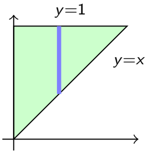

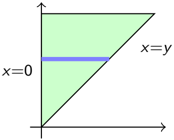

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We now substitute $u = 1 + y^4$, so $du/dy = 4y^3$

Reversing the order of integration

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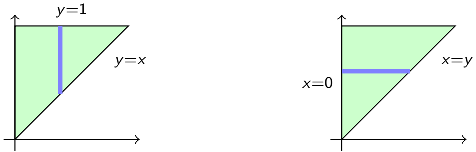

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We now substitute $u = 1 + y^4$, so $du/dy = 4y^3$, so $y^3\, dy = du/4$

Reversing the order of integration

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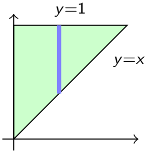

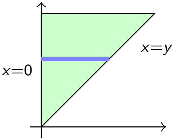

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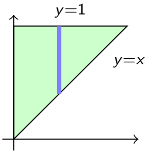
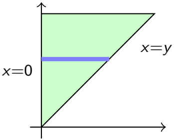
We now substitute $u = 1 + y^4$, so $du/dy = 4y^3$, so $y^3\, dy = du/4$ and $\sqrt{1+y^4} = u^{1/2}$.

Reversing the order of integration

Consider $I = \int_{x=0}^{1} \int_{y=x}^{1} \frac{xy}{\sqrt{1+y^4}}\, dy\, dx$.





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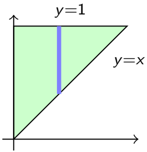

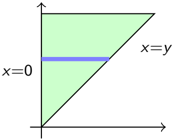

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$$I = \int_{u=1}^{2} \frac{du/4}{2u^{1/2}}$$

Reversing the order of integration

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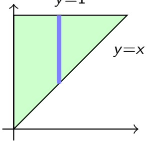

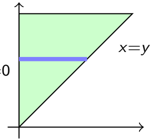

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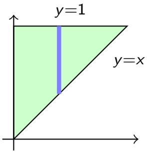
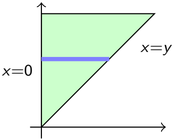
$$I = \int_{u=1}^{2}\frac{du/4}{2u^{1/2}} = \frac{1}{8}\int_{u=1}^{2} u^{-1/2}du$$

Reversing the order of integration

Consider $I = \int_{x=0}^{1} \int_{y=x}^{1} \frac{xy}{\sqrt{1+y^4}} \, dy \, dx$.





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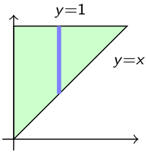
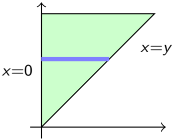
We now substitute $u = 1 + y^4$, so $du/dy = 4y^3$, so $y^3 \, dy = du/4$ and $\sqrt{1+y^4} = u^{1/2}$. The limits $y = 0$ and $y = 1$ correspond to $u = 1$ and $u = 2$. This gives

$$I = \int_{u=1}^{2} \frac{du/4}{2u^{1/2}} = \frac{1}{8} \int_{u=1}^{2} u^{-1/2} du = \frac{1}{8} \left[2u^{1/2} \right]_{u=1}^{2}$$

Reversing the order of integration

Consider $I = \int_{x=0}^{1} \int_{y=x}^{1} \frac{xy}{\sqrt{1+y^4}} \, dy \, dx$.





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$$I = \int_{u=1}^{2} \frac{du/4}{2u^{1/2}} = \frac{1}{8} \int_{u=1}^{2} u^{-1/2} du = \frac{1}{8} \left[2u^{1/2} \right]_{u=1}^{2} = (2\sqrt{2} - 2)/8$$

Consider $I = \int_{x=0}^{1} \int_{y=x}^{1} \frac{xy}{\sqrt{1+y^4}} \, dy \, dx.$

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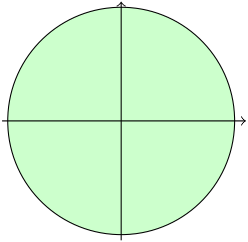
$$I = \int_{u=1}^{2} \frac{du/4}{2u^{1/2}} = \frac{1}{8} \int_{u=1}^{2} u^{-1/2} du = \frac{1}{8} \left[2u^{1/2} \right]_{u=1}^{2} = (2\sqrt{2} - 2)/8$$
$$= (\sqrt{2} - 1)/4 \simeq 0.1036.$$

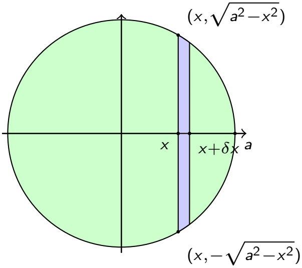
Integral over a disk

$$I = \iint_D x^2 \, dA$$



Integral over a disk

$$I = \iint_D x^2 \, dA = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} x^2 \, dy \, dx.$$



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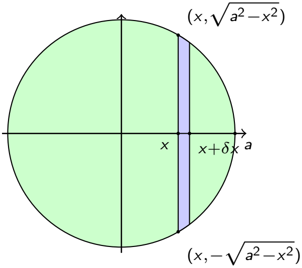
$$I = \iint_D x^2 \, dA = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} x^2 \, dy \, dx.$$

In the inner integral x^2 counts as a constant:

$$\int_{y=-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} x^2 \, dy = \left[x^2 y \right]_{y=-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}}$$



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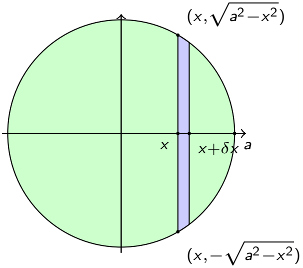
$$\int_{y=-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} x^2\, dy = \left[x^2 y\right]_{y=-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} = 2x^2\sqrt{a^2-x^2}.$$



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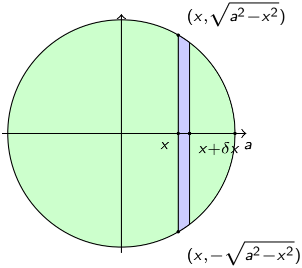

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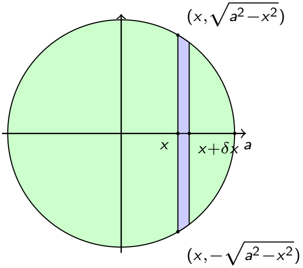

Now $I = \int_{x=-a}^{a} 2x^2\sqrt{a^2 - x^2} \, dx.$

Integral over a disk

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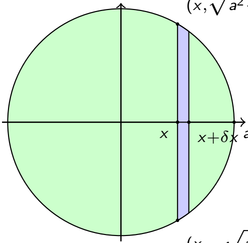

Now $I = \int_{x=-a}^{a} 2x^2\sqrt{a^2-x^2}\, dx$. Substitute $x = a\sin(\theta)$

Integral over a disk

$$I = \iint_D x^2\, dA = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} x^2\, dy\, dx.$$

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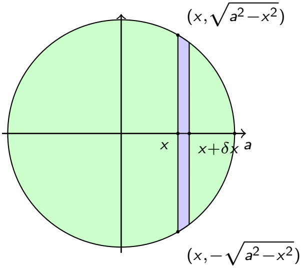

Now $I = \int_{x=-a}^{a} 2x^2\sqrt{a^2-x^2}\, dx$. Substitute $x = a\sin(\theta)$, so $-a \le x \le a$ becomes $-\pi/2 \le \theta \le \pi/2$.

$$I = \iint_D x^2\, dA = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} x^2\, dy\, dx.$$

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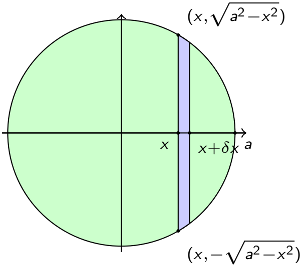


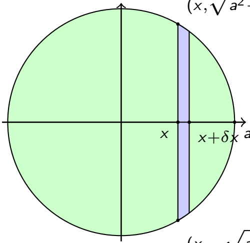
Now $I = \int_{x=-a}^{a} 2x^2\sqrt{a^2-x^2}\, dx$. Substitute $x = a\sin(\theta)$, so $-a \leq x \leq a$ becomes $-\pi/2 \leq \theta \leq \pi/2$. Also $dx/d\theta = a\cos(\theta)$

Integral over a disk

$$I = \iint_D x^2\,dA = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} x^2\,dy\,dx.$$

In the inner integral x^2 counts as a constant:

$$\int_{y=-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} x^2\,dy = \left[x^2 y\right]_{y=-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} = 2x^2\sqrt{a^2-x^2}.$$



Now $I = \int_{x=-a}^{a} 2x^2\sqrt{a^2-x^2}\,dx$. Substitute $x = a\sin(\theta)$, so $-a \le x \le a$ becomes $-\pi/2 \le \theta \le \pi/2$. Also $dx/d\theta = a\cos(\theta)$, so $dx = a\cos(\theta)d\theta$

Integral over a disk

$$I = \iint_D x^2\, dA = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} x^2\, dy\, dx.$$

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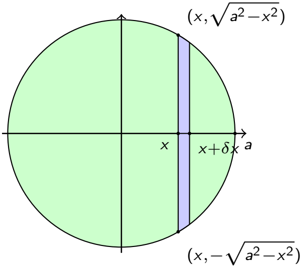

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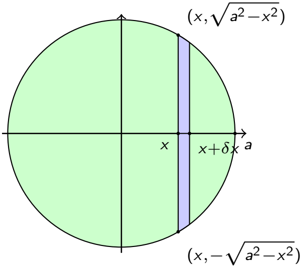

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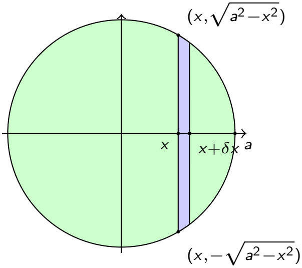

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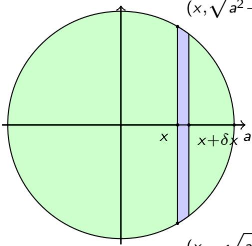

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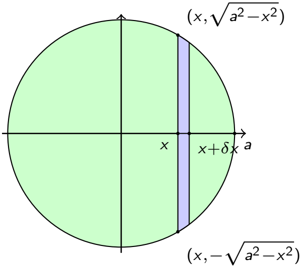

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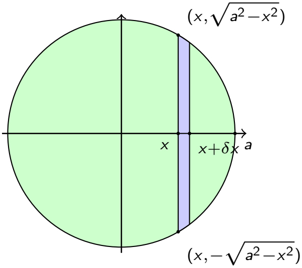

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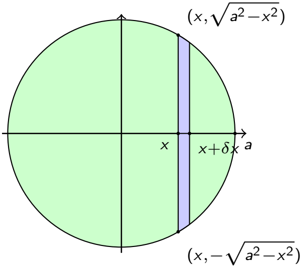

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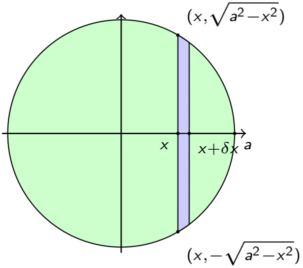

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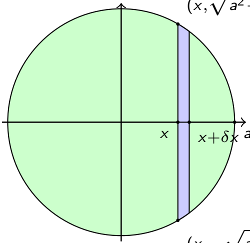

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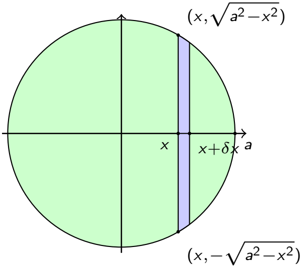

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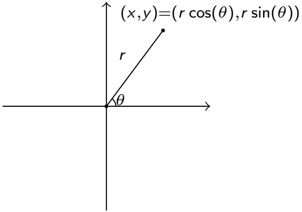
Polar coordinates

We describe points using the distance r from the origin and the angle θ anticlockwise from the x-axis.



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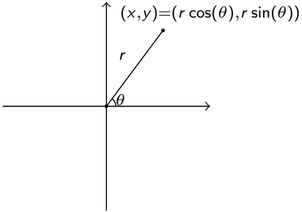

Polar coordinates are related to ordinary (cartesian) coordinates by the formulae

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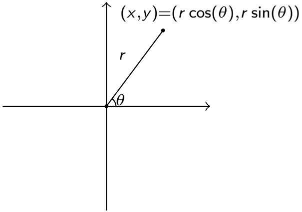

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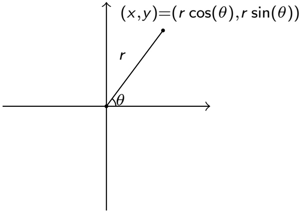

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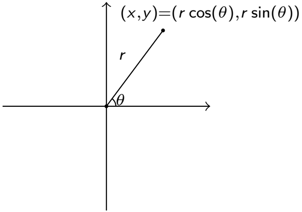

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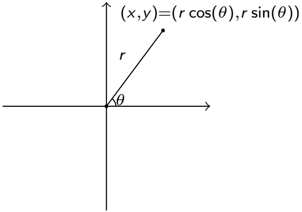

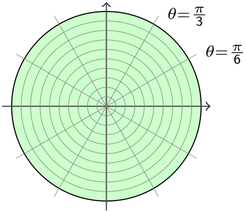

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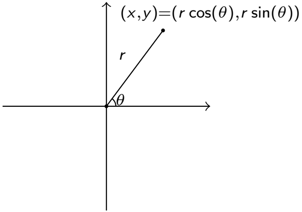

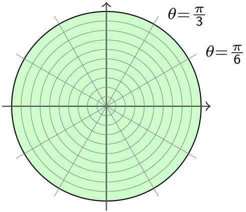

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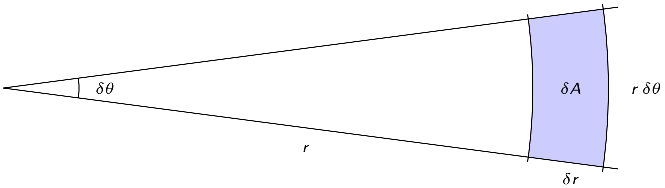
(Care is needed to choose the right value of $\arctan(y/x)$.) In the diagram on the right above, we have divided a disk into small pieces using lines of constant θ and circles of constant r. To use this kind of subdivision for integration, we need to know the area of the small pieces.

The polar area element

Consider a piece of angular width $\delta\theta$, where the radius runs from r to $r + \delta r$.



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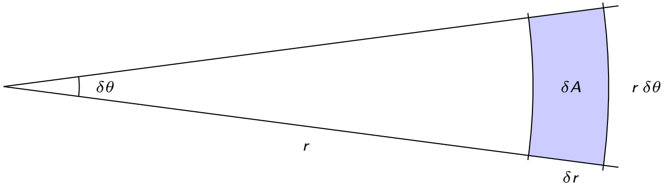

Provided that $\delta\theta$ is small this will be approximately rectangular.

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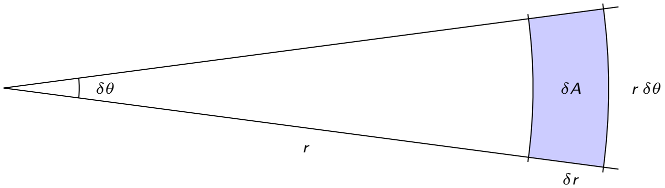

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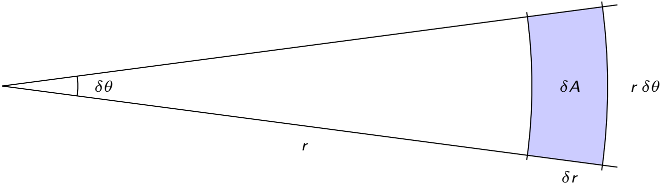

Provided that $\delta\theta$ is small this will be approximately rectangular. If we measure angles in radians (as we always will) then the length of the curved side will be $r\,\delta\theta$, and the straight side has length δr

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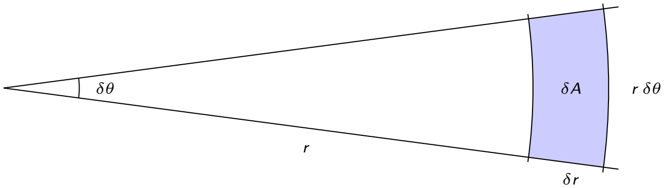

Provided that $\delta\theta$ is small this will be approximately rectangular. If we measure angles in radians (as we always will) then the length of the curved side will be $r\,\delta\theta$, and the straight side has length δr, so the area is approximately $\delta A = r\,\delta r\,\delta\theta$.

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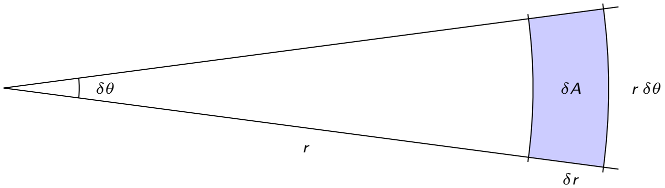
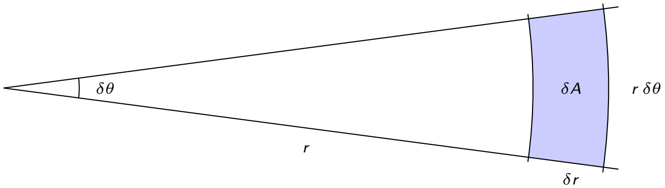


Provided that $\delta\theta$ is small this will be approximately rectangular. If we measure angles in radians (as we always will) then the length of the curved side will be $r\,\delta\theta$, and the straight side has length δr, so the area is approximately $\delta A = r\,\delta r\,\delta\theta$. In the limit this becomes $dA = r\,dr\,d\theta$

The polar area element

Consider a piece of angular width $\delta\theta$, where the radius runs from r to $r + \delta r$.



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$$\iint_D f(x,y)\,dA = \int_{\theta=\cdots}^{\cdots}\int_{r=\cdots}^{\cdots} f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta,$$

where the limits need to be filled in in accordance with the geometry of the region.

Disk integral of x^2, again

Consider again $\iint_D x^2\, dA$, where D is a disk of radius a around the origin.

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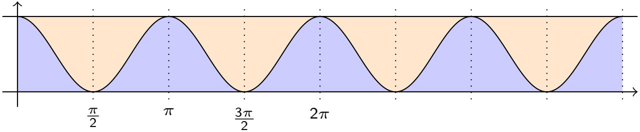
The following picture shows why $\int_0^{2\pi} \cos^2(\theta)\, d\theta = \pi$:



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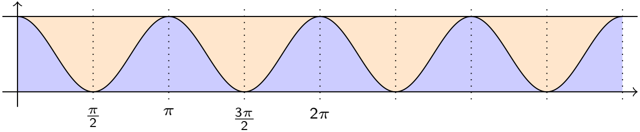
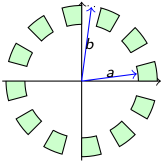


Each region has the same area, namely $\pi/4$.

A slotted rotor

Moment of inertia $I = \iint_D (x^2 + y^2) dA$.



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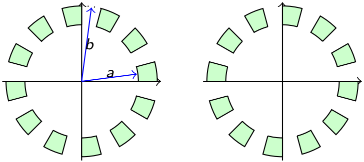

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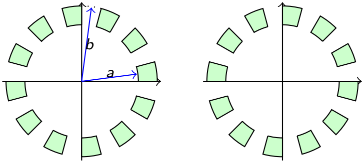

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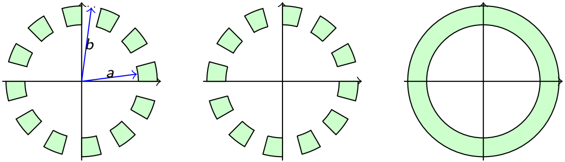

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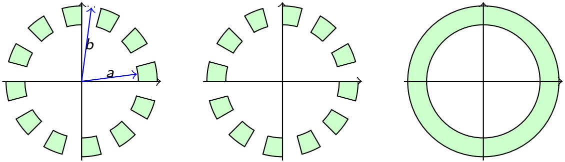

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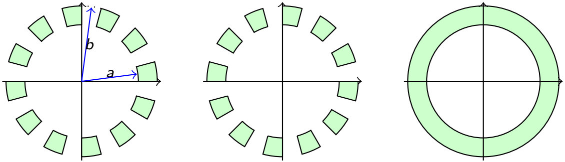

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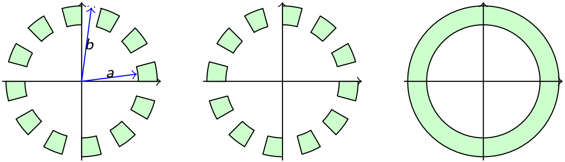

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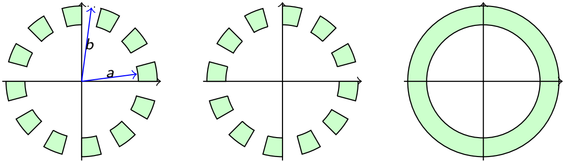

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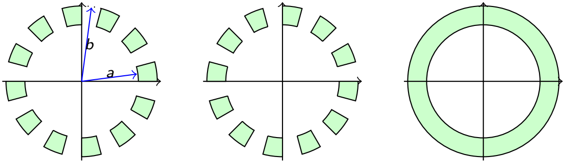

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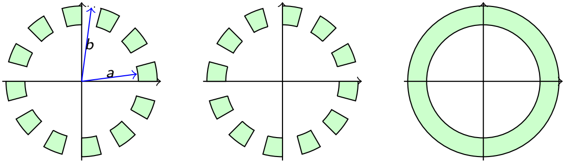

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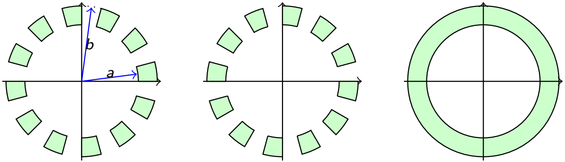

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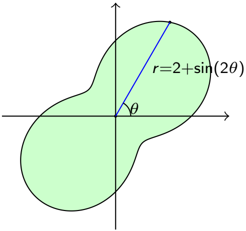
Area of a curved region

The picture shows the region D given in polar coordinates by $0 \leq r \leq 2 + \sin(2\theta)$.



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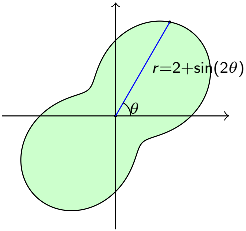

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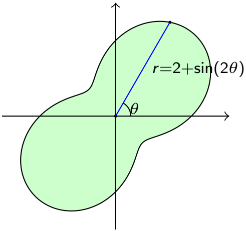

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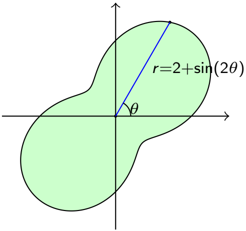

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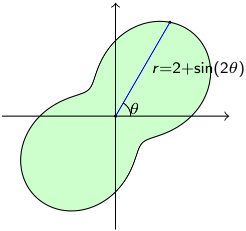

$$A = \int_{\theta=0}^{2\pi} \int_{r=0}^{2+\sin(2\theta)} r\, dr\, d\theta$$

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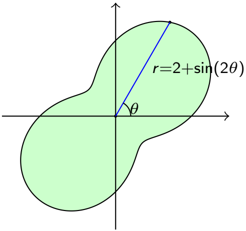

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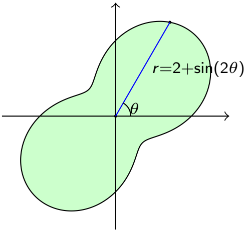

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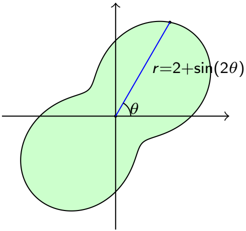

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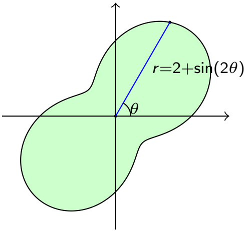

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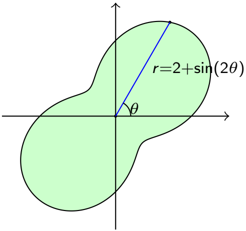

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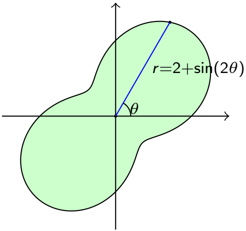

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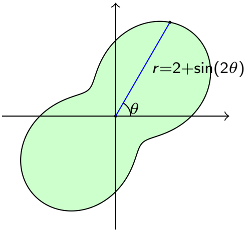

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$$A = \tfrac{1}{2}(2\pi.(4+\tfrac{1}{2})) = 9\pi/2.$$

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so $I = \sqrt{\pi}$ as claimed.

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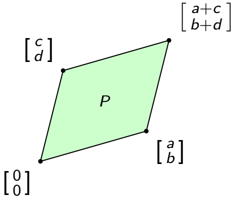

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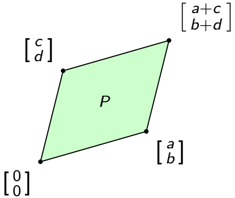

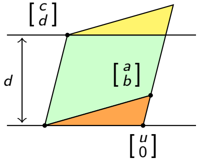

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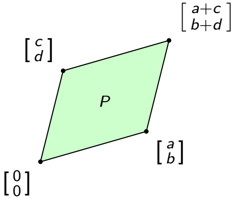

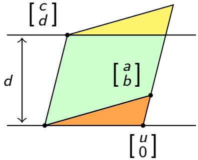

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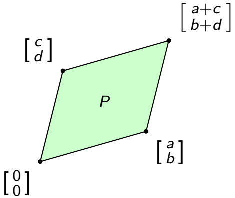

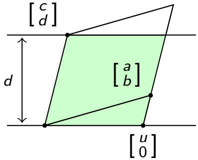

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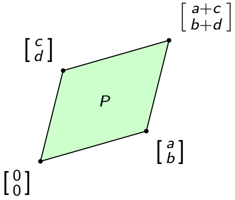

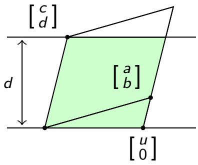

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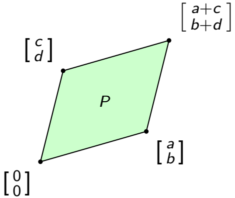

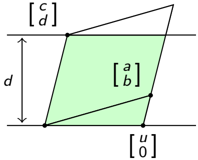

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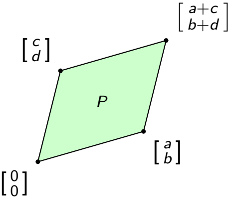

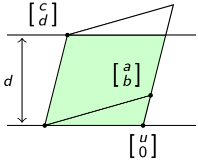

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Area of a parallelogram

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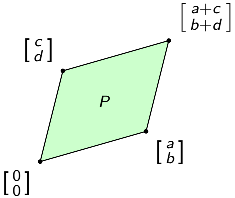

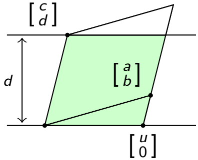

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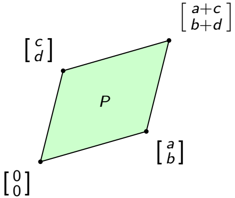

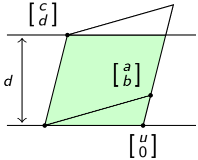

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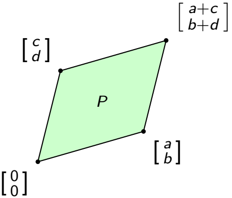

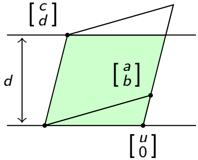

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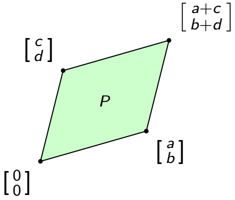

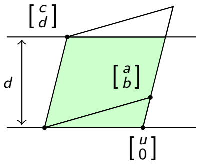

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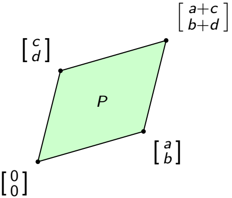

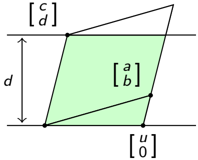

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Change of variables in a double integral

Suppose x and y can be expressed in terms of some other variables u and v.

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Using this: $$dA = \left|\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right| du\,dv.$$

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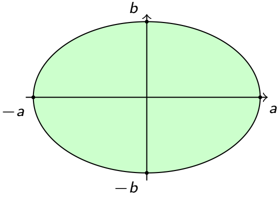
Using this: $$dA = \left|\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right| du\,dv.$$

This is a key ingredient for double integrals by substitution.

Area of an ellipse

We will find the area of an ellipse E with equation $x^2/a^2 + y^2/b^2 \leq 1$ (for some $a, b > 0$).



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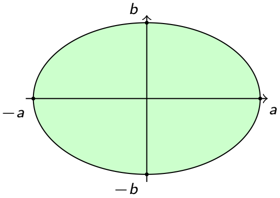
$$x = ar\cos(\theta) \qquad\qquad y = br\sin(\theta).$$



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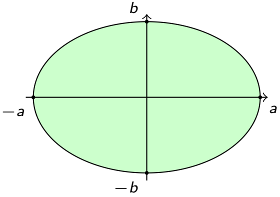
Then $x^2/a^2 + y^2/b^2 = r^2\cos^2(\theta) + r^2\sin^2(\theta) = r^2$



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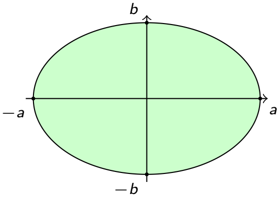

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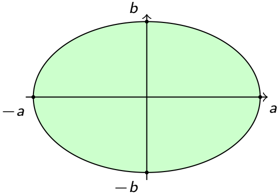

$$x_r = a\cos(\theta) \qquad x_\theta = -ar\sin(\theta) \qquad y_r = b\sin(\theta) \qquad y_\theta = br\cos(\theta),$$

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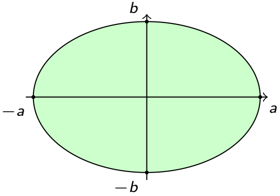

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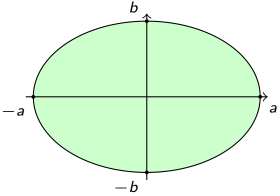

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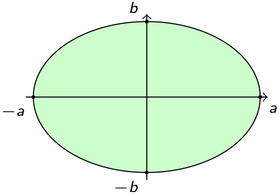

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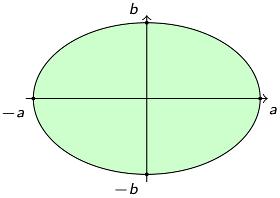

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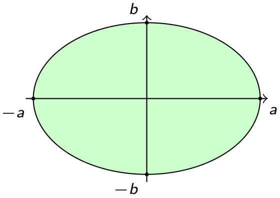

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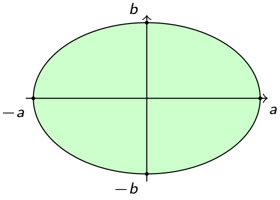

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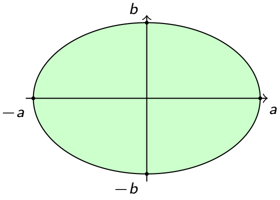

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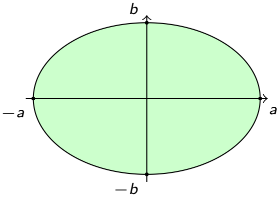

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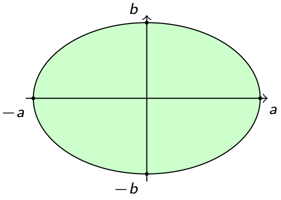

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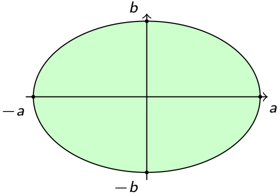

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(d) Centre of mass $(\overline{x}, \overline{y}, \overline{z})$ of an object: $\overline{x} = \left(\iiint_E x\, dV\right) / \left(\iiint_E 1\, dV\right)$ and similarly for $\overline{y}$ and $\overline{z}$.

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We can integrate over y and then over x in the same way, giving

$$I = \int_{x=0}^{a} \sin^2(k\pi x)\frac{b}{2}\frac{c}{2}\, dx. = \frac{a}{2}.\frac{b}{2}.\frac{c}{2} = \frac{abc}{8}.$$

Moment of inertia of a cube

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$$\begin{aligned}\iiint_E (x^2+y^2)dV &= \int_{x=-1}^{1}\int_{y=-1}^{1}\int_{z=-1}^{1}(x^2+y^2)dz\,dy\,dx \\ &= \int_{x=-1}^{1}\int_{y=-1}^{1}\left[x^2z+y^2z\right]_{z=-1}^{1}dy\,dx \\ &= \int_{x=-1}^{1}\int_{y=-1}^{1}2(x^2+y^2)dy\,dx \\ &= \int_{x=-1}^{1}\left[2x^2y+2y^3/3\right]_{y=-1}^{1}dx\end{aligned}$$

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&= \int_{x=-1}^{1} \int_{y=-1}^{1} 2(x^2 + y^2) dy\, dx \\
&= \int_{x=-1}^{1} \left[2x^2 y + 2y^3/3 \right]_{y=-1}^{1} dx \\
&= \int_{x=-1}^{1} 4x^2 + 4/3\, dx
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&= \int_{x=-1}^{1} 4x^2 + 4/3\, dx \\
&= \left[4x^3/3 + 4x/3 \right]_{x=-1}^{1} = 8/3 + 8/3 = 16/3.
\end{aligned}
$$

Integral over a tetrahedron

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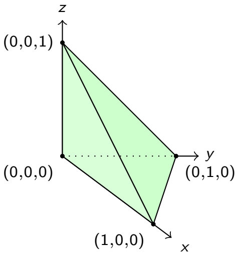

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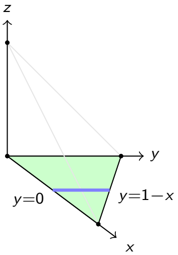

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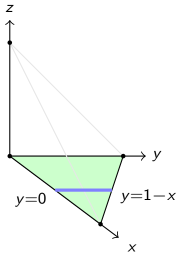

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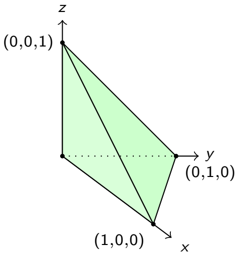

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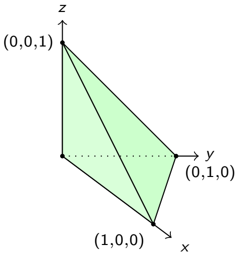

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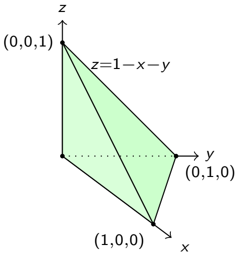

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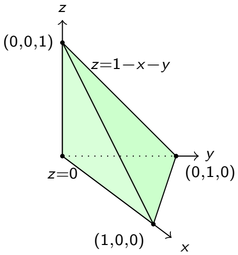

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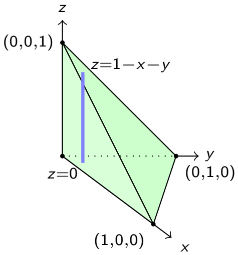

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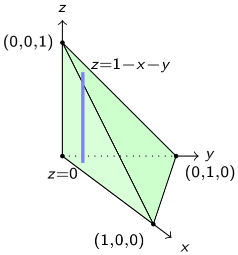

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$$\iiint_E f(x,y,z)\,dV = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} f(x,y,z)\,dz\,dy\,dx.$$

Volume of a tetrahedron

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Finally, the outermost integral (with respect to x) is

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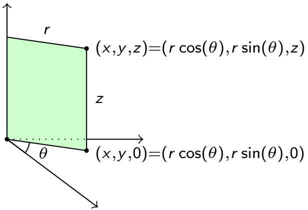
We conclude that the volume of the tetrahedron is $1/6$.

Cylindrical polar coordinates

When using cylindrical polar coordinates we describe points in terms of the distance r from the z-axis, the angle θ anticlockwise from the (x, z)-plane, and the height z above the (x, y)-plane.



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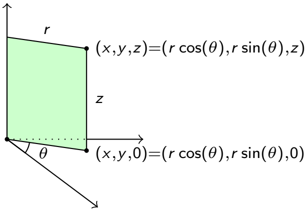

Just as in the two-dimensional case, r and θ are related to x and y by the equations

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

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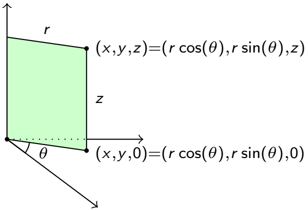

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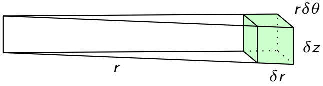
Applications: rotating machines, fibre-optic cables, dish-shaped antennas.

Cylindrical polar volume element

If we allow r, θ and z to vary by small amounts δr, $\delta\theta$ and δz, then the corresponding region is approximately a right-angled box with sides of length δr, δz and $r\delta\theta$. The volume is thus $\delta V \simeq r\delta r\,\delta\theta\,\delta z$.



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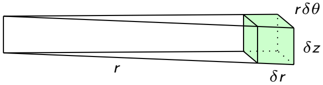

This means that for a function f on a 3-dimensional region E, we have

$$\iiint_E f(x,y,z)\,dV = \int_{z=\cdots}^{\cdots}\int_{\theta=\cdots}^{\cdots}\int_{r=\cdots}^{\cdots} f(r\cos(\theta), r\sin(\theta), z)\,{\color{red}r}\,dr\,d\theta\,dz,$$

where the limits must be determined using the geometry of the region.

Jacobian method

$$\iiint_E f(x,y,z)\,dV = \int_{z=\cdots}^{\cdots}\int_{\theta=\cdots}^{\cdots}\int_{r=\cdots}^{\cdots} f(r\cos(\theta), r\sin(\theta), z)\, r\, dr\, d\theta\, dz,$$

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The formula for dV can also be obtained using a three-dimensional of the Jacobian matrix.

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$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{bmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{bmatrix}$$

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The formula for dV can also be obtained using a three-dimensional of the Jacobian matrix. We have

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$$\det(J) = \cos(\theta)\ \det\begin{bmatrix} r\cos(\theta) & 0 \\ 0 & 1 \end{bmatrix} \cdots$$

Jacobian method

$$\iiint_E f(x,y,z)\,dV = \int_{z=\cdots}^{\cdots}\int_{\theta=\cdots}^{\cdots}\int_{r=\cdots}^{\cdots} f(r\cos(\theta), r\sin(\theta), z)\, r\, dr\, d\theta\, dz,$$

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Jacobian method

$$\iiint_E f(x,y,z)\,dV = \int_{z=\cdots}^{\cdots}\int_{\theta=\cdots}^{\cdots}\int_{r=\cdots}^{\cdots} f(r\cos(\theta), r\sin(\theta), z)\, r\, dr\, d\theta\, dz,$$

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Jacobian method

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(Tidier approach: expand along the bottom row.)

Jacobian method

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Now $|\det(J)| = |r| = r$ as $r \geq 0$.

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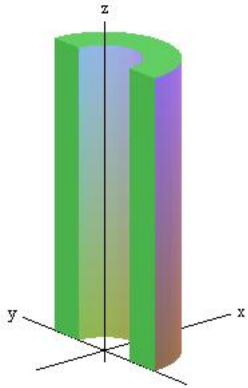
(Tidier approach: expand along the bottom row.)
Now $|\det(J)| = |r| = r$ as $r \geq 0$. We conclude that

$$dV = dx\,dy\,dz = \left|\det\left(\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right)\right| dr\,d\theta\,dz = |\det(J)|dr\,d\theta\,dz = r\,dr\,d\theta\,dz,$$

just as we saw before by a more geometric argument.

Centre of mass of a half-pipe

Consider a region E as shown on the right.
(Inner radius 1, outer radius 2, height 8.)



Centre of mass of a half-pipe

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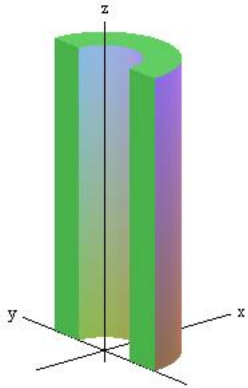

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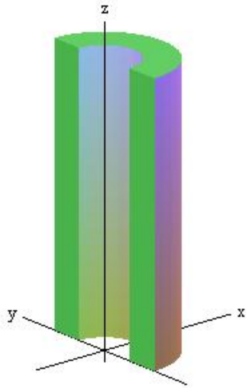
Centre of mass $(\overline{x}, 0, 4)$



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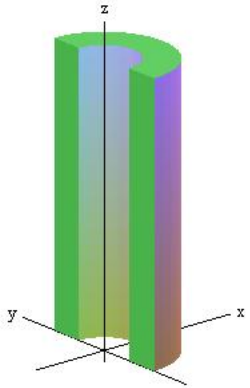

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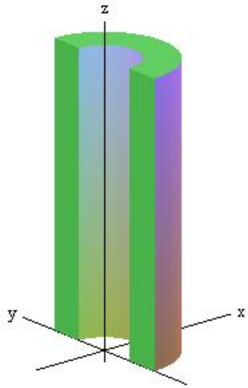

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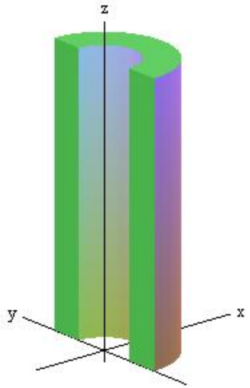

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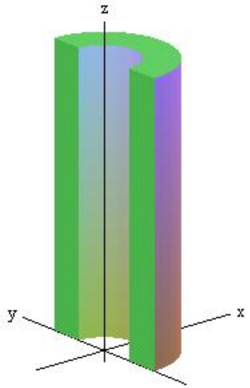
$$= \int_{z=0}^{8} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3}{2}\, d\theta\, dz$$



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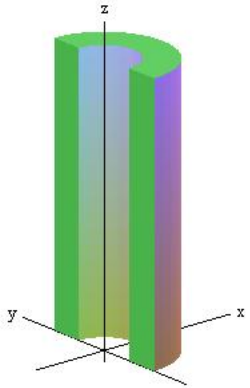

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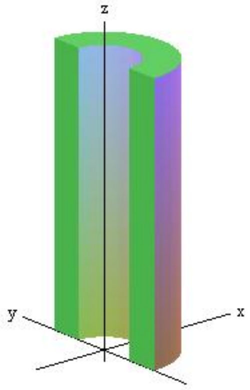

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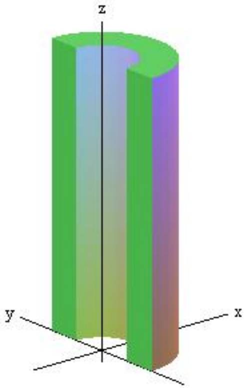

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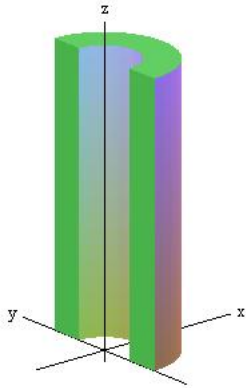

$$V = \int_{z=0}^{8} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^{2} r\, dr\, d\theta\, dz$$

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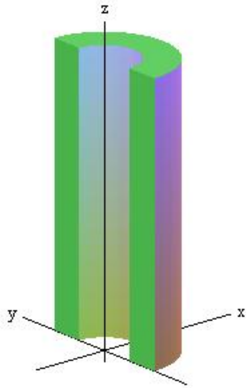

$$V = \int_{z=0}^{8} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^{2} r\, dr\, d\theta\, dz$$

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$$= \frac{7}{3} \int_{z=0}^{8} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta)\, d\theta\, dz$$

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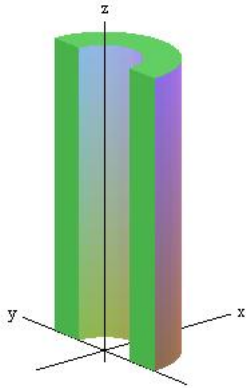

$$V = \int_{z=0}^{8} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^{2} r\, dr\, d\theta\, dz$$

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$$= \frac{7}{3} \int_{z=0}^{8} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta)\, d\theta\, dz = \frac{7}{3} \int_{z=0}^{8} \Big[\sin(\theta)\Big]_{-\pi/2}^{\pi/2} dz$$

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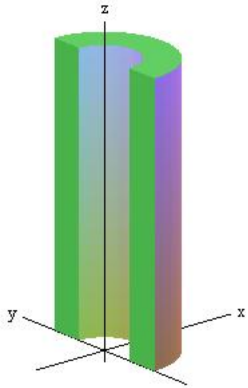

$$V = \int_{z=0}^{8} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^{2} r\, dr\, d\theta\, dz$$

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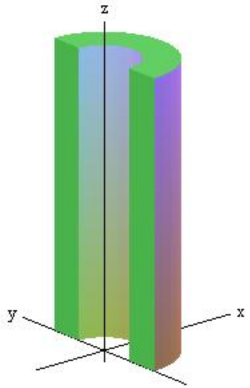

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$$X = \int_{z=0}^{8} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^{2} r\cos(\theta).r\, dr\, d\theta\, dz$$

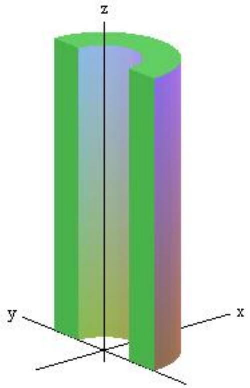
$$= \int_{z=0}^{8} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{3} r^3 \cos(\theta)\right]_{r=1}^{2} d\theta\, dz$$

$$= \frac{7}{3} \int_{z=0}^{8} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta)\, d\theta\, dz = \frac{7}{3} \int_{z=0}^{8} \Big[\sin(\theta)\Big]_{-\pi/2}^{\pi/2} dz = \frac{14}{3} \int_{z=0}^{8} 1\, dz = \frac{112}{3}$$

$$\overline{x} = \frac{112}{3 \times 12\pi}$$

Centre of mass of a half-pipe

Consider a region E as shown on the right.
(Inner radius 1, outer radius 2, height 8.)
$0 \leq z \leq 8; \qquad -\pi/2 \leq \theta \leq \pi/2; \qquad 1 \leq r \leq 2.$
Centre of mass $(\overline{x}, 0, 4)$, where $\overline{x} = X/V$,
$X = \iiint_E x \, dV$ and $V = \iiint_E 1 \, dV$.



$$V = \int_{z=0}^{8} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^{2} r \, dr \, d\theta \, dz$$

$$= \int_{z=0}^{8} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3}{2} \, d\theta \, dz = \int_{z=0}^{8} \frac{3\pi}{2} \, dz = 12\pi$$

$$X = \int_{z=0}^{8} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^{2} r\cos(\theta).r \, dr \, d\theta \, dz$$

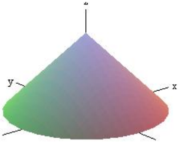
$$= \int_{z=0}^{8} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{3} r^3 \cos(\theta) \right]_{r=1}^{2} d\theta \, dz$$

$$= \frac{7}{3} \int_{z=0}^{8} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta) \, d\theta \, dz = \frac{7}{3} \int_{z=0}^{8} \left[\sin(\theta) \right]_{-\pi/2}^{\pi/2} dz = \frac{14}{3} \int_{z=0}^{8} 1 \, dz = \frac{112}{3}$$

$$\overline{x} = \frac{112}{3 \times 12\pi} = \frac{28}{9\pi} \simeq 0.99.$$

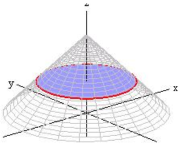
Centre of mass of a cone

Consider a cone E as on the right (base radius=height=1).



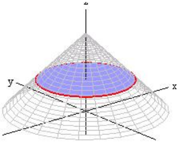
Centre of mass of a cone

Consider a cone E as on the right (base radius=height=1).
Radius at height z is $1 - z$.



Centre of mass of a cone

Consider a cone E as on the right (base radius=height=1).
Radius at height z is $1 - z$.
$0 \leq z \leq 1; \qquad 0 \leq \theta \leq 2\pi; \qquad 0 \leq r \leq 1 - z.$



Centre of mass of a cone

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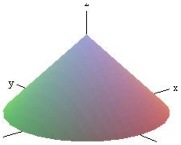
$0 \leq z \leq 1; \qquad 0 \leq \theta \leq 2\pi; \qquad 0 \leq r \leq 1 - z.$
Centre of mass $(0, 0, \overline{z})$



Centre of mass of a cone

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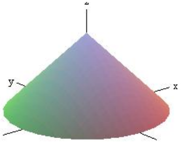
Radius at height z is $1 - z$.
$0 \leq z \leq 1; \qquad 0 \leq \theta \leq 2\pi; \qquad 0 \leq r \leq 1 - z.$
Centre of mass $(0, 0, \overline{z})$, where $\overline{z} = Z/V$



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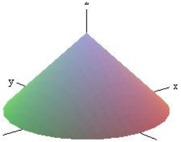

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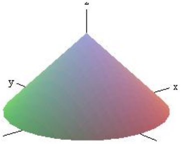

$$V = \int_{z=0}^{1} \int_{\theta=0}^{2\pi} \int_{r=0}^{1-z} r\, dr\, d\theta\, dz$$

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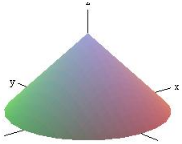

$$V = \int_{z=0}^{1} \int_{\theta=0}^{2\pi} \int_{r=0}^{1-z} r\,dr\,d\theta\,dz = \int_{z=0}^{1} \int_{\theta=0}^{2\pi} \frac{(1-z)^2}{2}\,d\theta\,dz$$

Centre of mass of a cone

Consider a cone E as on the right (base radius=height=1).
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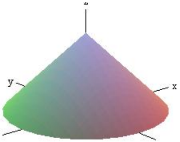

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$$= \int_{z=0}^{1} \pi(1-z)^2\, dz = \pi \int_{z=0}^{1} 1 - 2z + z^2\, dz$$

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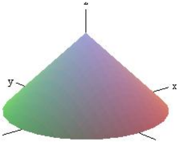

$$V = \int_{z=0}^{1}\int_{\theta=0}^{2\pi}\int_{r=0}^{1-z} r\,dr\,d\theta\,dz = \int_{z=0}^{1}\int_{\theta=0}^{2\pi} \frac{(1-z)^2}{2}\,d\theta\,dz$$

$$= \int_{z=0}^{1} \pi(1-z)^2\,dz = \pi\int_{z=0}^{1} 1-2z+z^2\,dz = \pi\left[z - z^2 + \tfrac{1}{3}z^3\right]_{z=0}^{1} = \pi/3$$

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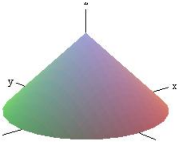

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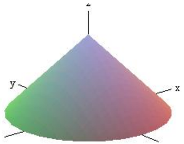

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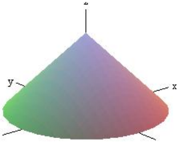

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Centre of mass of a cone

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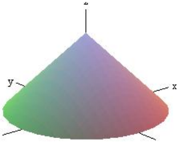

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Centre of mass of a cone

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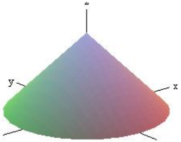

$$V = \int_{z=0}^{1} \int_{\theta=0}^{2\pi} \int_{r=0}^{1-z} r\,dr\,d\theta\,dz = \int_{z=0}^{1} \int_{\theta=0}^{2\pi} \frac{(1-z)^2}{2}\,d\theta\,dz$$
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$$= \int_{z=0}^{1} \pi z(1-z)^2\,dz = \pi \int_{z=0}^{1} z - 2z^2 + z^3\,dz$$
$$= \pi \left[\tfrac{1}{2} z^2 - \tfrac{2}{3} z^3 + \tfrac{1}{4} z^4 \right]_{z=0}^{1}$$

Centre of mass of a cone

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Radius at height z is $1 - z$.
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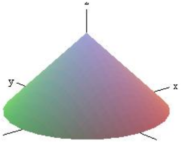

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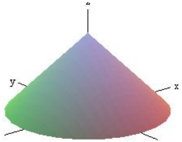

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$$\overline{z} = Z/V$$

Centre of mass of a cone

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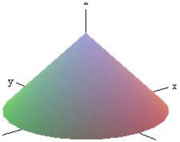

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$$\overline{z} = Z/V = \frac{\pi}{12} \Big/ \frac{\pi}{3}$$

Centre of mass of a cone

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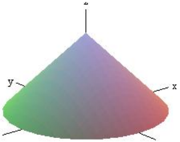

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$$= \int_{z=0}^{1} \pi(1-z)^2\, dz = \pi\int_{z=0}^{1} 1-2z+z^2\, dz = \pi\left[z - z^2 + \tfrac{1}{3}z^3\right]_{z=0}^{1} = \pi/3$$

$$Z = \int_{z=0}^{1}\int_{\theta=0}^{2\pi}\int_{r=0}^{1-z} zr\, dr\, d\theta\, dz = \int_{z=0}^{1}\int_{\theta=0}^{2\pi} z\frac{(1-z)^2}{2}\, d\theta\, dz$$

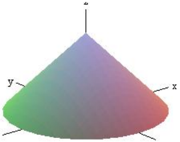
$$= \int_{z=0}^{1} \pi z(1-z)^2\, dz = \pi\int_{z=0}^{1} z - 2z^2 + z^3\, dz$$

$$= \pi\left[\tfrac{1}{2}z^2 - \tfrac{2}{3}z^3 + \tfrac{1}{4}z^4\right]_{z=0}^{1} = \pi/12$$

$$\overline{z} = Z/V = \frac{\pi}{12}\Big/\frac{\pi}{3} = \frac{\pi}{12}\Big/\frac{4\pi}{12}$$

Centre of mass of a cone

Consider a cone E as on the right (base radius=height=1).
Radius at height z is $1-z$.
$0 \le z \le 1; \qquad 0 \le \theta \le 2\pi; \qquad 0 \le r \le 1-z.$
Centre of mass $(0,0,\bar{z})$, where $\bar{z} = Z/V$,
$Z = \iiint_E z\,dV$ and $V = \iiint_E 1\,dV$.



$$V = \int_{z=0}^{1}\int_{\theta=0}^{2\pi}\int_{r=0}^{1-z} r\,dr\,d\theta\,dz = \int_{z=0}^{1}\int_{\theta=0}^{2\pi} \frac{(1-z)^2}{2}\,d\theta\,dz$$
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Mass of a parabolic mirror

Telescope mirrors always have a parabolic cross-section.

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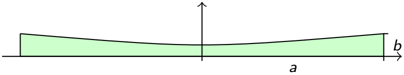

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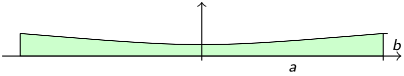

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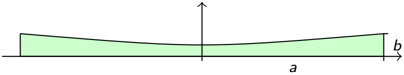

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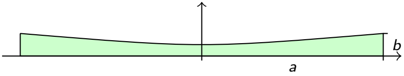

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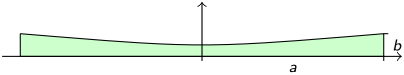

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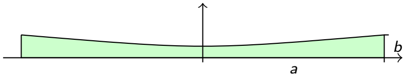

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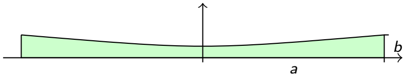

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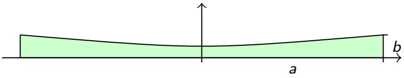

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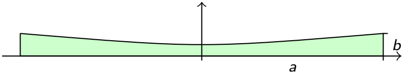

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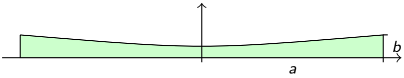

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&= \frac{b\rho\pi}{a^2} \int_{r=0}^{a} r^3 + a^2 r\, dr = \frac{b\rho\pi}{a^2} \left[\frac{r^4}{4} + \frac{a^2 r^2}{2} \right]_{r=0}^{a}
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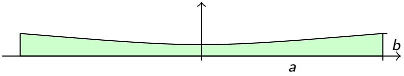

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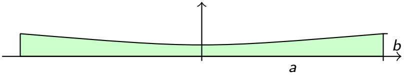

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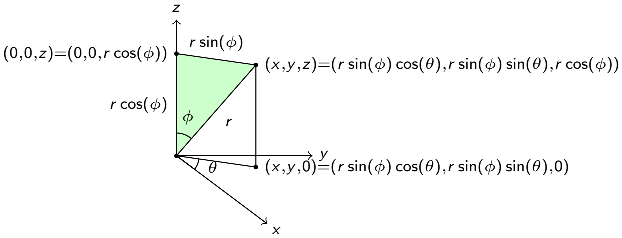
Spherical polar coordinates

In spherical polar coordinates we describe a point (x, y, z) by giving the distance r from the origin, the angle θ anticlockwise from the xz plane, and the angle ϕ from the z-axis.



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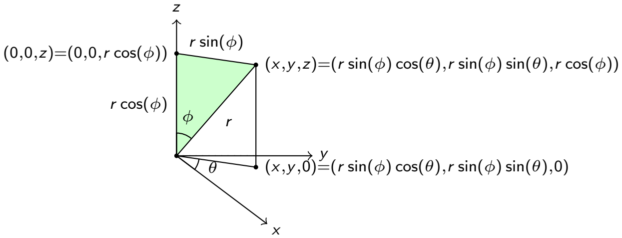

The variables r, θ and ϕ are related to x and y by the equations

$$x = r\sin(\phi)\cos(\theta) \qquad y = r\sin(\phi)\sin(\theta) \qquad z = r\cos(\phi)$$

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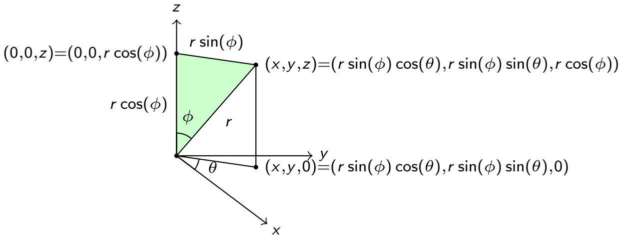

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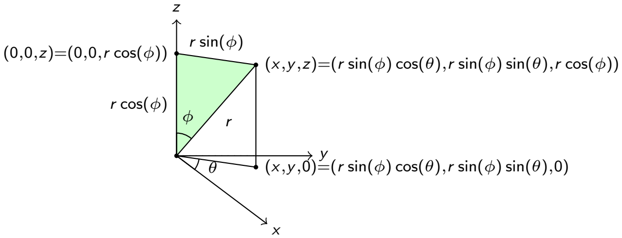

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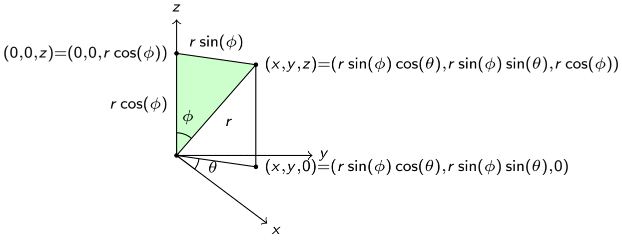

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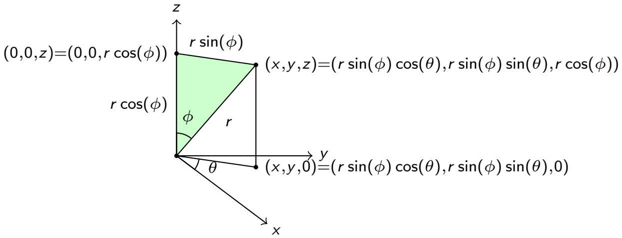

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For these coordinates it is easiest to find the area element using the Jacobian.

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$$\det(J) = \cos(\phi)\det(A) - 0\det(B) + (-r\sin(\phi))\det(C),$$

where

$$A = \begin{bmatrix} -r\sin(\phi)\sin(\theta) & r\cos(\phi)\cos(\theta) \\ r\sin(\phi)\cos(\theta) & r\cos(\phi)\sin(\theta) \end{bmatrix} \quad B = \begin{bmatrix} \sin(\phi)\cos(\theta) & r\cos(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) & r\cos(\phi)\sin(\theta) \end{bmatrix} \quad C = \begin{bmatrix} \sin(\phi)\cos(\theta) & -r\sin(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) & r\sin(\phi)\cos(\theta) \end{bmatrix}.$$

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$$dV = |\det(J)|\,dr\,d\theta\,d\phi = r^2\sin(\phi)\,dr\,d\theta\,d\phi.$$

Spherical polar volume element

$$x = r\sin(\phi)\cos(\theta) \qquad y = r\sin(\phi)\sin(\theta) \qquad z = r\cos(\phi)$$

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This means that for a function f on a 3-dimensional region E, we have

$$\iiint_E f(x,y,z)\,dV =$$

$$\int_{\phi=\cdots}^{\cdots}\int_{\theta=\cdots}^{\cdots}\int_{r=\cdots}^{\cdots} f(r\cos(\theta)\sin(\phi), r\sin(\theta)\sin(\phi), r\cos(\phi))\,r^2\sin(\phi)\,dr\,d\theta\,d\phi,$$

where the limits must be determined using the geometry of the region.

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Here the three different variables do not interact in any interesting way so we can rewrite the integral as

$$I = \left(\int_{\phi=0}^{\pi} \sin(\phi)^3\, d\phi \right) \left(\int_{\theta=0}^{2\pi} 1\, d\theta \right) \left(\int_{r=0}^{a} r^4\, dr \right) \rho.$$

Volume and moment of a sphere

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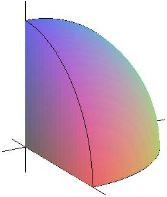
Combining this with the r and θ integrals gives $I = \frac{4}{3}.2\pi.\frac{a^5}{5}.\rho = \frac{8\pi a^5 \rho}{15}$.

Mass centre of an octant

Let E be the part of a sphere of radius 1 where $x \geq 0$, $y \geq 0$ and $z \geq 0$.



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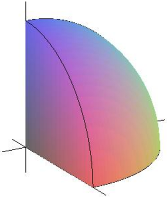

The centre of mass of E (assuming constant density) is $(\overline{x}, \overline{y}, \overline{z})$, where $\overline{x} = (\iiint_E x \, dV)/(\iiint_E 1 \, dV)$ and so on.

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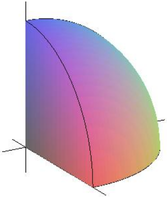

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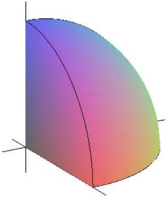

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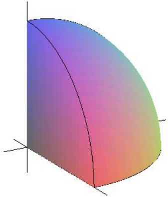

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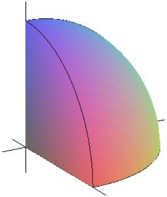

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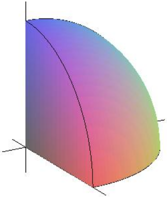

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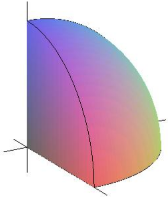

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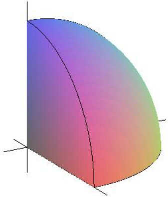

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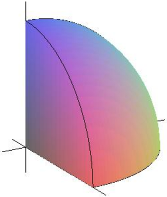

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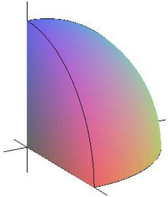

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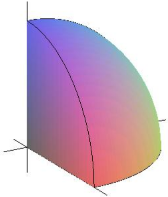

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We conclude that the centre of mass is $(\frac{3}{8}, \frac{3}{8}, \frac{3}{8})$.

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When answering questions in vector algebra or vector calculus, you should always ask yourself whether your answer should be a scalar or a vector, and make sure that what you have written has the right type. This simple check will detect a substantial fraction of incorrect answers.

Numerical vectors

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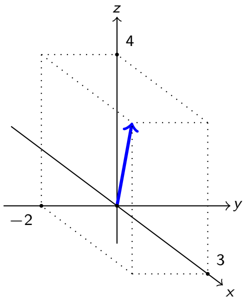

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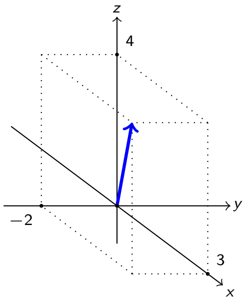

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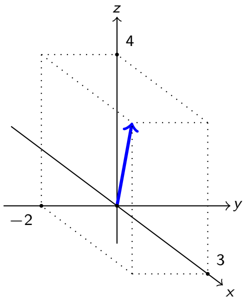

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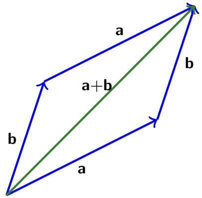

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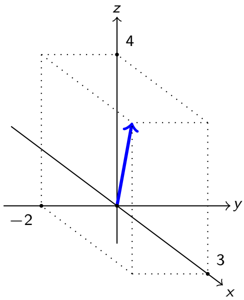

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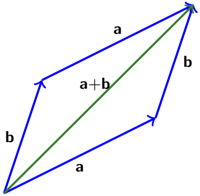

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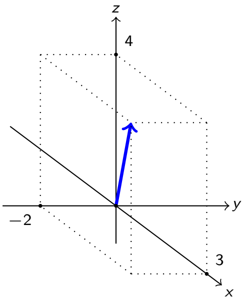

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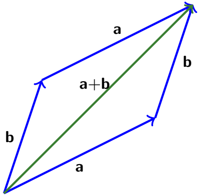

Similarly, we can multiply a vector by a scalar to get a new vector, for example $3(3, -2, 4) = (9, -6, 12)$. The new vector has the same direction as the old one (if the scalar is positive) or the opposite direction (if the scalar is negative).

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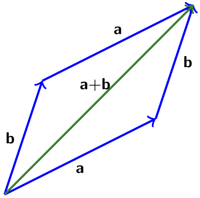

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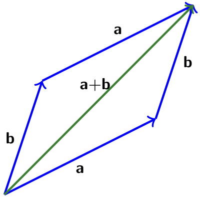

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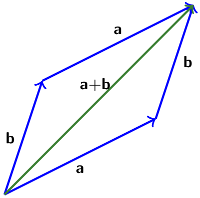

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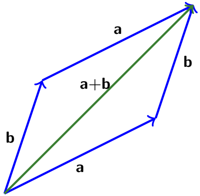

The distance from the origin to $\mathbf{a} + \mathbf{b}$ in a straight line is $|\mathbf{a} + \mathbf{b}|$, whereas the distance via $\mathbf{a}$ is $|\mathbf{a}| + |\mathbf{b}|$. The inequality just says that it is shorter to go in a straight line.

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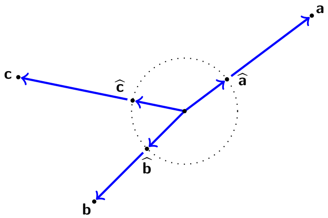

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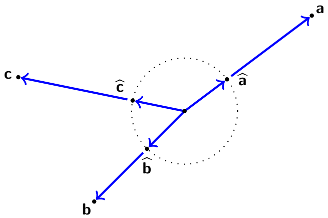

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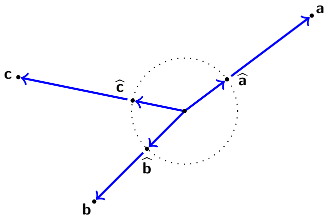

This is given by

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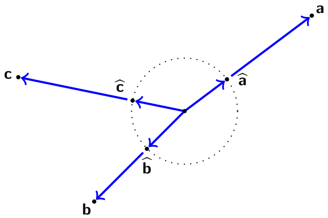

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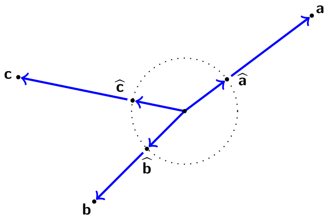

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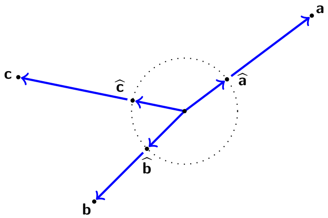

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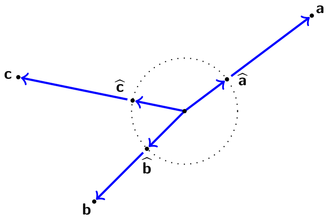

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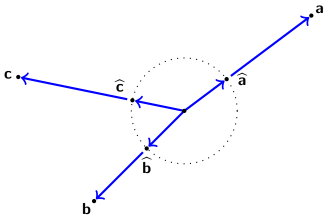

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Note that $|\mathbf{a}|$ is a scalar, and $\widehat{\mathbf{a}}$ is a vector.

Vectors along the coordinate axes

The unit vectors along the three coordinate axes are denoted by **i**, **j** and **k**:

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For example, the vector $(10, 0, -20)$ can also be expressed as $10\mathbf{i} - 20\mathbf{k}$.

Dot products

The dot product of vectors $\mathbf{a} = (x, y, z)$ and $\mathbf{b} = (u, v, w)$ is given by

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$$(1, 2, 3).(10, 100, 1000) = 10 + 200 + 3000 = 3210.$$

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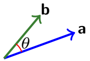
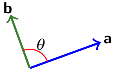


$\theta < \frac{\pi}{2}$; $\quad \mathbf{a}.\mathbf{b} > 0$



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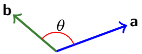

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Angle example

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This means that $\theta = \arccos(0.933)$, which is 0.367 radians or 21.04 degrees.

Methane

The hydrogen atoms in a molecule of methane lie at the following positions:

$$\mathbf{a} = (0, 0, 1) \qquad \mathbf{b} = \left(\frac{2\sqrt{2}}{3}, 0, -\frac{1}{3}\right)$$

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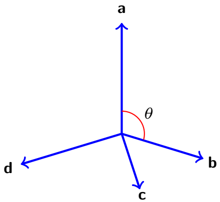

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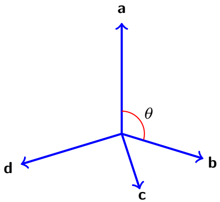

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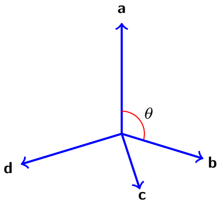

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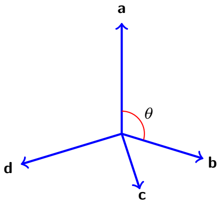

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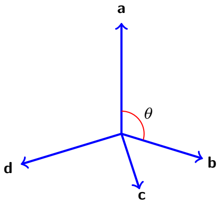

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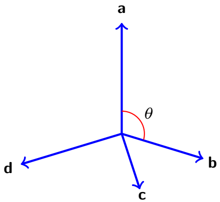

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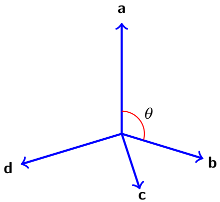

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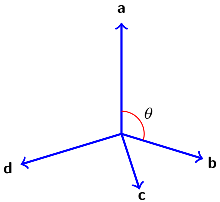

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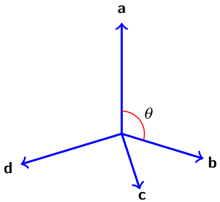

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so **b** and **c** are unit vectors, and **d** is also a unit vector by the same calculation as for **c**. It is also clear that $\mathbf{a}.\mathbf{b} = \mathbf{a}.\mathbf{c} = \mathbf{a}.\mathbf{d} = -1/3$.

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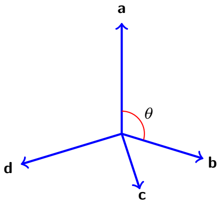
The hydrogen atoms in a molecule of methane lie at the following positions:

$$\mathbf{a} = (0,0,1) \qquad \mathbf{b} = \left(\frac{2\sqrt{2}}{3}, 0, -\frac{1}{3}\right)$$

$$\mathbf{c} = \left(-\frac{\sqrt{2}}{3}, \frac{\sqrt{6}}{3}, -\frac{1}{3}\right) \qquad \mathbf{d} = \left(-\frac{\sqrt{2}}{3}, -\frac{\sqrt{6}}{3}, -\frac{1}{3}\right).$$



It is clear that **a** is a unit vector. We also have

$$|\mathbf{b}|^2 = \left(\frac{2\sqrt{2}}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{4\times 2}{9} + \frac{1}{9} = 1$$

$$|\mathbf{c}|^2 = \left(\frac{\sqrt{2}}{3}\right)^2 + \left(\frac{\sqrt{6}}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{2}{9} + \frac{6}{9} + \frac{1}{9} = 1$$

so **b** and **c** are unit vectors, and **d** is also a unit vector by the same calculation as for **c**. It is also clear that $\mathbf{a}.\mathbf{b} = \mathbf{a}.\mathbf{c} = \mathbf{a}.\mathbf{d} = -1/3$. In fact, we also have $\mathbf{b}.\mathbf{c} = \mathbf{b}.\mathbf{d} = \mathbf{c}.\mathbf{d} = -1/3$.

Methane

$$\mathbf{a} = (0,0,1) \qquad \mathbf{b} = \left(\frac{2\sqrt{2}}{3}, 0, -\frac{1}{3}\right)$$
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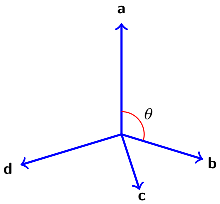

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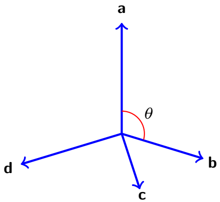

$$\mathbf{b}.\mathbf{c} = \frac{2\sqrt{2}}{3}.\left(-\frac{\sqrt{2}}{3}\right) + 0.\frac{\sqrt{6}}{3} + \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)$$

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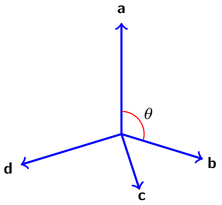

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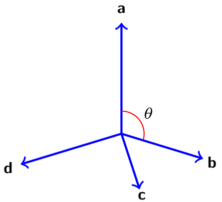

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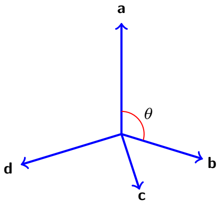

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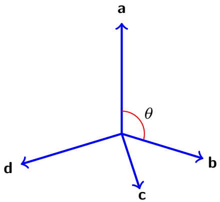

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$$\mathbf{c.d} = \left(-\frac{\sqrt{2}}{3}\right)\left(-\frac{\sqrt{2}}{3}\right) + \left(\frac{\sqrt{6}}{3}\right)\left(-\frac{\sqrt{6}}{3}\right) + \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)$$

Methane

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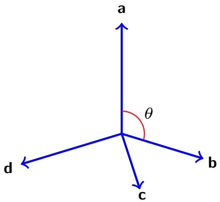

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$$\mathbf{c}.\mathbf{d} = \left(-\frac{\sqrt{2}}{3}\right)\left(-\frac{\sqrt{2}}{3}\right) + \left(\frac{\sqrt{6}}{3}\right)\left(-\frac{\sqrt{6}}{3}\right) + \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right) = \frac{2}{9} - \frac{6}{9} + \frac{1}{9} = -\frac{1}{3}.$$

Methane

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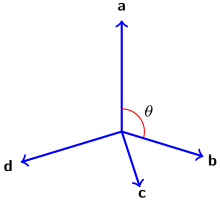

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If θ is the angle between **a** and **b**, then we have

$$\cos(\theta) = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{-1/3}{1 \times 1} = -\frac{1}{3}$$

Methane

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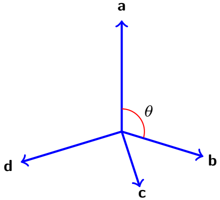

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so θ is $\arccos(-1/3)$

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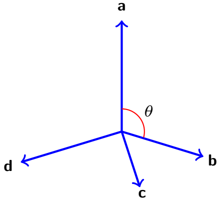

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If θ is the angle between **a** and **b**, then we have

$$\cos(\theta) = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{-1/3}{1 \times 1} = -\frac{1}{3},$$

so θ is $\arccos(-1/3)$, which is 1.911 radians or 109.5 degrees.

Methane

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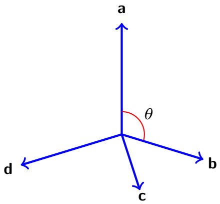

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If θ is the angle between **a** and **b**, then we have

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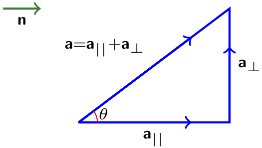
so θ is $\arccos(-1/3)$, which is 1.911 radians or 109.5 degrees. By the same calculation, the angle between any two of the atoms is 109.5 degrees.

Parallel and perpendicular components

Now suppose we have a vector **a** and a unit vector **n**.

Parallel and perpendicular components

Now suppose we have a vector **a** and a unit vector **n**. We can write **a** as $\mathbf{a}_{||} + \mathbf{a}_{\perp}$, where $\mathbf{a}_{||}$ is the part parallel to **n**, and $\mathbf{a}_{\perp}$ is the part perpendicular to **n**.



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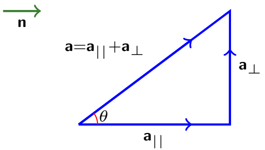

In the picture, θ is the angle between $\mathbf{a}$ and $\mathbf{a}_{||}$, which is the same as the angle between $\mathbf{a}$ and $\mathbf{n}$.

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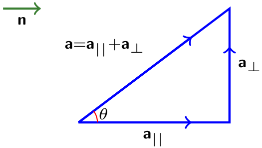

In the picture, θ is the angle between $\mathbf{a}$ and $\mathbf{a}_{||}$, which is the same as the angle between $\mathbf{a}$ and $\mathbf{n}$. From this (and the fact that $|\mathbf{n}| = 1$) it follows that

$$\mathbf{a}.\mathbf{n} = |\mathbf{a}||\mathbf{n}|\cos(\theta) = |\mathbf{a}|\cos(\theta) = |\mathbf{a}_{||}|.$$

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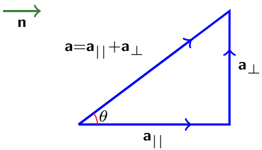

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(Equation of scalars, valid for $\theta \leq \frac{\pi}{2}$; for all θ we have $|\mathbf{a}.\mathbf{n}| = |\mathbf{a}_{||}|$.)

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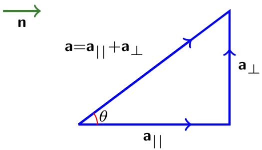

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$$\mathbf{a}_{||} = (\mathbf{a}.\mathbf{n})\mathbf{n}$$
$$\mathbf{a}_{\perp} = \mathbf{a} - (\mathbf{a}.\mathbf{n})\mathbf{n}.$$

Parallel and perpendicular example

$$\mathbf{a}_{||} = (\mathbf{a}.\mathbf{n})\mathbf{n} \qquad\qquad \mathbf{a}_{\perp} = \mathbf{a} - (\mathbf{a}.\mathbf{n})\mathbf{n}.$$

Consider the vector $\mathbf{a} = (3, 6, 9)$ and the unit vector $\mathbf{n} = (2/3, 2/3, -1/3)$.

Parallel and perpendicular example

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Consider the vector $\mathbf{a} = (3, 6, 9)$ and the unit vector $\mathbf{n} = (2/3, 2/3, -1/3)$. We have

$$\mathbf{a}.\mathbf{n} = 3.\frac{2}{3} + 6.\frac{2}{3} + 9.\frac{-1}{3} = 2 + 4 - 3 = 3$$

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$$\mathbf{a}_{||} = (\mathbf{a}.\mathbf{n})\mathbf{n} = 3\mathbf{n} = (2, 2, -1)$$

Parallel and perpendicular example

$$\mathbf{a}_{||} = (\mathbf{a}.\mathbf{n})\mathbf{n} \qquad \mathbf{a}_{\perp} = \mathbf{a} - (\mathbf{a}.\mathbf{n})\mathbf{n}.$$

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$$\begin{aligned}
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\mathbf{a}_{||} &= (\mathbf{a}.\mathbf{n})\mathbf{n} = 3\mathbf{n} = (2, 2, -1) \\
\mathbf{a}_{\perp} &= \mathbf{a} - \mathbf{a}_{||}
\end{aligned}$$

Parallel and perpendicular example

$$\mathbf{a}_{||} = (\mathbf{a}.\mathbf{n})\mathbf{n} \qquad \mathbf{a}_{\perp} = \mathbf{a} - (\mathbf{a}.\mathbf{n})\mathbf{n}.$$

Consider the vector $\mathbf{a} = (3, 6, 9)$ and the unit vector $\mathbf{n} = (2/3, 2/3, -1/3)$. We have

$$\begin{aligned} \mathbf{a}.\mathbf{n} &= 3.\frac{2}{3} + 6.\frac{2}{3} + 9.\frac{-1}{3} = 2 + 4 - 3 = 3 \\ \mathbf{a}_{||} &= (\mathbf{a}.\mathbf{n})\mathbf{n} = 3\mathbf{n} = (2, 2, -1) \\ \mathbf{a}_{\perp} &= \mathbf{a} - \mathbf{a}_{||} = (3, 6, 9) - (2, 2, -1) = (1, 4, 10). \end{aligned}$$

The cross product

We next recall the cross product operation. For vectors $\mathbf{a} = (x, y, z)$ and $\mathbf{b} = (u, v, w)$, we define

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Example: For the standard unit vectors you can check that

$$\begin{array}{lll} \mathbf{i} \times \mathbf{i} = 0 & \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{j} \times \mathbf{j} = 0 & \mathbf{j} \times \mathbf{k} = \mathbf{i} \\ \mathbf{k} \times \mathbf{i} = \mathbf{j} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{k} \times \mathbf{k} = 0. \end{array}$$

Cross product geometry

Geometrically, it can be shown that $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, and that

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\theta) = \text{ area of the parallelogram spanned by } \mathbf{a} \text{ and } \mathbf{b},$$

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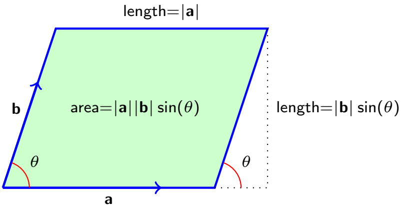

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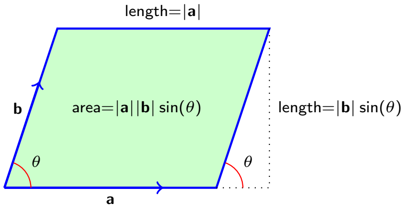

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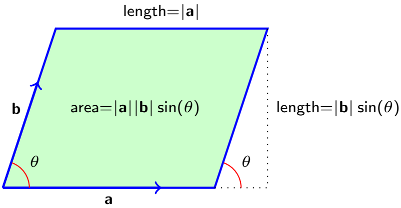

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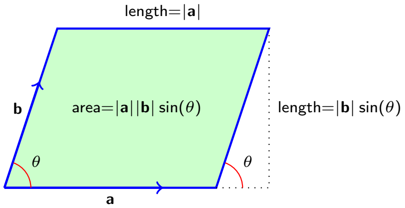

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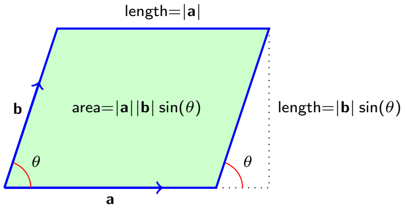

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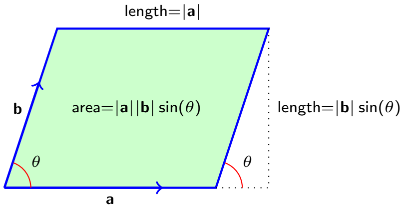

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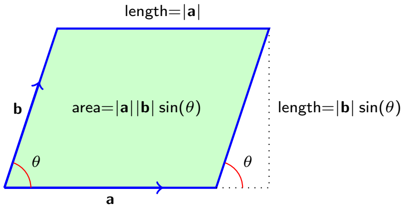

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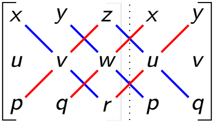

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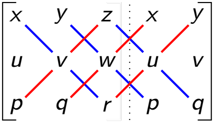

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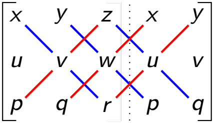

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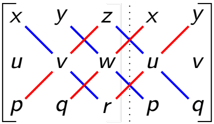

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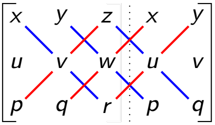
Suppose we have vectors $\mathbf{a} = (x, y, z)$, $\mathbf{b} = (u, v, w)$ and $\mathbf{c} = (p, q, r)$. We can take the cross product $\mathbf{b} \times \mathbf{c}$, which is a vector, and then take the dot product of that vector with $\mathbf{a}$ to get a scalar $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$, which is called the *scalar triple product* of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Using the determinant formula for $\mathbf{b} \times \mathbf{c}$ we find that $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$ is also a determinant:

$$\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = \det \begin{bmatrix} x & y & z \\ u & v & w \\ p & q & r \end{bmatrix}.$$



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$$\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = \det \begin{bmatrix} x & y & z \\ u & v & w \\ p & q & r \end{bmatrix} = xvr + ywp + zuq - zvp - xwq - yur.$$

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We also have $\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = (\mathbf{b} \times \mathbf{c}).\mathbf{a}$ and so on, just because $\mathbf{u}.\mathbf{v} = \mathbf{v}.\mathbf{u}$ for any vectors $\mathbf{u}$ and $\mathbf{v}$.

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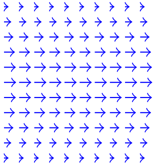
Although we will mainly be concerned with scalar and vector fields in three-dimensional space, we will sometimes use two-dimensional examples because they are easier to visualise.

Example vector fields in two dimensions



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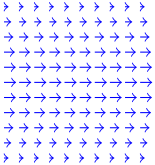

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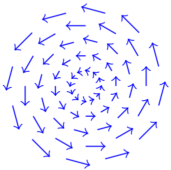

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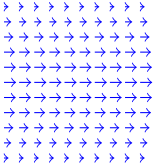

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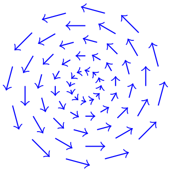

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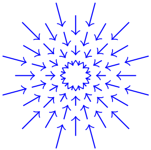

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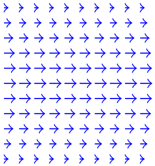

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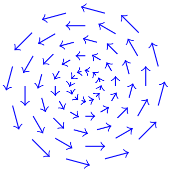

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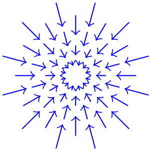

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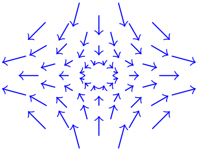

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More generally, for any n we have

$$(r^n)_x = nr^{n-1}r_x = nr^{n-1}x/r = nr^{n-2}x.$$

The other two derivatives work in the same way, so

$$\nabla(r^n) = nr^{n-2}\,(x, y, z).$$

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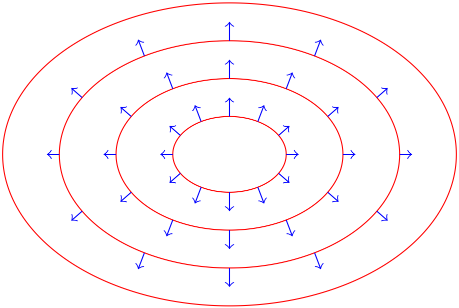

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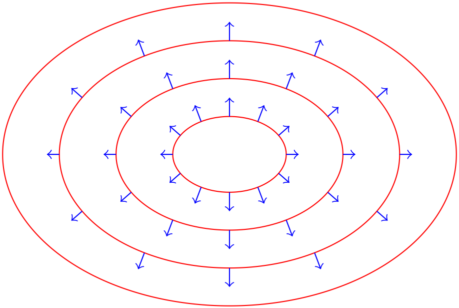

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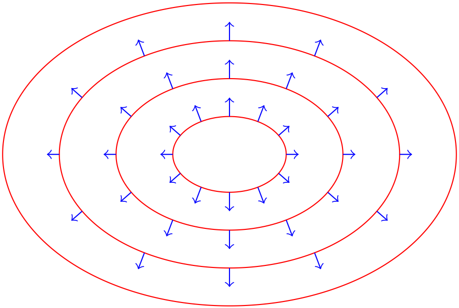
The picture below illustrates the two-dimensional version of this fact in the case where $f = \sqrt{x^2/9 + y^2/4}$.



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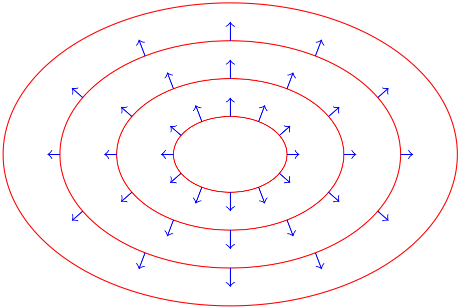

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(c) The net force on a particle of air involves $\nabla(p)$, where p is the pressure.

Electric field of a point charge

If we have a single charge at the origin, then the resulting electric potential function is $\phi = Ar^{-1}$ for some constant A, where $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ as usual.

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div and curl

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This means we can make sense of the dot product $\nabla.\mathbf{u}$ and the cross product $\nabla \times \mathbf{u}$ as follows:

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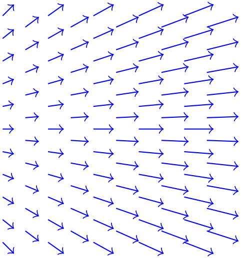
(c) For a vector field $\mathbf{u} = (p, q)$ in two dimensions,

$$\text{curl}(\mathbf{u}) = \det \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ p & q \end{bmatrix} = q_x - p_y$$

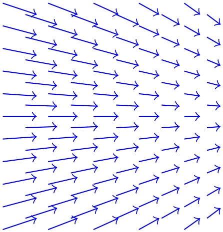
(a *scalar* field, not a vector field as in three dimensions).

Geometric interpretation of div($\mathbf{u}$)

It works out that the divergence div($\mathbf{u}$) $= \nabla.\mathbf{u}$ is positive when the vectors $\mathbf{u}$ are spreading out, and negative when they are coming together.



diverging: $\nabla.\mathbf{u} > 0$

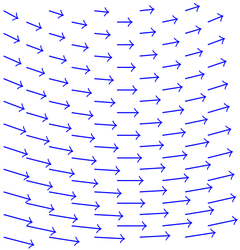


converging: $\nabla.\mathbf{u} < 0$

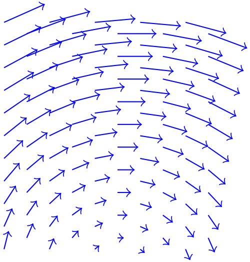
For the velocity field of an incompressible fluid we will have $\nabla.\mathbf{u} = 0$.

Geometric interpretation of curl($\mathbf{u}$)

In two dimensions, it works out that curl($\mathbf{u}$) > 0 in regions where the field is curling anticlockwise, and curl($\mathbf{u}$) < 0 in regions where it is curling clockwise, and the absolute value of curl($\mathbf{u}$) is determined by the strength of the curling.



curl($\mathbf{u}$) > 0, smaller



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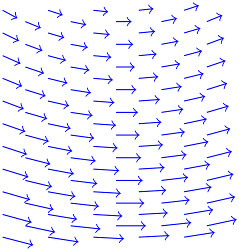

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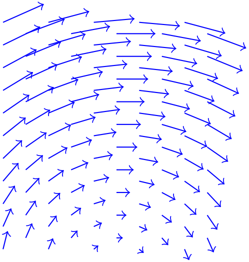

curl($\mathbf{u}$) < 0, larger

In three dimensions, the field $\mathbf{u}$ can curl around any axis. In this context, curl($\mathbf{u}$) is also a vector field, and it will point along the axis of the curling.

These involve:

Maxwell's equations

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The quantities $\mathbf{E}$, $\mathbf{B}$, $\mathbf{J}$ and ρ may also depend on time; we write $\dot{\mathbf{E}}$ for $\partial \mathbf{E}/\partial t$ and so on.

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The quantities $\mathbf{E}$, $\mathbf{B}$, $\mathbf{J}$ and ρ may also depend on time; we write $\dot{\mathbf{E}}$ for $\partial \mathbf{E}/\partial t$ and so on. The various fields are related by the following equations:

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This means that:

- The electric field diverges in regions where there is positive charge, and converges in regions where there is negative charge.

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- The electric field diverges in regions where there is positive charge, and converges in regions where there is negative charge.
- The magnetic field never diverges or converges.

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This means that:

- The electric field diverges in regions where there is positive charge, and converges in regions where there is negative charge.
- The magnetic field never diverges or converges.
- Changing magnetic fields cause the electric field to curl.

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$$\nabla.\mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dot{\mathbf{E}}.$$

This means that:

- The electric field diverges in regions where there is positive charge, and converges in regions where there is negative charge.
- The magnetic field never diverges or converges.
- Changing magnetic fields cause the electric field to curl.
- Currents cause the magnetic field to curl.

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The quantities $\mathbf{E}$, $\mathbf{B}$, $\mathbf{J}$ and ρ may also depend on time; we write $\dot{\mathbf{E}}$ for $\partial\mathbf{E}/\partial t$ and so on. The various fields are related by the following equations:

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$$\nabla.\mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\dot{\mathbf{E}}.$$

This means that:

- The electric field diverges in regions where there is positive charge, and converges in regions where there is negative charge.
- The magnetic field never diverges or converges.
- Changing magnetic fields cause the electric field to curl.
- Currents cause the magnetic field to curl. Changing electric fields also cause the magnetic field to curl, but the effect is usually much weaker, because ϵ_0 is small.

Plane wave solution to Maxwell's equations

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One class of solutions to Maxwell's equations is as follows. Put $c = 1/\sqrt{\mu_0 \epsilon_0} \simeq 3 \times 10^8 ms^{-1}$ (which turns out to be the speed of light), and let α be any constant.

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One class of solutions to Maxwell's equations is as follows. Put $c = 1/\sqrt{\mu_0\epsilon_0} \simeq 3 \times 10^8 ms^{-1}$ (which turns out to be the speed of light), and let α be any constant. We can take $\mathbf{J} = 0$ and $\rho = 0$ and

$$\mathbf{E} = (0, \sin(\alpha(x - ct)), 0) \qquad \mathbf{B} = (0, 0, \sin(\alpha(x - ct))/c).$$

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Plane wave solution to Maxwell's equations

One class of solutions to Maxwell's equations is as follows. Put $c = 1/\sqrt{\mu_0 \epsilon_0} \simeq 3 \times 10^8 ms^{-1}$ (which turns out to be the speed of light), and let α be any constant. We can take $\mathbf{J} = 0$ and $\rho = 0$ and

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$$\nabla.\mathbf{E} = \frac{\partial}{\partial y}\sin(\alpha(x - ct)) = 0 = \rho/\epsilon_0 \qquad \dot{\mathbf{E}} = (0, -\alpha c\cos(\alpha(x - ct)), 0)$$

$$\nabla.\mathbf{B} = \frac{\partial}{\partial z}\sin(\alpha(x - ct))/c = 0 \qquad \dot{\mathbf{B}} = (0, 0, -\alpha\cos(\alpha(x - ct))$$

$$\nabla \times \mathbf{E} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \sin(\alpha(x - ct)) & 0 \end{bmatrix} = (0, 0, \alpha\cos(\alpha(x - ct))) = -\dot{\mathbf{B}}$$

$$\nabla \times \mathbf{B} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \sin(\alpha(x - ct))/c \end{bmatrix} = (0, -\alpha\cos(\alpha(x - ct))/c, 0) = \dot{\mathbf{E}}/c^2 = \mu_0\epsilon_0\dot{\mathbf{E}}.$$

This shows that we do indeed have a solution to the equations.

Plane wave solution to Maxwell's equations

One class of solutions to Maxwell's equations is as follows. Put $c = 1/\sqrt{\mu_0\epsilon_0} \simeq 3\times 10^8 ms^{-1}$ (which turns out to be the speed of light), and let α be any constant. We can take $\mathbf{J} = 0$ and $\rho = 0$ and

$$\mathbf{E} = (0, \sin(\alpha(x - ct)), 0) \qquad\qquad \mathbf{B} = (0, 0, \sin(\alpha(x - ct))/c).$$

We find that

$$\nabla.\mathbf{E} = \frac{\partial}{\partial y}\sin(\alpha(x-ct)) = 0 = \rho/\epsilon_0 \qquad \dot{\mathbf{E}} = (0, -\alpha c\cos(\alpha(x-ct)), 0)$$

$$\nabla.\mathbf{B} = \frac{\partial}{\partial z}\sin(\alpha(x-ct))/c = 0 \qquad \dot{\mathbf{B}} = (0, 0, -\alpha\cos(\alpha(x-ct))$$

$$\nabla\times\mathbf{E} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \sin(\alpha(x-ct)) & 0 \end{bmatrix} = (0, 0, \alpha\cos(\alpha(x-ct))) = -\dot{\mathbf{B}}$$

$$\nabla\times\mathbf{B} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \sin(\alpha(x-ct))/c \end{bmatrix} = (0, -\alpha\cos(\alpha(x-ct))/c, 0) = \dot{\mathbf{E}}/c^2 = \mu_0\epsilon_0\dot{\mathbf{E}}.$$

This shows that we do indeed have a solution to the equations. It represents an electromagnetic wave of wavelength $1/\alpha$ moving at speed c in the x-direction.

Stationary charged particle

Another solution to Maxwell's equations has $\mathbf{E} = (-xr^{-3}, -yr^{-3}, -zr^{-3})$ with all other fields ($\mathbf{B}$, $\mathbf{J}$ and ρ) being zero.

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$$(-xr^{-3})_x = 3x^2r^{-5} - r^{-3}$$

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$$(-xr^{-3})_x = 3x^2r^{-5} - r^{-3} \qquad (-xr^{-3})_y = 3xyr^{-5}$$

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$$
\begin{aligned}
(-xr^{-3})_x &= 3x^2r^{-5} - r^{-3} & (-xr^{-3})_y &= 3xyr^{-5} & (-xr^{-3})_z &= 3xzr^{-5} \\
(-yr^{-3})_x &= 3xyr^{-5} & (-yr^{-3})_y &= 3y^2r^{-5} - r^{-3} & (-yr^{-3})_z &= 3yzr^{-5} \\
(-zr^{-3})_x &= 3xzr^{-5} & (-zr^{-3})_y &= 3yzr^{-5} & (-zr^{-3})_z &= 3z^2r^{-5} - r^{-3}.
\end{aligned}
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$$\begin{aligned}
(-xr^{-3})_x &= 3x^2r^{-5} - r^{-3} & (-xr^{-3})_y &= 3xyr^{-5} & (-xr^{-3})_z &= 3xzr^{-5} \\
(-yr^{-3})_x &= 3xyr^{-5} & (-yr^{-3})_y &= 3y^2r^{-5} - r^{-3} & (-yr^{-3})_z &= 3yzr^{-5} \\
(-zr^{-3})_x &= 3xzr^{-5} & (-zr^{-3})_y &= 3yzr^{-5} & (-zr^{-3})_z &= 3z^2r^{-5} - r^{-3}.
\end{aligned}$$

$$\nabla.\mathbf{E} = (-xr^{-3})_x + (-yr^{-3})_y + (-zr^{-3})_z$$

Stationary charged particle

Another solution to Maxwell's equations has $\mathbf{E} = (-xr^{-3}, -yr^{-3}, -zr^{-3})$ with all other fields ($\mathbf{B}$, $\mathbf{J}$ and ρ) being zero. It is clear that $\dot{\mathbf{E}} = 0$ and $\dot{\mathbf{B}} = 0$, so the only equations that we need to check are that $\nabla.\mathbf{E} = 0$ and $\nabla \times \mathbf{E} = 0$. For this we recall that $r_x = x/r$, so $(r^{-3})_x = -3r^{-4}r_x = -3xr^{-5}$. In the same way, we have $(r^{-3})_y = -3yr^{-5}$ and $(r^{-3})_z = -3zr^{-5}$. Using this we find that

$$\begin{aligned}
(-xr^{-3})_x &= 3x^2r^{-5} - r^{-3} & (-xr^{-3})_y &= 3xyr^{-5} & (-xr^{-3})_z &= 3xzr^{-5} \\
(-yr^{-3})_x &= 3xyr^{-5} & (-yr^{-3})_y &= 3y^2r^{-5} - r^{-3} & (-yr^{-3})_z &= 3yzr^{-5} \\
(-zr^{-3})_x &= 3xzr^{-5} & (-zr^{-3})_y &= 3yzr^{-5} & (-zr^{-3})_z &= 3z^2r^{-5} - r^{-3}.
\end{aligned}$$

$$\begin{aligned}
\nabla.\mathbf{E} &= (-xr^{-3})_x + (-yr^{-3})_y + (-zr^{-3})_z \\
&= 3x^2r^{-5} - r^{-3} + 3y^2r^{-5} - r^{-3} + 3z^2r^{-5} - r^{-3}
\end{aligned}$$

Another solution to Maxwell's equations has $\mathbf{E} = (-xr^{-3}, -yr^{-3}, -zr^{-3})$ with all other fields ($\mathbf{B}$, $\mathbf{J}$ and ρ) being zero. It is clear that $\dot{\mathbf{E}} = 0$ and $\dot{\mathbf{B}} = 0$, so the only equations that we need to check are that $\nabla.\mathbf{E} = 0$ and $\nabla \times \mathbf{E} = 0$. For this we recall that $r_x = x/r$, so $(r^{-3})_x = -3r^{-4}r_x = -3xr^{-5}$. In the same way, we have $(r^{-3})_y = -3yr^{-5}$ and $(r^{-3})_z = -3zr^{-5}$. Using this we find that

$$\begin{aligned}
(-xr^{-3})_x &= 3x^2r^{-5} - r^{-3} & (-xr^{-3})_y &= 3xyr^{-5} & (-xr^{-3})_z &= 3xzr^{-5} \\
(-yr^{-3})_x &= 3xyr^{-5} & (-yr^{-3})_y &= 3y^2r^{-5} - r^{-3} & (-yr^{-3})_z &= 3yzr^{-5} \\
(-zr^{-3})_x &= 3xzr^{-5} & (-zr^{-3})_y &= 3yzr^{-5} & (-zr^{-3})_z &= 3z^2r^{-5} - r^{-3}.
\end{aligned}$$

$$\begin{aligned}
\nabla.\mathbf{E} &= (-xr^{-3})_x + (-yr^{-3})_y + (-zr^{-3})_z \\
&= 3x^2r^{-5} - r^{-3} + 3y^2r^{-5} - r^{-3} + 3z^2r^{-5} - r^{-3} \\
&= 3(x^2 + y^2 + z^2)r^{-5} - 3r^{-3}
\end{aligned}$$

Another solution to Maxwell's equations has $\mathbf{E} = (-xr^{-3}, -yr^{-3}, -zr^{-3})$ with all other fields ($\mathbf{B}$, $\mathbf{J}$ and ρ) being zero. It is clear that $\dot{\mathbf{E}} = 0$ and $\dot{\mathbf{B}} = 0$, so the only equations that we need to check are that $\nabla.\mathbf{E} = 0$ and $\nabla \times \mathbf{E} = 0$. For this we recall that $r_x = x/r$, so $(r^{-3})_x = -3r^{-4}r_x = -3xr^{-5}$. In the same way, we have $(r^{-3})_y = -3yr^{-5}$ and $(r^{-3})_z = -3zr^{-5}$. Using this we find that

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\begin{aligned}
(-xr^{-3})_x &= 3x^2r^{-5} - r^{-3} & (-xr^{-3})_y &= 3xyr^{-5} & (-xr^{-3})_z &= 3xzr^{-5} \\
(-yr^{-3})_x &= 3xyr^{-5} & (-yr^{-3})_y &= 3y^2r^{-5} - r^{-3} & (-yr^{-3})_z &= 3yzr^{-5} \\
(-zr^{-3})_x &= 3xzr^{-5} & (-zr^{-3})_y &= 3yzr^{-5} & (-zr^{-3})_z &= 3z^2r^{-5} - r^{-3}.
\end{aligned}
$$

$$
\begin{aligned}
\nabla.\mathbf{E} &= (-xr^{-3})_x + (-yr^{-3})_y + (-zr^{-3})_z \\
&= 3x^2r^{-5} - r^{-3} + 3y^2r^{-5} - r^{-3} + 3z^2r^{-5} - r^{-3} \\
&= 3(x^2 + y^2 + z^2)r^{-5} - 3r^{-3} = 3r^2r^{-5} - 3r^{-3}
\end{aligned}
$$

Stationary charged particle

Another solution to Maxwell's equations has $\mathbf{E} = (-xr^{-3}, -yr^{-3}, -zr^{-3})$ with all other fields ($\mathbf{B}$, $\mathbf{J}$ and ρ) being zero. It is clear that $\dot{\mathbf{E}} = 0$ and $\dot{\mathbf{B}} = 0$, so the only equations that we need to check are that $\nabla.\mathbf{E} = 0$ and $\nabla \times \mathbf{E} = 0$. For this we recall that $r_x = x/r$, so $(r^{-3})_x = -3r^{-4}r_x = -3xr^{-5}$. In the same way, we have $(r^{-3})_y = -3yr^{-5}$ and $(r^{-3})_z = -3zr^{-5}$. Using this we find that

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(-xr^{-3})_x &= 3x^2r^{-5} - r^{-3} & (-xr^{-3})_y &= 3xyr^{-5} & (-xr^{-3})_z &= 3xzr^{-5} \\
(-yr^{-3})_x &= 3xyr^{-5} & (-yr^{-3})_y &= 3y^2r^{-5} - r^{-3} & (-yr^{-3})_z &= 3yzr^{-5} \\
(-zr^{-3})_x &= 3xzr^{-5} & (-zr^{-3})_y &= 3yzr^{-5} & (-zr^{-3})_z &= 3z^2r^{-5} - r^{-3}.
\end{aligned}$$

$$\begin{aligned}
\nabla.\mathbf{E} &= (-xr^{-3})_x + (-yr^{-3})_y + (-zr^{-3})_z \\
&= 3x^2r^{-5} - r^{-3} + 3y^2r^{-5} - r^{-3} + 3z^2r^{-5} - r^{-3} \\
&= 3(x^2 + y^2 + z^2)r^{-5} - 3r^{-3} = 3r^2r^{-5} - 3r^{-3} = 0.
\end{aligned}$$

Another solution to Maxwell's equations has $\mathbf{E} = (-xr^{-3}, -yr^{-3}, -zr^{-3})$ with all other fields ($\mathbf{B}$, $\mathbf{J}$ and ρ) being zero. It is clear that $\dot{\mathbf{E}} = 0$ and $\dot{\mathbf{B}} = 0$, so the only equations that we need to check are that $\nabla.\mathbf{E} = 0$ and $\nabla \times \mathbf{E} = 0$. For this we recall that $r_x = x/r$, so $(r^{-3})_x = -3r^{-4}r_x = -3xr^{-5}$. In the same way, we have $(r^{-3})_y = -3yr^{-5}$ and $(r^{-3})_z = -3zr^{-5}$. Using this we find that

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(-yr^{-3})_x &= 3xyr^{-5} & (-yr^{-3})_y &= 3y^2r^{-5} - r^{-3} & (-yr^{-3})_z &= 3yzr^{-5} \\
(-zr^{-3})_x &= 3xzr^{-5} & (-zr^{-3})_y &= 3yzr^{-5} & (-zr^{-3})_z &= 3z^2r^{-5} - r^{-3}.
\end{aligned}
$$

$$
\begin{aligned}
\nabla.\mathbf{E} &= (-xr^{-3})_x + (-yr^{-3})_y + (-zr^{-3})_z \\
&= 3x^2r^{-5} - r^{-3} + 3y^2r^{-5} - r^{-3} + 3z^2r^{-5} - r^{-3} \\
&= 3(x^2 + y^2 + z^2)r^{-5} - 3r^{-3} = 3r^2r^{-5} - 3r^{-3} = 0.
\end{aligned}
$$

$$
\nabla \times \mathbf{E} = \left((-zr^{-3})_y - (-yr^{-3})_z, (-xr^{-3})_z - (-zr^{-3})_x, (-yr^{-3})_x - (-xr^{-3})_y\right)
$$

Stationary charged particle

Another solution to Maxwell's equations has $\mathbf{E} = (-xr^{-3}, -yr^{-3}, -zr^{-3})$ with all other fields ($\mathbf{B}$, $\mathbf{J}$ and ρ) being zero. It is clear that $\dot{\mathbf{E}} = 0$ and $\dot{\mathbf{B}} = 0$, so the only equations that we need to check are that $\nabla.\mathbf{E} = 0$ and $\nabla \times \mathbf{E} = 0$. For this we recall that $r_x = x/r$, so $(r^{-3})_x = -3r^{-4}r_x = -3xr^{-5}$. In the same way, we have $(r^{-3})_y = -3yr^{-5}$ and $(r^{-3})_z = -3zr^{-5}$. Using this we find that

$$
\begin{aligned}
(-xr^{-3})_x &= 3x^2r^{-5} - r^{-3} & (-xr^{-3})_y &= 3xyr^{-5} & (-xr^{-3})_z &= 3xzr^{-5} \\
(-yr^{-3})_x &= 3xyr^{-5} & (-yr^{-3})_y &= 3y^2r^{-5} - r^{-3} & (-yr^{-3})_z &= 3yzr^{-5} \\
(-zr^{-3})_x &= 3xzr^{-5} & (-zr^{-3})_y &= 3yzr^{-5} & (-zr^{-3})_z &= 3z^2r^{-5} - r^{-3}.
\end{aligned}
$$

$$
\begin{aligned}
\nabla.\mathbf{E} &= (-xr^{-3})_x + (-yr^{-3})_y + (-zr^{-3})_z \\
&= 3x^2r^{-5} - r^{-3} + 3y^2r^{-5} - r^{-3} + 3z^2r^{-5} - r^{-3} \\
&= 3(x^2 + y^2 + z^2)r^{-5} - 3r^{-3} = 3r^2r^{-5} - 3r^{-3} = 0.
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times \mathbf{E} &= \left((-zr^{-3})_y - (-yr^{-3})_z, (-xr^{-3})_z - (-zr^{-3})_x, (-yr^{-3})_x - (-xr^{-3})_y\right) \\
&= \left(3yzr^{-3} - 3yzr^{-3}, 3xzr^{-3} - 3xzr^{-3}, 3xyr^{-3} - 3xyr^{-3}\right)
\end{aligned}
$$

Stationary charged particle

Another solution to Maxwell's equations has $\mathbf{E} = (-xr^{-3}, -yr^{-3}, -zr^{-3})$ with all other fields ($\mathbf{B}$, $\mathbf{J}$ and ρ) being zero. It is clear that $\dot{\mathbf{E}} = 0$ and $\dot{\mathbf{B}} = 0$, so the only equations that we need to check are that $\nabla.\mathbf{E} = 0$ and $\nabla \times \mathbf{E} = 0$. For this we recall that $r_x = x/r$, so $(r^{-3})_x = -3r^{-4}r_x = -3xr^{-5}$. In the same way, we have $(r^{-3})_y = -3yr^{-5}$ and $(r^{-3})_z = -3zr^{-5}$. Using this we find that

$$
\begin{aligned}
(-xr^{-3})_x &= 3x^2r^{-5} - r^{-3} & (-xr^{-3})_y &= 3xyr^{-5} & (-xr^{-3})_z &= 3xzr^{-5} \\
(-yr^{-3})_x &= 3xyr^{-5} & (-yr^{-3})_y &= 3y^2r^{-5} - r^{-3} & (-yr^{-3})_z &= 3yzr^{-5} \\
(-zr^{-3})_x &= 3xzr^{-5} & (-zr^{-3})_y &= 3yzr^{-5} & (-zr^{-3})_z &= 3z^2r^{-5} - r^{-3}.
\end{aligned}
$$

$$
\begin{aligned}
\nabla.\mathbf{E} &= (-xr^{-3})_x + (-yr^{-3})_y + (-zr^{-3})_z \\
&= 3x^2r^{-5} - r^{-3} + 3y^2r^{-5} - r^{-3} + 3z^2r^{-5} - r^{-3} \\
&= 3(x^2 + y^2 + z^2)r^{-5} - 3r^{-3} = 3r^2r^{-5} - 3r^{-3} = 0.
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times \mathbf{E} &= \left((-zr^{-3})_y - (-yr^{-3})_z, (-xr^{-3})_z - (-zr^{-3})_x, (-yr^{-3})_x - (-xr^{-3})_y\right) \\
&= \left(3yzr^{-3} - 3yzr^{-3}, 3xzr^{-3} - 3xzr^{-3}, 3xyr^{-3} - 3xyr^{-3}\right) = (0,0,0).
\end{aligned}
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Stationary charged particle

Another solution to Maxwell's equations has $\mathbf{E} = (-xr^{-3}, -yr^{-3}, -zr^{-3})$ with all other fields ($\mathbf{B}$, $\mathbf{J}$ and ρ) being zero. It is clear that $\dot{\mathbf{E}} = 0$ and $\dot{\mathbf{B}} = 0$, so the only equations that we need to check are that $\nabla.\mathbf{E} = 0$ and $\nabla \times \mathbf{E} = 0$. For this we recall that $r_x = x/r$, so $(r^{-3})_x = -3r^{-4}r_x = -3xr^{-5}$. In the same way, we have $(r^{-3})_y = -3yr^{-5}$ and $(r^{-3})_z = -3zr^{-5}$. Using this we find that

$$\begin{aligned}
(-xr^{-3})_x &= 3x^2r^{-5} - r^{-3} & (-xr^{-3})_y &= 3xyr^{-5} & (-xr^{-3})_z &= 3xzr^{-5} \\
(-yr^{-3})_x &= 3xyr^{-5} & (-yr^{-3})_y &= 3y^2r^{-5} - r^{-3} & (-yr^{-3})_z &= 3yzr^{-5} \\
(-zr^{-3})_x &= 3xzr^{-5} & (-zr^{-3})_y &= 3yzr^{-5} & (-zr^{-3})_z &= 3z^2r^{-5} - r^{-3}.
\end{aligned}$$

$$\begin{aligned}
\nabla.\mathbf{E} &= (-xr^{-3})_x + (-yr^{-3})_y + (-zr^{-3})_z \\
&= 3x^2r^{-5} - r^{-3} + 3y^2r^{-5} - r^{-3} + 3z^2r^{-5} - r^{-3} \\
&= 3(x^2 + y^2 + z^2)r^{-5} - 3r^{-3} = 3r^2r^{-5} - 3r^{-3} = 0.
\end{aligned}$$

$$\begin{aligned}
\nabla \times \mathbf{E} &= \left((-zr^{-3})_y - (-yr^{-3})_z, (-xr^{-3})_z - (-zr^{-3})_x, (-yr^{-3})_x - (-xr^{-3})_y\right) \\
&= \left(3yzr^{-3} - 3yzr^{-3}, 3xzr^{-3} - 3xzr^{-3}, 3xyr^{-3} - 3xyr^{-3}\right) = (0,0,0).
\end{aligned}$$

This shows that we have a solution to the equations, as claimed.

Stationary charged particle

Another solution to Maxwell's equations has $\mathbf{E} = (-xr^{-3}, -yr^{-3}, -zr^{-3})$ with all other fields ($\mathbf{B}$, $\mathbf{J}$ and ρ) being zero. It is clear that $\dot{\mathbf{E}} = 0$ and $\dot{\mathbf{B}} = 0$, so the only equations that we need to check are that $\nabla.\mathbf{E} = 0$ and $\nabla \times \mathbf{E} = 0$. For this we recall that $r_x = x/r$, so $(r^{-3})_x = -3r^{-4}r_x = -3xr^{-5}$. In the same way, we have $(r^{-3})_y = -3yr^{-5}$ and $(r^{-3})_z = -3zr^{-5}$. Using this we find that

$$
\begin{array}{lll}
(-xr^{-3})_x = 3x^2r^{-5} - r^{-3} & (-xr^{-3})_y = 3xyr^{-5} & (-xr^{-3})_z = 3xzr^{-5} \\
(-yr^{-3})_x = 3xyr^{-5} & (-yr^{-3})_y = 3y^2r^{-5} - r^{-3} & (-yr^{-3})_z = 3yzr^{-5} \\
(-zr^{-3})_x = 3xzr^{-5} & (-zr^{-3})_y = 3yzr^{-5} & (-zr^{-3})_z = 3z^2r^{-5} - r^{-3}.
\end{array}
$$

$$
\begin{aligned}
\nabla.\mathbf{E} &= (-xr^{-3})_x + (-yr^{-3})_y + (-zr^{-3})_z \\
&= 3x^2r^{-5} - r^{-3} + 3y^2r^{-5} - r^{-3} + 3z^2r^{-5} - r^{-3} \\
&= 3(x^2 + y^2 + z^2)r^{-5} - 3r^{-3} = 3r^2r^{-5} - 3r^{-3} = 0.
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times \mathbf{E} &= \left((-zr^{-3})_y - (-yr^{-3})_z, (-xr^{-3})_z - (-zr^{-3})_x, (-yr^{-3})_x - (-xr^{-3})_y\right) \\
&= \left(3yzr^{-3} - 3yzr^{-3}, 3xzr^{-3} - 3xzr^{-3}, 3xyr^{-3} - 3xyr^{-3}\right) = (0,0,0).
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This shows that we have a solution to the equations, as claimed. This one represents the electric field of a single stationary particle at the origin, with no magnetic field.

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It is straightforward but somewhat lengthy to check this; we will not give the details.

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$$\operatorname{curl}(\mathbf{u}) = \frac{\partial}{\partial x}(x^2 - y^2 - 2xy) - \frac{\partial}{\partial y}(x^2 - y^2 + 2xy)$$

Incompressible and irrotational fields

We will say that a vector field $\mathbf{u}$ is *incompressible* (or *solenoidal*) if $\operatorname{div}(\mathbf{u}) = 0$, and that it is *irrotational* (or *conservative*) if $\operatorname{curl}(\mathbf{u}) = 0$.

Example

(a) For any scalar field f (in two or three dimensions) we have a vector field $\nabla(f) = \operatorname{grad}(f)$. The rule $\operatorname{curl}(\operatorname{grad}(f)) = 0$ tells us that $\operatorname{grad}(f)$ is irrotational.

(b) For any vector field $\mathbf{v}$ in three dimensions we have another vector field $\operatorname{curl}(\mathbf{v})$. The rule $\operatorname{div}(\operatorname{curl}(\mathbf{v})) = \nabla.(\nabla \times \mathbf{v}) = 0$ tells us that $\operatorname{curl}(\mathbf{v})$ is incompressible.

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so it is both incompressible and irrotational.

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Another potential example

Consider again the two-dimensional vector field

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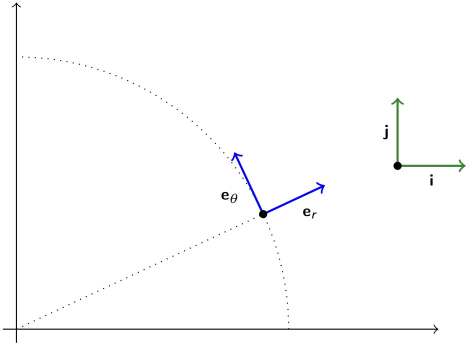
Two dimensions

At any point in the plane, we can define vectors $\mathbf{r}_r$ and $\mathbf{e}_\theta$ as shown:



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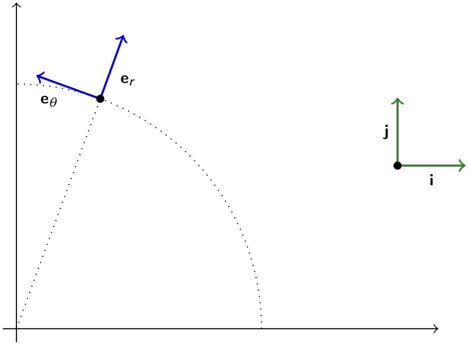

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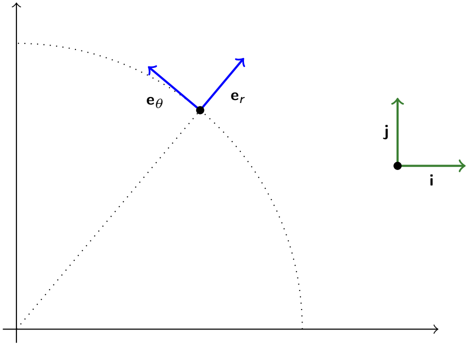

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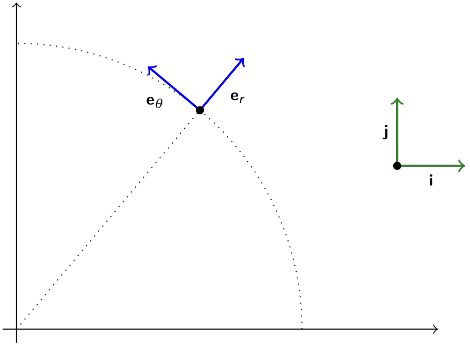

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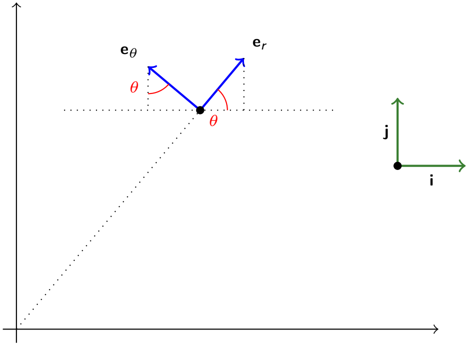

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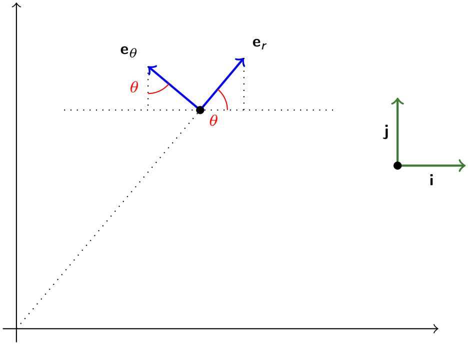

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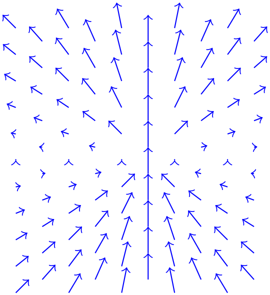
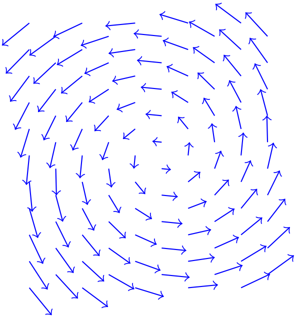
$$\mathbf{i} = \cos(\theta)\mathbf{e}_r - \sin(\theta)\mathbf{e}_\theta \qquad \mathbf{j} = \sin(\theta)\mathbf{e}_r + \cos(\theta)\mathbf{e}_\theta.$$

Examples

Here are two examples of vector fields described in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$:



$\mathbf{u} = \sin(\theta)\mathbf{e}_r$



$\mathbf{u} = \sqrt{r}(\mathbf{e}_\theta + \mathbf{e}_r/10)$

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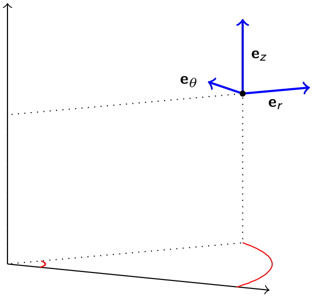
Cylindrical polar coordinates

In cylindrical polar coordinates we use unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ as shown below:



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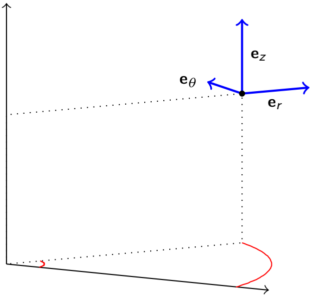

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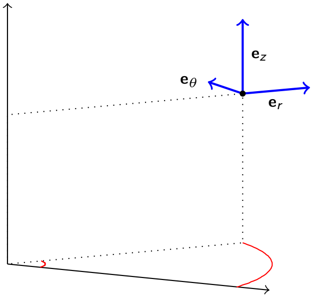

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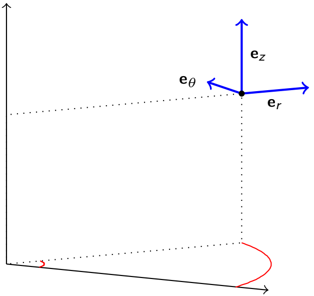

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$$\begin{aligned}
&\operatorname{curl}(\mathbf{u}) \\
&= \frac{1}{r} \det \begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & r^2 & r \end{bmatrix} \\
&= \frac{1}{r} \left(\left(\frac{\partial}{\partial \theta}(r) - \frac{\partial}{\partial z}(r^2) \right) \mathbf{e}_r - \left(\frac{\partial}{\partial r}(r) - \frac{\partial}{\partial z}(0) \right) r\mathbf{e}_\theta + \left(\frac{\partial}{\partial r}(r^2) - \frac{\partial}{\partial \theta}(0) \right) \mathbf{e}_z \right)
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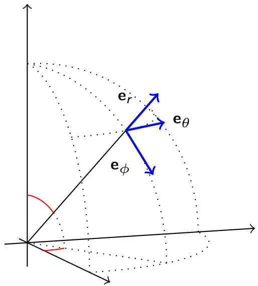
&= \frac{1}{r}\left(\left(\frac{\partial}{\partial \theta}(r) - \frac{\partial}{\partial z}(r^2)\right)\mathbf{e}_r - \left(\frac{\partial}{\partial r}(r) - \frac{\partial}{\partial z}(0)\right) r\mathbf{e}_\theta + \left(\frac{\partial}{\partial r}(r^2) - \frac{\partial}{\partial \theta}(0)\right)\mathbf{e}_z\right) \\
&= \frac{1}{r}\left(-r\mathbf{e}_\theta + 2r\mathbf{e}_z\right) = 2\mathbf{e}_z - \mathbf{e}_\theta .
\end{aligned}$$

Spherical polar coordinates

In spherical polar coordinates we use unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ as on the right:



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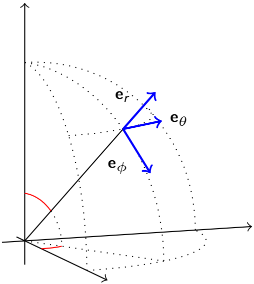

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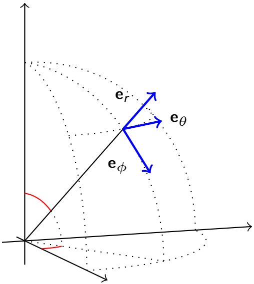

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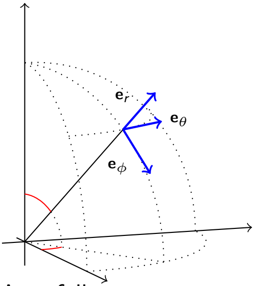

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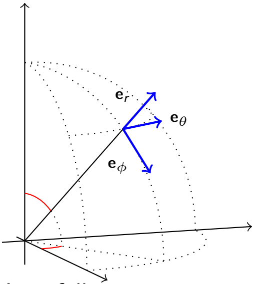

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\mathbf{e}_r &= \sin(\phi)\cos(\theta)\mathbf{i} + \sin(\phi)\sin(\theta)\mathbf{j} + \cos(\phi)\mathbf{k} \\
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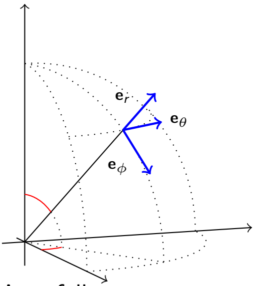

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\mathbf{i} &= \sin(\phi)\cos(\theta)\mathbf{e}_r + \cos(\phi)\cos(\theta)\mathbf{e}_\phi - \sin(\theta)\mathbf{e}_\theta \\
\mathbf{j} &= \sin(\phi)\sin(\theta)\mathbf{e}_r + \cos(\phi)\sin(\theta)\mathbf{e}_\phi + \cos(\theta)\mathbf{e}_\theta \\
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Example of div, grad and curl in spherical polar coordinates

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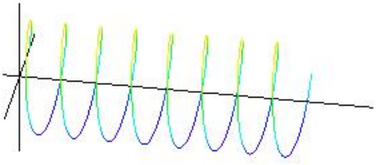
Helix

The equation

$$\mathbf{r} = (x, y, z) = (at,\ b\cos(t),\ b\sin(t))$$

describes a helix winding around the x-axis.



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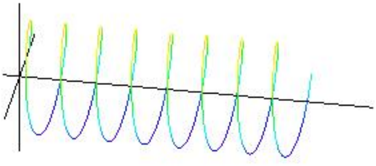

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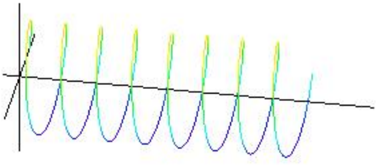

This is the path followed by an electron moving in a uniform magnetic field. It could also describe a wire wound round a cylinder.

Suppose that a car with axles of length a and wheels of radius b drives at constant speed c along the x-axis.

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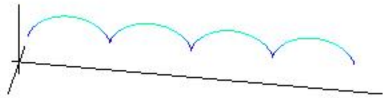

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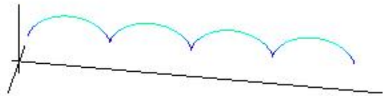

The first term $(ct, a/2, b)$ reflects the overall motion of the car, and the second term comes from the rotation of the wheel.

Projectile

A thrown ball will follow a parabolic path like

$$\mathbf{r} = (at, bt, ct - dt^2)$$

for some constants $a, \ldots, d$.

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Example of integration along a curve

Let C be the curve given by

$$\mathbf{r} = (x, y, z) = (6t, 3\sqrt{2}t^2, 2t^3)$$

for $0 \leq t \leq 1$.

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Example calculation using a potential

If C goes from $\mathbf{a}$ to $\mathbf{b}$ and $\mathbf{F} = \nabla(p)$ then

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The reason why this method works is that both $\int_C \mathbf{F}.d\mathbf{r}$ and $\int_{C'} \mathbf{F}.d\mathbf{r}$ are equal to $p(\mathbf{b}) - p(\mathbf{a})$, where p is the potential function.

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The reason why this method works is that both $\int_C \mathbf{F}.d\mathbf{r}$ and $\int_{C'} \mathbf{F}.d\mathbf{r}$ are equal to $p(\mathbf{b}) - p(\mathbf{a})$, where p is the potential function. For this to be valid, we need to know that p exists (so we must check that $\mathbf{F}$ is conservative) but we do not actually need to find p.

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As $\mathbf{F}$ is conservative, we can replace C by a simpler path without changing the integral.

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A non-conservative example

L : straight line $\mathbf{r} = (t, 1, 0)$ from $\mathbf{a} = (0, 1, 0)$ at $t = 0$ to $\mathbf{b} = (1, 1, 0)$ at $t = 1$

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As $\mathbf{G}$ is not conservative, the integrals $\int_C \mathbf{G}.d\mathbf{r}$ and $\int_L \mathbf{G}.d\mathbf{r}$ need not be the same. On C we have

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Thus, the integrals over C and L are different, as expected.

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Note again that this is only valid for conservative fields. Fields that are not conservative do not have a potential function.

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Example of finding a potential

We checked previously that the vector field $\mathbf{F} = (yz, xz, xy)$ has $\text{curl}(\mathbf{F}) = 0$, so $\mathbf{F}$ is conservative. It follows that there is a potential function p with $\text{grad}(p) = \mathbf{F}$. To find $p(a, b, c)$, we evaluate $\int_L \mathbf{F}.d\mathbf{r}$, where L is the straight line from $(0, 0, 0)$ to (a, b, c). This can be parametrised by $\mathbf{r} = (x, y, z) = (ta, tb, tc)$ for $0 \leq t \leq 1$, which gives

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It is convenient to write this calculation in terms of a, b and c, to avoid confusion between the end of the path (where $(x, y, z) = (a, b, c)$) and the points along the path (where $(x, y, z) = (ta, tb, tc)$).

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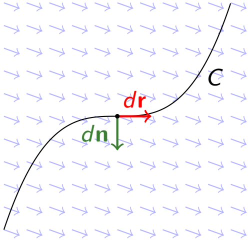
It is convenient to write this calculation in terms of a, b and c, to avoid confusion between the end of the path (where $(x, y, z) = (a, b, c)$) and the points along the path (where $(x, y, z) = (ta, tb, tc)$). However, we can restate the final answer as $p(x, y, z) = xyz$, which is more convenient for later use.

Flux across a curve

The picture shows a vector field $\mathbf{F}$ and a curve C, with the vector $d\mathbf{r}$ pointing along the curve, and another vector $d\mathbf{n}$ of the same length perpendicular to $d\mathbf{r}$.



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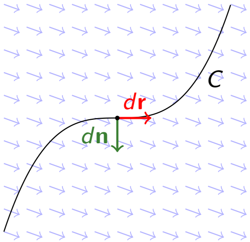

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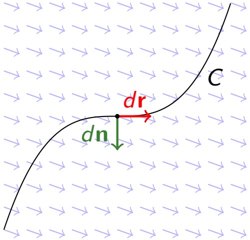

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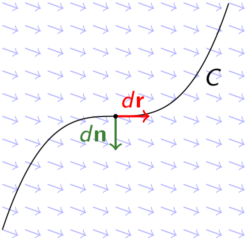

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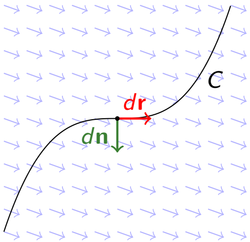

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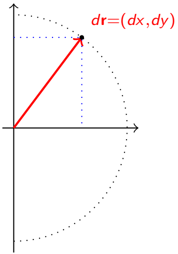

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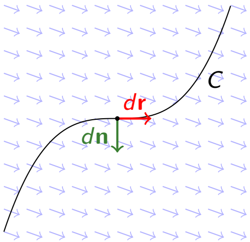

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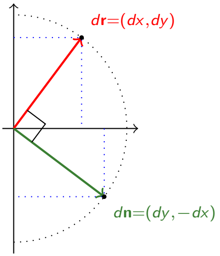

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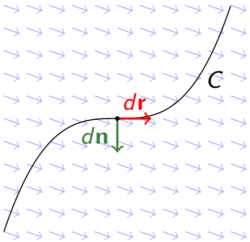

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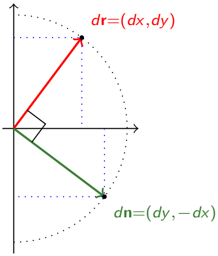

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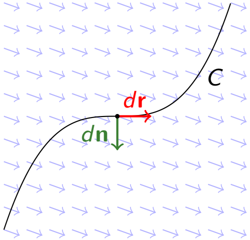

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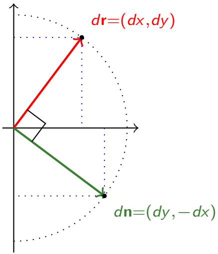

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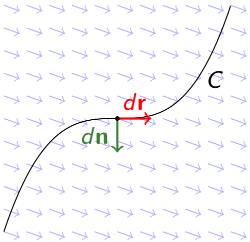

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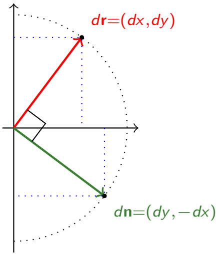

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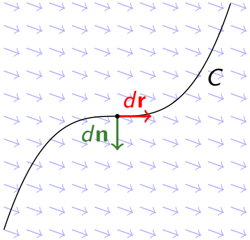

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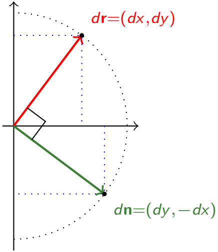
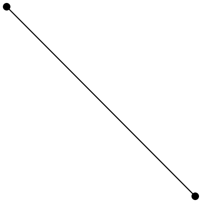


Example of flux across a curve



Let L be the straight line from $(1, 0)$ to $(0, 1)$

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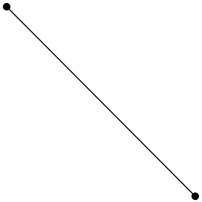

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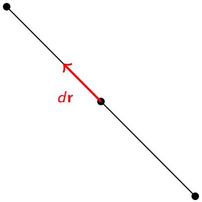

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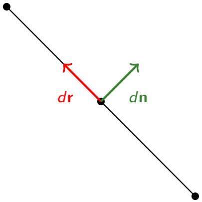

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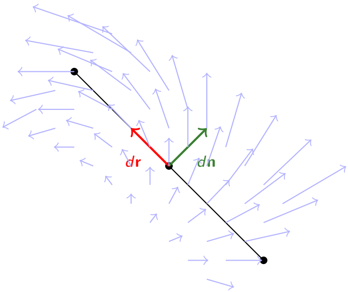

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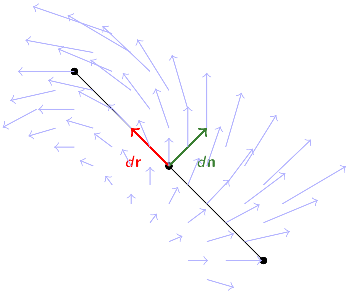

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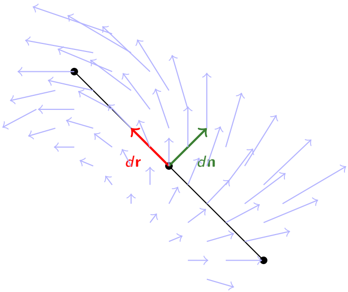

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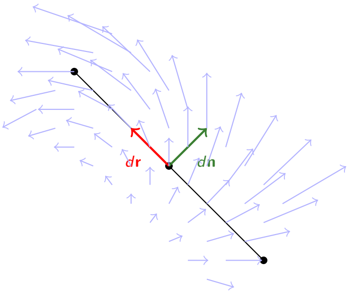

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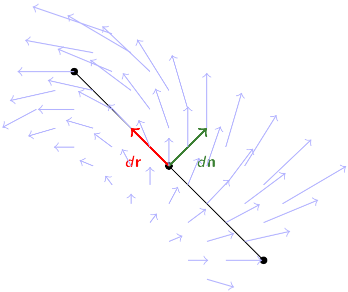

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so $\mathbf{F}.d\mathbf{n} = ((1-2t) + (2t - 2t^2))dt$

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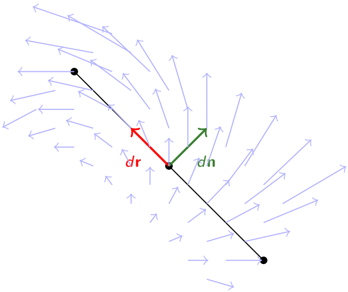

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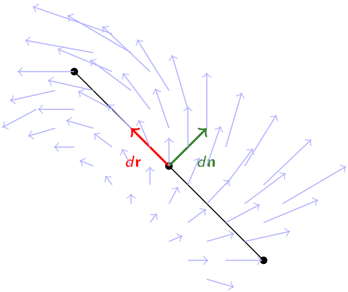

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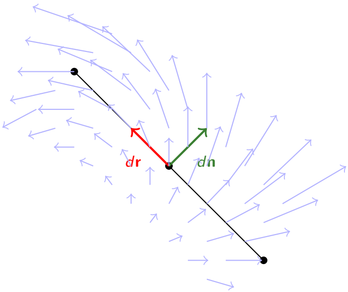

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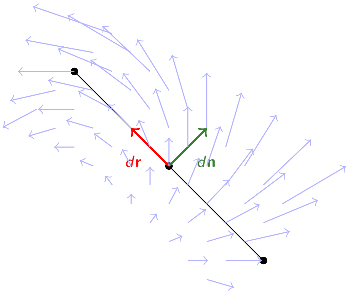
so $\mathbf{F}.d\mathbf{n} = ((1-2t)+(2t-2t^2))dt = (1-2t^2)dt$ so

$$\int_C \mathbf{F}.d\mathbf{n} = \int_{t=0}^{1} (1-2t^2)\,dt = \left[t - \tfrac{2}{3}t^3\right]_{t=0}^{1}$$

Example of flux across a curve



Let L be the straight line from $(1,0)$ to $(0,1)$, so $\mathbf{r} = (x,y) = (1-t,t)$ for $0 \leq t \leq 1$, so $d\mathbf{r} = (-1,1)dt$, so $d\mathbf{n} = (1,1)dt$. Let $\mathbf{F}$ be the vector field $(x^2 - y^2, 2xy)$, so on L we have

$$\mathbf{F} = ((1-t)^2 - t^2, 2t(1-t)) = (1 - 2t + t^2 - t^2, 2t - 2t^2) = (1-2t, 2t-2t^2),$$

so $\mathbf{F}.d\mathbf{n} = ((1-2t) + (2t-2t^2))dt = (1-2t^2)dt$ so

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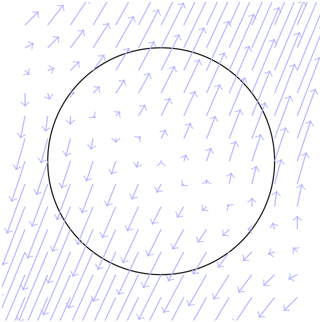
Flow out of a circle

We will calculate the flow of the field

$$\mathbf{F} = (x + 2y, 3x + 4y)$$

out of the unit circle C.



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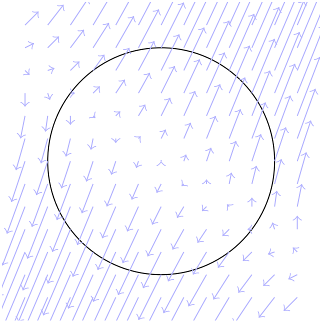

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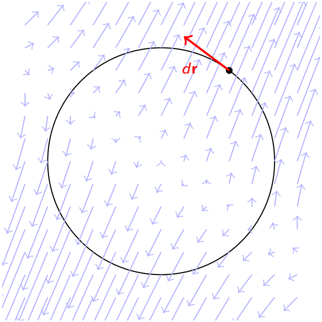

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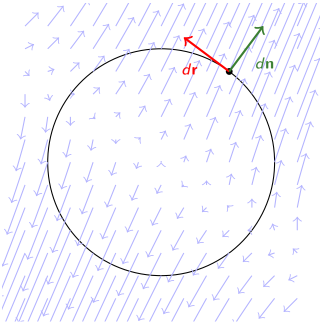

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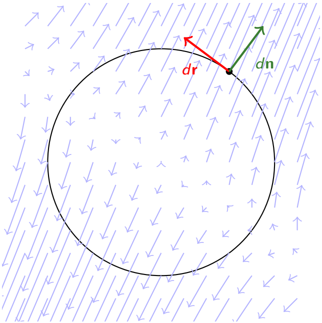

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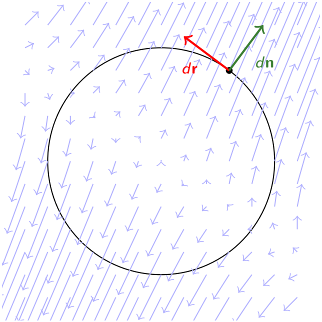

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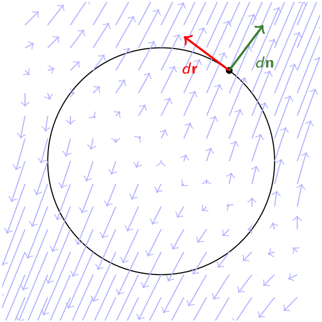

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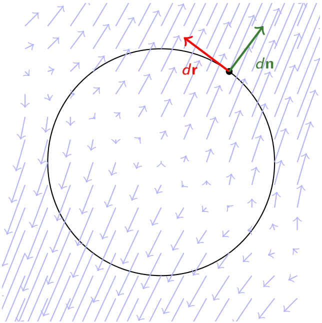

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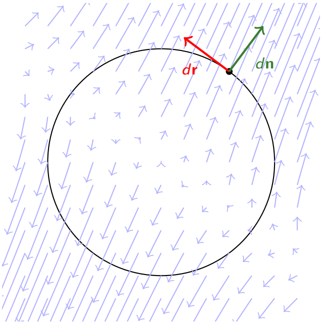

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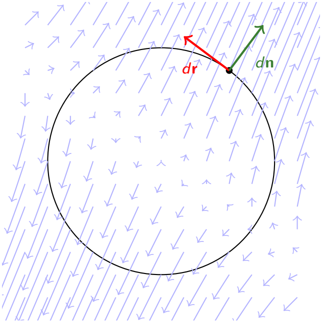

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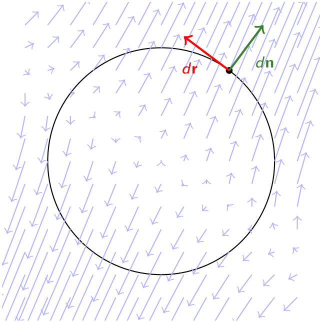

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Surfaces

As well as considering curved paths, we also need to consider curved surfaces in three-dimensional space.

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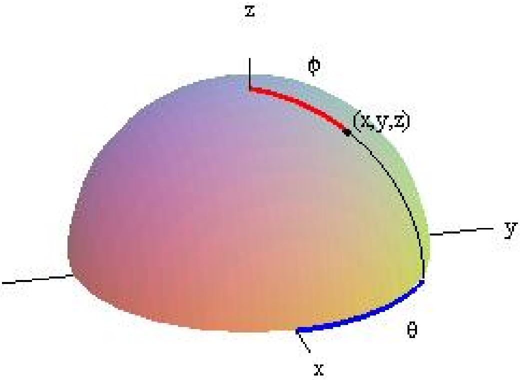
A hemisphere

The upper half of a spherical shell of radius 2 can be described in terms of parameters ϕ and θ by

$$(x, y, z) = (2\sin(\phi)\cos(\theta),\ 2\sin(\phi)\sin(\theta),\ 2\cos(\phi))$$

(for $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi/2$).

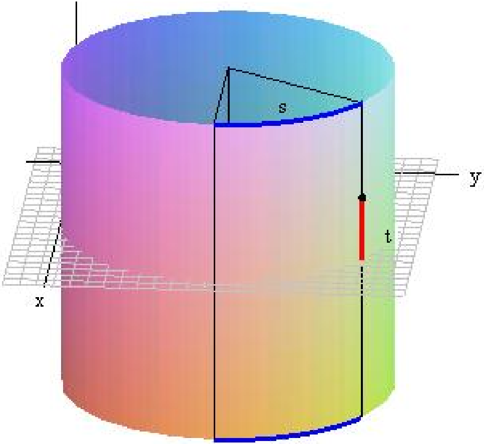


An off-centre cylinder

Let S be a cylindrical surface of radius 1, centred on the line joining $(1,1,-1)$ to $(1,1,1)$. Then S can be described in terms of parameters s and t by

$$(x, y, z) = (1 + \cos(s), 1 + \sin(s), t)$$

(for $0 \le s \le 2\pi$ and $-1 \le t \le 1$).



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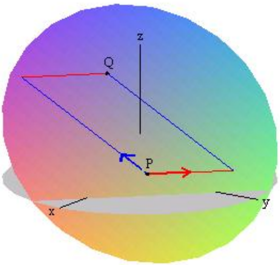

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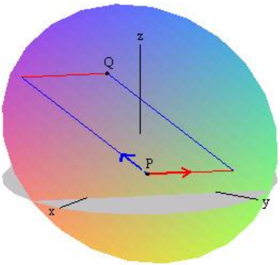

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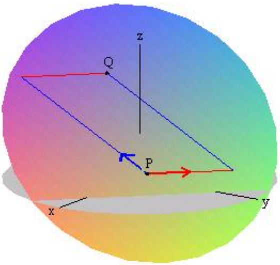
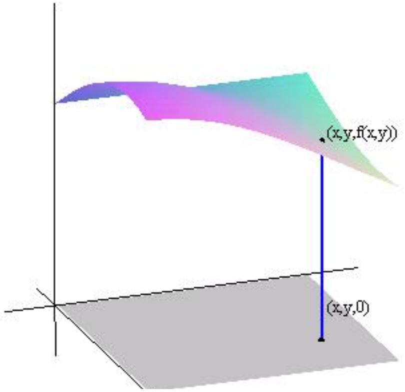


The picture shows the point $P=(1,1,1)$, which lies on S. Any other point on S (such as Q) can be reached from P by adding a multiple of the red vector $(-1,1,0)$ and a multiple of the blue vector $(0,-1,1)$.

The graph of $f(x, y)$

For any function $f(x, y)$, the equation $z = f(x, y)$ defines a surface.



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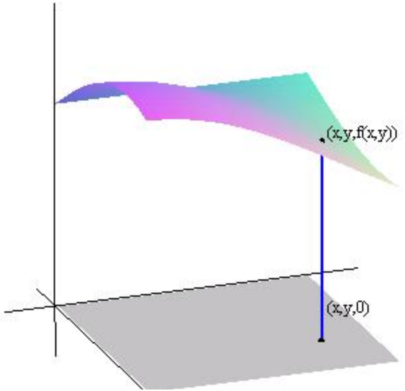

We can use the variables x and y themselves as parameters, and then the full parametrisation is

$$(x, y, z) = (x, y, f(x, y)).$$

Integration over surfaces

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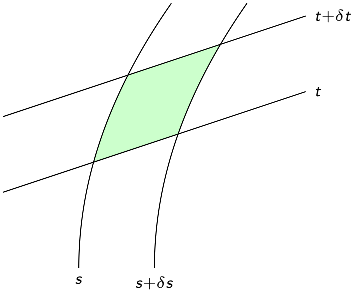

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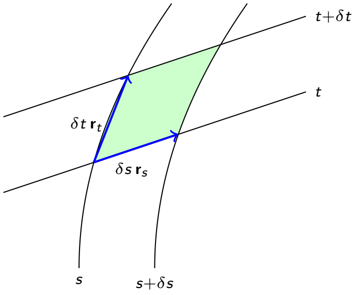

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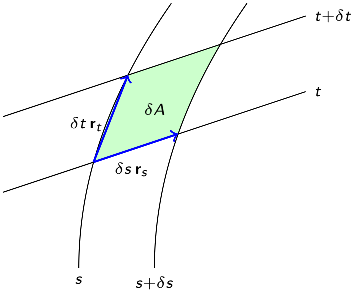

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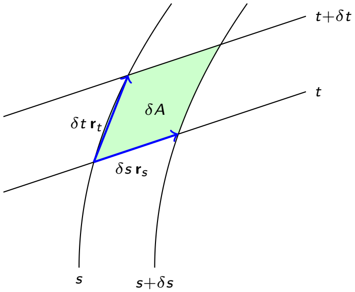

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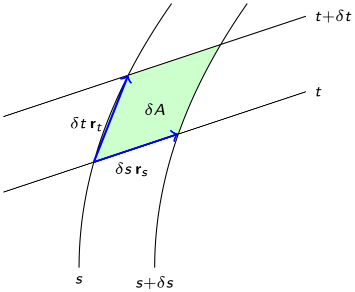

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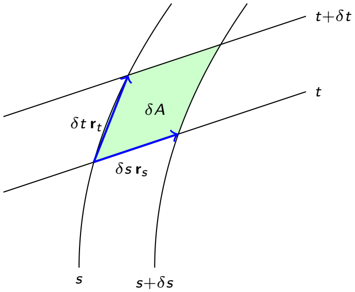

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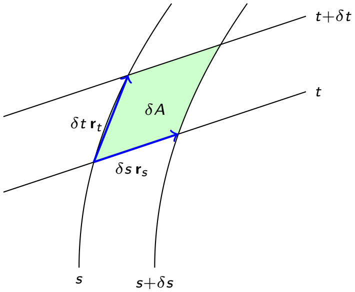

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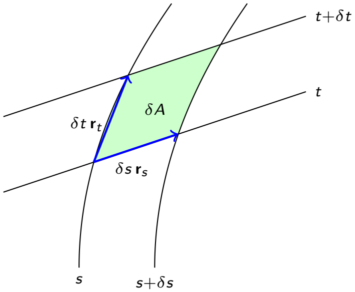

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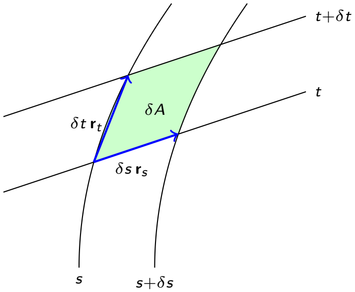

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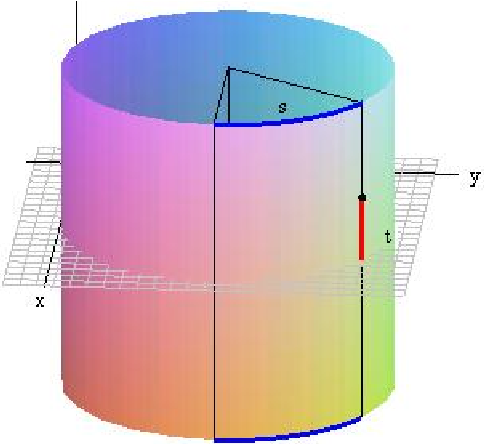

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The area of the surface $z = \cosh(x+y)/\sqrt{2}$

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$$= \cosh(2) - 2\cosh(1) + 1$$

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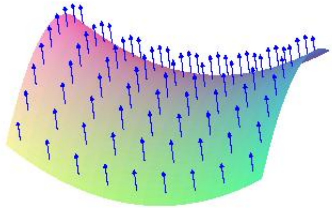
&= \cosh(2) - 2\cosh(1) + 1 \simeq 1.676
\end{aligned}$$

Consider a surface S in a region where there is a vector field $\mathbf{F}$.



We want to calculate the flow of $\mathbf{F}$ across S.

Flow across a surface

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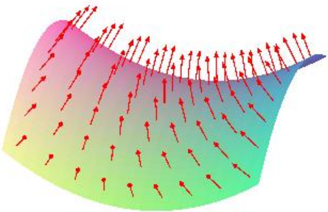

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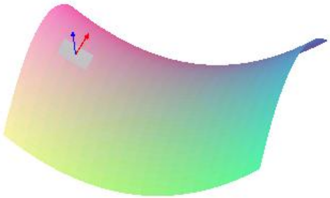

We want to calculate the flow of $\mathbf{F}$ across S. At each point on S there is a normal vector $\mathbf{n}$. The flow across S only involves the component of $\mathbf{F}$ in the normal direction, or in other words $\mathbf{F}.\mathbf{n}$.

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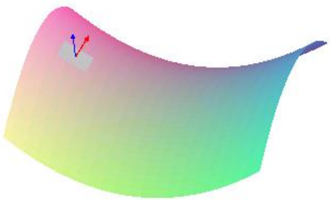

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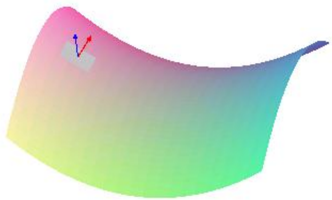

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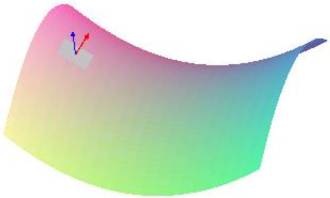

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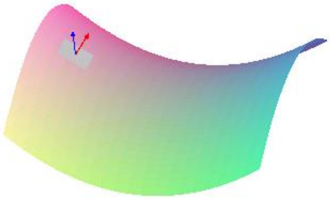

We want to calculate the flow of $\mathbf{F}$ across S. At each point on S there is a normal vector $\mathbf{n}$. The flow across S only involves the component of $\mathbf{F}$ in the normal direction, or in other words $\mathbf{F}.\mathbf{n}$. We need to integrate this with respect to area, giving $\iint_S \mathbf{F}.\mathbf{n}dA$. However, $\mathbf{n}dA$ is the same as the vector $d\mathbf{A}$ considered earlier, which can be calculated from a parametrisation by the rule $d\mathbf{A} = (\mathbf{r}_s \times \mathbf{r}_t)ds\,dt$. Note that if $\mathbf{F} = (P, Q, R)$ then the scalar triple product $\mathbf{F}.d\mathbf{A} = \mathbf{F}.(\mathbf{r}_s \times \mathbf{r}_t)ds\,dt$ can be expressed as a determinant:

$$\mathbf{F}.d\mathbf{A} = \det \begin{bmatrix} P & Q & R \\ x_s & y_s & z_s \\ x_t & y_t & z_t \end{bmatrix} ds\,dt.$$

Example

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Integral theorems — Introduction

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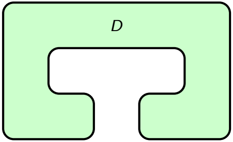
The sign convention for closed curves

Let D be a region in the plane.



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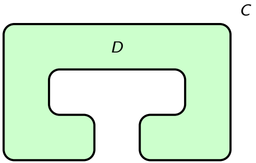

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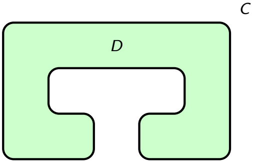

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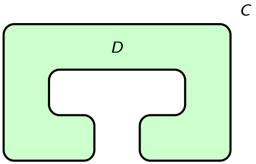

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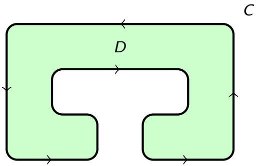

In this direction
we keep the region on the left

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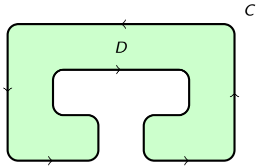

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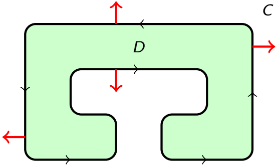

In this direction
we keep the region on the left
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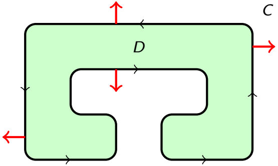

In this direction
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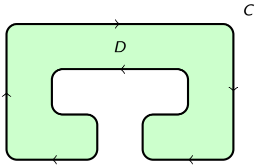

In the opposite direction
we keep the region on the right

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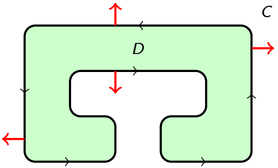

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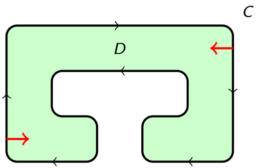
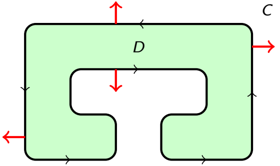


In the opposite direction
we keep the region on the right
so $d\mathbf{n}$ points inwards

The sign convention for closed curves

Let D be a region in the plane. The edge of the region will be a curve, which we call C. For any vector field $\mathbf{u}$, we can consider the integral $\int_C \mathbf{u}.d\mathbf{n}$ measuring the flux of $\mathbf{u}$ across C. This kind of integral depends on the direction in which we traverse the curve. We will always traverse in the direction which keeps the region D on our left. This means that we are basically going anticlockwise, although it may not always seem that way if C has a complicated shape.



In this direction
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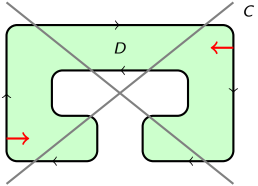

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The two-dimensional divergence theorem

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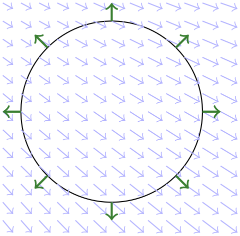

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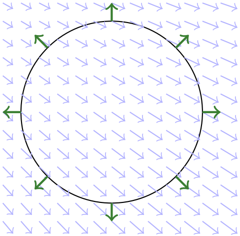

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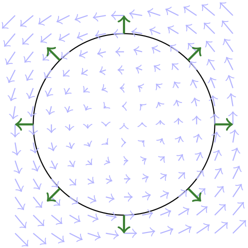

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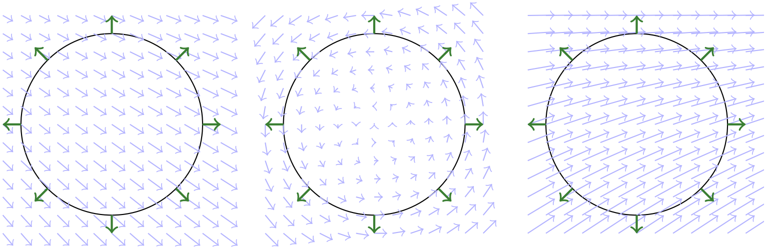

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$$\begin{aligned} \mathrm{div}(\mathbf{u}) &= p_x + q_y \\ \mathbf{u}.d\mathbf{n} &= (p, q).(dy, -dx) = p\,dy - q\,dx. \end{aligned}$$

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It will be enough to show that

$$\iint_D q_y\, dA = -\int_C q\, dx \qquad \text{(A)}$$
$$\iint_D p_x\, dA = \int_C p\, dy \qquad \text{(B)}$$

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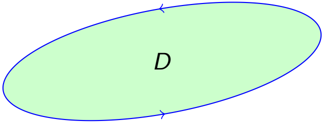
Claim (A): $$\iint_D q_y \, dA = -\int_C q \, dx \qquad (C = \text{ boundary of } D, \text{ anticlockwise })$$



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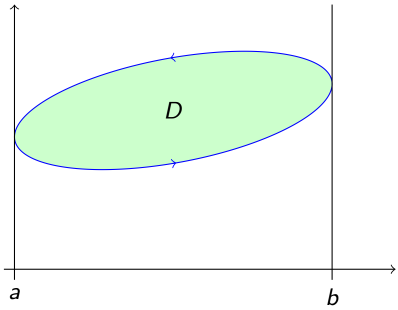

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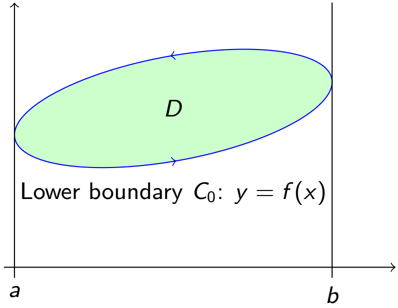

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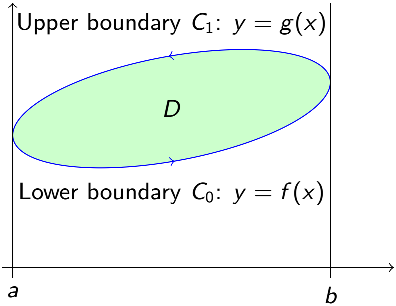

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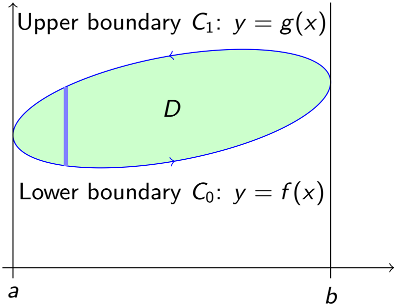

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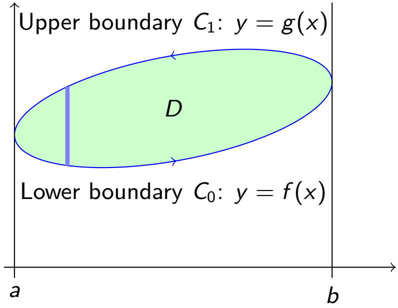

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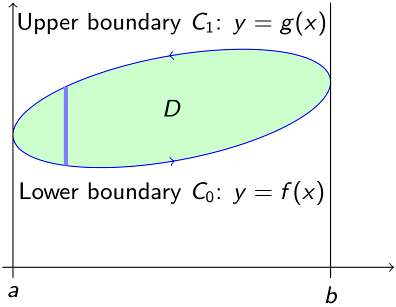

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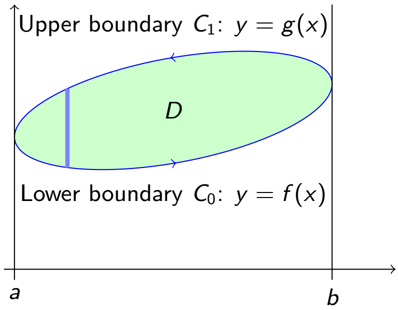

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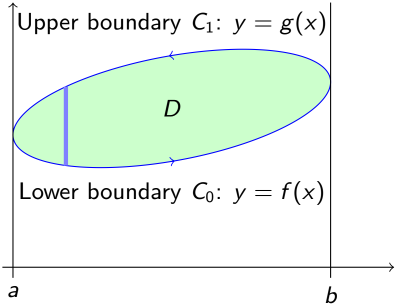

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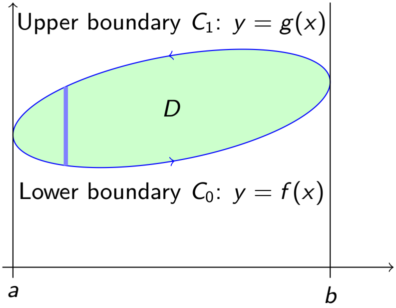
Add (B) and (C) and compare with (A):



$$\iint_D q_y\, dA \overset{(A)}{=} \int_{x=a}^{b} q(x,g(x))\, dx - \int_{x=a}^{b} q(x,f(x))\, dx$$

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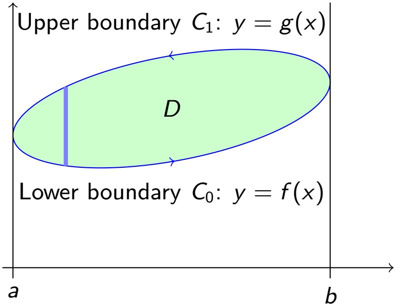

$$\iint_D q_y\, dA \overset{(A)}{=} \int_{x=a}^{b} q(x,g(x))\, dx - \int_{x=a}^{b} q(x,f(x))\, dx \overset{(B),(C)}{=} -\int_{C_1} q\, dx - \int_{C_0} q\, dx$$

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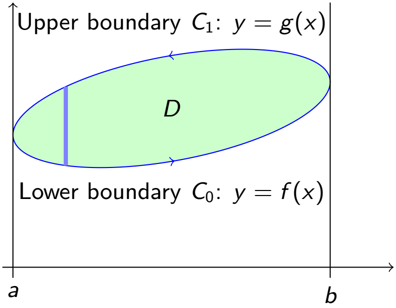

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$= -\int_C q \, dx$ as claimed.

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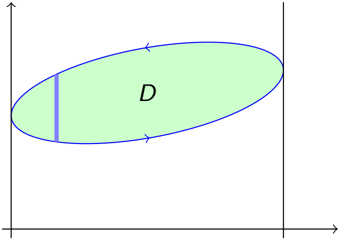
We just proved using vertical strips that $\iint_D q_y \, dA = -\int_C q \, dx$



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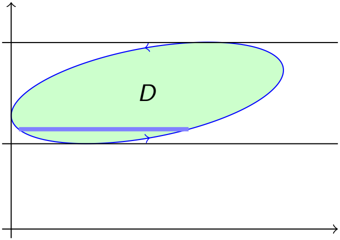

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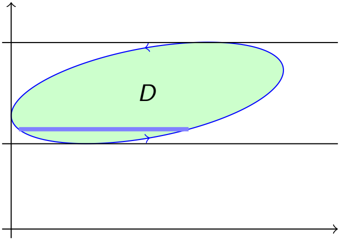

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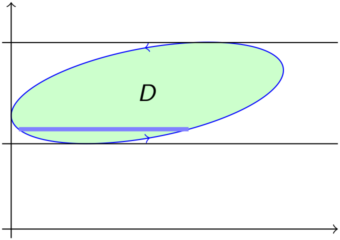

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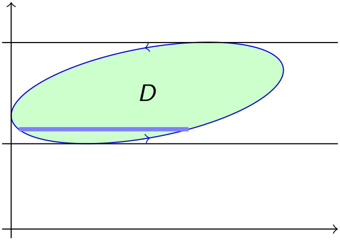

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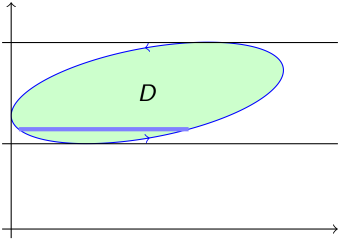

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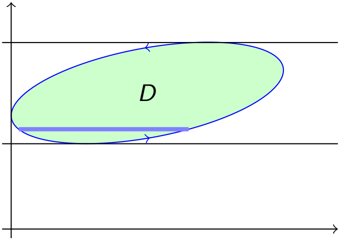

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which is the two-dimensional divergence theorem.

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Let D be the disc where $x^2 + y^2 \leq m^2$, so C is a circle of radius m.

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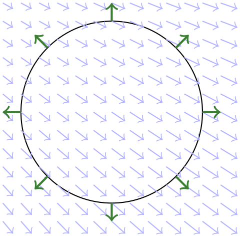

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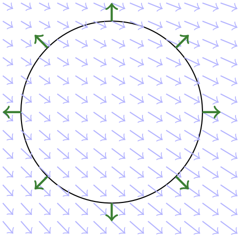

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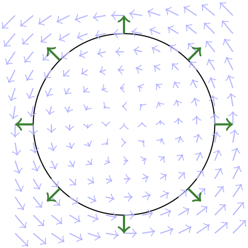

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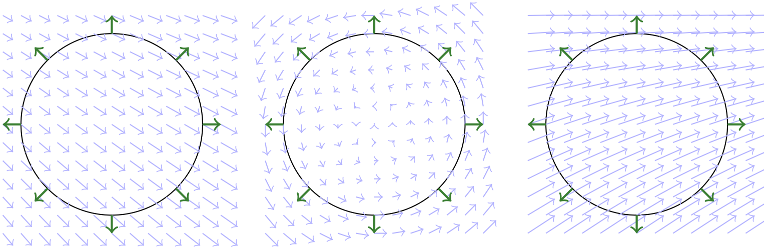

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Example

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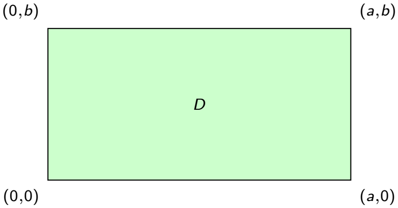

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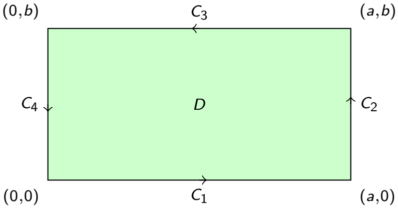

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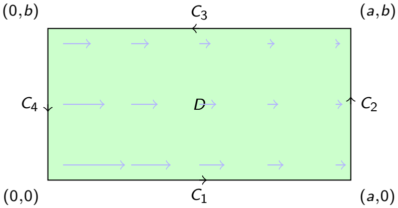

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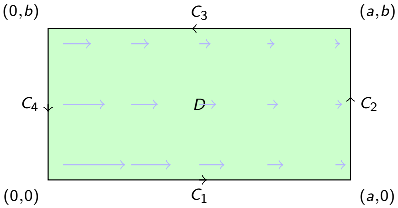

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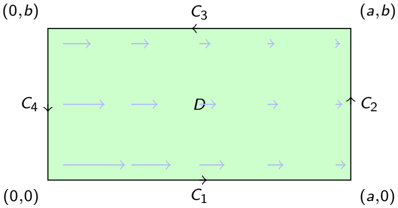

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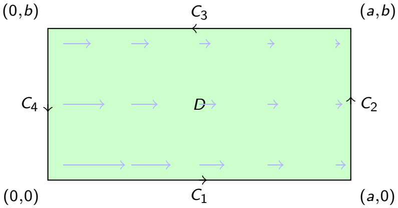

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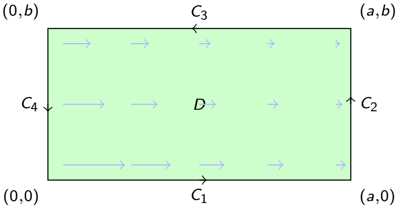

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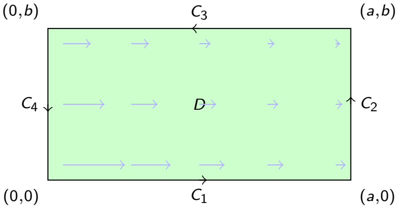

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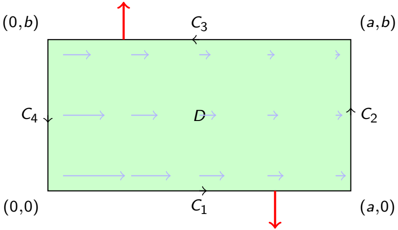

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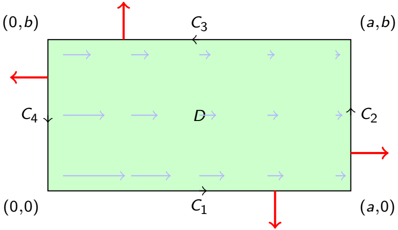

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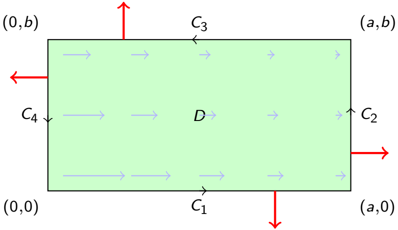

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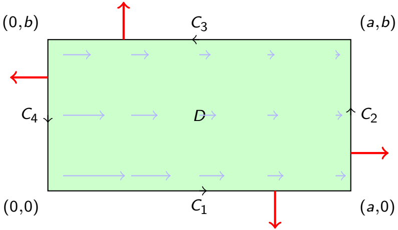

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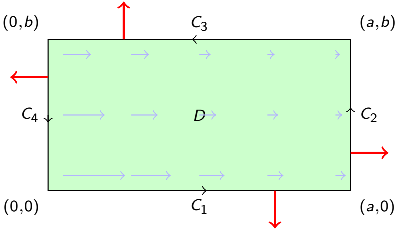

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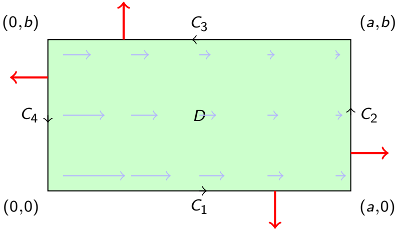

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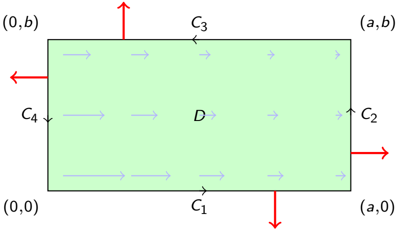

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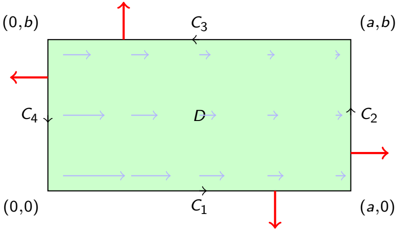

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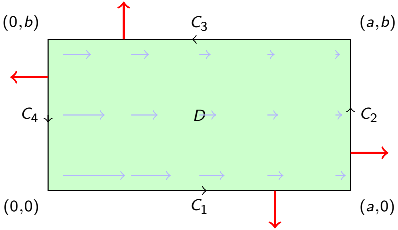

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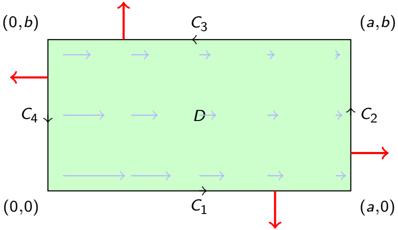

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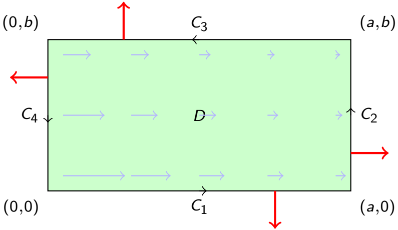

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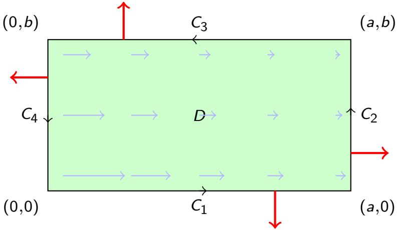

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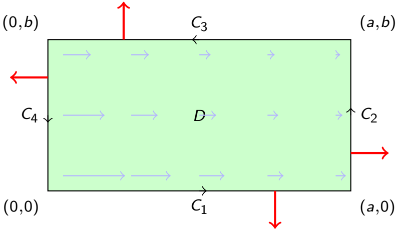

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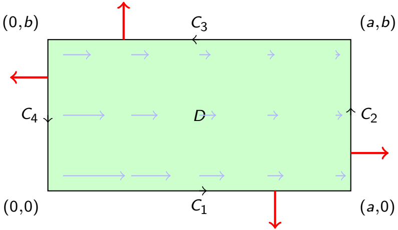

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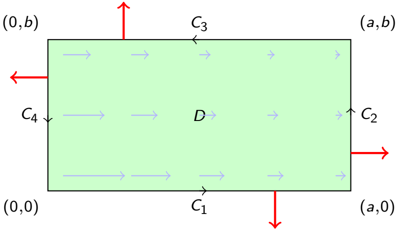

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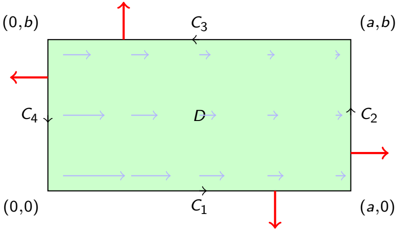

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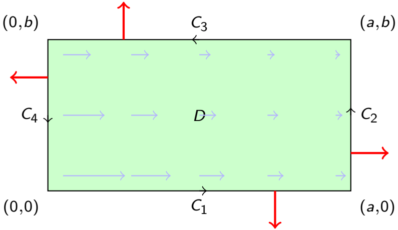

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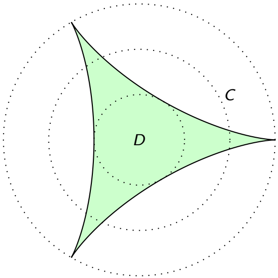
Area of a deltoid

The picture shows the deltoid curve C:

$x = 2\cos(t) + \cos(2t) \qquad y = 2\sin(t) - \sin(2t).$



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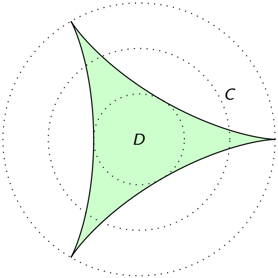

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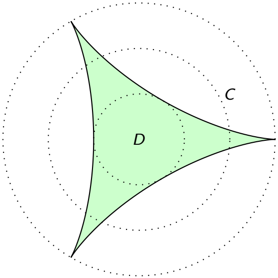

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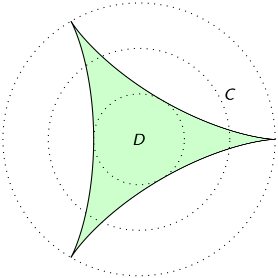

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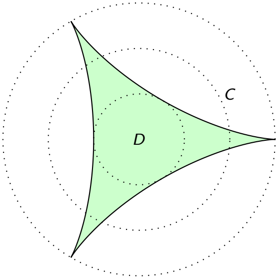

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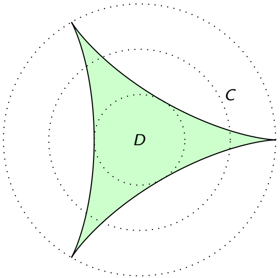

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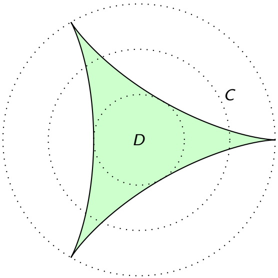

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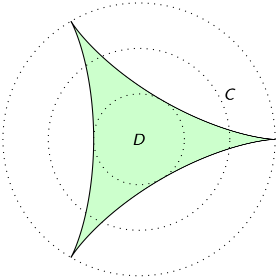

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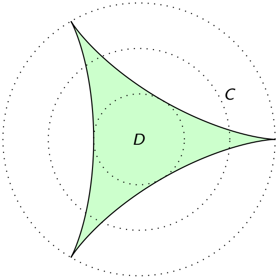

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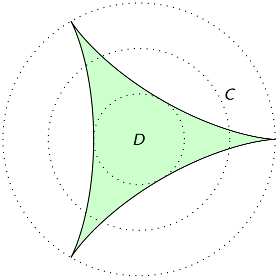

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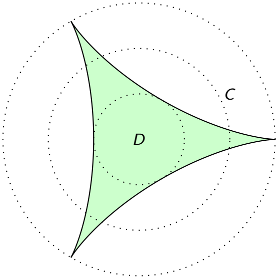

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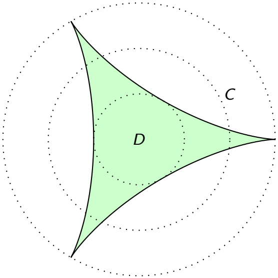

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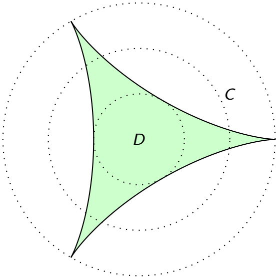

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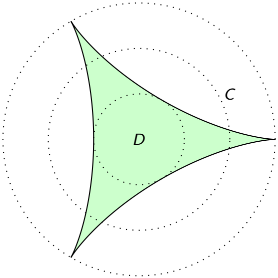

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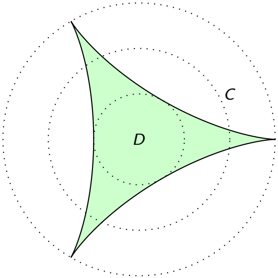

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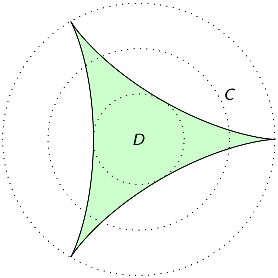

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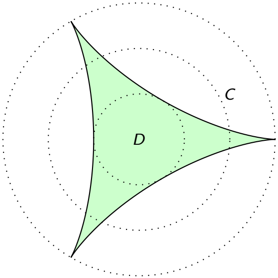

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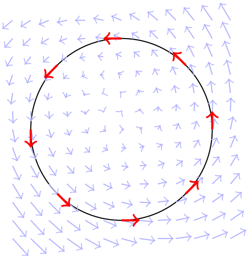

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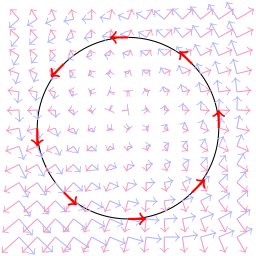

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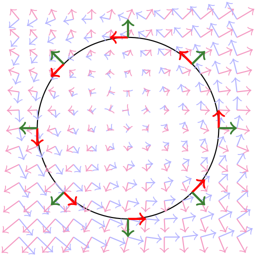

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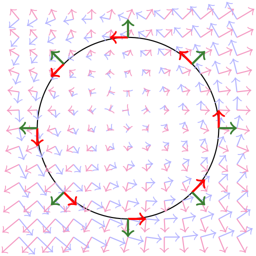

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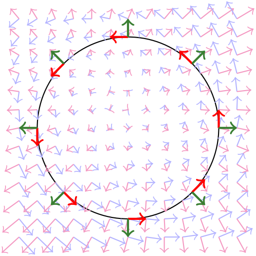

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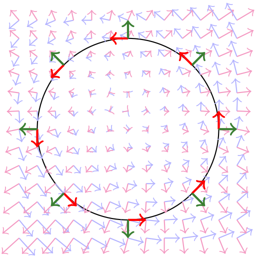

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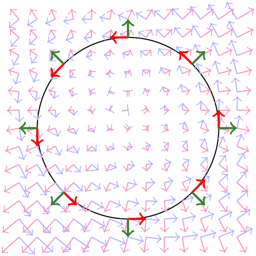

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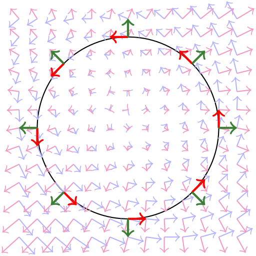

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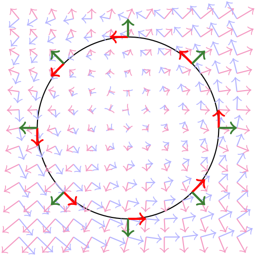

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Let D be the unit disc, so the boundary curve C is the unit circle. Let $\mathbf{u}$ be the vector field (x^3, x^3). Green's Theorem tells us that $\iint_D \text{curl}(\mathbf{u})\, dA = \int_C \mathbf{u}.d\mathbf{r}$. We will check this by evaluating both sides. First, we have

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As expected, this is the same as $\iint_D \text{curl}(\mathbf{u})\, dA$.

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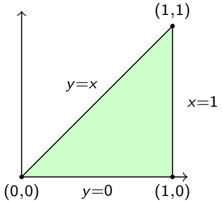
D : triangle, vertices $(0,0)$, $(1,0)$ and $(1,1)$.



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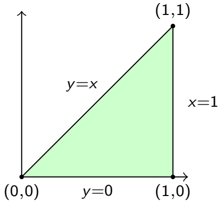

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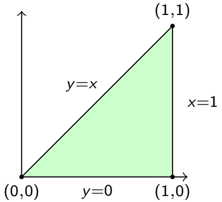

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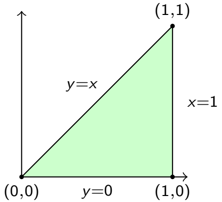

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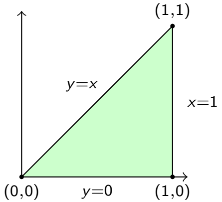

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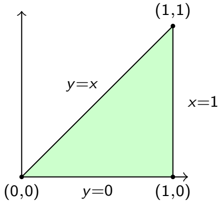

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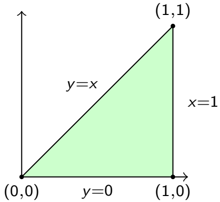

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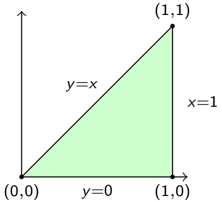

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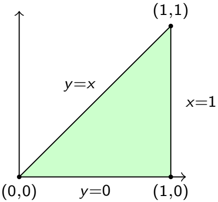

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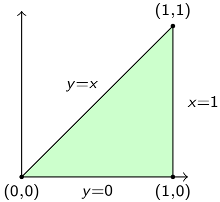

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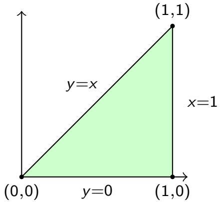

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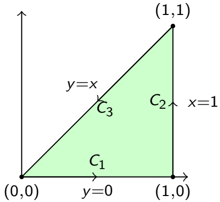

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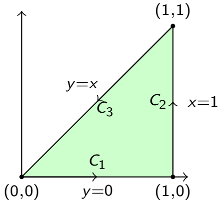

On C_1: $y = 0$

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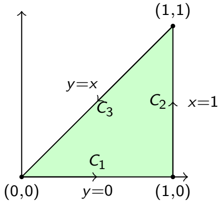

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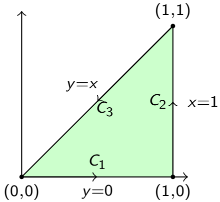

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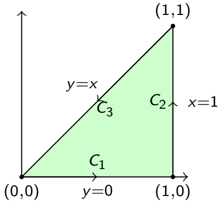

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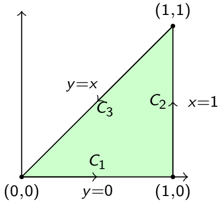

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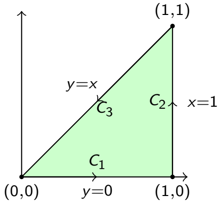

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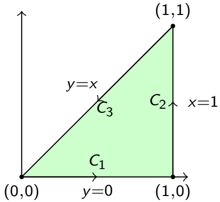

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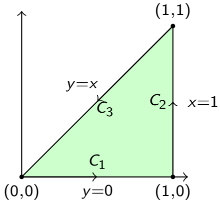

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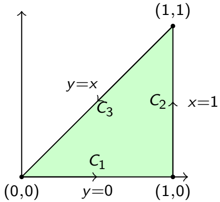

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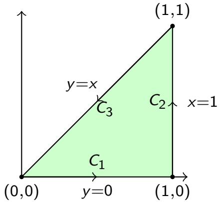

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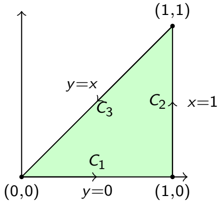

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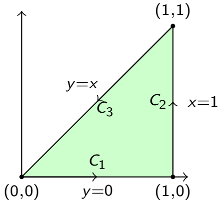

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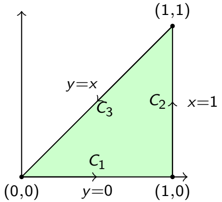

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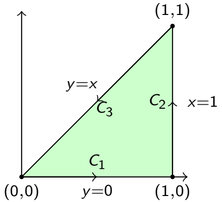

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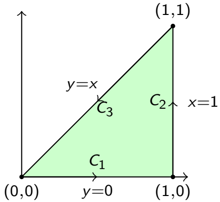

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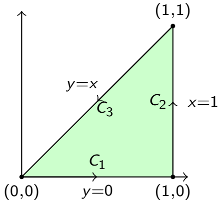

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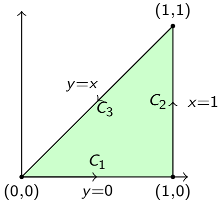

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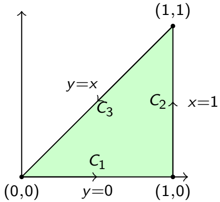

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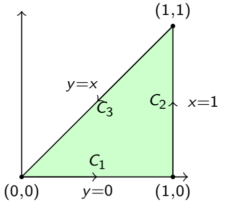

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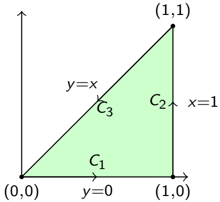

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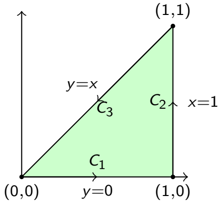

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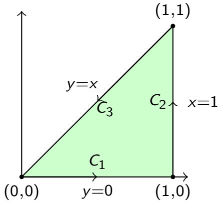

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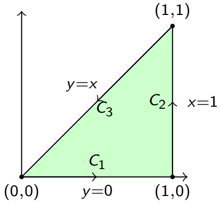

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As expected, this is the same as $\iint_D \text{curl}(\mathbf{u})\, dA$.

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$$\mathbf{u} = (x, 0, 0) = (\sin(\phi)\cos(\theta),\ 0,\ 0).$$

The unit normal vector is $\mathbf{n} = \mathbf{e}_r = \mathbf{r}$, so $\mathbf{u}.\mathbf{n} = x^2 = \sin^2(\phi)\cos^2(\theta)$.
We have also seen before that $dA = \sin(\phi)\, d\phi\, d\theta$, so

$$\iint_S \mathbf{u}.d\mathbf{A} = \iint_S \mathbf{u}.\mathbf{n}\, dA = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \sin^3(\phi)\cos^2(\theta)\, d\phi\, d\theta$$
$$= \left(\int_{\theta=0}^{2\pi} \cos^2(\theta)\, d\theta \right) \left(\int_{\phi=0}^{\pi} \sin^3(\phi)\, d\phi \right) = \pi \int_{\phi=0}^{\pi} \frac{1}{4}(3\sin(\phi) - \sin(3\phi)) d\phi$$

Divergence Theorem Example 1

Let S be the unit sphere, and let E be the solid ball enclosed by S. Consider the vector field $\mathbf{u} = (x, 0, 0)$. This has $\text{div}(\mathbf{u}) = \partial x/\partial x + \partial 0/\partial y + \partial 0/\partial z = 1$, so

$$\iiint_E \text{div}(\mathbf{u}) dV = \iiint_E dV = \text{ volume of } E = 4\pi/3.$$

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As expected, this is the same as $\iiint_E \text{div}(\mathbf{u}) dV$.

Divergence Theorem Example 2

Let E be the solid vertical cylinder of radius a and height $2b$ centred at the origin, and let S be the surface of E.

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$$\operatorname{div}(\mathbf{u}) = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z^3) = 0 + 0 + 3z^2 = 3z^2.$$

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The region E can be described in cylindrical polar coordinates by $0 \leq r \leq a$ and $-b \leq z \leq b$ (with $0 \leq \theta \leq 2\pi$ as usual).

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The region E can be described in cylindrical polar coordinates by $0 \leq r \leq a$ and $-b \leq z \leq b$ (with $0 \leq \theta \leq 2\pi$ as usual). Moreover, the volume element in those coordinates is $dV = r\,dr\,d\theta\,dz$.

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$$\iiint_E \operatorname{div}(\mathbf{u})\,dV = \int_{z=-b}^{b} \int_{\theta=0}^{2\pi} \int_{r=0}^{a} 3z^2 r\,dr\,d\theta\,dz$$

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$$= 2\pi \left(\int_{r=0}^{a} r\,dr \right) \left(\int_{z=-b}^{b} 3z^2\,dz \right)$$

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$$\begin{aligned}
\iiint_E \operatorname{div}(\mathbf{u})\,dV &= \int_{z=-b}^{b} \int_{\theta=0}^{2\pi} \int_{r=0}^{a} 3z^2 r\,dr\,d\theta\,dz \\
&= 2\pi \left(\int_{r=0}^{a} r\,dr \right) \left(\int_{z=-b}^{b} 3z^2\,dz \right) \\
&= 2\pi \left[\tfrac{1}{2} r^2 \right]_{r=0}^{a} \left[z^3 \right]_{z=-b}^{b}
\end{aligned}$$

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Divergence Theorem Example 2

$E: \qquad 0 \leq r \leq a, \qquad -b \leq z \leq b, \qquad 0 \leq \theta \leq 2\pi.$
$\mathbf{u} = (-y, x, z^3)$. $\iiint_E \text{div}(\mathbf{u}) dV = 2\pi a^2 b^3$.

Now consider instead $\iint_S \mathbf{u}.d\mathbf{A} = \iint_S \mathbf{u}.\mathbf{n}\, dA$.

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Now consider instead $\iint_S \mathbf{u}.d\mathbf{A} = \iint_S \mathbf{u}.\mathbf{n}\, dA$. Let S_1 be the bottom end of E, where $z = -b$. Let S_2 be the top end, where $z = b$.

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$$\iint_{S_1} \mathbf{u}.\mathbf{n}\, dA = \iint_{S_1} b^3\, dA = b^3 \times (\text{ area of } S_1)$$

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On S_2 we have $\mathbf{n} = (0, 0, 1)$ and $\mathbf{u} = (-y, x, b^3)$, and it follows easily that $\iint_{S_2} \mathbf{u}.\mathbf{n}\, dA$ is also equal to $\pi a^2 b^3$.

Divergence Theorem Example 2

$E: \qquad 0 \leq r \leq a, \qquad -b \leq z \leq b, \qquad 0 \leq \theta \leq 2\pi.$
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For S_3 it is convenient to work in cylindrical polar coordinates again.

$E: \quad 0 \leq r \leq a, \quad -b \leq z \leq b, \quad 0 \leq \theta \leq 2\pi.$
$\mathbf{u} = (-y, x, z^3)$. $\iiint_E \operatorname{div}(\mathbf{u}) dV = 2\pi a^2 b^3$.

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For S_3 it is convenient to work in cylindrical polar coordinates again. The outward unit normal is $\mathbf{n} = \mathbf{e}_r = (\cos(\theta), \sin(\theta), 0)$

Divergence Theorem Example 2

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For S_3 it is convenient to work in cylindrical polar coordinates again. The outward unit normal is $\mathbf{n} = \mathbf{e}_r = (\cos(\theta), \sin(\theta), 0)$, and the vector field is

$$\mathbf{u} = (-y, x, z^3) = (-r\sin(\theta), r\cos(\theta), z^3).$$

Divergence Theorem Example 2

$E: \quad 0 \leq r \leq a, \quad -b \leq z \leq b, \quad 0 \leq \theta \leq 2\pi.$
$\mathbf{u} = (-y, x, z^3)$. $\iiint_E \mathrm{div}(\mathbf{u}) dV = 2\pi a^2 b^3$.

Now consider instead $\iint_S \mathbf{u}.d\mathbf{A} = \iint_S \mathbf{u}.\mathbf{n}\, dA$. Let S_1 be the bottom end of E, where $z = -b$. Let S_2 be the top end, where $z = b$. Let S_3 be the curved outer surface, where $r = a$. On S_1, the outward unit normal is clearly $\mathbf{n} = -\mathbf{k} = (0, 0, -1)$. We also have $z = -b$, so $\mathbf{u} = (-y, x, -b^3)$, so $\mathbf{u}.\mathbf{n} = b^3$. As this is constant, it follows that

$$\iint_{S_1} \mathbf{u}.\mathbf{n}\, dA = \iint_{S_1} b^3\, dA = b^3 \times (\text{ area of } S_1) = \pi a^2 b^3.$$

On S_2 we have $\mathbf{n} = (0, 0, 1)$ and $\mathbf{u} = (-y, x, b^3)$, and it follows easily that $\iint_{S_2} \mathbf{u}.\mathbf{n}\, dA$ is also equal to $\pi a^2 b^3$.
For S_3 it is convenient to work in cylindrical polar coordinates again. The outward unit normal is $\mathbf{n} = \mathbf{e}_r = (\cos(\theta), \sin(\theta), 0)$, and the vector field is

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which is the same as $\iiint_E \mathrm{div}(\mathbf{u})\, dV$, as expected.

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Let E be the solid region where $-1 \leq x, y \leq 1$ and $0 \leq z \leq (1 - x^2)(1 - y^2)$.

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$$\iiint_E \text{div}(\mathbf{u})\, dV = 2 \times \tfrac{4}{3} \times \tfrac{4}{3} = 32/9.$$

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$E: \qquad -1 \leq x, y \leq 1, \qquad 0 \leq z \leq f(x, y) = (1 - x^2)(1 - y^2)$

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$$d\mathbf{A} = (-f_x, -f_y, 1)\, dx\, dy$$

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$$\iint_{S_2} \mathbf{u}.d\mathbf{A} = \int_{y=-1}^{1} \int_{x=-1}^{1} (2x^2 + 2y^2 - 4x^2y^2)\, dx\, dy.$$

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$$\mathbf{r}_r = (\cos(s), \sin(s), 2r\cos(2s))$$
$$\mathbf{r}_s = (-r\sin(s),\ r\cos(s),\ -2r^2\sin(2s))$$
$$\mathbf{r}_r \times \mathbf{r}_s = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(s) & \sin(s) & 2r\cos(2s)) \\ -r\sin(s) & r\cos(s) & -2r^2\sin(2s) \end{bmatrix}$$

$$= (-2r^2\sin(s)\sin(2s) - 2r^2\cos(s)\cos(2s),\ 2r^2\cos(s)\sin(2s) - 2r^2\sin(s)\cos(2s),\ r\cos^2(s) + r\sin^2(s))$$

Using

$$\sin(a)\sin(b) + \cos(a)\cos(b) = \cos(a-b) = \cos(b-a)$$
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Stokes's Theorem Example 1

Consider the surface S given by $z = x^2 - y^2$ with $x^2 + y^2 \leq 1$. We will check Stokes's Theorem for the vector field $(-y, x, 0)$. We parametrise S as

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$$\begin{aligned}
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this becomes $\mathbf{r}_r \times \mathbf{r}_s = (-2r^2\cos(s),\ 2r^2\sin(s),\ r)$

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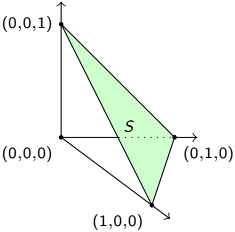
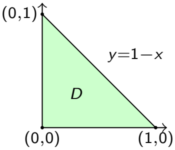
$$\int_C \mathbf{u}.d\mathbf{r} = \int_{s=0}^{2\pi} ds = 2\pi.$$

As expected, this is the same as $\iint_S \text{curl}(\mathbf{u}).d\mathbf{A}$.

Stokes's Theorem Example 2

Let S be the triangular surface shown on the left below, given by $x + y + z = 1$ with $x, y, z \geq 0$.





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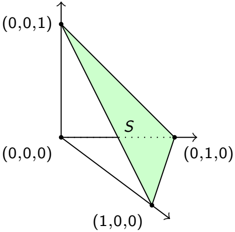

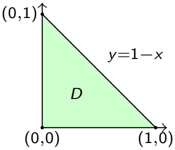

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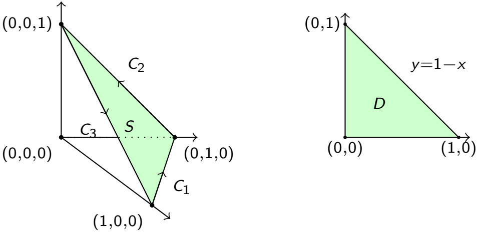

The boundary consists of the edges C_1, C_2 and C_3.

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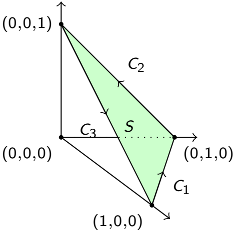

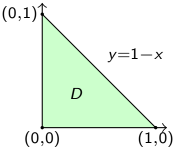

The boundary consists of the edges C_1, C_2 and C_3. We can parametrise C_1 by $\mathbf{r} = (x, y, z) = (1 - t, t, 0)$ for $0 \leq t \leq 1$.

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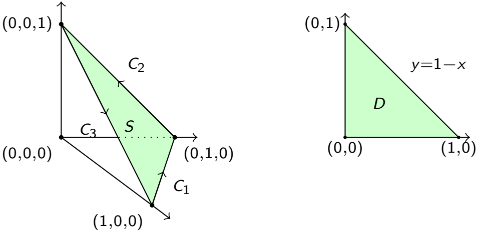

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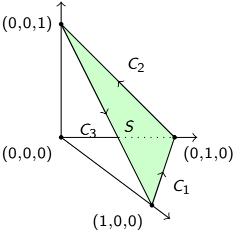

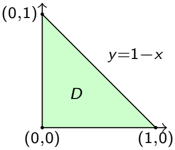

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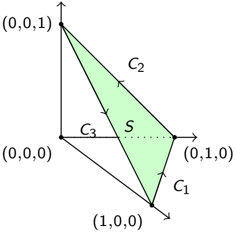

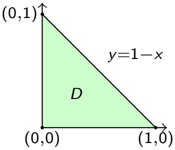

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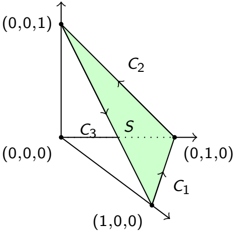

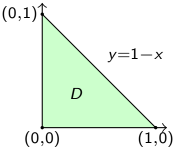

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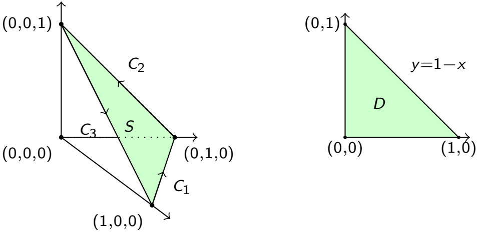

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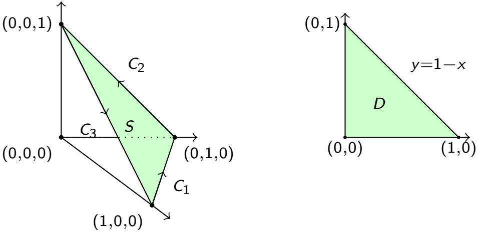

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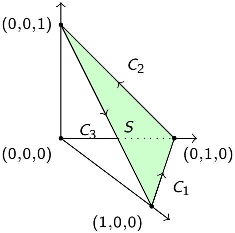

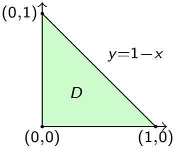

The other edges work in the same way, as in the following table:

edge	C_1	C_2	C_3
$\mathbf{r}$	$(1-t, t, 0)$	$(0, 1-t, t)$	$(t, 0, 1-t)$
$d\mathbf{r}$	$(-1, 1, 0)dt$	$(0, -1, 1)dt$	$(1, 0, -1)dt$
$\mathbf{u}$	$(0, 1-t, t)$	$(t, 0, 1-t)$	$(1-t, t, 0)$
$\mathbf{u}.d\mathbf{r}$	$(1-t)dt$	$(1-t)dt$	$(1-t)dt$
$\int \mathbf{u}.d\mathbf{r}$	$1/2$	$1/2$	$1/2$

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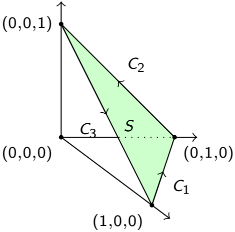

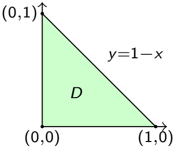

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Altogether, we have $\int_C \mathbf{u}.d\mathbf{r} = 3/2$.

Stokes's Theorem Example 2

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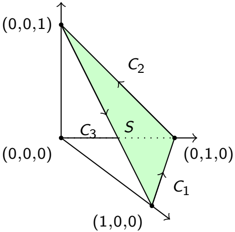

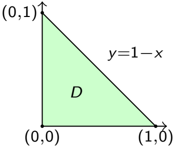

On the other hand, we have $\operatorname{curl}(\mathbf{u}) = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{bmatrix}$

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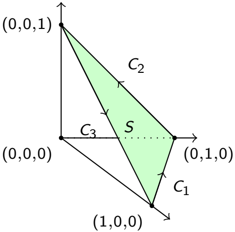

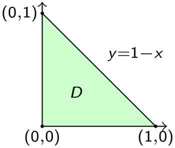

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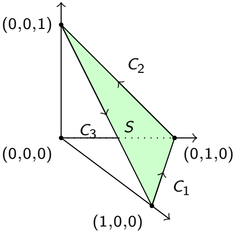

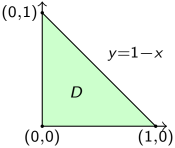

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The shadow of S in the xy-plane is the triangle D shown on the right.

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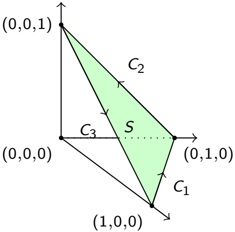

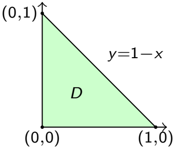

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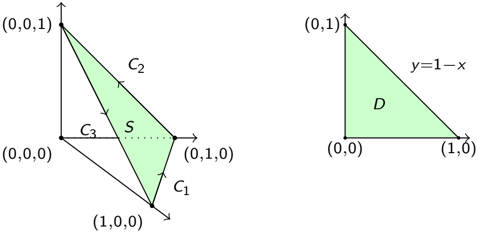

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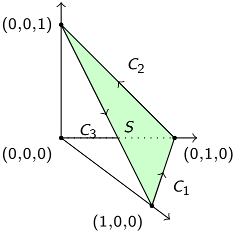

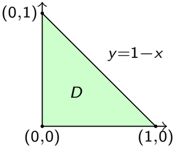

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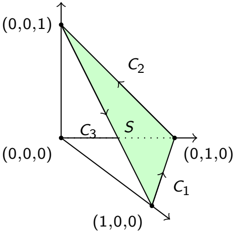

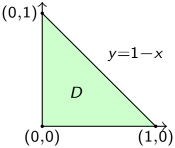

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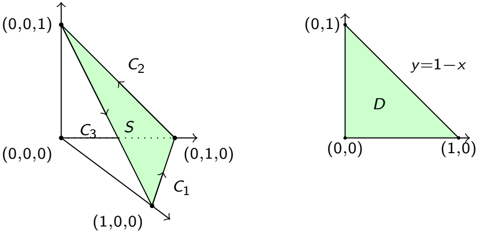

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$$\iint_S \text{curl}(\mathbf{u}).d\mathbf{A} = \int_D (1, 1, 1).(1, 1, 1)\, dx\, dy = 3 \int_{x=0}^{1} \int_{y=0}^{1-x} dy\, dx$$

Stokes's Theorem Example 2

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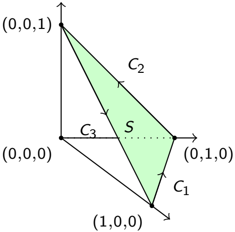

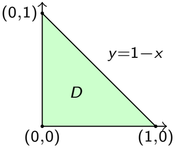

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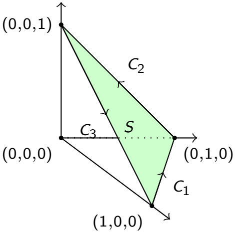

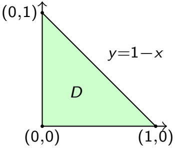

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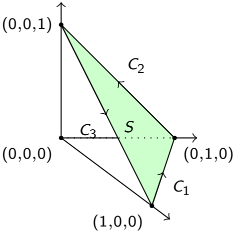

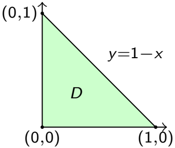

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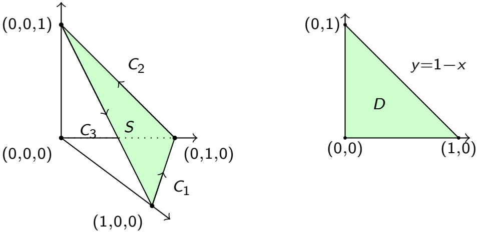

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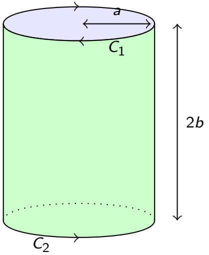
$$= 3 \int_{x=0}^{1} (1 - x)\, dx = 3 \left[x - \tfrac{1}{2} x^2 \right]_{x=0}^{1} = 3/2 = \int_C \mathbf{u}.d\mathbf{r}.$$

Stokes's Theorem Example 3

$S =$ cylinder: $r = a$ with $-b \leq z \leq b$ and $0 \leq \theta \leq 2\pi$. Check Stokes's Theorem for the vector field $\mathbf{u} = (-zy, zx, z^2)$.



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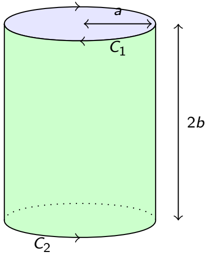

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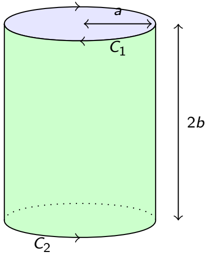

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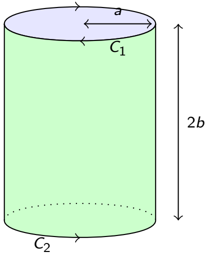

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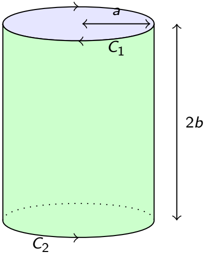

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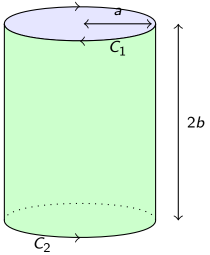

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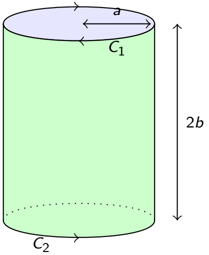

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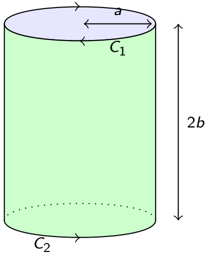

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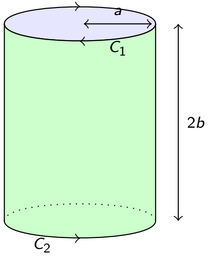

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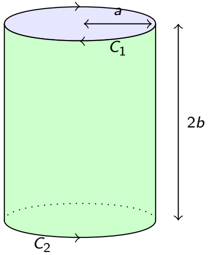

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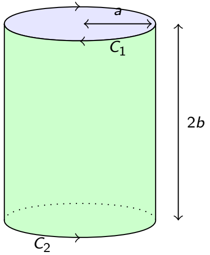

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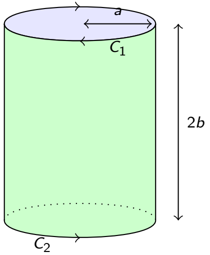

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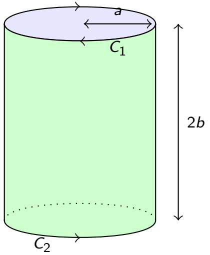

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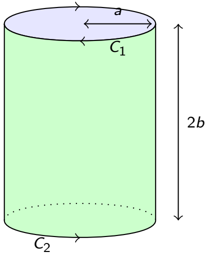

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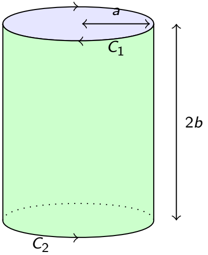

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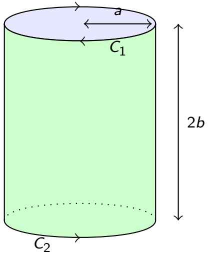

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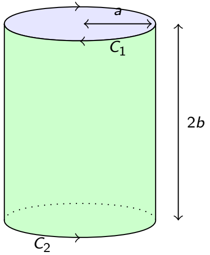

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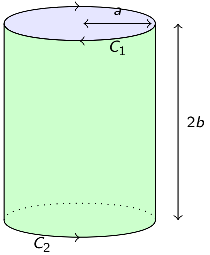

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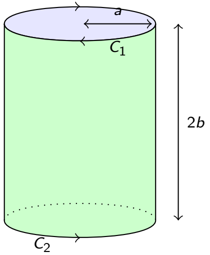

Boundary of S: C_1 and C_2.

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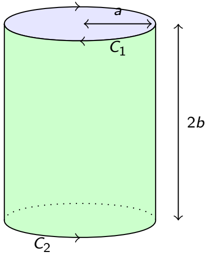

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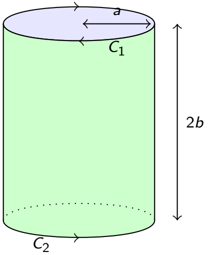

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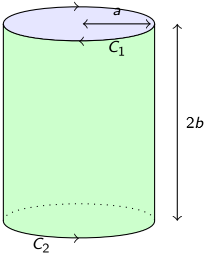

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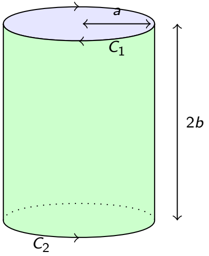

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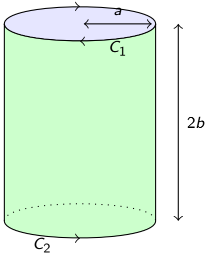

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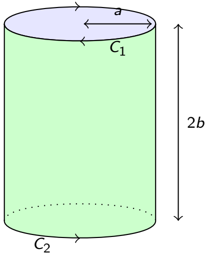

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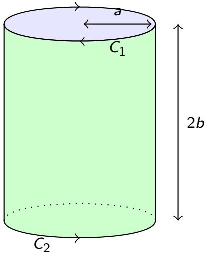

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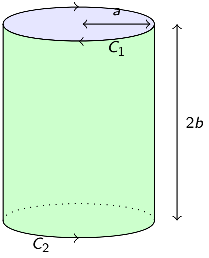

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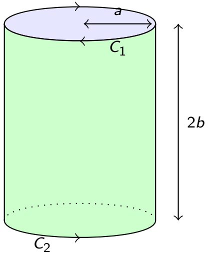

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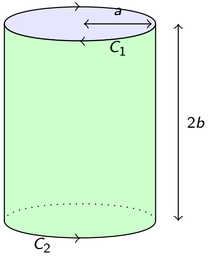

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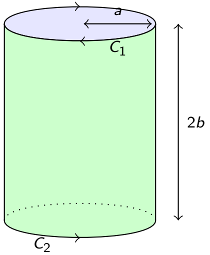

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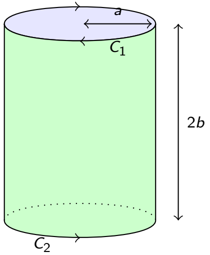

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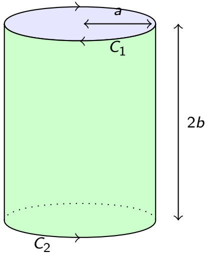

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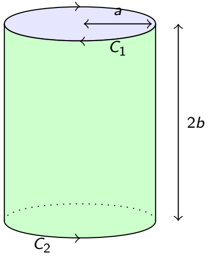

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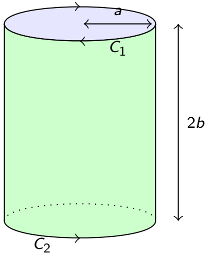

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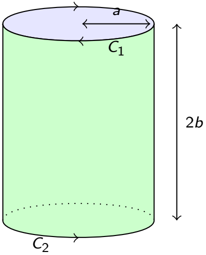

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