MAS243 MATHEMATICS IV (ELECTRICAL)

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1. Introduction

There are two main themes in this course, with some mathematical techniques shared between them.

The first is the problem of optimisation: we have some function f of one or more variables, and we want to choose the values of those variables so as to make f as large as possible. (Sometimes we want to make f as small as possible instead, but that is the same as making $-f$ as large as possible, so the techniques are essentially the same.) There are many applications for this. For example, we may have designed some kind of device, and we want to adjust the sizes of various parts or the values of various electrical components to make it function as effectively as possible; this can be formulated mathematically as an optimisation problem of the type that we will discuss. In another class of applications, we may not actually want f to be maximised, but we need to know the maximum possible value for other reasons. For example, it can happen that the force on a component of our device is given by a function f of certain inputs. To make sure that the component is strong enough, we need to know the maximum force that could be imposed on it, or in other words the maximum possible value of f.

The second main theme is the theory of vector calculus. This is the main mathematical language used in formulating the laws of physics. In electrical engineering the most important example is Maxwell's system of equations, which govern the behaviour of time-varying electric and magnetic fields. These are central for understanding the behaviour of electric generators and motors, radio transmitters and receivers, and so on. The same mathematical ideas are also used in continuum mechanics to understand stresses and strains in elastically deformed solids, and they appear again in the Navier-Stokes equations for the flow of liquids and gasses, and the magnetohydrodynamic equations for the behaviour of the solar plasma.

One of the ingredients in Maxwell's equations is the electric field $\mathbf{E}$. This will typically depend on the three spatial coordinates x, y and z and also on the time t. Moreover, $\mathbf{E}$ is a vector quantity, with a direction as well as a magnitude. Another ingredient is the charge density ρ, which is a scalar quantity, again depending on position and time. There are various different ways in which we can differentiate a scalar or vector quantity with respect to position or time to get another scalar or vector quantity. Three of the most important are called the gradient (written $\mathrm{grad}(f)$ or ∇f), the divergence ($\mathrm{div}(\mathbf{u})$ or $\nabla\cdot\mathbf{u}$) and the curl ($\mathrm{curl}(\mathbf{u})$ or $\nabla\times\mathbf{u}$). A large part of our task will be to understand the geometric and physical meaning of these operators, and their mathematical properties.

Next, in applications we do not just want to know the value of the electric potential or fluid pressure at particular points; we also want to calculate bulk quantities like the total energy stored in the electric field in a certain region, or the total flow of fluid through a pipe. To do these calculations, we need to perform some kind of integral. In multivariable calculus we have several different kinds of integration to go along with the several different kinds of differentiation. We can integrate along a curve, for example to find the total magnetic force on a wire carrying a current. We can integrate a charge density over a curved surface to find the total charge. We can perform a different kind of integral to find the total magnetic flux crossing a curved surface. We can also integrate the square of the electric field strength over a three-dimensional region to find the total energy. We will need to understand all these different kinds of integrals.

With functions of one variable, there is a simple relationship between integration and differentiation, expressed by the so-called Fundamental Theorem of Calculus:

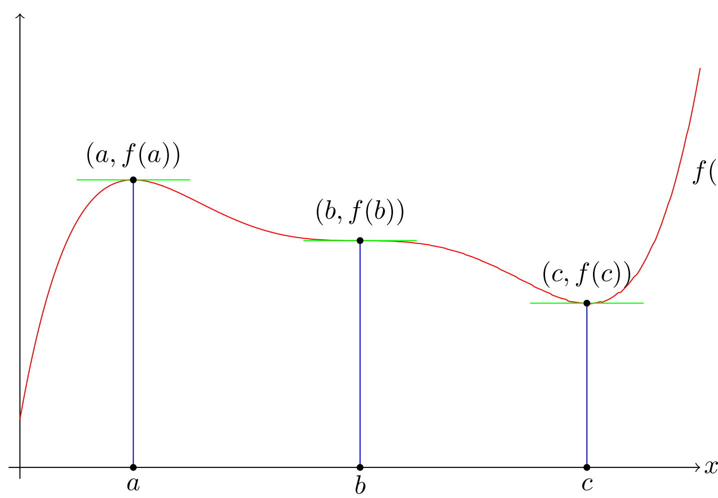
$$\int_a^b f'(x)\, dx = f(b) - f(a).$$

In the multivariable context, the relationship is equally important but more complicated to describe. It is encapsulated by results known as Stokes's theorem and the Divergence theorem, which we will study at the end of this course. As we will explain, they are not too hard to understand in the context of electromagnetism.

2. Optimisation

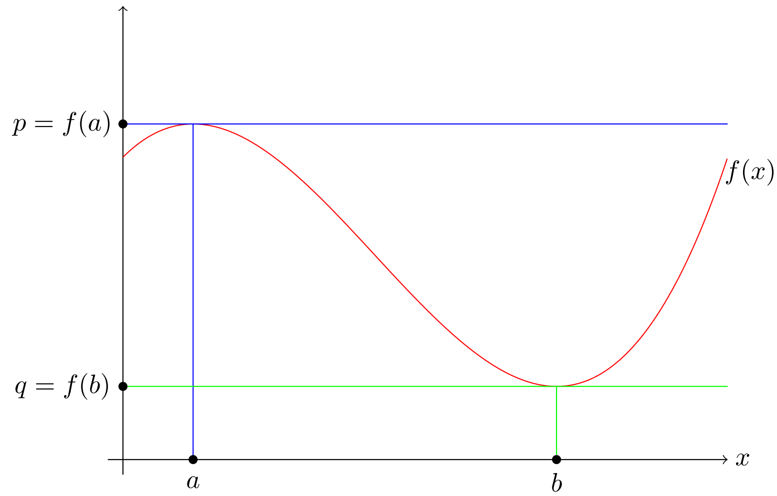
Often we have a function f of one or more variables, and we want to know the maximum and minimum possible values of f. In section 2.1 we review the familiar case where there is only one variable. Later, our main task will be to understand the more complicated case where there are two or more variables. This will have many applications. For example, suppose we have some device involving strong electric fields, with the field strength at each point (x, y, z) being given by a function $E(x, y, z)$. If we want to know whether arcing is likely to happen, we need to find the maximum value of E. For another kind of example, suppose we have designed a circuit involving resistors R_1, R_2 and R_3 and capacitors C_1 and C_2, and we want to choose the values of these components to make the circuit work as well as possible. We first need to come up with some kind of numerical measure Q of the quality of the circuit. This will depend on the component values, so we can think of it as a function $Q(R_1, R_2, R_3, C_1, C_2)$, and the general theory of circuits should give us a formula for this function. We then have the problem of finding values of R_i and C_j that make Q as large as possible. This is an optimisation problem of the type that we will study in this course.

2.1. **Optimisation in one variable.** Given a function $f(x)$, the *critical points* are the values of x where $f'(x) = 0$. Recall that $f'(x)$ is the slope of the graph at the point $(x, f(x))$, so the condition $f'(x) = 0$ means that the tangent line at that point is horizontal. In the picture below, a, b and c are critical points.



In the simplest case, the maximum and minimum values of $f(x)$ will occur at critical points. Indeed, if $f'(x) > 0$ then we can make $f(x)$ larger by increasing x slightly, and if $f'(x) < 0$ then we can make $f(x)$ bigger by decreasing x slightly. If $f(x)$ is already as large as possible then neither of these cases can occur, so we must have $f'(x) = 0$. The same kind of argument works for the minimum.

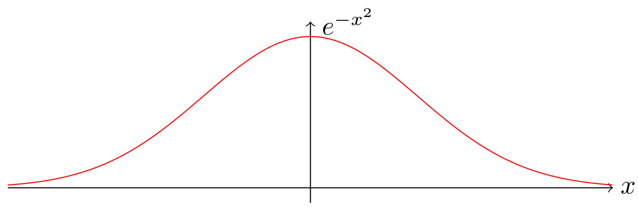
In the picture below, the critical points are $x = a$ and $x = b$. The maximum value of $f(x)$ is p, which occurs at $x = a$. The minimum value is q, which occurs at $x = b$.



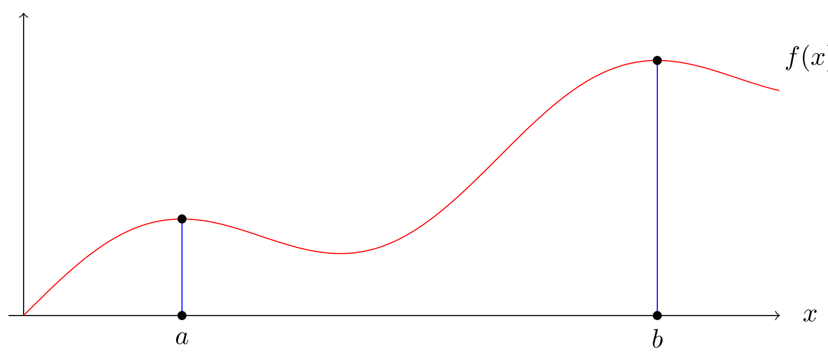
There are a number of wrinkles in this picture.

(a) The function $f(x)$ need not have a maximum or minimum. For example, the function $f(x) = x^2 + 1$ has a minimum (namely 1, which occurs at $x = 0$) but it does not have a maximum; it can be as large as you like. The function $f(x) = x^3$ has neither a maximum nor a minimum.

(b) More subtly, there are functions like $f(x) = e^{-x^2}$ (which is important in statistics and also in the theory of diffusion of heat). The maximum value of $f(x)$ is one, which occurs at $x = 0$, which is the only critical point. In a sense, the minimum value is zero. However, $f(x)$ is never actually equal to zero. Instead, it is always strictly positive, but it approaches zero arbitrarily closely when x is large. Because of this, we cannot find the minimum value by looking for critical points.

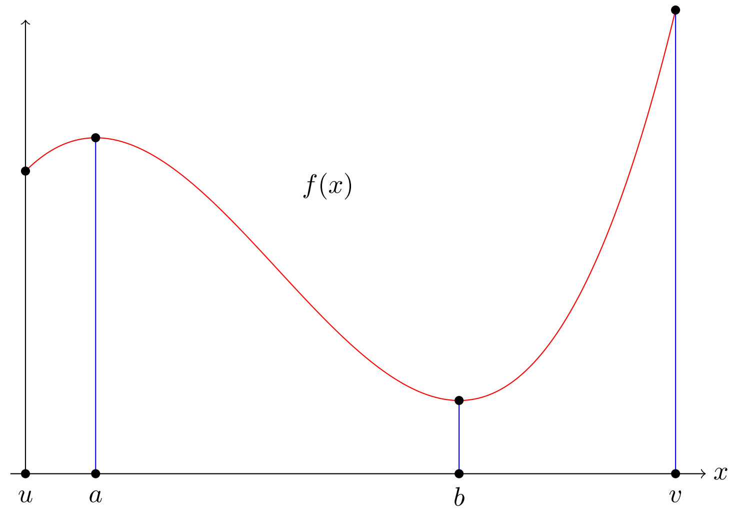


(c) The function $f(x)$ can have local maxima or minima that need not be global maxima or minima. For example, consider the following picture:



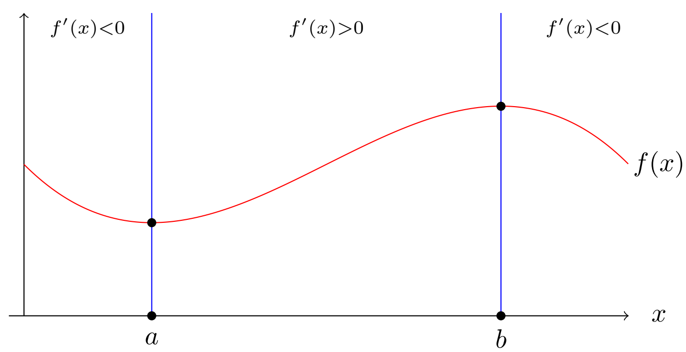
The function has a local maximum at $x = a$, in the sense that $f(a) \geq f(x)$ for all x close to a. However, this is not a global maximum, because there are points x far from $x = a$ with $f(x) > f(a)$. For example, we have $f(b) > f(a)$. Note that local maxima and minima are still critical points.

(d) If only a finite range of values of x is relevant (say $u \leq x \leq v$) then the maximum or minimum value might occur at $x = u$ or $x = v$ even if these are not critical points. For example, in this picture the minimum occurs at $x = b$ (which is a critical point) but the maximum occurs at the endpoint $x = v$ (which is not a critical point).

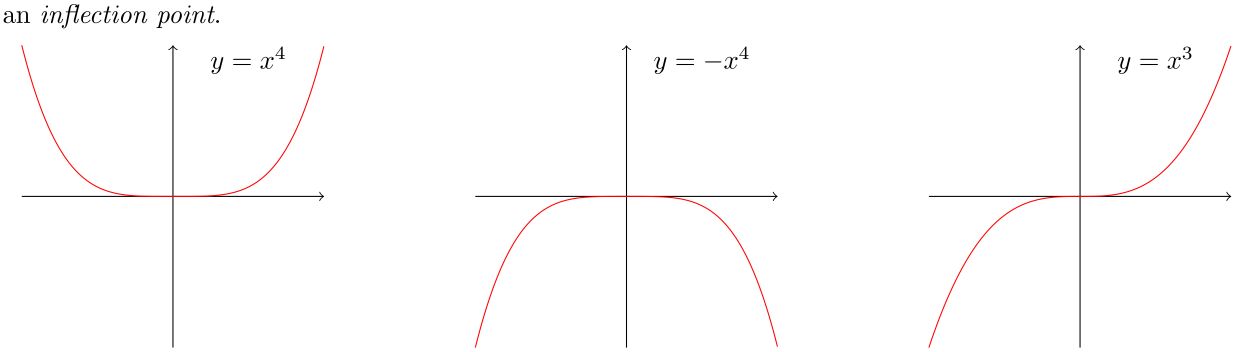


(e) If $f(x)$ jumps discontinuously (for example $f(x) = \tan(x)$) then the situation becomes much more complex; we will not discuss it here. Similar remarks apply if $f(x)$ does not have a well-defined derivative. For example the function $f(x) = |x|$ has $f'(x) = 1$ for $x > 0$ and $f'(x) = -1$ for $x < 0$ but $f'(0)$ is undefined.

Another important question is how to recognise whether a critical point is a (local) maximum or a minimum. The simplest case is illustrated in the following picture:



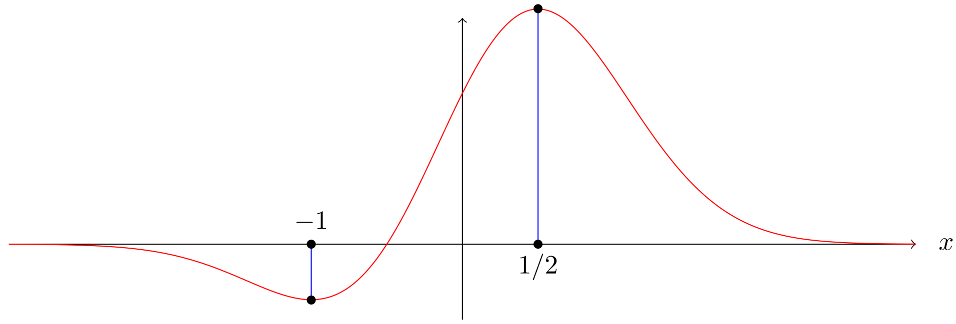
The function has a minimum at $x = a$. To the left of $x = a$ the function is decreasing so $f'(x) < 0$. To the right of $x = a$ the function is increasing so $f'(x) > 0$. Thus, at $x = a$ the function $f'(x)$ changes from being negative to being positive, so $f'(x)$ is increasing at a, so $f''(a) > 0$. Similarly, $f'(x)$ switches from being positive to negative at $x = b$, so $f''(b) < 0$. This is the usual situation: at a local minimum we have $f'' > 0$, and at a local minimum we have $f'' < 0$. It can also happen that we have a critical point a where $f''(a)$ is zero (as well as $f'(a)$ being zero, which is true by definition for a critical point). In this case we may have either a local minimum (as with $f(x) = x^4$ at $x = 0$) or a local minimum (as with $f(x) = -x^4$) or neither (as with $f(x) = x^3$). A critical point that is neither a local minimum nor a local maximum is called an *inflection point*.



Example 2.1. Suppose we want to understand the critical points of the function $f(x) = (2x+1)e^{-x^2}$. Using the product rule and the chain rule, we see that

$$\begin{aligned}
\tfrac{d}{dx}e^{-x^2} &= e^{-x^2}\tfrac{d}{dx}(-x^2) = -2xe^{-x^2} \\
f'(x) &= (2x+1)\tfrac{d}{dx}e^{-x^2} + e^{-x^2}\tfrac{d}{dx}(2x+1) = -2x(2x+1)e^{-x^2} + 2e^{-x^2} \\
&= -(4x^2+2x-2)e^{-x^2}.
\end{aligned}$$

Note that e^{-x^2} is never zero, so the critical points are just the roots of the quadratic $4x^2+2x-2$, which are $x=-1$ and $x=1/2$. The values of f at these points are $f(-1) = -1/e \simeq -0.37 < 0$ and $f(1/2) = 2e^{-1/4} \simeq 1.56 > 0$. It is also a standard fact that e^{-x^2} decays very rapidly for large x, more than enough to wipe out the growth of $2x+1$, so $f(x)$ is small for large x. From this it follows that we have a global maximum at $x=1/2$ and a global minimum at $x=-1$. The picture is as follows:



We could alternatively classify the critical points using the second derivative, as follows. We have

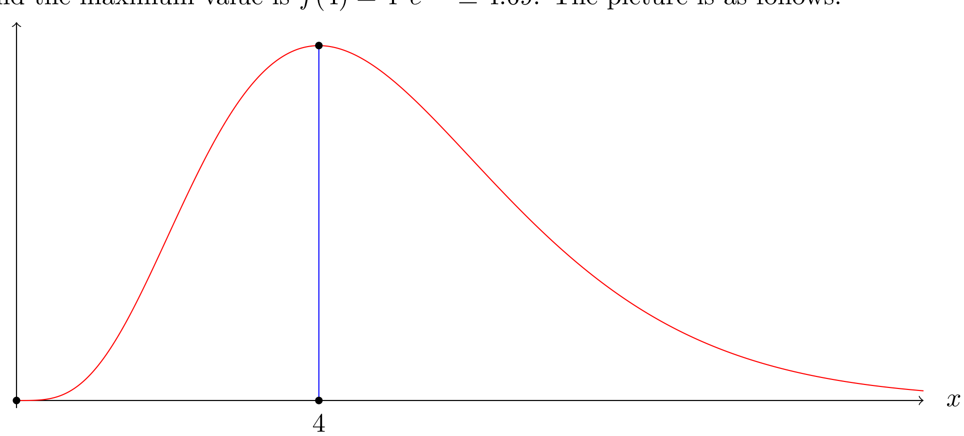
$$\begin{aligned}
f''(x) &= -e^{-x^2}\tfrac{d}{dx}(4x^2+2x-2) - (4x^2+2x-2)\tfrac{d}{dx}(e^{-x^2}) \\
&= (-8x-2)e^{-x^2} - (4x^2+2x-2)(-2x)e^{-x^2} \\
&= (8x^3+4x^2-12x-2)e^{-x^2}.
\end{aligned}$$

This gives $f''(-1) = 6e^{-1} > 0$ and $f''(1/2) = -6e^{-1/4} < 0$, so there is a local minimum at $x=-1$ and a local maximum at $x=1/2$, as we have already seen.

Example 2.2. Consider instead the function $f(x) = x^4e^{-x}$ for $x \geq 0$. Note that $f(x) \geq 0$ for all x. The derivative is

$$f'(x) = 4x^3e^{-x} - x^4e^{-x} = (4-x)x^3e^{-x},$$

which is zero for $x=0$ and $x=4$, so these are the critical points. When $0 < x < 4$ we see that all the terms $4-x$, x^3 and e^{-x} are positive, so $f'(x) > 0$, so $f(x)$ is increasing. For $x > 4$ we have $4-x < 0$ and the other factors are still positive so $f'(x) < 0$ and $f(x)$ is decreasing. It follows that the minimum value of $f(x)$ is $f(0) = 0$, and the maximum value is $f(4) = 4^4e^{-4} \simeq 4.69$. The picture is as follows:

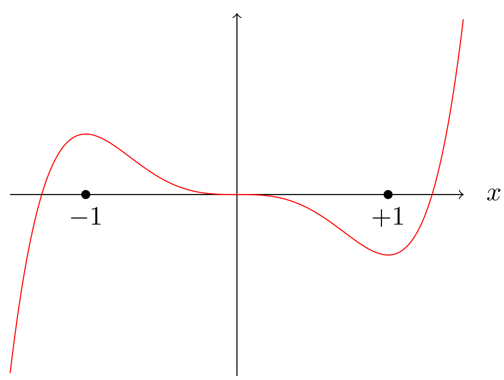


Note also that

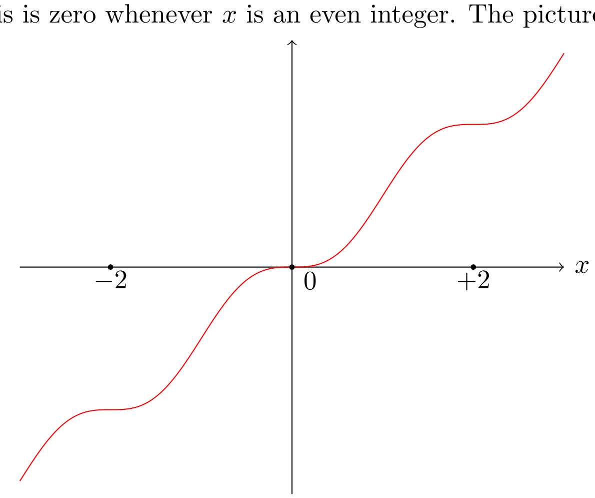
$$f''(x) = e^{-x}\frac{d}{dx}(4x^3 - x^4) + (4x^3 - x^4)\frac{d}{dx}(e^{-x}) = (12x^2 - 8x^3 + x^4)e^{-x},$$

which gives $f''(4) = -64e^{-4}$. This is negative, as we expect for a local maximum.

Example 2.3. Suppose we want to understand the critical points of the function $f(x) = x^5/5 - x^3/3$. The derivative is $f'(x) = x^4 - x^2 = x^2(x^2 - 1) = x^2(x-1)(x+1)$. The critical points are the values of x where $f'(x) = 0$, namely $x = 0$, $x = 1$ and $x = -1$. We can classify these by looking at the second derivative $f''(x) = 4x^3 - 2x$. We have $f''(1) = 2 > 0$, so there is a local minimum at $x = 1$. We have $f''(-1) = -2$, so there is a local maximum at $x = -1$. We have $f''(0) = 0$, so it is not clear what happens at $x = 0$. However, from the formula $f'(x) = x^4 - x^2$ we can see that $f'(x) \leq 0$ for $-1 \leq x \leq 1$, so $f(x)$ is decreasing all through that range, so $x = 0$ cannot be a local maximum or minimum; it must be an inflection point. Note also that when x is large, the term $x^5/5$ will be much bigger than $x^3/3$ and so $f(x)$ will be very large and positive. Similarly, when x is large and negative, the same will be true of $f(x)$. This means that there is no global maximum or minimum. The picture is as follows:

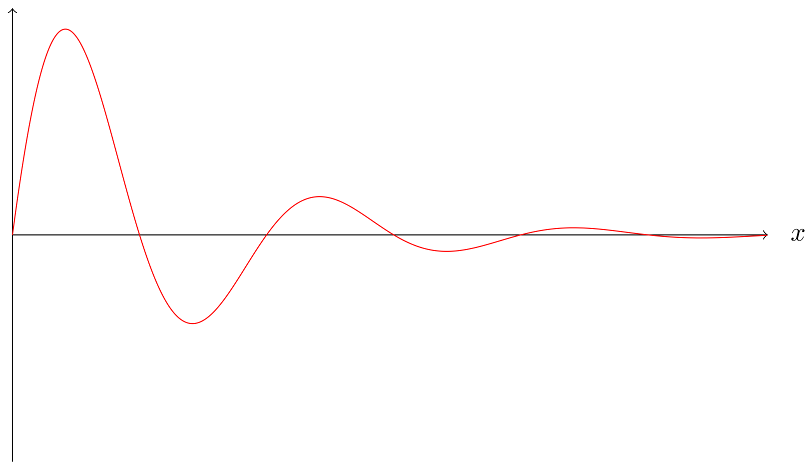


Example 2.4. We now consider the function $f(x) = \pi x - \sin(\pi x)$. The first term πx increases linearly, and the second term $-\sin(\pi x)$ adds an oscillating wiggle. You should be able to see that if we take a linearly increasing function and add a small slow wiggle then the resulting function will still be strictly increasing and so will not have any local maxima or minima. If instead we add a large fast wiggle then we will create lots of local maxima and minima. Which of these cases applies to our function $f(x)$? To find out, we note that the derivative is $f'(x) = \pi - \pi\cos(\pi x) = \pi(1 - \cos(\pi x))$. It is standard that $-1 \leq \cos(\theta) \leq 1$ for all θ, so $f'(x) \geq 0$. The critical points occur where $f'(x) = 0$, or in other words, where $\cos(\pi x) = 1$. It is also also standard that $\cos(\theta)$ is equal to one precisely when θ is a multiple of 2π, so $\cos(\pi x) = 1$ precisely when x is an even integer. In other words, the critical points are $x = \ldots, -4, -2, 0, 2, 4, \ldots$. As $f'(x) \geq 0$ for all x we see that none of these can be local maxima or minima. Another way to see this is to observe that $f''(x) = \pi^2\sin(\pi x)$, and this is zero whenever x is an even integer. The picture is as follows:



We are just on the boundary between the two cases that we mentioned previously. If we added a wiggle that was any smaller, we would not have any critical points. If we added a wiggle that was any larger, we would have local maxima and minima.

Example 2.5. Functions of the form $f(t) = e^{-\lambda t}\sin(\omega t)$ represent oscillations that die down over time, so they occur very frequently in physics and engineering. The function $\sin(\omega t)$ has local maxima at $t = (2n+\frac{1}{2})\pi/\omega$ and local minima at $t = (2n-\frac{1}{2})\pi/\omega$. If the oscillations are reasonably fast relative to the decay then $f(t)$ will have local maxima and minima close to those of $\sin(\omega t)$, but shifted slightly.



We can find an exact formula as follows. We have

$$f'(t) = -\lambda e^{-\lambda t}\sin(\omega t) + e^{-\lambda t}\omega\cos(\omega t),$$

so $f'(t) = 0$ when $\lambda e^{-\lambda t}\sin(\omega t) = e^{-\lambda t}\omega\cos(\omega t)$, which can be rearranged as $\tan(\omega t) = \omega/\lambda$. The obvious solution is to take $\omega t = \arctan(\omega/\lambda)$ so $t = \arctan(\omega/\lambda)/\omega$. More generally, as $\tan(\theta)$ is a periodic function of period π it is also valid to take $\omega t = n\pi + \arctan(\omega/\lambda)$ for any integer n, so the critical points are the numbers $t_n = (n\pi + \arctan(\omega/\lambda))/\omega$. Note that if the decay is slow relative to the oscillations then ω/λ will be large, so $\arctan(\omega/\lambda)$ will be close to $\pi/2$, so $t_n \simeq (n+\frac{1}{2})\pi/\omega$. This means that t_{2n} is close to the local maximum of $\sin(\omega t)$ at $(2n+\frac{1}{2})\pi/\omega$, and t_{2n-1} is close to the local minimum at $(2n-\frac{1}{2})\pi/\omega$, as expected.

2.2. **Reminder on partial derivatives.** Throughout this course we will be considering functions of several variables, such as $u = x^2y + y^3z$ for example. Such a function can be differentiated with respect to any of the variables involved. If we treat y and z as constants and differentiate u with respect to x, we just get

$$\frac{\partial u}{\partial x} = 2xy.$$

If instead we hold x and z constant and differentiate with respect to y we get

$$\frac{\partial u}{\partial y} = x^2 + 3y^2z.$$

Finally, we can hold x and y constant and differentiate with respect to z to get

$$\frac{\partial u}{\partial z} = y^3.$$

We have followed the usual practice of writing $\partial u/\partial x$ rather than du/dx and so on to indicate that u depends on other variables as well as x. We call these functions the *partial derivatives* of u. We can interpret them as measures of the sensitivity of u to small changes in x, y and z. More precisely, if we change x, y and z by small amounts δx, δy and δz then the resulting change δu in u is approximately given by

$$\delta u \simeq \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z.$$

Recall that in the single variable case, there are two different kinds of notation in common use. We can either write things like $u = x^3$ and $du/dx = 3x^2$, or we can write $f(x) = x^3$ and $f'(x) = 3x^2$. In the case of several variables we need some way of indicating which of the different partial derivatives we want, so we cannot simply use a dash. The usual convention is to write u_x or $u_x(x, y, z)$ for $\partial u/\partial x$, and so on.

In the single variable case, we often need to consider the second derivative u'', obtained by taking the first derivative u' and differentiating it again. When there are several variables, there are several different iterated derivatives, such as u_{xx}, u_{xy}, u_{yx} and u_{yy}. In the other notation these would be written as $\partial^2 u/\partial x^2$, $\partial^2 u/\partial y\partial x$, $\partial^2 u/\partial x\partial y$ and $\partial^2 u/\partial y^2$. If we differentiate with respect to two different variables, it turns out not to matter which one we do first. In symbols, of u is a function of x and y (and possibly some other variables as well) then $u_{xy} = u_{yx}$, or $\partial^2 u/\partial y\partial x = \partial^2 u/\partial x\partial y$. For example, if we take $u = x^2 y + y^3 z$ then we have seen that

$$u_x = 2xy \qquad u_y = x^2 + 3y^2 z \qquad u_z = y^3.$$

To find u_{yx} we differentiate the function $u_y = x^2 + 3y^2 z$ with respect to x, treating y and z as constants. This gives $u_{yx} = 2x$. We can calculate the other double derivatives in the same way, giving the following table:

$$\begin{array}{lll} u_{xx} = 2y & u_{yx} = 2x & u_{zx} = 0 \\ u_{xy} = 2x & u_{yy} = 6yz & u_{zy} = 3y^2 \\ u_{xz} = 0 & u_{yz} = 3y^2 & u_{zz} = 0. \end{array}$$

We find that $u_{xy} = u_{yx}$, $u_{xz} = u_{zx}$ and $u_{yz} = u_{zy}$ as expected. We can collect these answers together as a matrix:

$$H = \begin{bmatrix} u_{xx} & u_{yx} & u_{zx} \\ u_{xy} & u_{yy} & u_{zy} \\ u_{xz} & u_{yz} & u_{zz} \end{bmatrix} = \begin{bmatrix} 2y & 2x & 0 \\ 2x & 6yz & 3y^2 \\ 0 & 3y^2 & 0 \end{bmatrix}.$$

This is called the *Hessian matrix* for u. The fact that $u_{xy} = u_{yx}$ and so on means that it is a symmetric matrix.

Many of the most important mathematical relationships in physics and engineering can be expressed as equations relating partial derivatives of various functions. For example, Maxwell's equations for electromagnetic fields are of this type, as are the Navier-Stokes equations for fluid flow.

Example 2.6. Suppose we have a voltage V across a resistor R; then the power dissipated in the resistor is $P = V^2/R$. If we treat R as a constant and let V vary, the derivative is

$$P_V = \frac{\partial P}{\partial V} = 2V/R.$$

If instead we treat V as a constant and let R vary, we get

$$P_R = \frac{\partial P}{\partial R} = -V^2/R^2.$$

We can now calculate P_{VV} by looking at $P_V = 2V/R$, treating R as a constant and differentiating with respect to V again. The result is just $P_{VV} = 2/R$. Similarly, we can calculate P_{RR} by looking at $P_R = -V^2/R^2$, treating V as a constant and differentiating again with respect to R. The result is $P_{RR} = 2V^2/R^3$. For the mixed derivatives, we can calculate P_{VR} by differentiating $P_V = 2V/R$ with respect to R to get $P_{VR} = -2V/R^2$. Alternatively, we can differentiate $P_R = -V^2/R^2$ with respect to V to get $P_{RV} = -2V/R^2$. As expected, we have $P_{VR} = P_{RV}$. In summary, we have

$$\begin{aligned} P_{VV} &= \frac{\partial^2 P}{\partial V^2} = 2/R \\ P_{RR} &= \frac{\partial^2 P}{\partial R^2} = 2V^2/R^3 \\ P_{VR} = P_{RV} &= \frac{\partial^2 P}{\partial V\partial R} = \frac{\partial^2 P}{\partial R\partial V} = -2V/R^2. \end{aligned}$$

Example 2.7. Consider the function $u = a + ab^2 + ab^2c^3$. We have

$$\begin{aligned}
u_a &= 1 + b^2 + b^2c^3 \\
u_b &= 2ab + 2abc^3 \\
u_c &= 3ab^2c^2 \\
u_{aa} &= 0 \\
u_{ab} = u_{ba} &= 2b + 2bc^3 \\
u_{ac} = u_{ca} &= 3b^2c^2 \\
u_{bb} &= 2a + 2ac^3 \\
u_{bc} = u_{cb} &= 6abc^2 \\
u_{cc} &= 6ab^2c.
\end{aligned}$$

The Hessian is therefore

$$H = \begin{bmatrix} 0 & 2b + 2bc^3 & 3b^2c^2 \\ 2b + 2bc^3 & 2a + 2ac^3 & 6abc^2 \\ 3b^2c^2 & 6abc^2 & 6ab^2c \end{bmatrix}.$$

Example 2.8. Consider the function $f(x, y, z) = \ln(ax + by + cz)$, where a, b and c are constants. Using the chain rule, we obtain

$$\begin{aligned}
f_x(x, y, z) &= \ln'(ax + by + cz)\frac{\partial}{\partial x}(ax + by + cz) = \frac{a}{ax + by + cz} \\
f_y(x, y, z) &= \ln'(ax + by + cz)\frac{\partial}{\partial y}(ax + by + cz) = \frac{b}{ax + by + cz} \\
f_z(x, y, z) &= \ln'(ax + by + cz)\frac{\partial}{\partial z}(ax + by + cz) = \frac{c}{ax + by + cz}.
\end{aligned}$$

For the second derivatives, we have

$$\begin{aligned}
f_{xy}(x, y, z) &= \frac{\partial}{\partial y} f_x(x, y, z) = \frac{\partial}{\partial y}\left(\frac{a}{ax + by + cz}\right) \\
&= \frac{-a}{(ax + by + cz)^2}\frac{\partial}{\partial y}(ax + by + cz) = \frac{-ab}{(ax + by + cz)^2}.
\end{aligned}$$

Proceeding in the same way, we see that

$$\begin{aligned}
f_{xx}(x, y, z) &= \frac{-a^2}{(ax + by + cz)^2} & f_{xy}(x, y, z) &= \frac{-ab}{(ax + by + cz)^2} & f_{xz}(x, y, z) &= \frac{-ac}{(ax + by + cz)^2} \\
f_{yx}(x, y, z) &= \frac{-ab}{(ax + by + cz)^2} & f_{yy}(x, y, z) &= \frac{-b^2}{(ax + by + cz)^2} & f_{yz}(x, y, z) &= \frac{-bc}{(ax + by + cz)^2} \\
f_{zx}(x, y, z) &= \frac{-ac}{(ax + by + cz)^2} & f_{zy}(x, y, z) &= \frac{-bc}{(ax + by + cz)^2} & f_{zz}(x, y, z) &= \frac{-c^2}{(ax + by + cz)^2}
\end{aligned}$$

This means that the Hessian matrix is

$$H = \frac{-1}{(ax + by + cz)^2}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}.$$

Example 2.9. Consider a function of the form $f(x, y) = u(x) + v(y)$. Here we just

$$\begin{aligned}
f_x(x, y) &= u'(x) & & & f_y(x, y) &= v'(y) \\
f_{xx}(x, y) &= u''(x) & f_{xy}(x, y) &= 0 & f_{yy}(x, y) &= v''(y).
\end{aligned}$$

This means that the Hessian is $\begin{bmatrix} u''(x) & 0 \\ 0 & v''(y) \end{bmatrix}$.

Consider instead a function of the form $g(x,y) = u(x)v(y)$. Here we have

$$g_x(x,y) = u'(x)v(y) \qquad\qquad g_y(x,y) = u(x)v'(y)$$
$$g_{xx}(x,y) = u''(x)v(y) \qquad g_{xy}(x,y) = u'(x)v'(y) \qquad g_{yy}(x,y) = u(x)v''(y).$$

This means that the Hessian is $\begin{bmatrix} u''(x)v(y) & u'(x)v'(y) \\ u'(x)v'(y) & u(x)v''(y) \end{bmatrix}$.

2.3. **Optimisation with two variables.** Now suppose we have a function of two variables, say $f(x,y)$. We say that a point (a,b) is a *critical point* of f if $f_x(a,b) = f_y(a,b) = 0$.

Example 2.10. Consider the function $f(x,y) = 2x^2 + 2xy - 6x + y^2 - 4y + 5$. We have

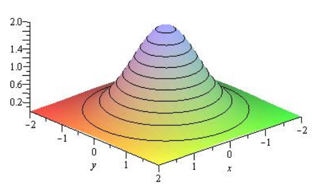
$$f_x(x,y) = 4x + 2y - 6$$
$$f_y(x,y) = 2x + 2y - 4.$$

Thus, for (a,b) to be a critical point, we must have $4a + 2b - 6 = 0$ and $2a + 2b - 4 = 0$. By subtracting these two equations we get $2a - 2 = 0$, so $a = 1$, and we can substitute this back into the first equation to get $b = 1$ as well. This means that there is precisely one critical point, namely $(1,1)$.

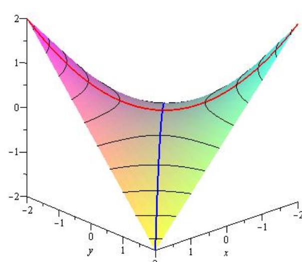
The key fact about critical points is that if f has a local maximum or a local minimum at (a,b), then (a,b) is a critical point. To see why, recall that if we make small changes to x and y, the resulting change in f is approximately $f_x.\delta x + f_y.\delta y$. If f_x or f_y is nonzero then by choosing δx and δy with the same sign as f_x and f_y we can arrange to have $\delta f > 0$. However, if we are already at a local maximum then it is impossible to increase f by a small change in x and y. The only way this can be consistent is if $f_x = f_y = 0$ at the local maximum. Essentially the same argument works for minima.

In the previous section we listed a number of issues that complicate the relationship between critical points and maxima/minima, in the case of a single variable. There are similar issues in the case of two variables, plus some extra ones. Although these points are important, we will not discuss them very much here. Instead, we will focus on finding and classifying critical points.

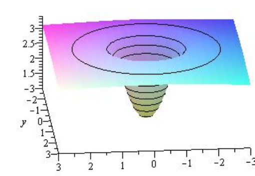
The main new ingredient is that we can have saddle points as well as local maxima and minima.



maximum



saddle point



minimum

At a saddle point we have a local maximum in some directions and a local minimum in other directions.

In the one variable case we can classify critical points by looking at the sign of the second derivative. In the two variable case, the analogous criterion involves the Hessian matrix

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}.$$

The criterion can be stated in terms of the eigenvalues of H or in terms of the entries of H. The first version is more precise and more illuminating, but computationally less efficient. We will state both versions but we will work mainly with the second one.

Method 2.11. Let (a,b) be a critical point of $f(x,y)$, and let H be the Hessian matrix at (a,b). Let s and t be the eigenvalues of H. Then s and t are always real numbers; there is never an imaginary part. Moreover:

(a) If $s, t < 0$ then we have a local maximum at (a, b).
(b) If $s < 0 < t$ or $t < 0 < s$ then we have a saddle point at (a, b).
(c) If $0 < s, t$ then we have a local minimum at (a, b).
(d) If one of s and t is zero, then there are many possibilities, and we will not consider them further. This situation rarely occurs in practice.

Method 2.12. Alternatively, we can put $A_1 = f_{xx}(a, b)$ and

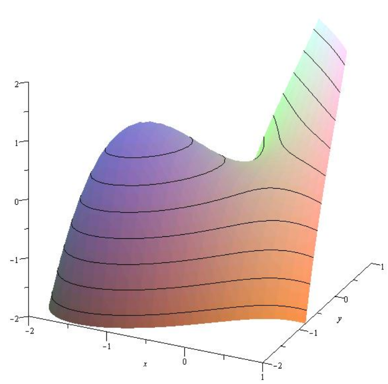
$$A_2 = \det(H) = \det \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{bmatrix} = f_{xx}(a,b) f_{yy}(a,b) - f_{xy}(a,b)^2.$$

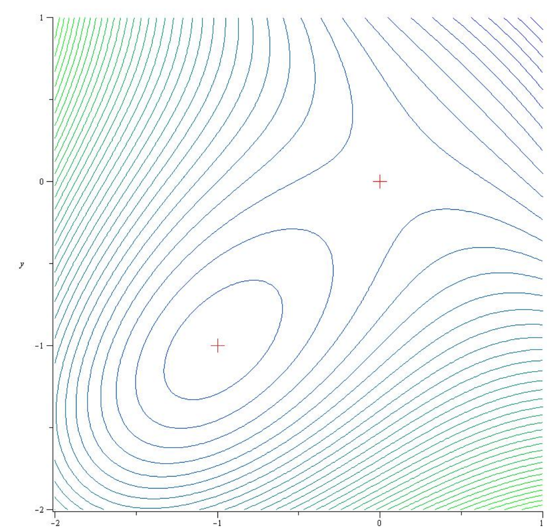
(a) If $A_1 < 0$ and $A_2 > 0$ then we have a local maximum at (a, b).
(b) If $A_2 < 0$ then we have a saddle point at (a, b).
(c) If $A_1 > 0$ and $A_2 > 0$ then we have a local minimum at (a, b).
(d) If $A_2 = 0$, then there are many possibilities, and we will not consider them further. This situation rarely occurs in practice.

Example 2.13. Consider the function $f(x, y) = x^3 + 3xy + y^3$. The derivatives are

$$f_x(x,y) = 3x^2 + 3y \qquad\qquad f_y(x,y) = 3x + 3y^2$$
$$f_{xx}(x,y) = 6x \qquad f_{xy}(x,y) = 3 \qquad f_{yy}(x,y) = 6y.$$

Thus, the critical points are the points (a, b) where $3a^2 + 3b = 0$ and $3a + 3b^2 = 0$, or equivalently $b = -a^2$ and $a = -b^2$. Substituting the first of these in the second gives $a = -a^4$, so $a^4 + a = 0$, so $(a^3 + 1)a = 0$. This can only happen if $a = -1$ (in which case $b = -a^2 = -1$) or $a = 0$ (in which case $b = -a^2 = 0$). Thus, there are two critical points, namely $p = (-1, -1)$ and $q = (0, 0)$. The Hessian matrix is $H = \left[\begin{smallmatrix} 6x & 3 \\ 3 & 6y \end{smallmatrix}\right]$, so $A_1 = 6x$ and the determinant is $A_2 = 36xy - 9 = 9(4xy - 1)$. At p we have $A_1 = -6 < 0$ and $A_2 = 27 > 0$ so we are in case (a) of method 2.12 and we have a local maximum. At q we have $A_1 = 0$ and $A_2 = -9 < 0$ so we are in case (b) so we have a saddle point. We can display this graphically as follows:





The picture on the left shows the surface $z = f(x, y)$. The one on the right is the corresponding contour plot, which is what you would see by looking vertically downwards at the picture on the left.

Example 2.14. Consider the function

$$f(x,y) = 3x^2 + 3y^2 + 2xy(x + y) = 3x^2 + 3y^2 + 2x^2y + 2xy^2.$$

The derivatives are

$$f_x(x,y) = 6x + 4xy + 2y^2 \qquad\qquad f_y(x,y) = 6y + 4xy + 2x^2$$
$$f_{xx}(x,y) = 6 + 4y \qquad f_{xy}(x,y) = 4x + 4y \qquad f_{yy} = 6 + 4x.$$

Thus, the critical points are the points (a, b) where $6a+4ab+2b^2 = 0$ and $6b+4ab+2a^2 = 0$. If we subtract these two equations we get $6a-6b+2b^2-2a^2 = 0$, which factors as $2(a-b)(3-a-b) = 0$. This means that either $a = b$ or $3 - a - b = 0$. If $a = b$ then our equations become $6a + 6a^2 = 0$, so $a(a + 1) = 0$, so $a = 0$ or $a = -1$. We now see that there are critical points at $p_0 = (0, 0)$ and $p_1 = (-1, -1)$. Suppose instead that $3 - a - b = 0$, so $b = 3 - a$. In this case we have

$$\begin{aligned} 0 &= 6a + 4ab + 2b^2 = 6a + 4a(3 - a) + 2(3 - a)^2 \\ &= 6a + 12a - 4a^2 + 2(9 - 6a + a^2) = -2a^2 + 6a + 18. \end{aligned}$$

By the standard quadratic formula this happens for $a = 3(1 \pm \sqrt{5})/2$, and because $b = 3 - a$ this gives $b = 3(1 \mp \sqrt{5})/2$. We therefore have two more critical points:

$$\begin{aligned} p_2 &= (3(1 + \sqrt{5})/2, 3(1 - \sqrt{5})/2) \\ p_3 &= (3(1 - \sqrt{5})/2, 3(1 + \sqrt{5})/2). \end{aligned}$$

Example 2.15. Consider the function $f(x, y) = \sin(x)\sin(y)$. The derivatives are

$$\begin{array}{lll} f_x(x, y) = \cos(x)\sin(y) & & f_y(x, y) = \sin(x)\cos(y) \\ f_{xx}(x, y) = -\sin(x)\sin(y) & f_{xy}(x, y) = \cos(x)\cos(y) & f_{yy}(x, y) = -\sin(x)\sin(y). \end{array}$$

For f_x to be zero, we must have either $\cos(x) = 0$ or $\sin(y) = 0$. For f_y to be zero, we must have either $\sin(x) = 0$ or $\cos(y) = 0$. At a critical point, both f_x and f_y must be zero, so we have one of the following four cases:

(p) $\cos(x) = \sin(x) = 0$
(q) $\cos(x) = \cos(y) = 0$
(r) $\sin(y) = \sin(x) = 0$
(s) $\sin(y) = \cos(y) = 0$.

However, case (p) cannot actually happen, because $\cos(x)^2+\sin(x)^2$ is always equal to one. Similarly, case (s) cannot happen because $\cos(y)^2 + \sin(y)^2$ is always equal to one.

In case (r) we have $x = n\pi$ and $y = m\pi$ for some integers n and m, and $f(x, y) = \sin(n\pi)\sin(m\pi) = 0$. After noting that $\cos(k\pi) = (-1)^k$ we see that the Hessian matrix is

$$H = \begin{bmatrix} -\sin(n\pi)\sin(m\pi) & \cos(n\pi)\cos(m\pi) \\ \cos(n\pi)\cos(m\pi) & -\sin(n\pi)\sin(m\pi) \end{bmatrix} = \begin{bmatrix} 0 & (-1)^{n+m} \\ (-1)^{n+m} & 0 \end{bmatrix}.$$

Thus, in Method 2.12 we have $A_1 = 0$ and

$$A_2 = 0 \times 0 - (-1)^{n+m} \times (-1)^{n+m} = -1,$$

so there is a saddle point at $(n\pi, m\pi)$

Similarly, in case (q) we have $x = (n + \frac{1}{2})\pi$ and $y = (m + \frac{1}{2})\pi$ for some n and m. From the graph of $\sin(x)$ we can observe that $\sin((n + \frac{1}{2})\pi) = (-1)^n$, so in these cases we have

$$f(x, y) = \sin((n + \tfrac{1}{2})\pi)\sin((m + \tfrac{1}{2})\pi) = (-1)^{n+m}.$$

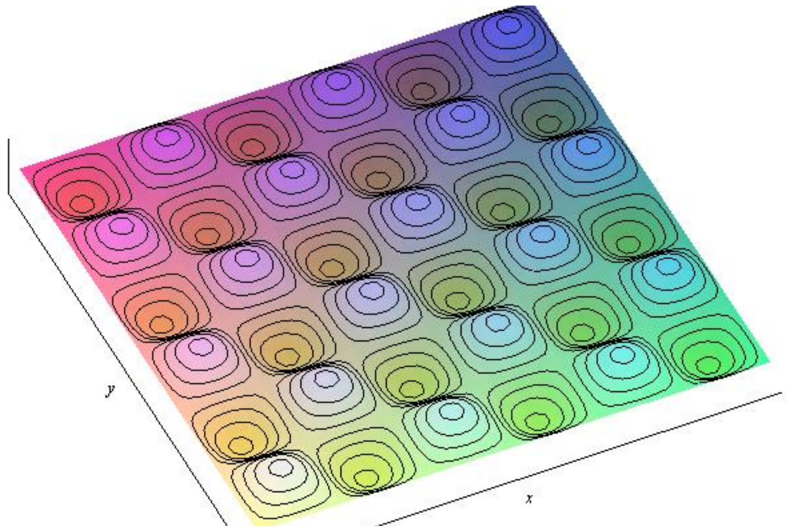
The Hessian matrix is

$$H = \begin{bmatrix} -\sin((n + \frac{1}{2})\pi)\sin((m + \frac{1}{2})\pi) & \cos((n + \frac{1}{2})\pi)\cos((m + \frac{1}{2})\pi) \\ \cos((n + \frac{1}{2})\pi)\cos((m + \frac{1}{2})\pi) & -\sin((n + \frac{1}{2})\pi)\sin((m + \frac{1}{2})\pi) \end{bmatrix} = \begin{bmatrix} (-1)^{n+m+1} & 0 \\ 0 & (-1)^{n+m+1} \end{bmatrix}.$$

This gives $A_1 = (-1)^{n+m+1}$ and

$$A_2 = (-1)^{n+m+1} \times (-1)^{n+m+1} - 0 \times 0 = 1.$$

If $n+m$ is even then $A_1 = -1 < 0$ so we have a local maximum (and in fact $f = 1$). If $n+m$ is odd then $A_1 = 1 > 0$ so we have a local minimum (and in fact $f = -1$).



Example 2.16. Consider the function $f(x,y) = e^{-x^2-y^2-2y}$. By the chain rule we have $f_x(x,y) = -2xe^{-x^2-y^2-2y}$. The second factor here is just $f(x,y)$ again, and it is convenient to write it that way, so that $f_x = -2xf$. For the second derivative we then get

$$\begin{aligned} f_{xx} &= (-2xf)_x = (-2x)_x\, f + (-2x)\, f_x \\ &= -2f + (-2x)\,(-2x)f = (4x^2-2)f. \end{aligned}$$

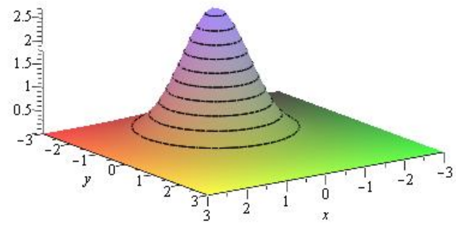
If we write all the other derivatives in terms of f in the same way we get

$$\begin{aligned} &f_x = -2xf & & & &f_y = (-2y-2)f \\ &f_{xx} = (4x^2-2)f & &f_{xy} = (4xy+4x)f & &f_{yy} = (4y^2+8y+2)f. \end{aligned}$$

Note that f itself is never zero, so for $f_x = -2xf$ to be zero we must have $-2x = 0$ or in other words $x = 0$. Similarly, for f_y to be zero we must have $(-2y-2)f = 0$ and so $y = -1$. Thus, there is only one critical point, namely $p = (0,-1)$. At this critical point we have $-x^2 - y^2 - 2y = -(0)^2 - (-1)^2 + 2 = 1$ so $f = e^1 = e$. The Hessian matrix is

$$H = \begin{bmatrix} 4x^2-2 & 4xy+4x \\ 4xy+4x & 4y^2+8y+2 \end{bmatrix} f = \begin{bmatrix} -2e & 0 \\ 0 & -2e \end{bmatrix}.$$

In Method 2.12 we therefore have $A_1 = -2e < 0$ and $A_2 = 4e^2 > 0$ so there is a local maximum.



2.4. **Functions of three or more variables.** If f is a function of several variables, then the maximum value of f (if there is one) will occur at a point where the partial derivatives with respect to all those variables are zero. The same condition will hold for the minimum value. For example, if f is a function of x, y and z, then the maximum and minimum will occur at a point where $f_x = f_y = f_z = 0$.

It is again possible to classify critical points using the Hessian. In the three variable case, the Hessian matrix is

$$H = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

In terms of eigenvalues, we can say the following:

Method 2.17. Let $p = (a, b, c)$ be a critical point of $f(x, y, z)$. Then all the eigenvalues of H at p are real.

(a) If all the eigenvalues are positive, then f has a local minimum at p.
(b) If all the eigenvalues are negative, then f has a local maximum at p.
(c) If some some eigenvalues are positive and some are negative, then we have something like a saddle point. In particular, we definitely do not have a local maximum or local minimum.
(d) If one or more eigenvalues is zero, then there are various possibilities that we will not explore further.

All this works in the same way if there are more than three variables.

There is also an alternative method involving determinants.

Method 2.18. Let $p = (a, b, c)$ be a critical point of $f(x, y, z)$. Put

$$A_1 = f_{xx} \qquad A_2 = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \qquad A_3 = \det \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix},$$

with everything evaluated at p.

(a) If $A_1 > 0$, $A_2 > 0$ and $A_3 > 0$ then f has a local minimum at p.
(b) If $A_1 < 0$, $A_2 > 0$ and $A_3 < 0$ then f has a local maximum at p.
(c) If A_1, A_2 and A_3 are nonzero but the pattern of signs is not as in (a) or (b), then we have something like a saddle point. In particular, we definitely do not have a local maximum or local minimum.
(d) If $A_1 = 0$ or $A_2 = 0$ or $A_3 = 0$, then there are various possibilities that we will not explore further.

All this generalises in a straightforward way if there are more than three variables. We take A_k to be the determinant of the top left $k \times k$ block in the Hessian matrix. If all the numbers A_k are positive we have a local minimum, and if all the numbers $(-1)^k A_k$ are positive then we have a local maximum.

Example 2.19. Consider the function $f(x,y,z) = 8(x^2+y^2+z^2) - (z+1)^3$. The first derivatives are $f_x = 16x$ and $f_y = 16y$ and

$$f_z = 16z - 3(z+1)^2 = 16z - 3z^2 - 6z + 3 = 3 + 10z - 3z^2 = (z-3)(1-3z).$$

This means that the critical points are where $x = y = 0$ and $(z-3)(3z-1) = 0$, so $z = 3$ or $z = 1/3$. The Hessian matrix is

$$H = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & & 10-6z \end{bmatrix},$$

so

$$A_1 = 16 \qquad A_2 = 256 \qquad A_3 = 256(10-6z).$$

At $(0,0,1/3)$ we get $A_3 = 1024$ so A_1, A_2 and A_3 are all positive, so we have a local minimum. At $(0,0,3)$ we have $A_3 = -1024$ so we are in case (c) of Method 2.18 and we have some kind of saddle.

3. Constrained optimisation

Often we want to find the maximum or minimum value of a function $f(x,y)$, where x and y cannot vary independently, but are linked by some kind of constraint, which we can write in the form $g(x,y) = 0$ for some other function g.

Example 3.1. We might want to find the closest point to the origin on the line with equation $3x + 4y = 5$. This amounts to minimising the function $f(x,y) = x^2 + y^2$ (the squared distance from the origin) subject to the constraint $g(x,y) = 0$, where $g(x,y) = 3x + 4y - 5$.

Example 3.2. Suppose we want to make a metal tank of volume $4m^3$. It will have length x, width y and height z (measured in metres), and it will have a base but no top, so there are five panels altogether, of areas xy, xz, xz, yz and yz. The total area of metal sheet that we need is thus $f(x,y,z) = xy + 2xz + 2yz$ (in square metres). As the volume needs to be $4m^3$, the function $g(x,y,z) = xyz - 4$ must be zero. To use as little metal as possible, we should minimise f subject to the constraint $g = 0$.

Method 3.3. To maximise $f(x,y,\dots)$ subject to the constraint $g(x,y,\dots) = 0$, simply find the unconstrained maximum of the function

$$L(\lambda, x, y, \dots) = f(x,y,\dots) - \lambda g(x,y,\dots)$$

and ignore the value of λ. The extra parameter λ is called a *Lagrange multiplier.*

We now explain in outline why this is usually valid. (If we were more careful we would discover that certain technical conditions are needed for the method to work, but we will not explore this in detail here. The conditions involved are usually satisfied.) We will focus on the case where we have just two variables (x and y); extra variables do not change the picture in any essential way. The solutions of $g(x,y) = 0$ will then form a curve in the plane, called the *constraint curve.* Note that

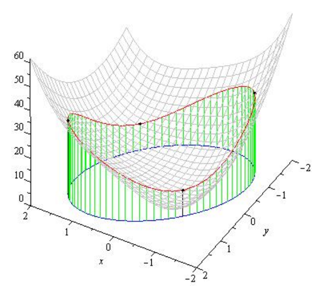
$$\begin{aligned} L_\lambda(\lambda,x,y) &= -g(x,y) \\ L_x(\lambda,x,y) &= f_x(x,y) - \lambda g_x(x,y) \\ L_y(\lambda,x,y) &= f_y(x,y) - \lambda g_y(x,y). \end{aligned}$$

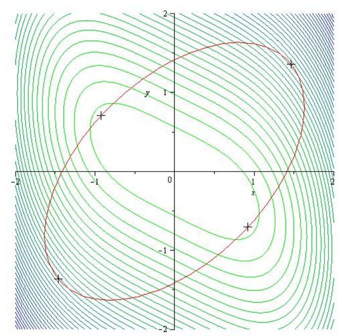
At a critical point of L, we must therefore have $g(x,y) = 0$ (so we are on the constraint curve) and $f_x = \lambda g_x$ and $f_y = \lambda g_y$. Now suppose we change x and y slightly, by δx and δy say, while leaving λ fixed. The resulting changes in f, g and h will be

$$\begin{aligned} \delta f &\simeq \delta x.f_x + \delta y.f_y \\ \delta g &\simeq \delta x.g_x + \delta y.g_y \\ \delta h &\simeq \delta x.L_x + \delta y.L_y = \delta x.(f_x - \lambda g_x) + \delta y.(f_y - \lambda g_y) \\ &= \delta f - \lambda \delta g. \end{aligned}$$

As (λ, x, y) is assumed to be a critical point of h, we have $\delta h = 0$, or equivalently $\delta h = \lambda\,\delta g$. This works for all small changes $(\delta x, \delta y)$, including those that move us off the constraint curve. We are mostly interested in changes that keep us on the constraint curve, which means that g remains equal to zero, so $\delta g = 0$. For such changes the equation $\delta f = \lambda\,\delta g$ becomes $\delta f = 0$. This means that (x, y) is a critical point for the variation of f along the constraint curve, so it will probably be a local maximum or minimum.

To understand this in another way, we can imagine a landscape where the height of the land at position (x, y) is given by $f(x, y)$. We can then think of the constraint curve as tracing out a road through this landscape. In this context, maximising f subject to $g = 0$ means finding the highest point on the road. Note that if the road is crossing the contours, then we can increase f by driving forwards or backwards a little, so we cannot already have the maximum possible value of f. Thus, to find the maximum, we need to look for places where the road is running along the contours, not crossing them. The normal vector to the contour through (a, b) is $\begin{bmatrix} f_x(a,b) \\ f_y(a,b) \end{bmatrix}$, and the normal vector to the road is $\begin{bmatrix} g_x(a,b) \\ g_y(a,b) \end{bmatrix}$. If the road is running along the contour then these two vectors will be multiples of each other, say $\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \lambda \begin{bmatrix} g_x \\ g_y \end{bmatrix}$ at (a, b). This means that $L_x(\lambda, a, b) = L_y(\lambda, a, b) = 0$, and as we are on the road we also have $L_\lambda(\lambda, a, b) = g(a, b) = 0$. This means that (λ, a, b) is a critical point of L, as expected.





The image on the left shows the surface $z = f(x, y)$ as a grey mesh. The constraint curve where $g(x, y) = 0$ is shown in blue in the xy-plane. The curve on the surface lying directly above it is in red. The problem is to find the highest and lowest points on the red curve. The image on the right is what we see when looking vertically downwards at the image on the left. The highest and lowest points are marked with crosses. These are where the red curve is tangent to the contours.

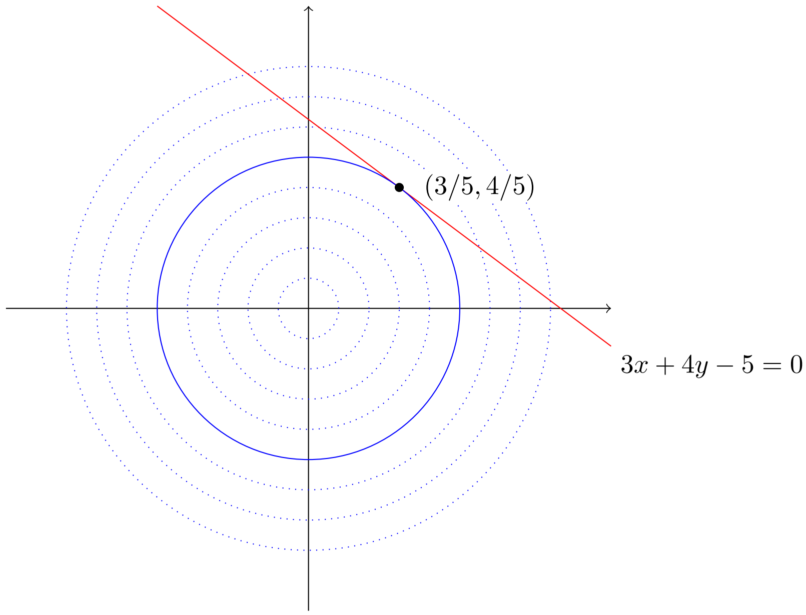
Example 3.4. In Example 3.1, we wanted to minimise $f(x, y) = x^2 + y^2$ subject to the constraint $g(x, y) = 3x + 4y - 5 = 0$. To do this, we need to find the unconstrained minimum of the function

$$L(\lambda, x, y) = x^2 + y^2 - \lambda(3x + 4y - 5).$$

The critical points are the places where the following partial derivatives are zero:

$$\begin{aligned} L_\lambda(\lambda, x, y) &= -g(x, y) = -3x - 4y + 5 \\ L_x(\lambda, x, y) &= 2x - 3\lambda \\ L_y(\lambda, x, y) &= 2y - 4\lambda. \end{aligned}$$

The equations $2x - 3\lambda = 2y - 4\lambda = 0$ give $x = 3\lambda/2$ and $y = 2\lambda$. Substituting these into the equation $3x + 4y - 5 = 0$ gives $9\lambda/2 + 8\lambda = 5$, which simplifies to $\lambda = 2/5$. Substituting this back into $x = 3\lambda/2$ and $y = 2\lambda$ gives $(x, y) = (3/5, 4/5)$. At this point we have $f(x, y) = (3/5)^2 + (4/5)^2 = (9 + 16)/25 = 1$. We conclude that the minimum value of f is 1, and that this occurs at the point $(3/5, 4/5)$.



Example 3.5. In Example 3.2 we wanted to minimise $f(x,y,z) = xy + 2xz + 2yz$ subject to the constraint $g(x,y,z) = xyz - 4 = 0$. To do this, we need to find the unconstrained minimum of the function

$$L(\lambda, x, y, z) = xy + 2xz + 2yz - \lambda(xyz - 4).$$

The critical points are the places where the following partial derivatives are zero:

$$\begin{aligned} L_\lambda(\lambda,x,y,z) &= -g(x,y,z) = 4 - xyz \\ L_x(\lambda,x,y,z) &= y + 2z - \lambda yz \\ L_y(\lambda,x,y,z) &= x + 2z - \lambda xz \\ L_z(\lambda,x,y,z) &= 2x + 2y - \lambda xy. \end{aligned}$$

We can rearrange these equations as follows:

$$xyz = 4 \tag{A}$$
$$z^{-1} + 2y^{-1} = \lambda \tag{B}$$
$$z^{-1} + 2x^{-1} = \lambda \tag{C}$$
$$2y^{-1} + 2x^{-1} = \lambda. \tag{D}$$

By subtracting equations (B) and (C) we see that $y^{-1} = x^{-1}$, so $x = y$. After substituting this in (D) we get $4x^{-1} = 4y^{-1} = \lambda$, so $x = y = 4/\lambda$. We can then substitute this in (C) to get $z^{-1} + \lambda/2 = \lambda$, so $z = 2/\lambda$. Now (A) becomes $(4/\lambda)(4/\lambda)(2/\lambda) = 4$, so $32 = 4\lambda^3$, so $\lambda = (32/4)^{1/3} = 2$. This gives $x = 4/\lambda = 2$, and similarly $y = 2$ and $z = 1$. For these values of x, y and z we have $f(x,y,z) = f(2,2,1) = 12$. Thus, the most efficient design is to make the dimensions of the tank $2 \times 2 \times 1$ metres, and we then need $12m^2$ of metal sheet.

(Note, incidentally, that in our initial rearrangement we divided the equation $y + 2z - \lambda yz = 0$ by yz to get $z^{-1} + 2y^{-1} = \lambda$. This would not be valid if yz were zero. Fortunately, yz cannot be zero, because we have $xyz = 4$. You should be careful to check this kind of thing when dividing.)

Example 3.6. Suppose we want to maximise $f(x,y) = x + y$ subject to the constraint $x^2/a + y^2/b = 1$ (for some constants $a, b > 0$). Here we must take $g(x,y) = x^2/a + y^2/b - 1$ and so

$$L(\lambda, x, y) = x + y - \lambda(x^2/a + y^2/b - 1).$$

The critical points are the places where the following partial derivatives are zero:

$$\begin{aligned}
L_\lambda(\lambda,x,y) &= g(x,y) = x^2/a + y^2/b - 1 \\
L_x(\lambda,x,y) &= 1 - 2x\lambda/a \\
L_y(\lambda,x,y) &= 1 - 2y\lambda/b.
\end{aligned}$$

The equations $1-2x\lambda/a = 1-2y\lambda/b = 0$ give $x = a/(2\lambda)$ and $y = b/(2\lambda)$. We can substitute these values in the equation $x^2/a + y^2/b = 1$ to get $a/(4\lambda^2) + b/(4\lambda^2) = 1$, which can be rearranged as $\lambda^2 = (a+b)/4$, so $\lambda = \pm\sqrt{a+b}/2$. As $x = a/(2\lambda)$ and $y = b/(2\lambda)$ we get

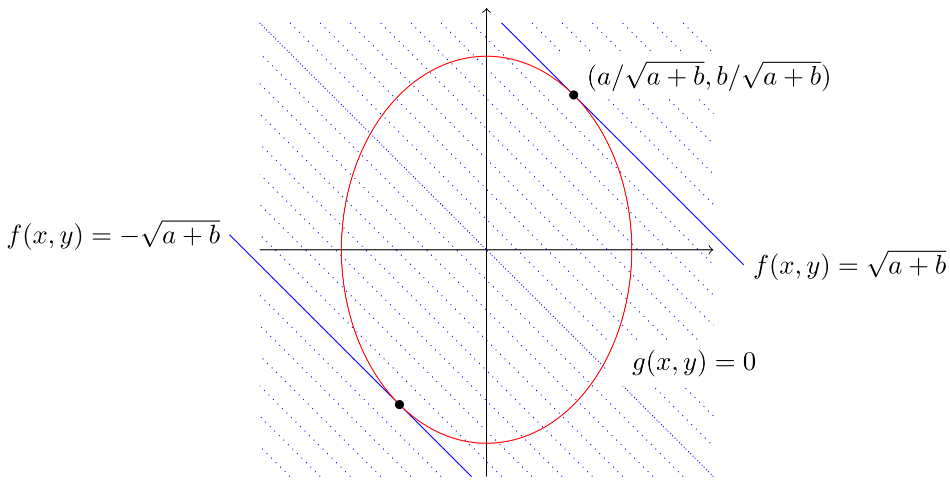
$$(x,y) = \pm\left(\frac{a}{\sqrt{a+b}}, \frac{b}{\sqrt{a+b}}\right).$$

For these points we have

$$f(x,y) = x + y = \pm(a+b)/\sqrt{a+b} = \pm\sqrt{a+b}.$$

This means that the maximum possible value of f (subject to the constraint) is $\sqrt{a+b}$, and the minimum is $-\sqrt{a+b}$.

This can be illustrated as follows:



We now consider the problem of optimisation when there are several constraints.

Method 3.7. To find the maximum and minimum values of $f(x,y,\dots)$ subject to two constraints $g=0$ and $h=0$, we find the unconstrained critical points of the function

$$L(\lambda,\mu,x,y,\dots) = f(x,y,\dots) - \lambda g(x,y,\dots) - \mu h(x,y,\dots)$$

(and then ignore the values of λ and μ). More generally, if there are constraints $g_1 = 0,\dots,g_r = 0$ then we use

$$L(\lambda_1,\dots,\lambda_r,x,y,\dots) = f(x,y,\dots) - \sum_{i=1}^{r} \lambda_i g_i(x,y,\dots).$$

Example 3.8. Suppose we want to maximise z subject to the constraints $x^2+y^2+z^2 = 9$ and $x+2y+4z = 3$. For this we need the unconstrained critical points of the function

$$L = z - \lambda(x^2+y^2+z^2-9) - \mu(x+2y+4z-3).$$

The partial derivatives L_λ, L_μ, L_x, L_y and L_z must all be zero, which gives the following equations:

$$x^2+y^2+z^2 = 9 \qquad \text{(A)}$$

$$x+2y+4z = 3 \qquad \text{(B)}$$

$$-2\lambda x - \mu = 0 \qquad \text{(C)}$$

$$-2\lambda y - 2\mu = 0 \qquad \text{(D)}$$

$$1 - 2\lambda z - 4\mu = 0. \qquad \text{(E)}$$

These equations are sufficiently complex that in practice you would probably use a computer to solve them. Nonetheless, it is possible to explain the solution by hand, as follows. We first multiply equation (C) by x, and equation (D) by y, and equation (E) by z, and add them together to get

$$z - 2\lambda(x^2 + y^2 + z^2) - \mu(x + 2y + 4z) = 0.$$

We can now substitute equations (A) and (B) in this to get

$$z - 18\lambda - 3\mu = 0. \tag{F}$$

Alternatively, we can add equation (C) to 2 times equation (D) and 4 times equation (E) to get

$$4 - 2\lambda(x + 2y + 4z) - 21\mu = 0.$$

We can now substitute equation (B) in this to get $4 - 6\lambda - 21\mu = 0$, so

$$\lambda = \tfrac{7}{2}\mu - \tfrac{2}{3}. \tag{G}$$

Subtracting 3 times (G) from (F) gives

$$z = 12 - 60\mu. \tag{H}$$

If we substitute (G) and (H) in (E) and expand out, we get $-15 + 160\mu - 420\mu^2 = 0$. Solving this quadratic equation in the usual way, we get $\mu = 1/6$ or $\mu = 3/14$. If $\mu = 1/6$ then (G) and (H) become $\lambda = 1/12$ and $z = 2$. After substituting these values into (C) and (D) we get $x = -1$ and $y = -2$. Thus, we have a critical point of L at

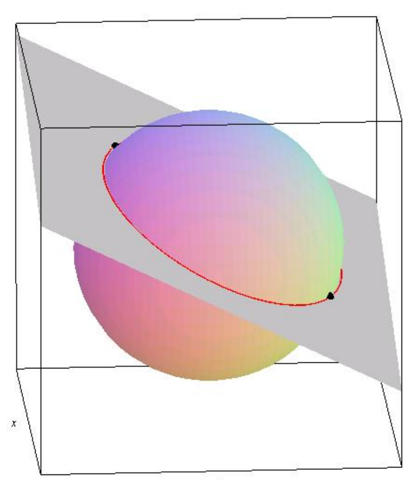
$$(\lambda, \mu, x, y, z) = (1/12, 1/6, -1, -2, 2).$$

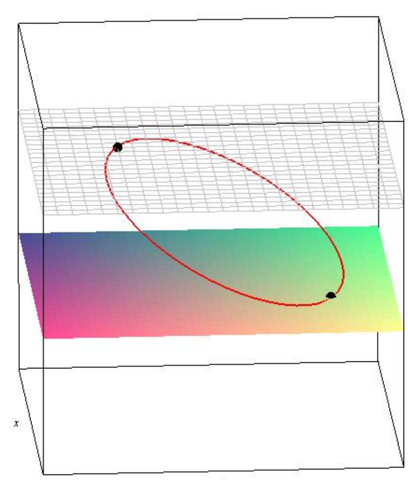
On the other hand, if $\mu = 3/14$ then (G) and (H) become $\lambda = -1/12$ and $z = -6/7$. After substituting these into (C) and (D) we get $x = 9/7$ and $y = 18/7$. Thus, we have a critical point of L at

$$(\lambda, \mu, x, y, z) = (-1/12, 3/14, 9/7, 18/7, -6/7).$$

Thus, the maximum value of z is 2 (at the first critical point) and the minimum is $-6/7$ (at the second critical point).

Geometrically, the constraint $x^2 + y^2 + z^2 = 9$ corresponds to a sphere of radius three around the origin, and the constraint $x + 2y + 4z = 3$ corresponds to a plane that cuts through the sphere. We are interested in points (x, y, z) where both constraints hold. These lie on the intersection of the sphere with the plane, which is the sloping circle in the picture on the left below. The picture on the right shows the same circle together with the planes $z = -6/7$ and $z = 2$, showing that these are indeed the maximum and minimum values of z on the curve.





4. Multiple integrals

4.1. **Integrals over plane regions.** Let D be a region in the plane, and let $f(x,y)$ be a function defined for points (x,y) in D. We define the integral $\iint_D f(x,y)dA$ as follows. First, we divide the region D into a large number of small regions $D_1,\dots,D_n$. As each region D_i is small, the value of f will not change much as we move around D_i, so it makes approximate sense to talk about the value of f on D_i as a single number. The integral is approximately defined by

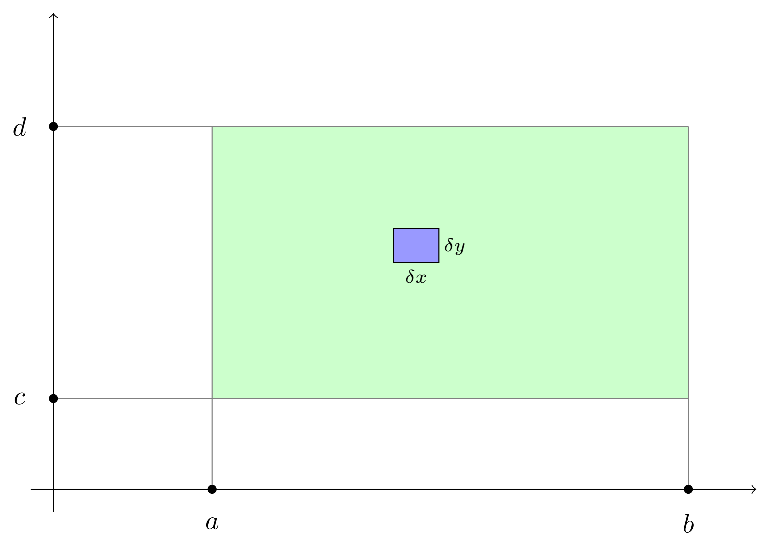
$$\iint_D f(x,y)\,dA = \sum_{i=1}^{N} (\text{ value of } f \text{ on } D_i\ ) \times (\text{ area of } D_i).$$

To get the exact value, we divide D into a larger and larger number of smaller and smaller pieces, and then pass to the limit. It takes considerable work to formulate this in a precise and rigorous way, but this general idea will be sufficient for our purposes.

Some applications of this kind of integration are as follows.

(a) Suppose that the region D is a charged plate, and that the charge density at a point (x,y) is $q(x,y)$; then the total charge is $Q=\iint_D q(x,y)\,dA$.
(b) Suppose that the region D represents a structure of constant density ρ and vertical thickness $f(x,y)$ attached to an axle passing vertically through the origin. Then the mass of the structure is $\iint_D \rho f(x,y)\,dA$, whereas the moment of inertia (which measures the difficulty of turning the axle) is $\iint_D \rho f(x,y)(x^2+y^2)dA$.
(c) Suppose that the region D represents a large solar cell, with the brightness of light arriving at (x,y) being given by the function $f(x,y)$. Then the total incident power on the cell will be (a constant times) $\iint_D f(x,y)\,dA$.
(d) The total area of a region D is just $\iint_D 1\,dA$.
(e) Suppose we have a solid region E in three-dimensional space. Often E can be described as follows: there is a two-dimensional region D that is the shadow of E in the (x,y)-plane, and the vertical line through (x,y) meets E at height $f(x,y)$ and leaves it at height $g(x,y)$ say. The total volume of E is then $\iint_D g(x,y)-f(x,y)\,dA$.

In the simplest case, the region D is a rectangle aligned with the axes, given by $a\leq x\leq b$ and $c\leq y\leq d$ say. In this case we can just divide the horizontal interval $[a,b]$ into small intervals of length δx, and divide the vertical interval $[c,d]$ into small intervals of length δy. This divides D into small rectangles of area $\delta A=\delta x.\delta y$.



Using this kind of subdivision, we see that the area integral is just obtained by integrating with respect to both variables x and y:

$$\iint_D f(x,y)\,dA = \int_{x=a}^{b}\left(\int_{y=c}^{d} f(x,y)\,dy\right)\,dx.$$

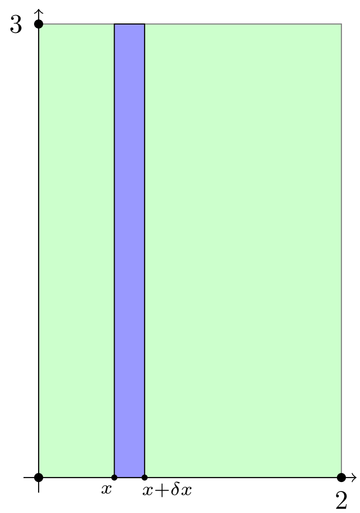
Example 4.1. Let D be the rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$. Then

$$\iint_D x^3 + y^2\,dA = \int_{x=0}^{2}\left(\int_{y=0}^{3} x^3 + y^2\,dy\right)dx$$

In the inner integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=0}^{3} x^3 + y^2\,dy = \Big[x^3 y + y^3/3\Big]_{y=0}^{3} = (3x^3 + 27/3) - (0) = 3x^3 + 9.$$

The meaning of this intermediate result is as follows: if we take a thin strip running horizontally from x to $x + \delta x$, and vertically all the way from 0 to 3, then the sum of the corresponding contributions is approximately $(3x^3 + 9)\delta x$ (and the approximation becomes exact in the limit as $\delta x \to 0$).



We can now perform the outer integral to add up the contributions from all such vertical strips.

$$\int_{x=0}^{2} 3x^3 + 9\,dx = \Big[3x^4/4 + 9x\Big]_{x=0}^{2} = (12 + 18) - (0) = 30.$$

The conclusion is that $\iint_D x^3 + y^2\,dA = 30$.

Example 4.2. Let D be the square where $0 \leq x \leq \pi$ and $-\pi/2 \leq y \leq \pi/2$. Then

$$\iint_D \sin(x)\cos(y)\,dA = \int_{x=0}^{\pi}\left(\int_{y=-\pi/2}^{\pi/2} \sin(x)\cos(y)\,dy\right)dx.$$

In the inner integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=-\pi/2}^{\pi/2} \sin(x)\cos(y)\,dy = \sin(x)\Big[\sin(y)\Big]_{y=-\pi/2}^{\pi/2} = \sin(x)(1 - (-1)) = 2\sin(x).$$

Again, this means that the contribution coming from a vertical strip of width δx is approximately $2\sin(x)\delta x$. We can now perform the outer integral to add up the contributions from all such vertical strips:

$$\int_{x=0}^{\pi} 2\sin(x)\,dx = 2\Big[-\cos(x)\Big]_0^{\pi} = 2(1 - (-1)) = 4.$$

The conclusion is that $\iint_D \sin(x)\cos(y)\,dA = 4$.

Note that in both the last two examples, the final answer is just a number, not a function of x or y. This is as it should be: we have added up all the contributions from all the different values of x and y, so there is no dependence on x and y left at the end. There are various common errors that lead to final answers depending on x and y. If you end up with such an answer, you should immediately realise that something has gone wrong.

We now consider Example 4.1 again. We previously did this by dividing the rectangle D into thin vertical strips, performing an inner integral over y to calculate the contribution from each strip, and then integrating over x to add up these contributions. We could equally well divide the rectangle into thin horizontal strips instead. We would then need to perform an inner integral over x to find the contribution from each horizontal strip, and then an outer integral over y to add up these contributions. In symbols, we have

$$\iint_D x^3 + y^2 \, dA = \int_{y=0}^{3} \left(\int_{x=0}^{2} x^3 + y^2 \, dx \right) dy.$$

In the inner integral, we treat y as a constant and x as a variable. This gives

$$\int_{x=0}^{2} x^3 + y^2 \, dx = \left[x^4/4 + xy^2 \right]_{x=0}^{2} = (16/4 + 2y^2) - (0) = 4 + 2y^2,$$

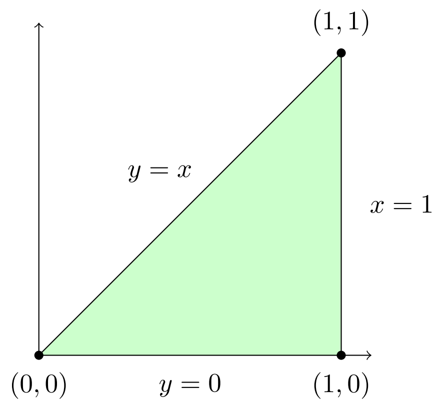
meaning that the contribution from a horizontal strip of width δy at height y is approximately $(4 + 2y^2)\delta y$. We can now perform the outer integral to add up the contributions from all such horizontal strips:

$$\int_{y=0}^{3} 4 + 2y^2 \, dy = \left[4y + 2y^3/3 \right]_{y=0}^{3} = (12 + 2 \times 27/3) - (0) = 30.$$

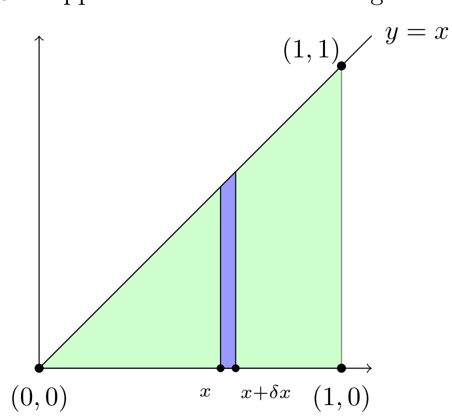
As expected, this gives the same answer $\iint_D x^3 + y^2 \, dA = 30$ as before.

The next simplest type of region that we want to consider is a triangle.

Example 4.3. Let D be the triangular region with vertices at $(0, 0)$, $(1, 0)$ and $(1, 1)$.



We will calculate $\iint_D e^{2x-2y} \, dA$. One approach is to divide the region into thin vertical strips:



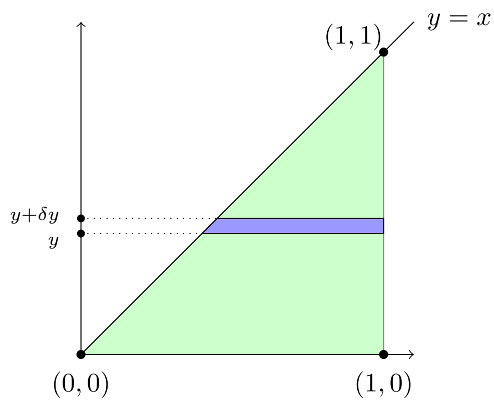
The strip shown runs from $y = 0$ to $y = x$, so the contribution from that strip is

$$\delta x. \int_{y=0}^{x} e^{2x-2y}\, dy = \delta x \left[e^{2x-2y}/(-2) \right]_{y=0}^{x} = \delta x (e^0 - e^{2x})/(-2) = \tfrac{1}{2}(e^{2x} - 1)\delta x.$$

To add up the contributions for all such strips, we need to integrate over all values of x that occur in D, which means from $x = 0$ to $x = 1$. This gives

$$\iint_D e^{2x-2y}\, dA = \int_{x=0}^{1} \tfrac{1}{2}(e^{2x} - 1)\, dx = \left[\tfrac{1}{2}(\tfrac{1}{2}e^{2x} - x) \right]_{x=0}^{1} = \tfrac{1}{2}(\tfrac{1}{2}e^2 - 1) - \tfrac{1}{2}(\tfrac{1}{2} - 0) = (e^2 - 3)/4.$$

Example 4.4. As an alternative, we could divide the triangle into horizontal strips:



The left hand end of the strip is at $x = y$, and the right hand end is at $x = 1$. Thus, the contribution from the strip is

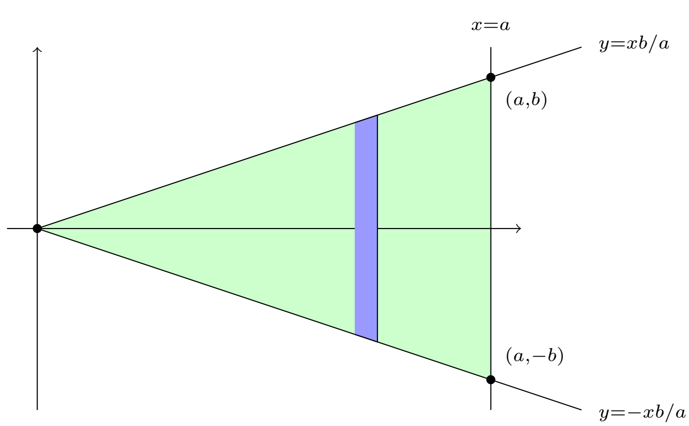
$$\delta y. \int_{x=y}^{1} e^{2x-2y} dx = \delta y. \left[\tfrac{1}{2}e^{2x-2y} \right]_{x=y}^{1} = \delta y.(\tfrac{1}{2}e^{2-2y} - \tfrac{1}{2}).$$

To add up the contributions for all such strips, we need to integrate over all values of y that occur in D, which means from $y = 0$ to $y = 1$. This gives

$$\iint_D e^{2x-2y}\, dA = \int_{y=0}^{1} \tfrac{1}{2}(e^{2-2y} - 1)\, dy = \left[\tfrac{1}{2}(-\tfrac{1}{2}e^{2-2y} - y) \right]_{y=0}^{1} = \tfrac{1}{2}(-\tfrac{1}{2} - 1) - \tfrac{1}{2}(-\tfrac{1}{2}e^2 - 0) = (e^2 - 3)/4,$$

just as before.

Example 4.5. Suppose we have a metal sheet D in the shape of a triangular wedge, with one vertex at the origin and the other two at (a, b) and $(a, -b)$.



To find the moment of inertia, we need to evaluate $\iint_D x^2 + y^2\, dA$. If we fix x with $0 \leq x \leq a$, then y will run from $-xb/a$ to $+xb/a$. We therefore have

$$\iint_D x^2 + y^2\, dA = \int_{x=0}^{a} \int_{y=-xb/a}^{xb/a} x^2 + y^2\, dy\, dx.$$

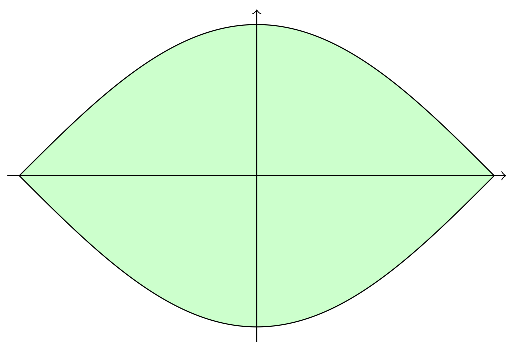
For the inner integral we have

$$\begin{aligned}
\int_{y=-xb/a}^{xb/a} x^2 + y^2\, dy &= \left[x^2 y + \tfrac{1}{3} y^3 \right]_{y=-xb/a}^{xb/a} \\
&= \left(x^2 . \frac{xb}{a} + \frac{1}{3} \left(\frac{xb}{a} \right)^3 \right) - \left(x^2 . \frac{-xb}{a} + \frac{1}{3} \left(\frac{-xb}{a} \right)^3 \right) \\
&= \frac{2x^3 b}{a} + \frac{2x^3 b^3}{3a^3} = \left(\frac{2b}{a} + \frac{2b^3}{3a^3} \right) x^3.
\end{aligned}$$

Using this we get

$$\iint_D x^2 + y^2\, dA = \left(\frac{2b}{a} + \frac{2b^3}{3a^3} \right) \int_{x=0}^{a} x^3\, dx = \left(\frac{2b}{a} + \frac{2b^3}{3a^3} \right) \frac{a^4}{4} = \tfrac{1}{2} a^3 b + \tfrac{1}{6} a b^3.$$

Example 4.6. Let D be the region where $-\pi/2 \leq x \leq \pi/2$ and $-\cos(x) \leq y \leq \cos(x)$.



We will find the area of D, or in other words the integral $\iint_D 1\, dA$. Using vertical strips we have

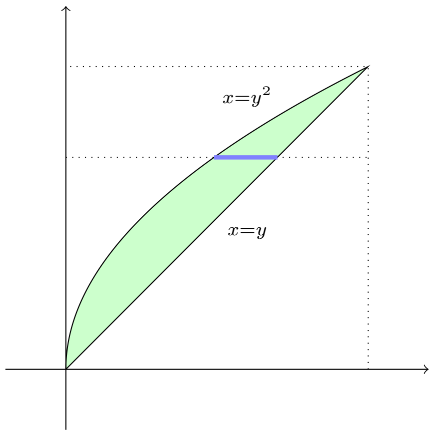
$$\begin{aligned}
\iint_D 1\, dA &= \int_{x=-\pi/2}^{\pi/2} \int_{y=-\cos(x)}^{\cos(x)} 1\, dy\, dx = \int_{x=-\pi/2}^{\pi/2} \Big[y \Big]_{-\cos(x)}^{\cos(x)} dx \\
&= \int_{x=-\pi/2}^{\pi/2} 2\cos(x)\, dx = \Big[2\sin(x) \Big]_{x=-\pi/2}^{\pi/2} \\
&= 2 - (-2) = 4.
\end{aligned}$$

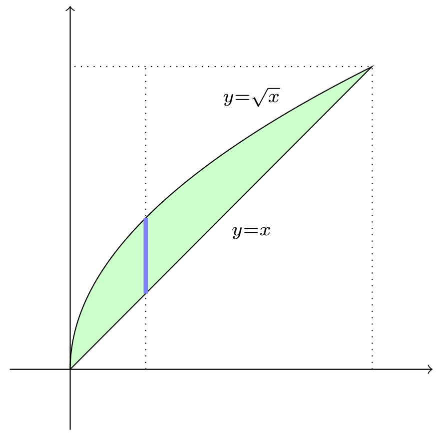
In the examples that we have considered so far, we were given a picture of the relevant region, and we had to find the corresponding limits of integration ourselves. Some natural examples work the other way around: we start with a formula for the limits of integration, and we need to draw the corresponding region. Once we have drawn the region, it may be possible to see different ways to approach the integral; in particular, we can try reversing the order of integration. This may (or may not) make it easier to evaluate the integral.

Example 4.7. Consider the integral

$$I = \int_{y=0}^{1} \int_{x=y^2}^{y} x^{-1} y e^x\, dx\, dy.$$

For the inner integral we would need to know $\int x^{-1} e^x\, dx$, but there is no formula for this integral in terms of familiar functions, so we appear to be stuck. However, we can sketch the relevant region D as shown on the left below:





In our original expression for I, the inner integral is with respect to x, and the outer one is with respect to y; this corresponds to dividing D into horizontal strips. We could instead divide it into vertical strips as shown on the right. The bottom end of the vertical strip at position x is given by $x = y$, or equivalently $y = x$. The top end of the vertical strip at position x is given by $x = y^2$, or equivalently $y = \sqrt{x}$. The overall limits on x are from 0 to 1. We therefore have

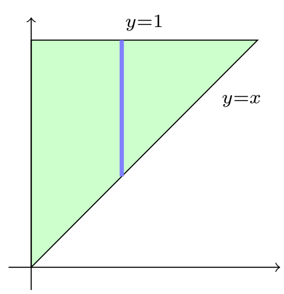
$$\begin{aligned} I &= \int_{x=0}^{1}\int_{y=x}^{\sqrt{x}} x^{-1}ye^x\,dy\,dx = \int_{x=0}^{1}\left[\tfrac{1}{2}x^{-1}y^2e^x\right]_{y=x}^{\sqrt{x}} dx \\ &= \frac{1}{2}\int_{x=0}^{1}(x^{-1}(\sqrt{x})^2e^x - x^{-1}x^2e^x)\,dx = \frac{1}{2}\int_{x=0}^{1}(e^x - xe^x)\,dx \\ &= \frac{1}{2}\Big[(2-x)e^x\Big]_{x=0}^{1} = (e-2)/2. \end{aligned}$$

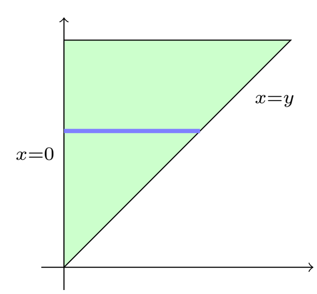
(Here we evaluated $\int x\,e^x\,dx$ using integration by parts. In more detail, we take $u = x$ and $dv/dx = e^x$, so $du/dx = 1$ and $v = e^x$. The rule $\int u\frac{dv}{dx}dx = uv - \int \frac{du}{dx}v\,dx$ gives $\int x\,e^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x$.)

Example 4.8. Consider the integral

$$I = \int_{x=0}^{1}\int_{y=x}^{1}\frac{xy}{\sqrt{1+y^4}}\,dy\,dx$$

It is possible to evaluate the inner integral starting with the substitution $u = y^2$, but it turns out to be easier to first change the order of integration. The relevant region is as follows:





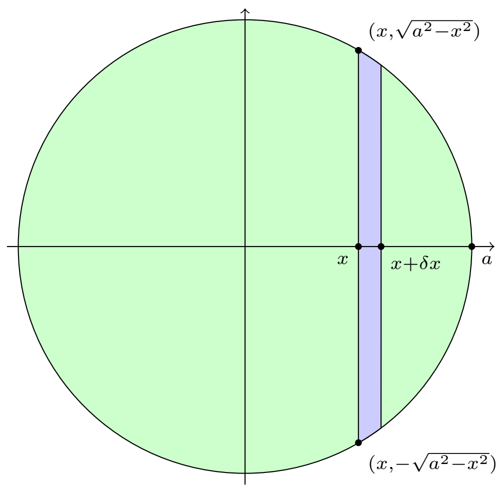
The original presentation (with the integral over x on the outside) corresponds to dividing the region into vertical strips running from $y = x$ to $y = 1$, as on the left. If we instead use horizontal strips running from $x = 0$ to $x = y$, we get the formula

$$\begin{aligned} I &= \int_{y=0}^{1}\int_{x=0}^{y}\frac{xy}{\sqrt{1+y^4}}\,dy\,dx = \int_{y=0}^{1}\left[\frac{x^2y}{2\sqrt{1+y^4}}\right]_{x=0}^{y} dy \\ &= \int_{y=0}^{1}\frac{y^3}{2\sqrt{1+y^4}}\,dy. \end{aligned}$$

We now substitute $u = 1 + y^4$, so $du/dy = 4y^3$, so $y^3\, dy = du/4$ and $\sqrt{1+y^4} = u^{1/2}$. The limits $y = 0$ and $y = 1$ correspond to $u = 1$ and $u = 2$. This gives

$$\begin{aligned} I &= \int_{u=1}^{2} \frac{du/4}{2u^{1/2}} = \frac{1}{8}\int_{u=1}^{2} u^{-1/2} du \\ &= \frac{1}{8}\Big[2u^{1/2}\Big]_{u=1}^{2} = (2\sqrt{2}-2)/8 \\ &= (\sqrt{2}-1)/4 \simeq 0.1036. \end{aligned}$$

4.2. **Polar coordinates.** We now consider integrals over circular disks and similar regions. One possibility is to follow the same method that we used for triangles, and divide the disk into vertical strips. If we consider a disk D of radius a centred at $(0,0)$, the vertical strip at position x will run from $y = -\sqrt{a^2-x^2}$ to $y = +\sqrt{a^2-x^2}$.



Consider the integral $I = \iint_D x^2\, dA$. This becomes

$$I = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} x^2\, dy\, dx.$$

In the inner integral x^2 counts as a constant, so we just have

$$\int_{y=-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} x^2\, dy = \Big[x^2 y\Big]_{y=-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} = 2x^2\sqrt{a^2-x^2}.$$

We therefore have

$$I = \int_{x=-a}^{a} 2x^2\sqrt{a^2-x^2}\, dx.$$

To evaluate this, we make the substitution $x = a\sin(\theta)$, so θ runs from $-\pi/2$ (corresponding to $x = -a$) to $\theta = \pi/2$ (corresponding to $x = +a$). We then have $dx/d\theta = a\cos(\theta)$, so $dx = a\cos(\theta)d\theta$, and $\sqrt{a^2-x^2} = a\sqrt{1-\sin^2(\theta)} = a\cos(\theta)$. This gives

$$I = \int_{\theta=-\pi/2}^{\pi/2} 2a^2\sin^2(\theta).a\cos(\theta).a\cos(\theta)\, d\theta = 2a^4\int_{-\pi/2}^{\pi/2} \sin^2(\theta)\cos^2(\theta)\, d\theta.$$

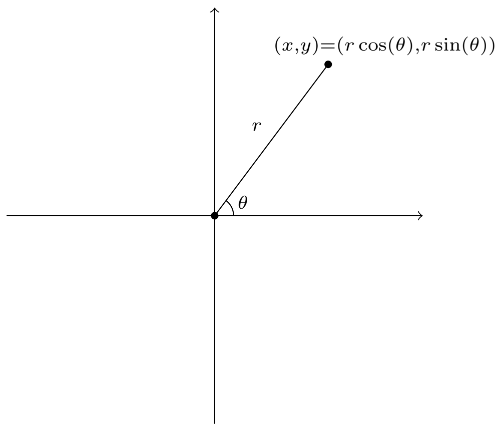
Here $\sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta)$, so

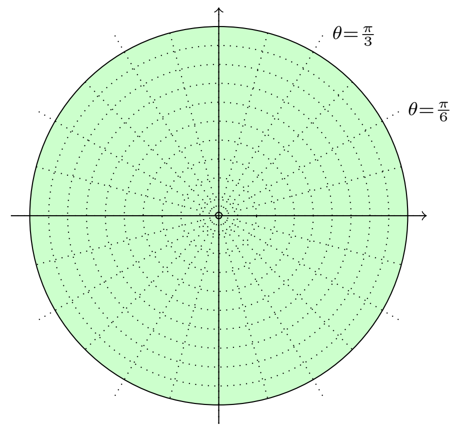
$$\sin^2(\theta)\cos^2(\theta) = \tfrac{1}{4}\sin^2(2\theta) = \tfrac{1}{4}\,\frac{1-\cos(4\theta)}{2}.$$

This gives

$$
\begin{aligned}
I &= \frac{a^4}{4}\int_{\theta=-\pi/2}^{\pi/2} 1-\cos(4\theta)\,d\theta \\
&= \frac{a^4}{4}\left[\theta - \tfrac{1}{4}\sin(4\theta)\right]_{-\pi/2}^{\pi/2} \\
&= \frac{a^4}{4}((\pi/2-0)-(-\pi/2-0)) = \frac{\pi a^4}{4}.
\end{aligned}
$$

However, a more natural approach for integrals with circular symmetry is to divide the region using polar coordinates. Recall that the basic setup of polar coordinates is as shown on the left below: we describe points using the distance r from the origin and the angle θ anticlockwise from the x-axis.



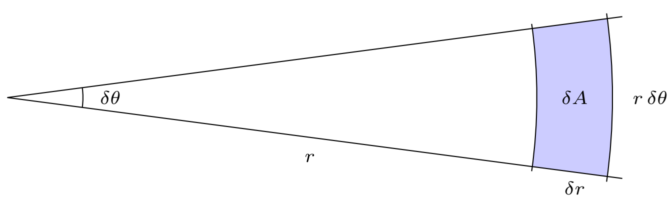


Polar coordinates are related to ordinary (rectangular) coordinates by the formulae

$$
\begin{aligned}
x &= r\cos(\theta) & y &= r\sin(\theta) \\
r &= \sqrt{x^2+y^2} & \theta &= \arctan(y/x).
\end{aligned}
$$

(The last of these formulae needs a little interpretation, because arctan() is a multivalued function and we need to choose the right value depending on the signs of x and y. However, we will not explore that in more detail here.) In the diagram on the right above, we have divided a disk into small pieces using lines of constant θ and circles of constant r. To use this kind of subdivision for integration, we need to know the area of the small pieces.

Consider a piece of angular width $\delta\theta$, where the radius runs from r to $r+\delta r$. Provided that $\delta\theta$ is small this will be approximately rectangular. If we measure angles in radians (as we always will) then the length of the curved side will be $r\,\delta\theta$, and the straight side has length δr, so the area is approximately $\delta A = r\,\delta r\,\delta\theta$.



In the limit this becomes $dA = r\,dr\,d\theta$, so we have the following prescription: if D is a region that is conveniently described in polar coordinates, then

$$
\iint_D f(x,y)\,dA = \int_{\theta=\cdots}^{\cdots}\int_{r=\cdots}^{\cdots} f(r\cos(\theta), r\sin(\theta))r\,dr\,d\theta,
$$

where the limits need to be filled in in accordance with the geometry of the region.

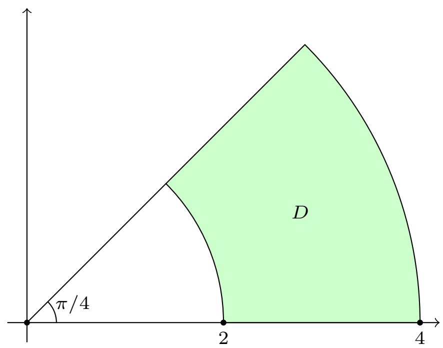
As an example, we can use this method to reevaluate the integral $\iint_D x^2\,dA$ over a disk of radius a, as we considered previously. Here the appropriate limits are just $0 \le \theta \le 2\pi$ and $0 \le r \le a$. The integral is just

$$\begin{aligned}
\iint_D x^2\,dA &= \int_{\theta=0}^{2\pi}\int_{r=0}^{a} r^2\cos^2(\theta)\,r\,dr\,d\theta \\
&= \int_{\theta=0}^{2\pi}\cos^2(\theta)\int_{r=0}^{a} r^3\,dr\,d\theta \\
&= \frac{a^4}{4}\int_{\theta=0}^{2\pi}\cos^2(\theta)\,d\theta = \frac{a^4}{4}\int_{\theta=0}^{2\pi}\frac{1+\cos(2\theta)}{2}\,d\theta \\
&= \frac{a^4}{4}\left[\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right]_0^{2\pi} = \frac{\pi a^4}{4}
\end{aligned}$$

as before.

We will now go through some similar calculations.

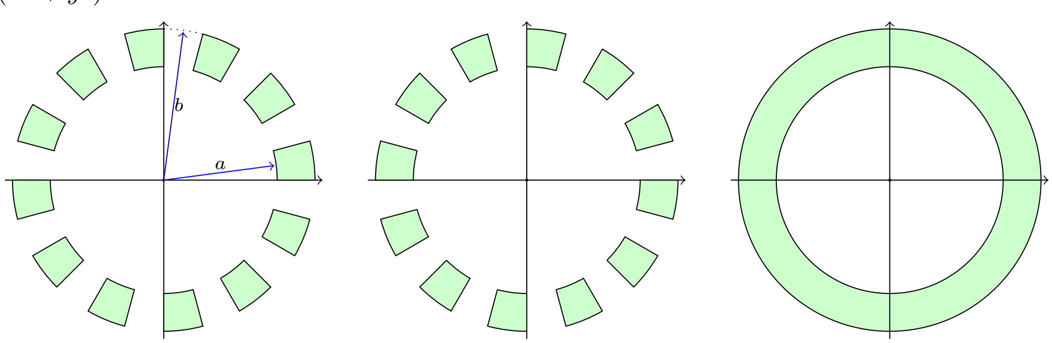
Example 4.9. We will calculate $\iint_D xy\,dA$ for the region D shown below.



In polar coordinates the integrand xy becomes $(r\cos(\theta)).(r\sin(\theta)) = \frac{1}{2}r^2\sin(2\theta)$. The limits are $2 \le r \le 4$ and $0 \le \theta \le \pi/4$, and the area element dA is $r\,dr\,d\theta$. We thus have

$$\begin{aligned}
\iint_D xy\,dA &= \int_{\theta=0}^{\pi/4}\int_{r=2}^{4}\tfrac{1}{2}r^3\sin(2\theta)\,dr\,d\theta = \int_{\theta=0}^{\pi/4}\left[\frac{r^4}{8}\sin(2\theta)\right]_{r=2}^{4}d\theta \\
&= \tfrac{1}{8}(256-16)\int_{\theta=0}^{\pi/4}\sin(2\theta)\,d\theta = 30\left[-\frac{\cos(2\theta)}{2}\right]_{\theta=0}^{\pi/4} \\
&= 15(-\cos(\pi/2)+\cos(0)) = 15(-0+1) = 15.
\end{aligned}$$

Example 4.10. Suppose we want to calculate the moment of inertia of a slotted rotor whose cross section is the region D shown on the left below. This will be the density times the length times the integral $I = \iint_D (x^2+y^2)dA$.



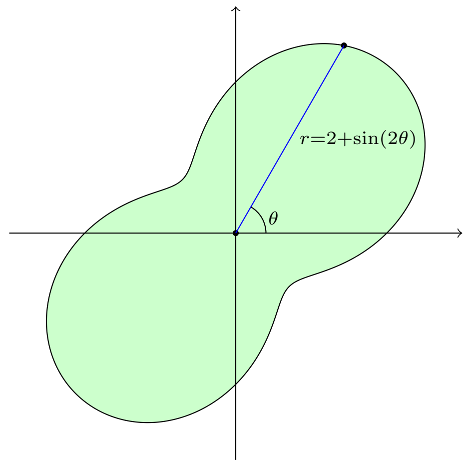
We first use a simplifying trick. Let D' be the region in the middle picture, and put $I' = \iint_{D'}(x^2+y^2)dA$. As D' is just obtained by turning D slightly, the moment of inertia will be the same, so $I' = I$. On the other hand, $2I = I + I'$ is just the integral over the simpler region D'' shown on the right. We thus have $I = \frac{1}{2}\iint_{D''}(x^2+y^2)dA$. For D'' the limits are just $0 \leq \theta \leq 2\pi$ and $a \leq r \leq b$. The integrand is

$$x^2 + y^2 = (r\cos(\theta))^2 + (r\sin(\theta))^2 = r^2,$$

and the area element is $dA = r\,dr\,d\theta$. We thus have

$$\begin{aligned} I &= \tfrac{1}{2}\int_{\theta=0}^{2\pi}\int_{r=a}^{b} r^3\,dr\,d\theta \\ &= \tfrac{1}{2}\int_{\theta=0}^{2\pi} \frac{b^4-a^4}{4}d\theta \\ &= \frac{1}{2}\,\frac{b^4-a^4}{4}\,2\pi = \pi(b^4-a^4)/4. \end{aligned}$$

Example 4.11. The picture below shows the region D given in polar coordinates by $0 \leq r \leq 2+\sin(2\theta)$.



We would like to find the area of D, or in other words $A = \iint_D 1\,dA$. Here $dA = r\,dr\,d\theta$ as usual, and the relevant limits are $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2 + \sin(\theta)$, so

$$\begin{aligned} A &= \int_{\theta=0}^{2\pi}\int_{r=0}^{2+\sin(2\theta)} r\,dr\,d\theta = \int_{\theta=0}^{2\pi}\left[\frac{r^2}{2}\right]_{r=0}^{2+\sin(2\theta)} d\theta \\ &= \frac{1}{2}\int_{\theta=0}^{2\pi}(2+\sin(2\theta))^2\,d\theta = \frac{1}{2}\int_{\theta=0}^{2\pi} 4 + 4\sin(2\theta) + \sin^2(2\theta)\,d\theta \\ &= \frac{1}{2}\int_{\theta=0}^{2\pi} 4 + 4\sin(2\theta) + \tfrac{1}{2} - \tfrac{1}{2}\cos(2\theta)\,d\theta. \end{aligned}$$

It is a standard fact that the integral of $\sin(k\theta)$ or $\cos(k\theta)$ over a whole number of complete cycles is zero. Thus, only the terms 4 and $\frac{1}{2}$ contribute to the integral, and we have

$$A = \tfrac{1}{2}(2\pi.(4+\tfrac{1}{2})) = 9\pi/2.$$

Example 4.12. It is an important fact (for the theory of the normal distribution in statistics, the analysis of heat flow, the pricing of financial derivatives, and other applications) that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. We will explain one way to calculate this. Put $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$. It obviously does not matter what we call the variable, so we also have $I = \int_{-\infty}^{\infty} e^{-y^2}\,dy$. We can now multiply these two expressions together to get

$$I^2 = \int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \iint_{\text{whole plane}} e^{-x^2-y^2}\,dA.$$

We can rewrite this using polar coordinates, noting that $x^2+y^2 = r^2$ and $dA = r\,dr\,d\theta$. We get

$$I^2 = \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} r\,e^{-r^2}\,d\theta\,dr = 2\pi\int_{r=0}^{\infty} r\,e^{-r^2} dr.$$

We now substitute $u = r^2$, so u also runs from 0 to ∞ and $du = 2r\,dr$. The integral becomes

$$I^2 = 2\pi \int_{u=0}^{\infty} e^{-u}.\tfrac{1}{2}du = \pi \left[-e^{-u} \right]_{u=0}^{\infty} = \pi((-0) - (-1)) = \pi,$$

so $I = \sqrt{\pi}$ as claimed.

4.3. **More general change of coordinates.** In this section we will discuss a version of integration by substitution that works for integrals over plane regions.

We first recall the method for integrals in a single variable. There we want to evaluate $\int_{x=a}^{b} f(x)\,dx$ say. We suppose that x can be expressed in terms of some other variable u. The values $x = a$ and $x = b$ will correspond to certain values of u, say $u = p$ and $u = q$. We can differentiate the formula giving x in terms of u to find dx/du, and then rearrange to express dx in terms of du. Using this we can express $f(x)\,dx$ as $g(u)\,du$ say, and we have

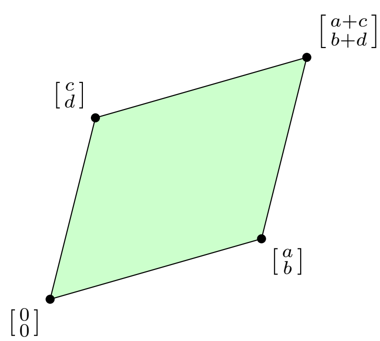
$$\int_{x=a}^{b} f(x)\,dx = \int_{u=p}^{q} g(u)\,du.$$

Example 4.13. Consider the integral $I = \int_{x=0}^{2} x^3\sqrt{9+x^4}\,dx$. Put $u = 9 + x^4$, so $x = 0$ corresponds to $u = 9$ and $x = 2$ corresponds to $u = 25$. We can differentiate $u = 9+x^4$ to get $du/dx = 4x^3$, or $x^3\,dx = \frac{1}{4}du$. We also have $\sqrt{9+x^4} = u^{1/2}$, so

$$I = \int_{x=0}^{2} \sqrt{9+x^4}.x^3\,dx = \int_{u=9}^{25} u^{1/2}\frac{du}{4} = \frac{1}{4}\left[\frac{2}{3}u^{3/2}\right]_{u=9}^{25} = \frac{1}{6}(125-27) = \frac{49}{3} \simeq 16.33.$$

Now consider an integral $I = \iint_D f(x,y)\,dA$ in two variables, and suppose we can express x and y in terms of some other variables u and v. We would like to rewrite I as an integral over u and v. The trickiest issue is what to do with the area element dA. For this we need the following observation:

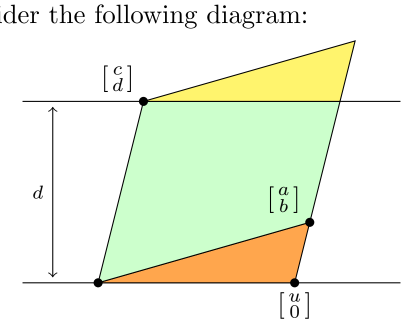
Fact 4.14. Let P be the parallelogram spanned by vectors $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$.



Then

$$\text{area}(P) = |ad - bc| = \left| \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right|.$$

To see why this is the case, consider the following diagram:



The parallelogram P consists of the top triangle (shown in yellow) together with the middle region (shown in green). It is clear that the top triangle has the same area as the bottom one, so we may as well consider the parallelogram P' consisting of the bottom triangle together with the middle region. This parallelogram has a base of length u and a perpendicular height of d, so the area is ud. To see what u is, note that the point $\left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right]$ is reached by starting at $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ and moving in the opposite direction to the vector $\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$, so $\left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] - t\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$ for some t. By comparing the y-coordinates we see that $t = b/d$, and by looking at the x-coordinates we deduce that $u = a - bc/d$, so the area is $ud = ad - bc$. This works whenever the vector $\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$ is anticlockwise from $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$. If $\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$ is clockwise from $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ it works out instead that $ad - bc < 0$ and the area is $-(ad - bc)$. In all cases we can say that the area is $|ad - bc|$.

Now consider again the situation where x and y can be expressed as functions of some other variables u and v. We then have partial derivatives $x_u = \partial x/\partial u$, $x_v = \partial x/\partial v$, $y_u = \partial y/\partial u$ and $y_v = \partial y/\partial v$. It is convenient to write these as a matrix, called the *Jacobian matrix*:

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}.$$

As we have mentioned before, if we make small changes δu and δv to u and v, then the resulting changes in x and y are approximately

$$\begin{aligned} \delta x &= x_u\, \delta u + x_v\, \delta v \\ \delta y &= y_u\, \delta u + y_v\, \delta v. \end{aligned}$$

These equations can be combined as a single matrix equation:

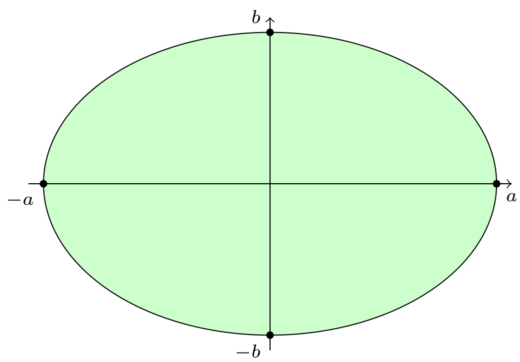
$$\begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \begin{bmatrix} \delta u \\ \delta v \end{bmatrix} = \frac{\partial(x, y)}{\partial(u, v)} \begin{bmatrix} \delta u \\ \delta v \end{bmatrix}.$$

Now suppose we let the change in u vary between 0 and δu, and let the change in v vary between 0 and δv. The resulting changes in $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ then cover a small parallelogram spanned by $\begin{bmatrix} x_u \\ y_u \end{bmatrix} \delta u$ and $\begin{bmatrix} x_v \\ y_v \end{bmatrix} \delta v$, and the area of this parallelogram is $|x_u y_v - x_v y_u| \delta u\, \delta v$, or in other words $\left|\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right| \delta u\, \delta v$. Using this, we see that the area element $dA = dx\, dy$ can be rewritten as

$$dA = \left|\det\left(\frac{\partial(x, y)}{\partial(u, v)}\right)\right| du\, dv.$$

This is one of the key ingredients that we need when we evaluate a double integral by substitution.

Example 4.15. We will find the area of an ellipse E with equation $x^2/a^2 + y^2/b^2 \leq 1$ (for some $a, b > 0$).



For this it is best to use a kind of distorted polar coordinates, by putting

$$x = ar\cos(\theta) \qquad\qquad y = br\sin(\theta).$$

We then have $x^2/a^2 + y^2/b^2 = r^2\cos^2(\theta) + r^2\sin^2(\theta) = r^2$, so the equation $x^2/a^2 + y^2/b^2 \leq 1$ just gives $0 \leq r \leq 1$. The first order partial derivatives are

$$\begin{aligned} x_r &= a\cos(\theta) & x_\theta &= -ar\sin(\theta) \\ y_r &= b\sin(\theta) & y_\theta &= br\cos(\theta), \end{aligned}$$

so the Jacobian matrix is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{bmatrix} a\cos(\theta) & -ar\sin(\theta) \\ b\sin(\theta) & br\cos(\theta) \end{bmatrix}.$$

This means that the absolute value of the determinant is

$$\left|\det\left(\frac{\partial(x,y)}{\partial(r,\theta)}\right)\right| = |abr\cos^2(\theta) - (-abr\sin^2(\theta))| = |abr| = abr,$$

so $dA = abr\,dr\,d\theta$. We therefore have

$$\begin{aligned} \text{area}(E) &= \iint_E 1\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} abr\,dr\,d\theta \\ &= ab\int_{\theta=0}^{2\pi}\left[\frac{r^2}{2}\right]_{r=0}^{1} d\theta \\ &= ab\int_{\theta=0}^{2\pi}\frac{1}{2}d\theta = \pi ab. \end{aligned}$$

In some cases it is easier to calculate the partial derivatives u_x, u_y, v_x and v_y instead of x_u, x_v, y_u and y_v. Because of the matrix equations

$$\begin{bmatrix}\delta x\\ \delta y\end{bmatrix} = \frac{\partial(x,y)}{\partial(u,v)}\begin{bmatrix}\delta u\\ \delta v\end{bmatrix}. \qquad \begin{bmatrix}\delta u\\ \delta v\end{bmatrix} = \frac{\partial(u,v)}{\partial(x,y)}\begin{bmatrix}\delta x\\ \delta y\end{bmatrix},$$

we see that $\partial(x,y)/\partial(u,v)$ is just the inverse matrix of $\partial(u,v)/\partial(x,y)$, so

$$\left|\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right| = \left|\det\left(\frac{\partial(u,v)}{\partial(x,y)}\right)\right|^{-1}.$$

4.4. **Three-dimensional regions.** Suppose we have a solid region E in 3-dimensional space, and a function $f(x,y,z)$. We can define the volume integral of f (written $\iiint_E f(x,y,z)\,dV$) in a very similar way to the area integrals that we discussed above: we divide E into a large number of small regions $E_1,\dots,E_N$, and then

$$\iiint_E f(x,y,z)\,dV \simeq \sum_{i=1}^{N}(\text{ value of } f \text{ on } E_i\ ) \times (\text{ area of } E_i),$$

with the approximation becoming exact in the limit where the size of the subregions tends to zero. As in the plane case, such integrals can usually be evaluated by integrating over three different variables with suitable limits depending on the geometry of the region E.

Some applications are as follows:

(a) To find the total energy of a magnetic field, integrate the square of the field strength.
(b) To find the moment of inertia of a rotor, integrate the square of the distance from the axis.
(c) To find the total mass of a star, integrate the density function over a spherical ball.
(d) For an object E of constant density, the centre of mass is $(\overline{x},\overline{y},\overline{z}) = (X/V, Y/V, Z/V)$ where

$$X = \iiint_E x\,dV \qquad Y = \iiint_E y\,dV \qquad Z = \iiint_E z\,dV \qquad V = \iiint_E 1\,dV.$$

Example 4.16. Let E be the inside of a microwave oven of length a, width b and height c, and suppose for simplicity that a, b and c are integers. To find the total energy of the microwaves in E, we need to calculate integrals like

$$I = \iiint_E (\sin(k\pi x)\sin(m\pi y)\sin(n\pi z))^2\,dV,$$

where k, n and m are also integers. This just reduces to

$$I = \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{c} \sin^2(k\pi x)\sin^2(m\pi y)\sin^2(n\pi z)\,dz\,dy\,dx$$

For the innermost integral, we have $\sin^2(n\pi z) = (1 - \cos(2n\pi z))/2$, so

$$\int_{z=0}^{c} \sin^2(n\pi z)\,dz = \left[\frac{z}{2} - \frac{\sin(2n\pi z)}{4n\pi}\right]_{z=0}^{c}$$

As n and c are integers, the sin() term is zero at both endpoints, and we just get $\int_{z=0}^{c} \sin^2(n\pi z)\,dz = c/2$. The terms $\sin^2(x)$ and $\sin^2(y)$ are just carried along as constants, so we get

$$I = \int_{x=0}^{a} \int_{y=0}^{b} \sin^2(k\pi x)\sin^2(m\pi y)\frac{c}{2}\,dy\,dx.$$

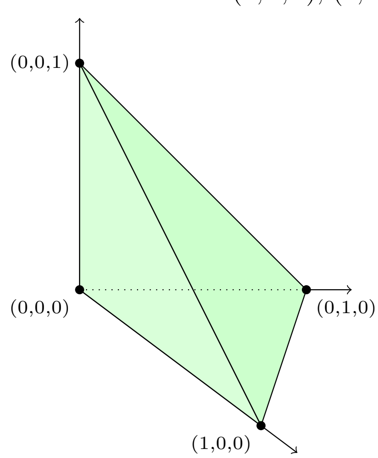
We can integrate over y and then over x in the same way, giving

$$I = \int_{x=0}^{a} \sin^2(k\pi x)\frac{b}{2}\frac{c}{2}\,dx. = \frac{a}{2}.\frac{b}{2}.\frac{c}{2} = \frac{abc}{8}.$$

Example 4.17. Let E be the cube given by $-1 \leq x, y, z \leq 1$. The moment of inertia about the z-axis is

$$\begin{aligned}
\iiint_E (x^2+y^2)dV &= \int_{x=-1}^{1} \int_{y=-1}^{1} \int_{z=-1}^{1} (x^2+y^2)dz\,dy\,dx \\
&= \int_{x=-1}^{1} \int_{y=-1}^{1} \left[x^2z + y^2z\right]_{z=-1}^{1} dy\,dx = \int_{x=-1}^{1} \int_{y=-1}^{1} 2(x^2+y^2)dy\,dx \\
&= \int_{x=-1}^{1} \left[2x^2y + 2y^3/3\right]_{y=-1}^{1} dx = \int_{x=-1}^{1} 4x^2 + 4/3\,dx \\
&= \left[4x^3/3 + 4x/3\right]_{x=-1}^{1} = 8/3 + 8/3 = 16/3.
\end{aligned}$$

Example 4.18. Let E be the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.



The shadow in the (x, y)-plane is the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$, which means that x varies from 0 to 1, and y varies from 0 to $1 - x$. Each of the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ satisfies $x+y+z = 1$, which means that the equation of the top face is $x+y+z = 1$, or in other words $z = 1-x-y$. The equation of the bottom face is $z = 0$, so overall z varies from 0 to $1 - x - y$. Thus, for any function $f(x, y, z)$ we have

$$\iiint_E f(x,y,z)\,dV = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} f(x,y,z)\,dz\,dy\,dx.$$

We can calculate the volume of E by taking $f(x,y,z)=1$. The innermost integral (with respect to z) is then

$$\int_{z=0}^{1-x-y} 1\,dz = 1-x-y.$$

Thus, the integral with respect to y is

$$\begin{aligned}\int_{y=0}^{1-x}(1-x-y)\,dy &= \Big[(1-x)y-y^2/2\Big]_{y=0}^{1-x} = ((1-x)(1-x)-(1-x)^2/2)-0\\ &= 1/2-x+x^2/2.\end{aligned}$$

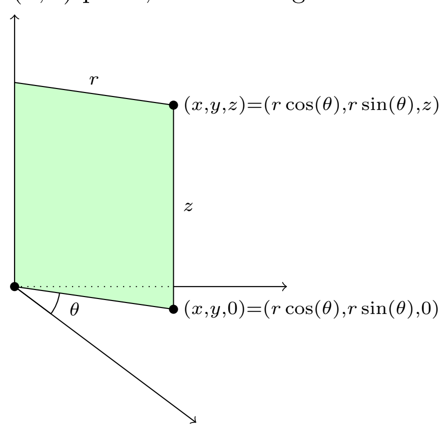
Finally, the outermost integral (with respect to x) is

$$\int_{x=0}^{1}(1/2-x+x^2/2)\,dx = \Big[x/2-x^2/2+x^3/6\Big]_{x=0}^{1} = 1/2-1/2+1/6=1/6.$$

We conclude that the volume of the tetrahedron is $1/6$.

4.5. **Three-dimensional polar coordinates.** In three dimensions there are two different kinds of polar coordinates, called *cylindrical polar coordinates* and *spherical polar coordinates*.

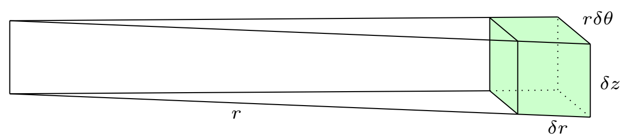
When using cylindrical polar coordinates we describe points in terms of the distance r from the z-axis, the angle θ anticlockwise from the (x,z)-plane, and the height z above the (x,y)-plane.



Just as in the two-dimensional case, r and θ are related to x and y by the equations

$$\begin{aligned} x &= r\cos(\theta) & y &= r\sin(\theta)\\ r &= \sqrt{x^2+y^2} & \theta &= \arctan(y/x).\end{aligned}$$

If we allow r, θ and z to vary by small amounts δr, $\delta\theta$ and δz, then the corresponding region is approximately a right-angled box with sides of length δr, δz and $r\delta\theta$. The volume is thus $\delta V \simeq r\delta r\,\delta\theta\,\delta z$.



This means that for a function f on a 3-dimensional region E, we have

$$\iiint_E f(x,y,z)\,dV = \int_{z=\cdots}^{\cdots}\int_{\theta=\cdots}^{\cdots}\int_{r=\cdots}^{\cdots} f(r\cos(\theta),r\sin(\theta),z)\,r\,dr\,d\theta\,dz,$$

where the limits must be determined using the geometry of the region.

The formula for dV can also be obtained using a three-dimensional of the Jacobian matrix. We have

$$J=\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{bmatrix} x_r & x_\theta & x_z\\ y_r & y_\theta & y_z\\ z_r & z_\theta & z_z\end{bmatrix} = \begin{bmatrix}\cos(\theta) & -r\sin(\theta) & 0\\ \sin(\theta) & r\cos(\theta) & 0\\ 0 & 0 & 1\end{bmatrix}.$$

If we expand out the determinant in the most obvious way, this gives

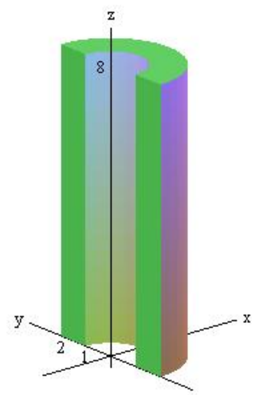
$$
\begin{aligned}
\det(J) &= \cos(\theta)\ \det\begin{bmatrix} r\cos(\theta) & 0 \\ 0 & 1 \end{bmatrix} - (-r\sin(\theta))\ \det\begin{bmatrix} \sin(\theta) & 0 \\ 0 & 1 \end{bmatrix} + 0\ \det\begin{bmatrix} \sin(\theta) & r\cos(\theta) \\ 0 & 0 \end{bmatrix} \\
&= r\cos^2(\theta) + r\sin^2(\theta) = r.
\end{aligned}
$$

(This could be made tidier if we remember that the determinant can also be calculated by expanding along the bottom row instead of the top row.) We now need to take the absolute value of $\det(J)$, but as r is always positive, this makes no difference. We conclude that

$$
dV = dx\,dy\,dz = \left|\det\left(\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right)\right| dr\,d\theta\,dz = |\det(J)| dr\,d\theta\,dz = r\,dr\,d\theta\,dz,
$$

just as we saw before by a more geometric argument.

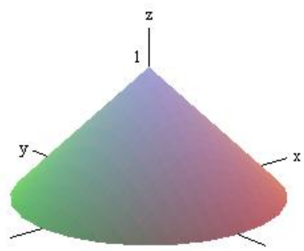
Example 4.19. Consider a region E as shown below:



Suppose that the inner radius is 1, the outer radius is 2 and the height is 8. The region is then described by the inequalities $0 \leq z \leq 8$ and $-\pi/2 \leq \theta \leq \pi/2$ and $1 \leq r \leq 2$. Suppose that we want to find the centre of mass of the region (assuming that the density is constant). By considering the symmetries of the region, we see that the center of mass is $(\overline{x}, 0, 4)$ for some number $\overline{x}$. In fact we have $\overline{x} = X/V$, where $X = \iiint_E x\,dV$ and $V = \iiint_E 1\,dV$. These integrals can be evaluated as follows:

$$
\begin{aligned}
V &= \int_{z=0}^{8} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^{2} r\,dr\,d\theta\,dz \\
&= \int_{z=0}^{8} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3}{2}\,d\theta\,dz = \int_{z=0}^{8} \frac{3\pi}{2}\,dz = 12\pi \\
X &= \int_{z=0}^{8} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^{2} r\cos(\theta).r\,dr\,d\theta\,dz \\
&= \int_{z=0}^{8} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{3}r^3\cos(\theta)\right]_{r=1}^{2} d\theta\,dz = \frac{7}{3}\int_{z=0}^{8} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta)\,d\theta\,dz \\
&= \frac{7}{3}\int_{z=0}^{8} \Big[\sin(\theta)\Big]_{-\pi/2}^{\pi/2} dz = \frac{14}{3}\int_{z=0}^{8} 1\,dz = \frac{112}{3} \\
\overline{x} &= \frac{112}{3\times 12\pi} = \frac{28}{9\pi} \simeq 0.99.
\end{aligned}
$$

Example 4.20. Let E be a cone as shown below:



Suppose that the radius of the base is 1, and the height is also 1. This means that the radius of the slice at height z will be $1-z$, so for any function $f(x,y,z)$ we have

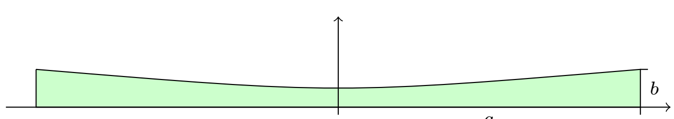
$$\iiint_E f(x,y,z)\,dV = \int_{z=0}^{1}\int_{\theta=0}^{2\pi}\int_{r=0}^{1-z} f(r\cos(\theta), r\sin(\theta), z) r\,dr\,d\theta\,dz.$$

Suppose that we again want to find the centre of mass, assuming constant density. By considering the symmetries of the region, we see that the center of mass is $(0,0,\overline{z})$ for some number $\overline{z}$. In fact we have $\overline{z} = Z/V$, where $Z = \iiint_E z\,dV$ and $V = \iiint_E 1\,dV$. These integrals can be evaluated as follows:

$$\begin{aligned}
V &= \int_{z=0}^{1}\int_{\theta=0}^{2\pi}\int_{r=0}^{1-z} r\,dr\,d\theta\,dz = \int_{z=0}^{1}\int_{\theta=0}^{2\pi} \frac{(1-z)^2}{2}\,d\theta\,dz \\
&= \int_{z=0}^{1} \pi(1-z)^2\,dz = \pi\int_{z=0}^{1} 1-2z+z^2\,dz \\
&= \pi\left[z - z^2 + \tfrac{1}{3}z^3\right]_{z=0}^{1} = \pi/3 \\
Z &= \int_{z=0}^{1}\int_{\theta=0}^{2\pi}\int_{r=0}^{1-z} zr\,dr\,d\theta\,dz = \int_{z=0}^{1}\int_{\theta=0}^{2\pi} z\frac{(1-z)^2}{2}\,d\theta\,dz \\
&= \int_{z=0}^{1} \pi z(1-z)^2\,dz = \pi\int_{z=0}^{1} z-2z^2+z^3\,dz \\
&= \pi\left[\tfrac{1}{2}z^2 - \tfrac{2}{3}z^3 + \tfrac{1}{4}z^4\right]_{z=0}^{1} = \pi/12 \\
\overline{z} &= Z/V = \frac{\pi}{12}/\frac{\pi}{3} = \frac{\pi}{12}/\frac{4\pi}{12} = \frac{1}{4}.
\end{aligned}$$

We conclude that the centre of mass is $(0,0,1/4)$.

Example 4.21. Telescope mirrors always have a parabolic cross-section. We could make such a mirror by starting with a large flat cylinder of radius a and thickness b, and grinding the top until it fits the surface $z = b(r^2+a^2)/(2a^2)$.



We will write E for the region filled by the remaining material. It is easiest to integrate over E using vertical strips. This means that the innermost integral should be an integral over z, and the outer integrals should be with respect to r and θ.

If the density of the material is ρ, the total mass of the mirror will be

$$\begin{aligned}
M &= \iiint_E \rho\, dV = \int_{r=0}^{a}\int_{\theta=0}^{2\pi}\int_{z=0}^{\frac{b(r^2+a^2)}{2a^2}} \rho r\, dz\, d\theta\, dr \\
&= \int_{r=0}^{a}\int_{\theta=0}^{2\pi} \frac{\rho r b(r^2+a^2)}{2a^2} d\theta\, dr = \frac{b\rho}{2a^2}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^3 + a^2 r d\theta\, dr \\
&= \frac{b\rho\pi}{a^2}\int_{r=0}^{a} r^3 + a^2 r\, dr = \frac{b\rho\pi}{a^2}\left[\frac{r^4}{4} + \frac{a^2r^2}{2}\right]_{r=0}^{a} \\
&= \frac{b\rho\pi}{a^2}\left(\frac{a^4}{4} + \frac{a^4}{2}\right) = \frac{3a^2b\rho\pi}{4}.
\end{aligned}$$

The moment of inertia about the x-axis is $I = \iiint_E \rho(y^2+z^2)\, dV$. It is convenient to write this as $I = \rho I_1 + \rho I_2$, where

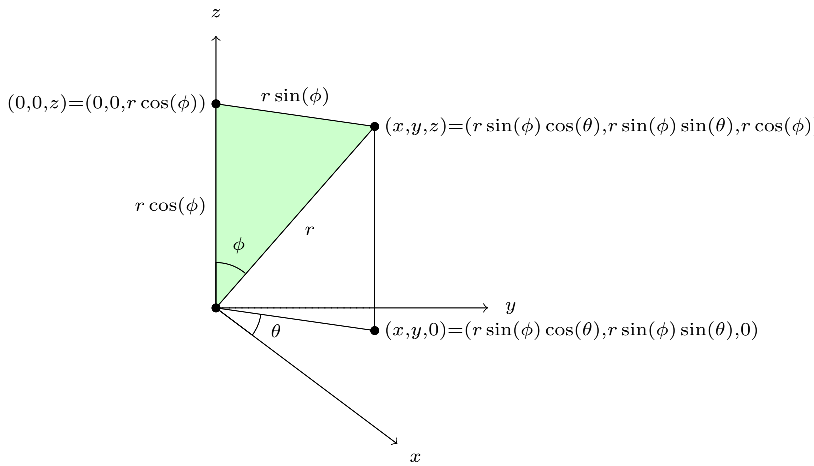
$$\begin{aligned}
I_1 &= \iiint_E y^2\, dV = \int_{r=0}^{a}\int_{\theta=0}^{2\pi}\int_{z=0}^{\frac{b(r^2+a^2)}{2a^2}} r^2\sin^2(\theta) r\, dz\, d\theta\, dr \\
&= \int_{r=0}^{a}\int_{\theta=0}^{2\pi} \frac{b(r^2+a^2)}{2a^2} r^3 \sin^2(\theta)\, d\theta\, dr \\
&= \int_{r=0}^{a} \frac{b(r^2+a^2)\pi}{2a^2} r^3\, dr = \frac{b\pi}{2a^2}\int_{r=0}^{a}(r^5 + a^2r^3)\, dr \\
&= \frac{b\pi}{2a^2}\left(\frac{a^6}{6} + a^2\frac{a^4}{4}\right) = \frac{5a^4b\pi}{24} \\
I_2 &= \iiint_E z^2\, dV = \int_{r=0}^{a}\int_{\theta=0}^{2\pi}\int_{z=0}^{\frac{b(r^2+a^2)}{2a^2}} z^2 r\, dz\, d\theta\, dr \\
&= \int_{r=0}^{a}\int_{\theta=0}^{2\pi} \frac{r}{3}\left(\frac{b(r^2+a^2)}{2a^2}\right)^3 d\theta\, dr = \frac{b^3}{24a^6}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r(r^2+a^2)^3\, d\theta\, dr \\
&= \frac{b^3\pi}{12a^6}\int_{r=0}^{a} r(r^2+a^2)^3\, dr = \frac{b^3\pi}{12a^6}\int_{r=0}^{a} r^7 + 3a^2r^5 + 3a^4r^3 + a^6 r\, dr \\
&= \frac{b^3\pi}{12a^6}\left(\frac{a^8}{8} + 3a^2\frac{a^6}{6} + 3a^4\frac{a^4}{4} + a^6\frac{a^2}{2}\right) = \frac{a^2b^3\pi}{12}\left(\frac{1}{8} + \frac{3}{6} + \frac{3}{4} + \frac{1}{2}\right) \\
&= \frac{a^2b^3\pi}{12}\frac{15}{8} = \frac{5a^2b^3\pi}{32}.
\end{aligned}$$

Putting these together, we get

$$I = \rho I_1 + \rho I_2 = 5a^2b\rho\pi\left(\frac{a^2}{24} + \frac{b^2}{32}\right) = \frac{5a^2b\rho\pi}{96}(4a^2 + 3b^2).$$

Note that in practice b will be much smaller than a, so I_2 will be much smaller than I_1, so $I \simeq I_1\rho = 5a^4b\rho\pi/24$.

We now turn to the other kind of three-dimensional polar coordinates, which are called *spherical*. In spherical polar coordinates we describe a point (x, y, z) by giving the distance r from the origin, the angle θ anticlockwise from the xz plane, and the angle ϕ from the z-axis.



The variables r, θ and ϕ are related to x and y by the equations

$$x = r\sin(\phi)\cos(\theta) \qquad y = r\sin(\phi)\sin(\theta) \qquad z = r\cos(\phi)$$

$$r = \sqrt{x^2+y^2+z^2} \qquad \theta = \arctan(y/x) \qquad \phi = \arctan(\sqrt{x^2+y^2}/z).$$

Note that ϕ ranges from 0 (on the positive z-axis) to π (on the negative z-axis), whereas θ ranges from 0 to 2π (or equivalently, from $-\pi$ to π). It is also useful to observe that $\sqrt{x^2+y^2} = r\sin(\phi)$.

For these coordinates it is easiest to find the area element using the Jacobian. We have

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{bmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{bmatrix} = \begin{bmatrix} \sin(\phi)\cos(\theta) & -r\sin(\phi)\sin(\theta) & r\cos(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) & r\sin(\phi)\cos(\theta) & r\cos(\phi)\sin(\theta) \\ \cos(\phi) & 0 & -r\sin(\phi) \end{bmatrix}.$$

We will expand the determinant along the bottom row. This gives

$$\det(J) = \cos(\phi)\det(A) - 0\det(B) + (-r\sin(\phi))\det(C),$$

where

$$A = \begin{bmatrix} -r\sin(\phi)\sin(\theta) & r\cos(\phi)\cos(\theta) \\ r\sin(\phi)\cos(\theta) & r\cos(\phi)\sin(\theta) \end{bmatrix} \qquad B = \begin{bmatrix} \sin(\phi)\cos(\theta) & r\cos(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) & r\cos(\phi)\sin(\theta) \end{bmatrix} \qquad C = \begin{bmatrix} \sin(\phi)\cos(\theta) & -r\sin(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) & r\sin(\phi)\cos(\theta) \end{bmatrix}.$$

(Here A is obtained from J by deleting the last row and the first column, B is obtained from J by deleting the last row and the middle column, and C is obtained from J by deleting the last row and the last column.) We need not calculate $\det(B)$ because it is multiplied by zero. We have

$$\begin{aligned} \det(A) &= -r^2\sin(\phi)\cos(\phi)\sin^2(\theta) - r^2\sin(\phi)\cos(\phi)\cos^2(\theta) \\ &= -r^2\sin(\phi)\cos(\phi) \\ \det(C) &= r\sin^2(\phi)\cos^2(\theta) - (-r\sin^2(\phi)\sin^2(\theta)) \\ &= r\sin^2(\phi) \\ \det(J) &= \cos(\phi)\det(A) - 0\det(B) + (-r\sin(\phi))\det(C) \\ &= -r^2\sin(\phi)\cos^2(\phi) - r^2\sin(\phi)\sin^2(\phi) \\ &= -r^2\sin(\phi). \end{aligned}$$

As $0 \leq \phi \leq \pi$ we have $\sin(\phi) \geq 0$ so $|-r^2\sin(\phi)| = r^2\sin(\phi)$. We conclude that

$$dV = |\det(J)| dr\, d\theta\, d\phi = r^2\sin(\phi)\, dr\, d\theta\, d\phi.$$

This means that for a function f on a 3-dimensional region E, we have

$$\iiint_E f(x,y,z)\, dV = \int_{\phi=\cdots}^{\cdots}\int_{\theta=\cdots}^{\cdots}\int_{r=\cdots}^{\cdots} f(r\cos(\theta)\sin(\phi), r\sin(\theta)\sin(\phi), r\cos(\phi))\, r^2\sin(\phi)\, dr\, d\theta\, d\phi,$$

where the limits must be determined using the geometry of the region.

Example 4.22. The volume of a sphere E of radius a is

$$\begin{aligned}
V &= \iiint_E 1\, dV = \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \int_{r=0}^{a} r^2 \sin(\phi) dr\, d\theta\, d\phi \\
&= \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \frac{a^3}{3} \sin(\phi)\, d\theta\, d\phi = \int_{\phi=0}^{\pi} \frac{2\pi a^3}{3} \sin(\phi)\, d\phi \\
&= \frac{2\pi a^3}{3} \Big[-\cos(\phi) \Big]_{\phi=0}^{\pi} = \frac{2\pi a^3}{3}(1-(-1)) = \frac{4\pi a^3}{3}.
\end{aligned}$$

Now suppose that the sphere has density ρ. The distance of a point from the z-axis is $r\,\sin(\phi)$, so the moment of inertia around that axis is

$$I = \iiint_E \rho.(r\sin(\phi))^2\, dV = \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \rho r^2 \sin^2(\phi) r^2 \sin(\phi) dr\, d\theta\, d\phi = \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \rho r^4 \sin^3(\phi) dr\, d\theta\, d\phi.$$

Here the three different variables do not interact in any interesting way so we can rewrite the integral as

$$I = \left(\int_{\phi=0}^{\pi} \sin(\phi)^3\, d\phi \right) \left(\int_{\theta=0}^{2\pi} 1\, d\theta \right) \left(\int_{r=0}^{a} r^4\, dr \right) \rho.$$

Two of these integrals are easy: we have $\int_{\theta=0}^{2\pi} 1\, d\theta = 2\pi$ and $\int_{r=0}^{a} r^4\, dr = a^5/5$. For the integral with respect to ϕ, we recall that $\sin(\phi) = (e^{j\phi} - e^{-j\phi})/(2j)$. We can cube this to get

$$\begin{aligned}
\sin^3(\phi) &= \frac{1}{8j^3}(e^{3j\phi} - 3e^{2j\phi}e^{-j\phi} + 3e^{j\phi}e^{-2j\phi} - e^{-3j\phi}) \\
&= \frac{-1}{8j}(e^{3j\phi} - 3e^{j\phi} + 3e^{-j\phi} - e^{-3j\phi}) \\
&= \frac{3}{4}\left(\frac{e^{j\phi} - e^{-j\phi}}{2j} \right) - \frac{1}{4}\left(\frac{e^{3j\phi} - e^{-3j\phi}}{2j} \right) \\
&= \frac{3}{4}\sin(\phi) - \frac{1}{4}\sin(3\phi).
\end{aligned}$$

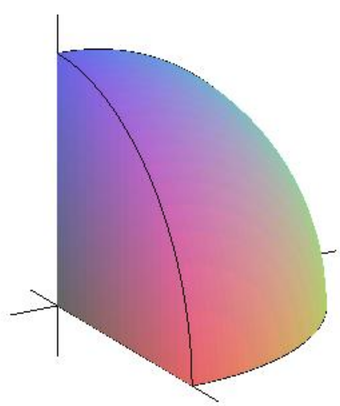
Integrating this, we get

$$\begin{aligned}
\int_{\phi=0}^{\pi} \sin^3(\phi)\, d\phi &= \Big[-\frac{3}{4}\cos(\phi) + \frac{1}{12}\cos(3\phi) \Big]_{\phi=0}^{\pi} \\
&= (3/4 - 1/12) - (-3/4 + 1/12) = 4/3.
\end{aligned}$$

Combining this with the r and θ integrals gives

$$I = \frac{4}{3}.2\pi.\frac{a^5}{5}.\rho = \frac{8\pi a^5 \rho}{15}.$$

Example 4.23. Let E be the part of a sphere of radius 1 where $x \geq 0$, $y \geq 0$ and $z \geq 0$.



The centre of mass of E (assuming constant density) is $(\overline{x},\overline{y},\overline{z})$, where $\overline{x}=(\iiint_E x\,dV)/(\iiint_E 1\,dV)$ and so on. It is clear by symmetry that $\overline{x}$, $\overline{y}$ and $\overline{z}$ are all the same, so we will just calculate $\overline{z}$. Note that if we cut a sphere along the three coordinate planes then it splits into eight pieces all congruent to E, so the volume $V=\iiint_E 1\,dV$ is just one eighth of the volume of the sphere. We saw in the previous example (with $a=1$) that the volume of the sphere is $4\pi/3$, so $V=\pi/6$. We now need to calculate the integral $Z=\iiint_E z\,dV$. The restriction $z\geq 0$ means that $0\leq\phi\leq\pi/2$, and the restrictions $x,y\geq 0$ mean that $0\leq\theta\leq\pi/2$. Recall also that $z=r\cos(\phi)$ and $dV=r^2\sin(\phi)\,dr\,d\theta\,d\phi$, so

$$z\,dV=r^3\sin(\phi)\cos(\phi)\,dr\,d\theta\,d\phi=\tfrac{1}{2}r^3\sin(2\phi)\,dr\,d\theta\,d\phi.$$

This gives

$$\begin{aligned}
Z&=\int_{\phi=0}^{\frac{\pi}{2}}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1}\tfrac{1}{2}r^3\sin(2\phi)\,dr\,d\theta\,d\phi\\
&=\frac{1}{2}\left(\int_{\phi=0}^{\frac{\pi}{2}}\sin(2\phi)\,d\phi\right)\left(\int_{\theta=0}^{\frac{\pi}{2}}1\,d\theta\right)\left(\int_{r=0}^{1}r^3\,dr\right)\\
&=\frac{1}{2}\left[-\frac{\cos(2\phi)}{2}\right]_{\phi=0}^{\frac{\pi}{2}}.\frac{\pi}{2}.\frac{1}{4}\\
&=\frac{1}{2}.\frac{2}{2}.\frac{\pi}{2}.\frac{1}{4}=\frac{\pi}{16},
\end{aligned}$$

so

$$\overline{z}=Z/V=\frac{\pi}{16}\Big/\frac{\pi}{6}=\frac{6}{16}=\frac{3}{8}.$$

We conclude that the centre of mass is $(\frac{3}{8},\frac{3}{8},\frac{3}{8})$.

5. Algebra and geometry of vectors

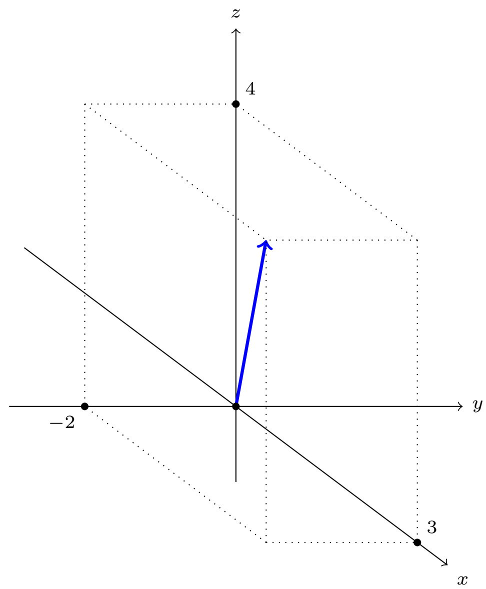
Recall that a *vector* is a quantity with both magnitude and direction. Examples include:

(a) The velocity and acceleration of a particle are vectors.
(b) The separation between two particles is a vector.
(c) If we have chosen a point to count as the origin, then the displacement of a particle from that origin is also a vector.
(d) The electric field at a point is a vector, and the magnetic field is another vector.

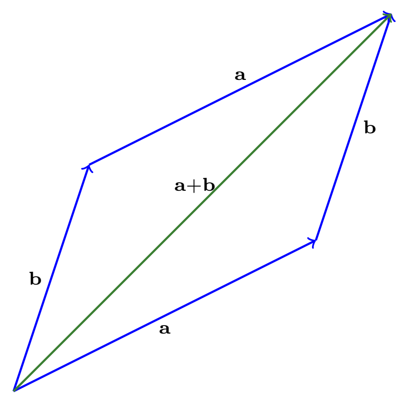
By contrast, a *scalar* is a quantity that has a magnitude, but not a direction. For example, the pressure, temperature and electric potential at a point are scalars.

When answering questions in vector algebra or vector calculus, you should always ask yourself whether your answer should be a scalar or a vector, and make sure that what you have written has the right type. This simple check will detect a substantial fraction of incorrect answers.

Normally we will fix a coordinate system, and use it to represent vectors as triples of numbers. For example, the triple $(3,-2,4)$ represents the vector that goes 3 steps along the x-axis, 2 steps backwards parallel to the y-axis, and 4 steps parallel to the z-axis.



We can add vectors in an obvious way, for example $(3,-2,4)+(1,1,1)=(4,-1,5)$. Geometrically, this corresponds to joining the vectors together nose to tail:



Similarly, we can multiply a vector by a scalar to get a new vector, for example $3(3,-2,4)=(9,-6,12)$. The new vector has the same direction as the old one (if the scalar is positive) or the opposite direction (if the scalar is negative).

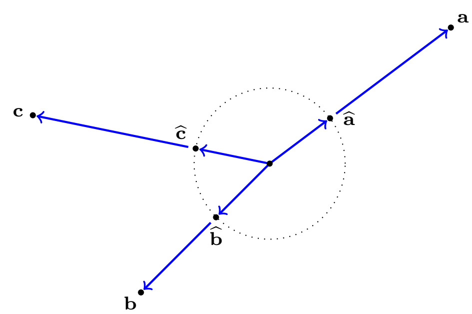
So far we have only discussed three-dimensional vectors, which is what we need for most applications in physics or engineering. One can also consider two-dimensional vectors, which are used for applications where everything is happening in a single plane. In some cases we will find it helpful to discuss the two-dimensional theory as a warm-up before turning to the more complicated three-dimensional case.

The length of a vector $\mathbf{a}=(x,y,z)$ is given by

$$|\mathbf{a}|=\sqrt{x^2+y^2+z^2}.$$

It is a useful fact that we always have $|\mathbf{a}+\mathbf{b}|\leq|\mathbf{a}|+|\mathbf{b}|$; this is called the *triangle inequality*. To see why it is true, consider the parallelogram in the previous picture. The distance from the origin to $\mathbf{a}+\mathbf{b}$ in a straight line is $|\mathbf{a}+\mathbf{b}|$, whereas the distance via $\mathbf{a}$ is $|\mathbf{a}|+|\mathbf{b}|$. The inequality just says that it is shorter to go in a straight line.

A *unit vector* is a vector of length one. We write $\widehat{\mathbf{a}}$ for the unit vector in the same direction as $\mathbf{a}$.



This is given by

$$\widehat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right).$$

For example, if $\mathbf{a} = (1, -2, 2)$ then

$$|\mathbf{a}| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{1+4+4} = 3$$
$$\widehat{\mathbf{a}} = \frac{\mathbf{a}}{3} = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \right).$$

Note that $|\mathbf{a}|$ is a scalar, and $\widehat{\mathbf{a}}$ is a vector.

The unit vectors along the three coordinate axes are denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$:

$$\mathbf{i} = (1,0,0) \qquad \mathbf{j} = (0,1,0) \qquad \mathbf{k} = (0,0,1).$$

Note that

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x,0,0) + (0,y,0) + (0,0,z) = (x,y,z).$$

The dot product of vectors $\mathbf{a} = (x,y,z)$ and $\mathbf{b} = (u,v,w)$ is given by

$$\mathbf{a}.\mathbf{b} = (x,y,z).(u,v,w) = xu + yv + zw.$$

Note that this is a scalar, and that $\mathbf{a}.\mathbf{b}$ is the same as $\mathbf{b}.\mathbf{a}$. For example, we have

$$(1,2,3).(10,100,1000) = 10 + 200 + 3000 = 3210.$$

Note also that

$$\mathbf{a}.\mathbf{a} = x^2 + y^2 + z^2 = |\mathbf{a}|^2.$$

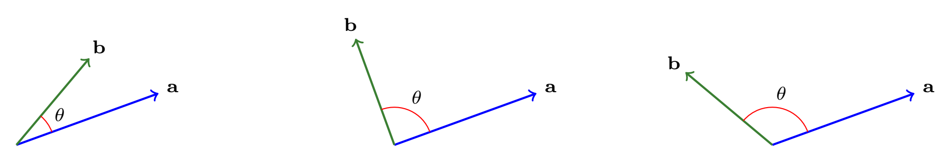
For the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ we have

$$\begin{array}{ccc} \mathbf{i}.\mathbf{i} = 1 & \mathbf{i}.\mathbf{j} = 0 & \mathbf{i}.\mathbf{k} = 0 \\ \mathbf{j}.\mathbf{i} = 0 & \mathbf{j}.\mathbf{j} = 1 & \mathbf{j}.\mathbf{k} = 0 \\ \mathbf{k}.\mathbf{i} = 0 & \mathbf{k}.\mathbf{j} = 0 & \mathbf{k}.\mathbf{k} = 1. \end{array}$$

Geometrically, it can be shown that

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta),$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. In particular, as $-1 \leq \cos(\theta) \leq 1$ this means that $-|\mathbf{a}||\mathbf{b}| \leq \mathbf{a}.\mathbf{b} \leq |\mathbf{a}||\mathbf{b}|$, or equivalently $|\mathbf{a}.\mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$. This is called the *Cauchy-Schwartz inequality*. We also see that $\mathbf{a}.\mathbf{b}$ is zero when $\theta = \pi/2$, which means that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other.



$\theta < \frac{\pi}{2}$; $\quad \mathbf{a}.\mathbf{b} > 0$ $\qquad$ $\theta = \frac{\pi}{2}$; $\quad \mathbf{a}.\mathbf{b} = 0$ $\qquad$ $\theta > \frac{\pi}{2}$; $\quad \mathbf{a}.\mathbf{b} < 0$

Example 5.1. Consider the vectors $\mathbf{a} = (3, 0, 4)$ and $\mathbf{b} = (2, -1, 2)$. We will find the angle θ between $\mathbf{a}$ and $\mathbf{b}$. The inner products are

$$|\mathbf{a}|^2 = \mathbf{a}.\mathbf{a} = 3^2 + 0^2 + 4^2 = 25$$
$$|\mathbf{b}|^2 = \mathbf{b}.\mathbf{b} = 2^2 + (-1)^2 + 2^2 = 9$$
$$|\mathbf{a}||\mathbf{b}|\cos(\theta) = \mathbf{a}.\mathbf{b} = 3 \times 2 + 0 \times (-1) + 4 \times 2 = 14.$$

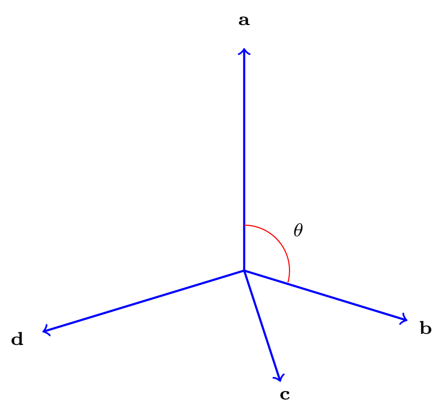
From this we see that $|\mathbf{a}| = \sqrt{25} = 5$ and $|\mathbf{b}| = \sqrt{9} = 3$, so

$$\cos(\theta) = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{14}{5 \times 3} = \frac{14}{15} \simeq 0.933.$$

This means that $\theta = \arccos(0.933)$, which is 0.367 radians or 21.04 degrees.

Example 5.2. The hydrogen atoms in a molecule of methane lie at the following positions:

$$\mathbf{a} = (0, 0, 1) \qquad \mathbf{b} = \left(\frac{2\sqrt{2}}{3}, 0, -\frac{1}{3}\right)$$
$$\mathbf{c} = \left(-\frac{\sqrt{2}}{3}, \frac{\sqrt{6}}{3}, -\frac{1}{3}\right) \qquad \mathbf{d} = \left(-\frac{\sqrt{2}}{3}, -\frac{\sqrt{6}}{3}, -\frac{1}{3}\right).$$



It is clear that $\mathbf{a}$ is a unit vector. We also have

$$|\mathbf{b}|^2 = \left(\frac{2\sqrt{2}}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{4 \times 2}{9} + \frac{1}{9} = 1$$
$$|\mathbf{c}|^2 = \left(\frac{\sqrt{2}}{3}\right)^2 + \left(\frac{\sqrt{6}}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{2}{9} + \frac{6}{9} + \frac{1}{9} = 1,$$

so $\mathbf{b}$ and $\mathbf{c}$ are unit vectors, and $\mathbf{d}$ is also a unit vector by the same calculation as for $\mathbf{c}$. It is also clear that

$$\mathbf{a}.\mathbf{b} = \mathbf{a}.\mathbf{c} = \mathbf{a}.\mathbf{d} = -1/3.$$

In fact, we also have

$$\mathbf{b}.\mathbf{c} = \mathbf{b}.\mathbf{d} = \mathbf{c}.\mathbf{d} = -1/3,$$

by the following calculations:

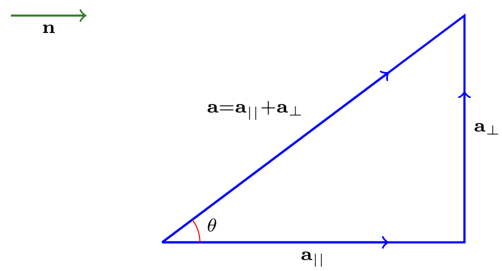
$$\mathbf{b}.\mathbf{c} = \frac{2\sqrt{2}}{3}.\left(-\frac{\sqrt{2}}{3}\right) + 0.\frac{\sqrt{6}}{3} + \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right) = \frac{-4}{9} + \frac{1}{9} = -\frac{1}{3}$$
$$\mathbf{b}.\mathbf{d} = \frac{2\sqrt{2}}{3}.\left(-\frac{\sqrt{2}}{3}\right) + 0.\left(-\frac{\sqrt{6}}{3}\right) + \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right) = \frac{-4}{9} + \frac{1}{9} = -\frac{1}{3}$$
$$\mathbf{c}.\mathbf{d} = \left(-\frac{\sqrt{2}}{3}\right)\left(-\frac{\sqrt{2}}{3}\right) + \left(\frac{\sqrt{6}}{3}\right)\left(-\frac{\sqrt{6}}{3}\right) + \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right) = \frac{2}{9} - \frac{6}{9} + \frac{1}{9} = -\frac{1}{3}.$$

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then we have

$$\cos(\theta) = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{-1/3}{1 \times 1} = -\frac{1}{3},$$

so θ is $\arccos(-1/3)$, which is 1.911 radians or 109.5 degrees. By the same calculation, the angle between any two of the atoms is 109.5 degrees.

Now suppose we have a vector $\mathbf{a}$ and a unit vector $\mathbf{n}$. We can write $\mathbf{a}$ as $\mathbf{a}_{||} + \mathbf{a}_{\perp}$, where $\mathbf{a}_{||}$ is the part parallel to $\mathbf{n}$, and $\mathbf{a}_{\perp}$ is the part perpendicular to $\mathbf{n}$.



In the picture, θ is the angle between $\mathbf{a}$ and $\mathbf{a}_{||}$, which is the same as the angle between $\mathbf{a}$ and $\mathbf{n}$. From this (and the fact that $|\mathbf{n}| = 1$) it follows that

$$\mathbf{a}.\mathbf{n} = |\mathbf{a}||\mathbf{n}|\cos(\theta) = |\mathbf{a}|\cos(\theta) = |\mathbf{a}_{||}|.$$

Note that this is an equation between two scalars. It actually relies on the fact that $\theta \leq \pi/2$, which is the case in our picture. To be correct for all θ, we need to say that $|\mathbf{a}.\mathbf{n}| = |\mathbf{a}_{||}|$. Note that the bars on the left denote the absolute value of the scalar $\mathbf{a}.\mathbf{n}$, whereas the bars on the right denote the length of the vector $\mathbf{a}_{||}$.

Next, we can multiply the scalar $\mathbf{a}.\mathbf{n}$ by the vector $\mathbf{n}$ to get a vector $(\mathbf{a}.\mathbf{n})\mathbf{n}$, and we also see from the diagram above that

$$\begin{aligned} \mathbf{a}_{||} &= (\mathbf{a}.\mathbf{n})\mathbf{n} \\ \mathbf{a}_{\perp} &= \mathbf{a} - (\mathbf{a}.\mathbf{n})\mathbf{n}. \end{aligned}$$

These equations are valid for all θ. They are often useful in calculations. For example, suppose we have a satellite relaying signals from point A to point B, where A and B are both on the earth's surface. We have a unit vector $\mathbf{n}_1$ pointing vertically upwards at A, another unit vector $\mathbf{n}_2$ pointing vertically upwards at B, another unit vector $\mathbf{n}_3$ pointing from the satellite towards A and a fourth unit vector $\mathbf{n}_4$ pointing from the satellite towards B. If we want to calculate various things about the signals, we might want to resolve various vectors parallel and perpendicular to any of $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ or $\mathbf{n}_4$.

Example 5.3. Consider the vector $\mathbf{a} = (3, 6, 9)$ and the unit vector $\mathbf{n} = (2/3, 2/3, -1/3)$. We have

$$\begin{aligned} \mathbf{a}.\mathbf{n} &= 3.\frac{2}{3} + 6.\frac{2}{3} + 9.\frac{-1}{3} = 2 + 4 - 3 = 3 \\ \mathbf{a}_{||} &= (\mathbf{a}.\mathbf{n})\mathbf{n} = 3\mathbf{n} = (2, 2, -1) \\ \mathbf{a}_{\perp} &= \mathbf{a} - \mathbf{a}_{||} = (3, 6, 9) - (2, 2, -1) = (1, 4, 10). \end{aligned}$$

We next recall the cross product operation. For vectors $\mathbf{a} = (x, y, z)$ and $\mathbf{b} = (u, v, w)$, we define

$$\mathbf{a} \times \mathbf{b} = (x, y, z) \times (u, v, w) = (yw - zv, zu - xw, xv - yu).$$

Perhaps the simplest way to remember this is as a kind of determinant:

$$\mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ u & v & w \end{bmatrix} = \det \begin{bmatrix} y & z \\ v & w \end{bmatrix} \mathbf{i} - \det \begin{bmatrix} x & z \\ u & w \end{bmatrix} \mathbf{j} + \det \begin{bmatrix} x & y \\ u & v \end{bmatrix} \mathbf{k}.$$

Note that $\mathbf{a} \times \mathbf{b}$ is a vector, in contrast to $\mathbf{a}.\mathbf{b}$, which is a scalar. Note also that this definition only works for three-dimensional vectors; there is no way to define the cross product in two dimensions.

Example 5.4. Consider the vectors $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (3, 2, 1)$. We have

$$\mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} = \det \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \mathbf{i} - \det \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \mathbf{j} + \det \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \mathbf{k} = -4\mathbf{i} - (-8)\mathbf{j} + (-4)\mathbf{k} = (-4, 8, -4).$$

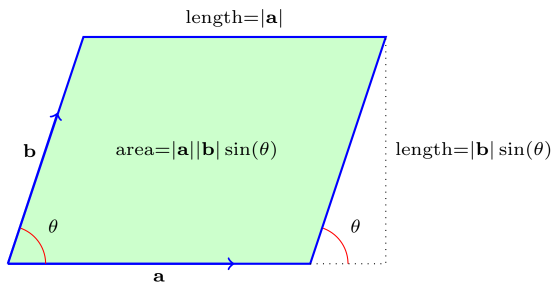
Example 5.5. For the standard unit vectors you can check that

$$\begin{aligned} &\mathbf{i} \times \mathbf{i} = 0 && \mathbf{i} \times \mathbf{j} = \mathbf{k} && \mathbf{i} \times \mathbf{k} = -\mathbf{j} \\ &\mathbf{j} \times \mathbf{i} = -\mathbf{k} && \mathbf{j} \times \mathbf{j} = 0 && \mathbf{j} \times \mathbf{k} = \mathbf{i} \\ &\mathbf{k} \times \mathbf{i} = \mathbf{j} && \mathbf{k} \times \mathbf{j} = -\mathbf{i} && \mathbf{k} \times \mathbf{k} = 0. \end{aligned}$$

Geometrically, it can be shown that $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, and that

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta) = \text{ area of the parallelogram spanned by } \mathbf{a} \text{ and } \mathbf{b},$$

where θ is again the angle between $\mathbf{a}$ and $\mathbf{b}$.



In particular, we see that $\mathbf{a} \times \mathbf{b}$ is zero when $\sin(\theta) = 0$, which means that $\theta = 0$ or $\theta = \pi$, so $\mathbf{a}$ and $\mathbf{b}$ have the same direction or opposite directions.

Algebraically, we have the following identities:

$$\begin{aligned} \mathbf{a} \times \mathbf{a} &= \mathbf{0} \\ \mathbf{b} \times \mathbf{a} &= -\mathbf{a} \times \mathbf{b} \\ \mathbf{a}.(\mathbf{a} \times \mathbf{b}) &= 0 \\ \mathbf{b}.(\mathbf{a} \times \mathbf{b}) &= 0. \end{aligned}$$

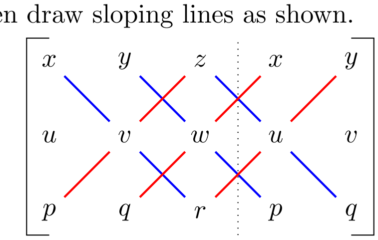
Next, suppose we have three different vectors $\mathbf{a} = (x, y, z)$, $\mathbf{b} = (u, v, w)$ and $\mathbf{c} = (p, q, r)$. There are several different ways to take the product of all three vectors. The simplest is to take the dot product of $\mathbf{a}$ and $\mathbf{b}$ to get a scalar $\mathbf{a}.\mathbf{b}$, and multiply that scalar by $\mathbf{c}$ to get a new vector

$$(\mathbf{a}.\mathbf{b})\mathbf{c} = (xu + yv + zw)\mathbf{c} = (xup + yvp + zwp, xuq + yvq + zwq, xur + yvr + zwr).$$

Alternatively, we could take the cross product $\mathbf{b} \times \mathbf{c}$, which is a vector, and then take the dot product of that vector with $\mathbf{a}$ to get a scalar $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$, which is called the *scalar triple product* of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Using the determinant formula for $\mathbf{b} \times \mathbf{c}$ we find that $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$ is also a determinant:

$$\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = \det \begin{bmatrix} x & y & z \\ u & v & w \\ p & q & r \end{bmatrix}.$$

A convenient trick for expanding such determinants is as follows. We first expand the matrix by repeating the first two columns at the end, then draw sloping lines as shown.



For each of the blue lines sloping down and to the right, we have a term with a plus sign. For example, the first blue line joins x, v and r, giving a term $+xvr$. Each of the red lines sloping up and to the right gives a term with a minus sign. Altogether, the determinant is

$$\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = \det\begin{bmatrix} x & y & z \\ u & v & w \\ p & q & r \end{bmatrix} = xvr + ywp + zuq - zvp - xwq - yur.$$

There are a number of slight variants of the scalar triple product, but they all turn out to be the same, at least up to a plus or minus sign. Specifically, we have

$$\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = \mathbf{b}.(\mathbf{c}\times\mathbf{a}) = \mathbf{c}.(\mathbf{a}\times\mathbf{b}) = -\mathbf{a}.(\mathbf{c}\times\mathbf{b}) = -\mathbf{b}.(\mathbf{a}\times\mathbf{c}) = -\mathbf{c}.(\mathbf{b}\times\mathbf{a}).$$

We also have $\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = (\mathbf{b}\times\mathbf{c}).\mathbf{a}$ and so on, just because $\mathbf{u}.\mathbf{v} = \mathbf{v}.\mathbf{u}$ for any vectors $\mathbf{u}$ and $\mathbf{v}$.

The third way we can combine $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is to take the cross product of the vector $\mathbf{a}$ with the vector $\mathbf{b}\times\mathbf{c}$ to get another vector $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$. By analogy with ordinary multiplication, you might think that this is the same as $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$, but in fact that is **not** true. However, both of these iterated cross products, and various variants, can be described in terms of dot products as follows:

$$\begin{aligned} \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c} \\ (\mathbf{a}\times\mathbf{b})\times\mathbf{c} &= (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{b}.\mathbf{c})\mathbf{a}. \end{aligned}$$

The following observations may help you remember the rules:

(a) The vector outside the brackets on the left occurs in both the dot products on the right.
(b) Each of the vectors inside the brackets on the left occurs in one of the dot products on the right.
(c) The dot product of the first vector with the last vector occurs with a plus sign. The other dot product occurs with a minus sign.

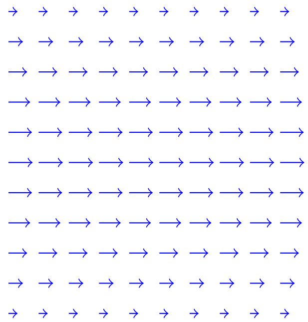
6. Fields and vector calculus

In many applications, we do not consider individual vectors or scalars, but functions that give a vector or scalar at every point. Such functions are called *vector fields* or *scalar fields*. For example:

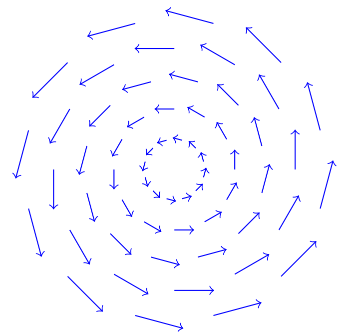
(a) Suppose we want to model the flow of air around an aeroplane. The velocity of the air flow at any given point is a vector. These vectors will be different at different points, so they are functions of position (and also of time). Thus, the air velocity is a vector field. Similarly, the pressure and temperature are scalar quantities that depend on position, or in other words, they are scalar fields.
(b) The magnetic field inside an electrical machine is a vector that depends on position, or in other words a vector field. The electric potential is a scalar field.

Although we will mainly be concerned with scalar and vector fields in three-dimensional space, we will sometimes use two-dimensional examples because they are easier to visualise.

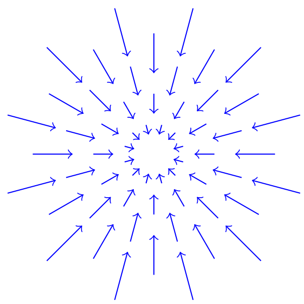
The following pictures illustrate some two-dimensional vector fields $\mathbf{u}$:



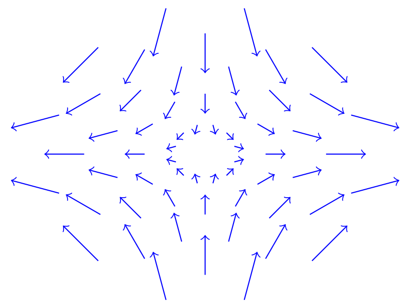
$\mathbf{u} = (1 - y^2, 0)$



$\mathbf{u} = (-y/3, x/3)$



$\mathbf{u} = (-x/3, -y/3)$



$\mathbf{u} = (x/3, -y/3)$

We next discuss various ways to differentiate scalar and vector fields. The most basic is as follows: if f is a scalar field, then we define

$$\nabla(f) = (f_x, f_y, f_z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

This is a vector field. It is called the *gradient* of f and is sometimes written $\mathrm{grad}(f)$ rather than $\nabla(f)$. In the two-dimensional case, there will only be two variables (x and y) and $\nabla(f)$ is defined to be (f_x, f_y).

Example 6.1.

(a) For the function $f = x^3 + y^4 + z^5$, we have $\nabla(f) = (3x^2, 4y^3, 5z^4)$.

(b) For the function $f = \sin(x)\sin(y)\sin(z)$ we have

$$\nabla(f) = (\cos(x)\sin(y)\sin(z), \sin(x)\cos(y)\sin(z), \sin(x)\sin(y)\cos(z)).$$

(c) For the function $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ we have

$$r_x = \tfrac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}.2x = \frac{x}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} = x/r,$$

and similarly $r_y = y/r$ and $r_z = z/r$. This means that

$$\nabla(r) = (x/r, y/r, z/r).$$

More generally, for any n we have

$$(r^n)_x = nr^{n-1}r_x = nr^{n-1}x/r = nr^{n-2}x.$$

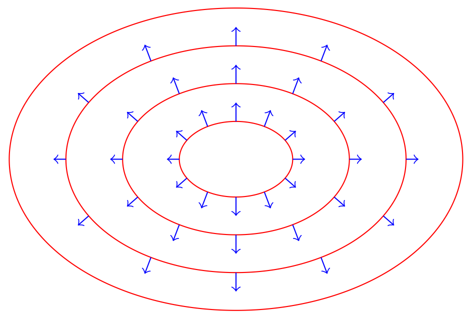
The other two derivatives work in the same way, so

$$\nabla(r^n) = nr^{n-2}\,(x, y, z).$$

The key point for interpreting the gradient geometrically is as follows:

Fact 6.2. The vector $\nabla(f)$ points in the direction of maximum increase of f. It is perpendicular to the surfaces where f is constant.

The picture below illustrates the two-dimensional version of this fact in the case where $f = \sqrt{x^2/9 + y^2/4}$. The four red ovals are given by $f = 1$, $f = 2$, $f = 3$ and $f = 4$. The blue arrows show the vector field $\nabla(f) = (2x/9, 2y/4)$, which is perpendicular to the red ovals as expected.



To see why the above fact is true, remember that if we make small changes δx, δy and δz to x, y and z, then the resulting change in f is approximately given by

$$\delta f = f_x\,\delta x + f_y\,\delta y + f_z\,\delta z.$$

If we write $\mathbf{r}$ for the vector (x, y, z), this becomes

$$\delta f = \nabla(f).\delta\mathbf{r} = |\nabla(f)||\delta\mathbf{r}|\cos(\theta),$$

where θ is the angle between $\delta\mathbf{r}$ and $\nabla(f)$. If we move along a surface where f is constant, then δf will be zero so we must have $\cos(\theta) = 0$, so $\theta = \pi/2$, so $\delta\mathbf{r}$ is perpendicular to $\nabla(f)$. This means that $\nabla(f)$ is perpendicular to the surfaces of constant f, as we stated before. On the other hand, to make δf as large as possible (for a fixed step size $|\delta\mathbf{r}|$) we need to maximise $\cos(\theta)$, which means taking $\theta = 0$, so that $\delta\mathbf{r}$ is in the same direction as $\nabla(f)$. In other words, $\nabla(f)$ points in the direction of maximum increase of f.

Two important physical applications of the gradient are as follows:

(a) We write $\mathbf{E}$ for the electric field (which is a vector field) and V for the electric potential (which is a scalar field). These are related by the equation $\mathbf{E} = \nabla(V)$. (All this is valid only when there are no significant time-varying magnetic fields.)
(b) Similarly, there is a gravitational potential function ψ, and the gravitational force field is proportional to $\nabla(\psi)$.
(c) The net force on a particle of air involves $\nabla(p)$, where p is the pressure.

Example 6.3. If we have a single charge at the origin, then the resulting electric potential function is $V = Ar^{-1}$ for some constant A, where $r = \sqrt{x^2 + y^2 + z^2}$ as usual. Using Example 6.1(c) we see that

$$\mathbf{E} = \nabla(V) = -Ar^{-3}(x, y, z) = -A\mathbf{r}/r^3.$$

Example 6.4. Suppose instead that we have a whole line of charges distributed along the z-axis. It works out that the corresponding electric potential function is $V = -\frac{1}{2}A\ln(x^2 + y^2)$ for some constant A. This is independent of z, so $V_z = 0$. On the other hand, we have

$$V_x = \frac{-\frac{1}{2}A}{x^2 + y^2}.2x = -\frac{Ax}{x^2 + y^2}.$$

By a similar calculation we have $V_y = -Ay/(x^2 + y^2)$, so

$$\mathbf{E} = \nabla(V) = \left(-\frac{Ax}{x^2 + y^2}, -\frac{Ay}{x^2 + y^2}, 0\right).$$

Example 6.5. Suppose we have an electric potential of the form $V = ax + by + cz$, where a, b and c are constant. The corresponding electric field is

$$\mathbf{E} = \nabla(V) = (a, b, c).$$

In other words, we have a uniform electric field everywhere. If we put $\mathbf{u} = (a, b, c)$ we can write the above in vector notation as $V = \mathbf{u}.\mathbf{r}$ and $\nabla(V) = \nabla(\mathbf{u}.\mathbf{r}) = \mathbf{u}$.

Example 6.6. Consider the function

$$\theta(x,y,z) = \text{ angle between the } x\text{-axis and } (x,y,0) \ = \arctan(y/x)$$

(as used in polar coordinates). It is a standard fact that $\arctan'(t) = 1/(1+t^2)$. Using this, we get

$$\begin{aligned}
\theta_x &= \arctan'\left(\frac{y}{x}\right)\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{1}{1+(y/x)^2}\frac{-y}{x^2} = \frac{-y}{x^2+y^2} \\
\theta_y &= \arctan'\left(\frac{y}{x}\right)\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{1+(y/x)^2}\frac{1}{x} = \frac{x}{x^2+y^2} \\
\theta_z &= 0,
\end{aligned}$$

so

$$\nabla(\theta) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right).$$

Now suppose we have a vector field $\mathbf{u} = (f,g,h)$, so f, g and h are all functions of x, y and z. We can think of ∇ as itself being a strange kind of vector, in which the entries are differential operators:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

This means we can make sense of the dot product $\nabla.\mathbf{u}$ and the cross product $\nabla \times \mathbf{u}$ as follows:

$$\begin{aligned}
\nabla.\mathbf{u} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).(f,g,h) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = f_x + g_y + h_z \\
\nabla \times \mathbf{u} &= \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{bmatrix} = (h_y - g_z, f_z - h_x, g_x - f_y).
\end{aligned}$$

Note that $\nabla.\mathbf{u}$ is a scalar field, and $\nabla \times \mathbf{u}$ is a vector field. The scalar field $\nabla.\mathbf{u}$ is called the *divergence* of $\mathbf{u}$, and is sometimes written as $\mathrm{div}(\mathbf{u})$. The vector field $\nabla \times \mathbf{u}$ is called the *curl* of $\mathbf{u}$, and is sometimes written $\mathrm{curl}(\mathbf{u})$.

In the two-dimensional case, a vector field has the form $\mathbf{u} = (f,g)$, where f and g are functions of x and y. In this context we define

$$\begin{aligned}
\mathrm{div}(\mathbf{u}) &= f_x + g_y \\
\mathrm{curl}(\mathbf{u}) &= \det\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ f & g \end{bmatrix} = g_x - f_y.
\end{aligned}$$

Note that here $\mathrm{curl}(\mathbf{u})$ is a scalar field, whereas in the three-dimensional case it is a vector field.

Example 6.7.

(a) For the vector field $\mathbf{u} = (x^2+y^2, y^2+z^2, z^2+x^2)$ we have

$$\begin{aligned}
\nabla.\mathbf{u} &= \frac{\partial}{\partial x}(x^2+y^2) + \frac{\partial}{\partial y}(y^2+z^2) + \frac{\partial}{\partial z}(z^2+x^2) = 2x+2y+2z \\
\nabla \times \mathbf{u} &= ((z^2+x^2)_y - (y^2+z^2)_z,\ (x^2+y^2)_z - (z^2+x^2)_x,\ (y^2+z^2)_x - (x^2+y^2)_y) \\
&= (-2z, -2x, -2y).
\end{aligned}$$

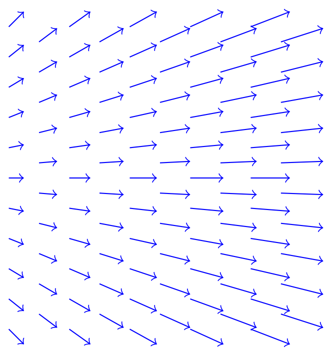
(b) For the vector field $\mathbf{u} = (\sin(x), \sin(x), \sin(x))$ we have

$$\begin{aligned}
\nabla.\mathbf{u} &= \frac{\partial}{\partial x}\sin(x) + \frac{\partial}{\partial y}\sin(x) + \frac{\partial}{\partial z}\sin(x) = \cos(x) + 0 + 0 = \cos(x) \\
\nabla \times \mathbf{u} &= (\sin(x)_y - \sin(x)_z,\ \sin(x)_z - \sin(x)_x,\ \sin(x)_x - \sin(x)_y) \\
&= (0, -\cos(x), \cos(x)).
\end{aligned}$$

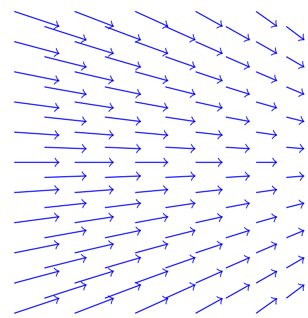
(c) For the vector field $\mathbf{u} = (-y, x, z)$ we have

$$\begin{aligned}\nabla.\mathbf{u} &= \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z) = 0 + 0 + 1 = 1 \\ \nabla \times \mathbf{u} &= (z_y - x_z,\ (-y)_z - z_x,\ x_x - (-y)_y) \\ &= (0, 0, 2).\end{aligned}$$

It works out that the divergence $\operatorname{div}(\mathbf{u}) = \nabla.\mathbf{u}$ is positive when the vectors $\mathbf{u}$ are spreading out, and negative when they are coming together.



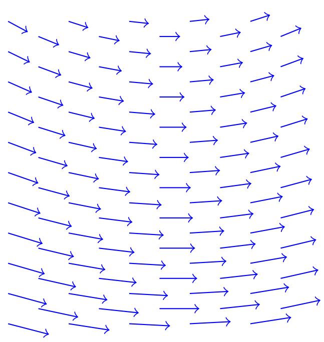
diverging: $\nabla.\mathbf{u} > 0$



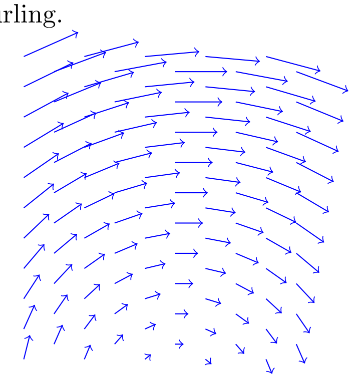
converging: $\nabla.\mathbf{u} < 0$

Now suppose that $\mathbf{u}$ is the velocity vector field for the flow of a fluid. In any region where $\nabla.\mathbf{u} > 0$, the flow is spreading outwards so the pressure and density must be decreasing. In any region where $\nabla.\mathbf{u} < 0$, the flow is coming together so the pressure and density must be increasing. If the fluid is water and the conditions are not too extreme then it is not possible to have significant changes in density through compression or decompression, so we must have $\nabla.\mathbf{u} = 0$ to a good approximation.

We next consider the geometric meaning of $\operatorname{curl}(\mathbf{u})$. In two dimensions, it works out that $\operatorname{curl}(\mathbf{u}) > 0$ in regions where the field is curling anticlockwise, and $\operatorname{curl}(\mathbf{u}) < 0$ in regions where it is curling clockwise, and the absolute value of $\operatorname{curl}(\mathbf{u})$ is determined by the strength of the curling.



$\operatorname{curl}(\mathbf{u}) > 0$, smaller



$\operatorname{curl}(\mathbf{u}) < 0$, larger

In three dimensions, the field $\mathbf{u}$ can curl around any axis. In this context, $\operatorname{curl}(\mathbf{u})$ is also a vector field, and it will point along the axis of the curling.

We now have all the ingredients needed to formulate Maxwell's equations for electromagnetism. These involve:

- The electric field $\mathbf{E}$, which is a vector field.
- The magnetic field $\mathbf{B}$, which is another vector field.
- The current density $\mathbf{J}$, which is also a vector field.
- The charge density ρ, which is a scalar field.

- Two constants: $\epsilon_0 \simeq 8.854 \times 10^{-12} F/m^2$ and $\mu_0 \simeq 1.257 \times 10^{-6} Hm^{-1}$.

The quantities $\mathbf{E}$, $\mathbf{B}$, $\mathbf{J}$ and ρ may also depend on time; we write $\dot{\mathbf{E}}$ for $\partial\mathbf{E}/\partial t$ and so on. The various fields are related by the following equations:

$$\nabla.\mathbf{E} = \rho/\epsilon_0 \qquad \nabla \times \mathbf{E} = -\dot{\mathbf{B}}$$
$$\nabla.\mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\dot{\mathbf{E}}.$$

This means that:

- The electric field diverges in regions where there is positive charge, and converges in regions where there is negative charge.
- The magnetic field never diverges or converges.
- Changing magnetic fields cause the electric field to curl.
- Currents cause the magnetic field to curl. Changing electric fields also cause the magnetic field to curl, but the effect is usually much weaker, because ϵ_0 is small.

Example 6.8. One class of solutions to Maxwell's equations is as follows. Put $c = 1/\sqrt{\mu_0\epsilon_0} \simeq 3 \times 10^8 ms^{-1}$ (which turns out to be the speed of light), and let α be any constant. We can take $\mathbf{J} = 0$ and $\rho = 0$ and

$$\begin{aligned}\mathbf{E} &= (0, \sin(\alpha(x - ct)), 0)\\ \mathbf{B} &= (0, 0, \sin(\alpha(x - ct))/c).\end{aligned}$$

We find that

$$\begin{aligned}
\nabla.\mathbf{E} &= \frac{\partial}{\partial y}\sin(\alpha(x-ct)) = 0 = \rho/\epsilon_0\\
\nabla.\mathbf{B} &= \frac{\partial}{\partial z}\sin(\alpha(x-ct))/c = 0\\
\dot{\mathbf{E}} &= (0, -\alpha c\cos(\alpha(x-ct)), 0)\\
\dot{\mathbf{B}} &= (0, 0, -\alpha\cos(\alpha(x-ct))\\
\nabla \times \mathbf{E} &= \det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & \sin(\alpha(x-ct)) & 0\end{bmatrix} = \left(0, 0, \frac{\partial}{\partial x}\sin(\alpha(x-ct))\right)\\
&= (0, 0, \alpha\cos(\alpha(x-ct))) = -\dot{\mathbf{B}}\\
\nabla \times \mathbf{B} &= \det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & 0 & \sin(\alpha(x-ct))/c\end{bmatrix} = \left(0, -\frac{\partial}{\partial x}\sin(\alpha(x-ct))/c, 0\right)\\
&= (0, -\alpha\cos(\alpha(x-ct))/c, 0) = \dot{\mathbf{E}}/c^2 = \mu_0\epsilon_0\dot{\mathbf{E}}.
\end{aligned}$$

This shows that we do indeed have a solution to the equations. It represents an electromagnetic wave of wavelength $1/\alpha$ moving at speed c in the x-direction.

Example 6.9. Another solution to Maxwell's equations has $\mathbf{E} = (-xr^{-3}, -yr^{-3}, -zr^{-3})$ with all other fields ($\mathbf{B}$, $\mathbf{J}$ and ρ) being zero. It is clear that $\dot{\mathbf{E}} = 0$ and $\dot{\mathbf{B}} = 0$, so the only equations that we need to check are that $\nabla.\mathbf{E} = 0$ and $\nabla \times \mathbf{E} = 0$. For this we recall that $r_x = x/r$, so

$$(r^{-3})_x = -3r^{-4}r_x = -3xr^{-5}.$$

In the same way, we have $(r^{-3})_y = -3yr^{-5}$ and $(r^{-3})_z = -3zr^{-5}$. Using this we find that

$$\begin{aligned}
(-xr^{-3})_x &= 3x^2r^{-5} - r^{-3} & (-xr^{-3})_y &= 3xyr^{-5} & (-xr^{-3})_z &= 3xzr^{-5}\\
(-yr^{-3})_x &= 3xyr^{-5} & (-yr^{-3})_y &= 3y^2r^{-5} - r^{-3} & (-yr^{-3})_z &= 3yzr^{-5}\\
(-zr^{-3})_x &= 3xzr^{-5} & (-zr^{-3})_y &= 3yzr^{-5} & (-zr^{-3})_z &= 3z^2r^{-5} - r^{-3}.
\end{aligned}$$

This gives

$$\begin{aligned}\nabla.\mathbf{E} &= (-xr^{-3})_x + (-yr^{-3})_y + (-zr^{-3})_z \\ &= 3x^2r^{-5} - r^{-3} + 3y^2r^{-5} - r^{-3} + 3z^2r^{-5} - r^{-3} \\ &= 3(x^2+y^2+z^2)r^{-5} - 3r^{-3} = 3r^2r^{-5} - 3r^{-3} = 0.\end{aligned}$$

It also gives

$$\begin{aligned}\nabla\times\mathbf{E} &= \big((-zr^{-3})_y - (-yr^{-3})_z, (-xr^{-3})_z - (-zr^{-3})_x, (-yr^{-3})_x - (-xr^{-3})_y\big) \\ &= \big(3yzr^{-3} - 3yzr^{-3}, 3xzr^{-3} - 3xzr^{-3}, 3xyr^{-3} - 3xyr^{-3}\big) = (0,0,0).\end{aligned}$$

This shows that we have a solution to the equations, as claimed. This one represents the electric field of a single stationary particle at the origin, with no magnetic field.

Example 6.10. Now suppose we have some positive and negative charges near the origin, such that the total charge is zero, but the charges are spread out a little so that the resulting electric fields do not cancel exactly. It works out that the resulting electric potential has the form

$$V = \frac{\mathbf{u}.\mathbf{r}}{r^3} = \frac{ax+by+cz}{(x^2+y^2+z^2)^{3/2}}$$

for some constant vector $\mathbf{u} = (a,b,c)$ (called the *dipole moment*). We will calculate the corresponding electric field $\mathbf{E} = \mathrm{grad}(V)$. Using the relation $(r^{-3})_x = -3xr^{-5}$ again, we obtain

$$\begin{aligned}V_x &= \frac{\partial}{\partial x}\big((ax+by+cz)r^{-3}\big) \\ &= r^{-3}\frac{\partial}{\partial x}(ax+by+cz) + (ax+by+cz)\frac{\partial}{\partial x}(r^{-3}) \\ &= r^{-3}a + (ax+by+cz)(-3xr^{-5}) \\ &= r^{-3}a - 3r^{-5}(\mathbf{u}.\mathbf{r})x.\end{aligned}$$

In the same way, we have

$$\begin{aligned}V_y &= r^{-3}b - 3r^{-5}(\mathbf{u}.\mathbf{r})y \\ V_z &= r^{-3}c - 3r^{-5}(\mathbf{u}.\mathbf{r})z,\end{aligned}$$

so

$$\begin{aligned}\mathrm{grad}(V) = (V_x,\ V_y,\ V_z) &= (r^{-3}a - 3r^{-5}(\mathbf{u}.\mathbf{r})x, r^{-3}b - 3r^{-5}(\mathbf{u}.\mathbf{r})y, r^{-3}c - 3r^{-5}(\mathbf{u}.\mathbf{r})z) \\ &= r^{-3}(a,b,c) - 3r^{-5}(\mathbf{u}.\mathbf{r})(x,y,z) \\ &= r^{-3}\mathbf{u} - 3r^{-5}(\mathbf{u}.\mathbf{r})\mathbf{r}.\end{aligned}$$

If we want we can rewrite $\mathbf{r}$ as $r\,\widehat{\mathbf{r}}$ and cancel an r^2 with an r^{-2} to get

$$\mathrm{grad}(V) = r^{-3}\mathbf{u} - 3r^{-3}(\mathbf{u}.\widehat{\mathbf{r}})\widehat{\mathbf{r}} = r^{-3}(\mathbf{u} - 3(\mathbf{u}.\widehat{\mathbf{r}})\widehat{\mathbf{r}}).$$

We next mention some identities that are useful for calculating or thinking about div, grad and curl. Let $\mathbf{u}$ and $\mathbf{v}$ be vector fields, let f be a scalar field, and let p be a function of one variable. Then:

$$\begin{aligned}\nabla(f+g) &= \nabla(f) + \nabla(g) \\ \nabla(fg) &= f\,\nabla(g) + g\,\nabla(f) \\ \nabla(p(f)) &= p'(f)\,\nabla(f) \\ \nabla.(\mathbf{u}+\mathbf{v}) &= \nabla.\mathbf{u} + \nabla.\mathbf{v} \\ \nabla.(f\mathbf{u}) &= f\nabla.\mathbf{u} + \nabla(f).\mathbf{u} \\ \nabla.(\mathbf{u}\times\mathbf{v}) &= \mathbf{v}.(\nabla\times\mathbf{u}) - \mathbf{u}.(\nabla\times\mathbf{u}) \\ \nabla\times(\mathbf{u}+\mathbf{v}) &= \nabla\times\mathbf{u} + \nabla\times\mathbf{v} \\ \nabla\times(f\mathbf{u}) &= f\nabla\times\mathbf{u} + \nabla(f)\times\mathbf{u}.\end{aligned}$$

All of these can be verified directly from the definitions. For example, we will check the equation
$$\nabla.(\mathbf{u}\times\mathbf{v}) = \mathbf{v}.(\nabla\times\mathbf{u}) - \mathbf{u}.(\nabla\times\mathbf{u}).$$
Suppose that $\mathbf{u}=(f,g,h)$ and $\mathbf{v}=(p,q,r)$. Then
$$\begin{aligned}
\mathbf{u}\times\mathbf{v} &= \det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ f & g & h\\ p & q & r\end{bmatrix} = (gr-hq, hp-fr, fq-gp)\\
\nabla.(\mathbf{u}\times\mathbf{v}) &= (gr-hq)_x + (hp-fr)_y + (fq-gp)_z\\
&= (g_xr+gr_x-h_xq-hq_x) + (h_yp+hp_y-f_yr-fr_y) + (f_zq+fq_z-g_zp-gp_z)\\
&= p(h_y-g_z) + q(f_z-h_x) + r(g_x-f_y) + f(q_z-r_y) + g(r_x-p_z) + h(p_y-q_x)\\
&= (p,q,r).(h_y-g_z, f_z-h_x, g_x-f_y) - (f,g,h).(r_y-q_z, p_z-r_x, q_x-p_y)\\
&= \mathbf{u}.(\nabla\times\mathbf{v}) - \mathbf{v}.(\nabla\times\mathbf{u}).
\end{aligned}$$

We now consider second-order derivatives of scalar and vector fields. There are several different ways to combine the div, grad and curl operators:

$$\text{scalar field} \xrightarrow{\text{grad}} \text{vector field} \xrightarrow{\text{div}} \text{scalar field}$$
$$\text{scalar field} \xrightarrow{\text{grad}} \text{vector field} \xrightarrow{\text{curl}} \text{vector field}$$
$$\text{vector field} \xrightarrow{\text{div}} \text{scalar field} \xrightarrow{\text{grad}} \text{vector field}$$
$$\text{vector field} \xrightarrow{\text{curl}} \text{vector field} \xrightarrow{\text{div}} \text{scalar field}$$
$$\text{vector field} \xrightarrow{\text{curl}} \text{vector field} \xrightarrow{\text{curl}} \text{vector field}\ .$$

(No other combinations make sense. For example, we cannot define curl(div($\mathbf{u}$)), because div($\mathbf{u}$) is a scalar field, and we can only take the curl of a vector field.)

It is important that two of above combinations are automatically zero.

Fact 6.11. (a) For any scalar field f we have $\mathrm{curl}(\mathrm{grad}(f)) = \nabla\times(\nabla(f)) = 0$.
(b) For any vector field $\mathbf{u}$ we have $\mathrm{div}(\mathrm{curl}(\mathbf{u})) = \nabla.(\nabla\times\mathbf{u}) = 0$.

These can be checked directly. For a scalar field f, we have $\nabla(f) = (f_x, f_y, f_z)$. After remembering that $f_{xy} = f_{yx}$ and so on, we find that
$$\nabla\times(\nabla(f)) = \det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_x & f_y & f_z\end{bmatrix} = (f_{zy}-f_{yz}, f_{xz}-f_{zx}, f_{yx}-f_{xy}) = (0,0,0).$$
Now consider instead a vector field $\mathbf{u}=(p,q,r)$. We have
$$\nabla\times\mathbf{u} = \det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ p & q & r\end{bmatrix} = (r_y-q_z, p_z-r_x, q_x-p_y),$$
so
$$\begin{aligned}
\nabla.(\nabla\times\mathbf{u}) &= (r_y-q_z)_x + (p_z-r_x)_y + (q_x-p_y)_z\\
&= r_{yx} - q_{zx} + p_{zy} - r_{xy} + q_{xz} - p_{yz}\\
&= p_{zy} - p_{yz} + q_{xz} - q_{zx} + r_{yx} - r_{xy} = 0.
\end{aligned}$$

The other combinations can be analysed as follows.

(a) For a scalar field f we have
$$\mathrm{div}(\mathrm{grad}(f)) = \nabla.(\nabla(f)) = f_{xx}+f_{yy}+f_{zz}.$$
This is usually written as $\nabla^2(f)$, and called the *Laplacian* of f. Note that the Laplacian of a scalar field is a scalar field. We can also define the Laplacian of a vector field by the rule
$$\nabla^2(p,q,r) = (\nabla^2(p), \nabla^2(q), \nabla^2(r)) = (p_{xx}+p_{yy}+p_{zz}, q_{xx}+q_{yy}+q_{zz}, r_{xx}+r_{yy}+r_{zz}).$$

Note that the Laplacian of a vector field is again a vector field.

(b) For a vector field $\mathbf{u} = (p, q, r)$ we have

$$\mathrm{grad}(\mathrm{div}(\mathbf{u})) = \nabla(\nabla.\mathbf{u}) = \nabla(p_x + q_y + r_z) = (p_{xx} + q_{yx} + r_{zx}, p_{xy} + q_{yy} + r_{zy}, p_{xz} + q_{yz} + r_{zz}).$$

(c) The last remaining combination can be expressed in terms of (a) and (b), by the equation

$$\mathrm{curl}(\mathrm{curl}(\mathbf{u})) = \nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla.\mathbf{u}) - \nabla^2(\mathbf{u}).$$

It is straightforward but somewhat lengthy to check this; we will not give the details.

In two dimensions, the situation is similar but simpler:

(a) For any scalar field f we have

$$\mathrm{div}(\mathrm{grad}(f)) = \nabla.(\nabla(f)) = f_{xx} + f_{yy},$$

which is again called the Laplacian and denoted by $\nabla^2(f)$.

(b) We also have

$$\mathrm{curl}(\mathrm{grad}(f)) = \mathrm{curl}(f_x, f_y) = f_{yx} - f_{xy} = 0.$$

We will say that a vector field $\mathbf{u}$ is *incompressible* (or *solenoidal*) if $\mathrm{div}(\mathbf{u}) = 0$, and that it is *irrotational* (or *conservative*) if $\mathrm{curl}(\mathbf{u}) = 0$.

Example 6.12. (a) For any scalar field f (in two or three dimensions) we have a vector field $\nabla(f) = \mathrm{grad}(f)$. The rule $\mathrm{curl}(\mathrm{grad}(f)) = 0$ tells us that $\mathrm{grad}(f)$ is irrotational.

(b) For any vector field $\mathbf{v}$ in three dimensions we have another vector field $\mathrm{curl}(\mathbf{v})$. The rule $\mathrm{div}(\mathrm{curl}(\mathbf{v})) = \nabla.(\nabla \times \mathbf{v}) = 0$ tells us that $\mathrm{curl}(\mathbf{v})$ is incompressible.

Example 6.13. Consider a vector field of the form

$$\mathbf{u} = (ax + by + cz, dx + ey + fz, gx + hy + iz)$$

(where $a, b, \ldots, i$ are constants). We have

$$\nabla.\mathbf{u} = (ax + by + cz)_x + (dx + ey + fz)_y + (gx + hy + iz)_z = a + e + i$$

$$\nabla \times \mathbf{u} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax + by + cz & dx + ey + fz & gx + hy + iz \end{bmatrix} = (h - f, c - g, d - b).$$

Thus, $\mathbf{u}$ is incompressible when $a + e + i = 0$, and it is irrotational when $h = f$, $g = c$ and $d = b$. In the irrotational case, we can rewrite the equation for $\mathbf{u}$ as

$$\mathbf{u} = (ax + by + cz, bx + ey + fz, cx + fy + iz).$$

If we put

$$p = \tfrac{1}{2}(ax^2 + ey^2 + iz^2) + bxy + cxz + fyz,$$

we find that

$$\begin{aligned} p_x &= ax + by + cz \\ p_y &= bx + ey + fz \\ p_z &= cx + fy + iz, \end{aligned}$$

so

$$\nabla(p) = (ax + by + cz, bx + ey + fz, cx + fy + iz) = \mathbf{u}.$$

Example 6.14. Consider the two-dimensional vector field

$$\mathbf{u} = (x^2 - y^2 + 2xy, x^2 - y^2 - 2xy).$$

This has

$$\begin{aligned} \mathrm{div}(\mathbf{u}) &= \frac{\partial}{\partial x}(x^2 - y^2 + 2xy) + \frac{\partial}{\partial y}(x^2 - y^2 - 2xy) = (2x + 2y) + (-2y - 2x) = 0 \\ \mathrm{curl}(\mathbf{u}) &= \frac{\partial}{\partial x}(x^2 - y^2 - 2xy) - \frac{\partial}{\partial y}(x^2 - y^2 + 2xy) = (2x - 2y) - (-2y + 2x) = 0, \end{aligned}$$

so it is both incompressible and irrotational.

If $\mathbf{u}$ is an irrotational vector field, a *potential function* for $\mathbf{u}$ is a scalar field p such that $\nabla(p) = \mathbf{u}$. (Because of Fact 6.11, only irrotational fields can have a potential.) Potential functions always exist (but they may be multi-valued), and it is often useful to find them.

Example 6.15. Consider the vector field $\mathbf{u} = (y+z, z+x, x+y)$. This has

$$\nabla \times \mathbf{u} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{bmatrix} = (1-1, 1-1, 1-1) = 0,$$

so it is irrotational. It therefore makes sense to look for a potential function, or in other words a function $p(x,y,z)$ with

$$p_x = y + z \tag{A}$$
$$p_y = z + x \tag{B}$$
$$p_z = x + y. \tag{C}$$

As we want $p_x = y + z$, we must have

$$p = \int y + z\, dx = xy + xz + \text{ arbitrary constant }.$$

We will call the constant here q. Note that as we integrated with respect to x, the quantity q does not need to be completely constant, it just needs to be independent of x (so $q_x = 0$). We thus have

$$p = xy + xz + q. \tag{D}$$

We can differentiate (D) with respect to y to get

$$p_y = x + q_y. \tag{E}$$

On the other hand, equation (B) says that $p_y = x + z$. By comparing (B) and (E) we see that $q_y = z$. Integrating this with respect to y gives

$$q = \int z\, dy = yz + r, \tag{F}$$

where r is "constant" in the sense that it is independent of both x and y, so it can only depend on z. We can substitute (F) in (D) to get

$$p = xy + xz + q = xy + xz + yz + r. \tag{G}$$

We can differentiate (G) with respect to z to get

$$p_z = x + y + r_z. \tag{H}$$

On the other hand, equation (C) says that $p_z = x + y$. By comparing (C) and (H) we see that $r_z = 0$. As r can only depend on z and we have $r_z = 0$ we see that r is a genuine constant. We can choose it to be zero, and we find that the function $p = xy + xz + yz$ is a potential function for $\mathbf{u}$.

Example 6.16. Consider the vector field $\mathbf{u} = (0, 0, x^2)$. This has

$$\nabla \times \mathbf{u} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & x^2 \end{bmatrix} = (0, -2x, 0) \neq 0,$$

so it is not irrotational, so it cannot have a potential function. We will nonetheless try to find one, and see what goes wrong. A potential function p would have to have $(p_x, p_y, p_z) = (0, 0, x^2)$. As $p_x = p_y = 0$, we see that p can only depend on z. That means that the derivative p_z also depends only on z, so we cannot have $p_z = x^2$. Thus, there is no potential function.

Example 6.17. Consider again the two-dimensional vector field

$$\mathbf{u} = (x^2 - y^2 + 2xy, x^2 - y^2 - 2xy).$$

We saw in Example 6.14 that this is irrotational, so it has a potential function p, satisfying $p_x = x^2-y^2+2xy$ and $p_y = x^2 - y^2 - 2xy$. Integrating the first of these gives

$$p = \int x^2 - y^2 + 2xy\,dx = \tfrac{1}{3}x^3 - xy^2 + x^2y + q,$$

where q depends only on y. This gives $p_y = -2xy+x^2+q_y$, but p_y is supposed to be equal to x^2-y^2-2xy, so we must have $q_y = -y^2$, which gives $q = -\frac{1}{3}y^3$ (plus a constant, which we may take to be zero). Altogether this gives

$$p = \tfrac{1}{3}x^3 - xy^2 + x^2y - \tfrac{1}{3}y^3.$$

Example 6.18. Consider the vector field $\mathbf{u} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$.

We have

$$\nabla \times \mathbf{u} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{bmatrix} = \left(0, 0, \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right)$$

The relevant partial derivatives are

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{1.(x^2+y^2) - x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$
$$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{1.(x^2+y^2) - y.2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2},$$

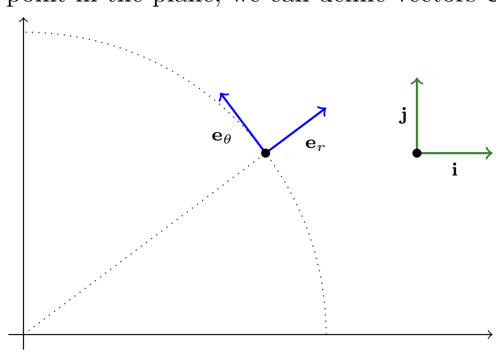
and when we add these together we get zero. This means that $\nabla \times \mathbf{u} = 0$, so $\mathbf{u}$ is irrotational. It therefore makes sense to look for a potential function p, which must satisfy

$$\nabla(p) = (p_x, p_y, p_z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right).$$

Looking back to Example 6.6, we see that the required function is $p = \theta = \arctan(y/x)$. This is most naturally thought of as a multivalued function: for example, the value at $(-1,0,0)$ could be any odd multiple of π. This is bound up with the fact that $\mathbf{u}$ is not well-defined on the z-axis (where the formula $x/(x^2+y^2)$ involves division by zero). There is much more that could be said about this kind of phenomenon (with applications to magnetic fields around superconductors, for example) but we will not explore that here.

7. Vector fields in polar coordinates

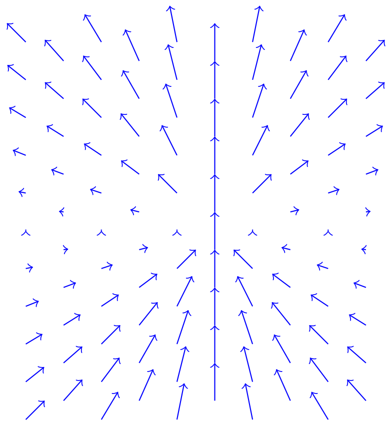
7.1. **Two dimensions.** At any point in the plane, we can define vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ as shown:

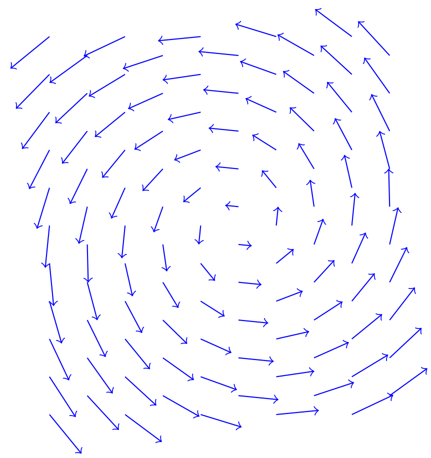


In situations with circular symmetry, it is often more natural to describe vector fields in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ rather than $\mathbf{i}$ and $\mathbf{j}$. One can translate between the two descriptions as follows:

$$\mathbf{e}_r = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} \qquad \mathbf{e}_\theta = -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}$$
$$\mathbf{i} = \cos(\theta)\mathbf{e}_r - \sin(\theta)\mathbf{e}_\theta \qquad \mathbf{j} = \sin(\theta)\mathbf{e}_r + \cos(\theta)\mathbf{e}_\theta.$$

Here are two examples of vector fields described in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$:





$\mathbf{u} = \sin(\theta)\mathbf{e}_r$ $\qquad\qquad$ $\mathbf{u} = \sqrt{r}(\mathbf{e}_\theta + \mathbf{e}_r/10)$

We will need to express the operators grad, div and curl in terms of polar coordinates.

Fact 7.1. (a) For any two-dimensional scalar field f (expressed as a function of r and θ) we have

$$\nabla(f) = \operatorname{grad}(f) = f_r\,\mathbf{e}_r + r^{-1} f_\theta\,\mathbf{e}_\theta.$$

(b) For any 2-dimensional vector field $\mathbf{u} = m\,\mathbf{e}_r + p\,\mathbf{e}_\theta$ (where m and p are expressed as functions of r and θ) we have

$$\operatorname{div}(\mathbf{u}) = r^{-1}m + m_r + r^{-1}p_\theta$$
$$\operatorname{curl}(\mathbf{u}) = r^{-1}p + p_r - r^{-1}m_\theta.$$

Note that the product rule gives $(rm)_r = m + r\,m_r$ and $(rp)_r = p + r\,p_r$. Using this, we can rewrite the above equations as

$$\operatorname{div}(\mathbf{u}) = r^{-1}\left((rm)_r + p_\theta\right)$$
$$\operatorname{curl}(\mathbf{u}) = r^{-1}\left((rp)_r - m_\theta\right) = \frac{1}{r}\det\begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} \\ m & rp \end{bmatrix}.$$

(c) For any two-dimensional scalar field f we have

$$\nabla^2(f) = r^{-1}f_r + f_{rr} + r^{-2}f_{\theta\theta} = r^{-1}(rf_r)_r + r^{-2}f_{\theta\theta}$$

We will explain part (a) of the above. Consider the field $\mathbf{u} = f_r\,\mathbf{e}_r + r^{-1}f_\theta\,\mathbf{e}_\theta$; we need to show that this is the same as $\operatorname{grad}(f)$. For this, we need a version of the two-variable chain rule. Suppose we make a small change of δr to r. This will cause a change of $\delta x \simeq x_r\,\delta r$ to x, which in turn causes a change of approximately $f_x\,\delta x \simeq f_x\,x_r\,\delta r$ to f. At the same time, our change in r also causes a change of $\delta y \simeq y_r\,\delta r$ to x, which in turn causes a change of approximately $f_y\,\delta y = f_y\,y_r\,\delta r$ to f. Altogether, the change in f is $\delta f \simeq (f_x x_r + f_y y_r)\delta r$. By passing to the limit $\delta r \to 0$, we see that $f_r = f_x x_r + f_y y_r$. Similarly, we have $f_\theta = f_x x_\theta + f_y y_\theta$. Moreover, we can differentiate the formulae

$$x = r\cos(\theta) \qquad\qquad y = r\sin(\theta)$$

to get

$$x_r = \cos(\theta) \qquad\qquad y_r = \sin(\theta)$$
$$x_\theta = -r\sin(\theta) \qquad\qquad y_\theta = r\cos(\theta),$$

so

$$\begin{aligned}
f_r &= f_x x_r + f_y y_r = \cos(\theta) f_x + \sin(\theta) f_y \\
f_\theta &= f_x x_\theta + f_y y_\theta = -r\sin(\theta) f_x + r\cos(\theta) f_y \\
\mathbf{u} &= f_r\,\mathbf{e}_r + r^{-1} f_\theta\,\mathbf{e}_\theta \\
&= f_x\cos(\theta)\mathbf{e}_r + f_y\sin(\theta)\mathbf{e}_r - f_x\sin(\theta)\mathbf{e}_\theta + f_y\cos(\theta)\mathbf{e}_\theta \\
&= f_x\left(\cos(\theta)\mathbf{e}_r - \sin(\theta)\mathbf{e}_\theta\right) + f_y\left(\sin(\theta)\mathbf{e}_r + \cos(\theta)\mathbf{e}_\theta\right) \\
&= f_x\mathbf{i} + f_y\mathbf{j} = \mathrm{grad}(f).
\end{aligned}$$

The formulae for $\mathrm{div}(\mathbf{u})$ and $\mathrm{curl}(\mathbf{u})$ in polar coordinates can be checked in a similar way, but the calculations are lengthy and not very illuminating, so we will omit them.

Example 7.2. Consider the function $f = r^n$. Clearly $f_r = nr^{n-1}$ and $f_\theta = 0$, so

$$\mathrm{grad}(f) = f_r\,\mathbf{e}_r + r^{-1} f_\theta\,\mathbf{e}_\theta = nr^{n-1}\mathbf{e}_r.$$

Note also that $\mathbf{r} = (x, y) = (r\cos(\theta), r\sin(\theta)) = r\,\mathbf{e}_r$, so $\mathbf{e}_r = \mathbf{r}/r$, so we can rewrite the above as $\mathrm{grad}(r^n) = nr^{n-2}\mathbf{r}$. This is the same as the formula we obtained in Example 6.1(c).

Example 7.3. Consider the function $f = \theta$. Clearly $f_r = 0$ and $f_\theta = 1$, so

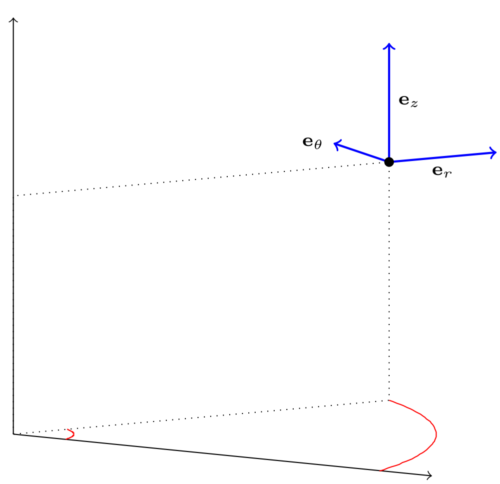
$$\mathrm{grad}(f) = f_r\,\mathbf{e}_r + r^{-1} f_\theta\,\mathbf{e}_\theta = r^{-1}\mathbf{e}_\theta = r^{-2}(-r\sin(\theta), r\cos(\theta)) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right).$$

This is the same as the formula we obtained in Example 6.6.

Example 7.4. Consider the vector field $\mathbf{u} = \sqrt{r}(\mathbf{e}_\theta + \mathbf{e}_r/10)$ from the plot above. This is $\mathbf{u} = p\mathbf{e}_r + q\mathbf{e}_\theta$ where $p = r^{\frac{1}{2}}/10$ and $q = r^{\frac{1}{2}}$, so $p_\theta = q_\theta = 0$ and $p_r = r^{-\frac{1}{2}}/20$ and $q_r = r^{-\frac{1}{2}}/2$. It follows that

$$\begin{aligned}
\mathrm{div}(\mathbf{u}) &= r^{-1}p + p_r + r^{-1}q_\theta \\
&= r^{-1}r^{\frac{1}{2}}/10 + r^{-\frac{1}{2}}/20 + 0 = 3r^{-\frac{1}{2}}/20 \\
\mathrm{curl}(\mathbf{u}) &= r^{-1}q + q_r - r^{-1}p_\theta \\
&= r^{-1}r^{-\frac{1}{2}} + r^{-\frac{1}{2}}/2 - 0 = 3r^{-\frac{1}{2}}/2.
\end{aligned}$$

7.2. **Cylindrical polar coordinates.** In cylindrical polar coordinates we use unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ as shown below:



Thus, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the same as for two-dimensional polar coordinates, and $\mathbf{e}_z$ is just the vertical unit vector $\mathbf{k}$. The equations are:

$$\begin{aligned}
\mathbf{e}_r &= \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} & \mathbf{e}_\theta &= -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j} & \mathbf{e}_z &= \mathbf{k} \\
\mathbf{i} &= \cos(\theta)\mathbf{e}_r - \sin(\theta)\mathbf{e}_\theta & \mathbf{j} &= \sin(\theta)\mathbf{e}_r + \cos(\theta)\mathbf{e}_\theta & \mathbf{k} &= \mathbf{e}_z.
\end{aligned}$$

The rules for div, grad and curl are as follows:

Fact 7.5. (a) For any three-dimensional scalar field f (expressed as a function of r, θ and z) we have

$$\nabla(f) = \mathrm{grad}(f) = f_r\,\mathbf{e}_r + r^{-1}f_\theta\,\mathbf{e}_\theta + f_z\mathbf{e}_z.$$

(b) For any three-dimensional vector field $\mathbf{u} = m\,\mathbf{e}_r + p\,\mathbf{e}_\theta + q\,e_z$ (where m, p and q are expressed as functions of r, θ and z) we have

$$\begin{aligned}
\mathrm{div}(\mathbf{u}) &= r^{-1}m + m_r + r^{-1}p_\theta + q_z = r^{-1}(rm)_r + r^{-1}p_\theta + q_z \\
\mathrm{curl}(\mathbf{u}) &= \frac{1}{r}\det\begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ m & rp & q \end{bmatrix}.
\end{aligned}$$

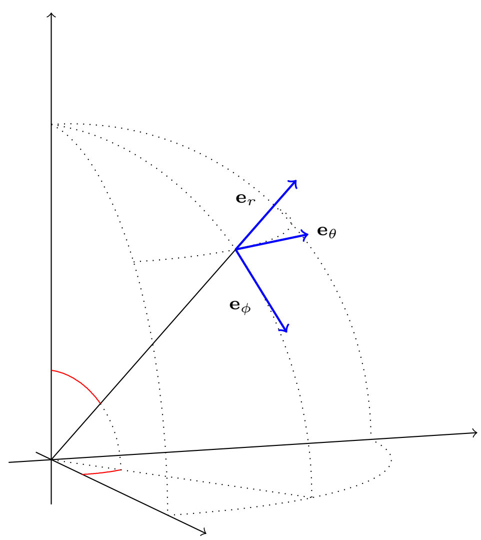
(c) For any three-dimensional scalar field f we have

$$\nabla^2(f) = r^{-1}f_r + f_{rr} + r^{-2}f_{\theta\theta} + f_{zz} = r^{-1}(rf_r)_r + r^{-2}f_{\theta\theta} + f_{zz}.$$

Example 7.6. Consider the vector field $\mathbf{u}$ given in cylindrical polar coordinates by $\mathbf{u} = r(\mathbf{e}_\theta + \mathbf{e}_z)$. Here $m = 0$ and $p = q = r$, so

$$\begin{aligned}
\mathrm{curl}(\mathbf{u}) &= \frac{1}{r}\det\begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & r^2 & r \end{bmatrix} \\
&= \frac{1}{r}\left(\left(\frac{\partial}{\partial\theta}(r) - \frac{\partial}{\partial z}(r^2)\right)\mathbf{e}_r - \left(\frac{\partial}{\partial r}(r) - \frac{\partial}{\partial z}(0)\right)r\mathbf{e}_\theta + \left(\frac{\partial}{\partial r}(r^2) - \frac{\partial}{\partial\theta}(0)\right)\mathbf{e}_z\right) \\
&= \frac{1}{r}\left(-r\mathbf{e}_\theta + 2r\mathbf{e}_z\right) = 2\mathbf{e}_z - \mathbf{e}_\theta.
\end{aligned}$$

7.3. **Spherical polar coordinates.** In spherical polar coordinates we use unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ as shown below:



Note that $\mathbf{e}_\theta$ has the same meaning as it did in the cylindrical case, but $\mathbf{e}_r$ has changed. It used to be the unit vector pointing horizontally away from the z-axis, but now it points directly away from the origin.

The vectors $\mathbf{e}_r$, $\mathbf{e}_\phi$ and $\mathbf{e}_\theta$ are related to $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as follows.

$$\begin{aligned}
\mathbf{e}_r &= \sin(\phi)\cos(\theta)\mathbf{i} + \sin(\phi)\sin(\theta)\mathbf{j} + \cos(\phi)\mathbf{k} \\
\mathbf{e}_\phi &= \cos(\phi)\cos(\theta)\mathbf{i} + \cos(\phi)\sin(\theta)\mathbf{j} - \sin(\phi)\mathbf{k} \\
\mathbf{e}_\theta &= -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j} \\
\mathbf{i} &= \sin(\phi)\cos(\theta)\mathbf{e}_r + \cos(\phi)\cos(\theta)\mathbf{e}_\phi - \sin(\theta)\mathbf{e}_\theta \\
\mathbf{j} &= \sin(\phi)\sin(\theta)\mathbf{e}_r + \cos(\phi)\sin(\theta)\mathbf{e}_\phi + \cos(\theta)\mathbf{e}_\theta \\
\mathbf{k} &= \cos(\phi)\mathbf{e}_r - \sin(\phi)\mathbf{e}_\phi.
\end{aligned}$$

The rules for div, grad and curl in spherical polar coordinates are as follows:

Fact 7.7. (a) For any three-dimensional scalar field f (expressed as a function of r, ϕ and θ) we have

$$\nabla(f) = \operatorname{grad}(f) = f_r\,\mathbf{e}_r + r^{-1} f_\phi\,\mathbf{e}_\phi + (r\,\sin(\phi))^{-1} f_\theta \mathbf{e}_\theta.$$

(b) For any three-dimensional vector field $\mathbf{u} = m\,\mathbf{e}_r + p\,\mathbf{e}_\phi + q\,e_\theta$ (where m, p and q are expressed as functions of r, ϕ and θ) we have

$$\operatorname{div}(\mathbf{u}) = r^{-2}(r^2 m)_r + (r\sin(\phi))^{-1}(\sin(\phi)p)_\phi + (r\sin(\phi))^{-1} q_\theta$$

$$\operatorname{curl}(\mathbf{u}) = \frac{1}{r^2\sin(\phi)} \det \begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\phi & r\sin(\phi)\mathbf{e}_\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ m & rp & r\sin(\phi)q \end{bmatrix}.$$

(c) For any three-dimensional scalar field f we have

$$\nabla^2(f) = r^{-2}(r^2 f_r)_r + (r^2\sin(\phi))^{-1}(\sin(\phi)f_\phi)_\phi + (r^2\sin^2(\phi))^{-1} f_{\theta\theta}.$$

Example 7.8. The electric potential created by a point charge at the origin is $V = A/r$, where A is a constant and $r = \sqrt{x^2+y^2+z^2}$ (as in spherical polar coordinates). The corresponding electric field is $\mathbf{E} = \operatorname{grad}(V)$. As there are no magnetic fields, and no charges away from the origin, we should have $\operatorname{div}(\mathbf{E}) = 0$ and $\operatorname{curl}(\mathbf{E}) = 0$. We will check that this all works out as expected.

First, we have $V_r = -A/r^2$ and $V_\phi = V_\theta = 0$, so the rule

$$\operatorname{grad}(V) = V_r\,\mathbf{e}_r + r^{-1}V_\phi\,\mathbf{e}_\phi + (r\,\sin(\phi))^{-1}V_\theta\mathbf{e}_\theta$$

just gives

$$\mathbf{E} = \operatorname{grad}(V) = -Ar^{-2}\mathbf{e}_r.$$

In other words, we have $\mathbf{E} = m\mathbf{e}_r + p\mathbf{e}_\phi + q\mathbf{e}_\theta$ with $m = -Ar^{-2}$ and $p = q = 0$. The general rule for the divergence is

$$\operatorname{div}(\mathbf{E}) = r^{-2}(r^2 m)_r + (r\sin(\phi))^{-1}(\sin(\phi)p)_\phi + (r\sin(\phi))^{-1}q_\theta.$$

As $p = q = 0$, the second and third terms are zero. In the first term, we have $r^2 m = -A$, which is constant, so $(r^2 m)_r = 0$ as well. This means that $\operatorname{div}(\mathbf{E}) = 0$ as expected.

Finally, we have

$$\operatorname{curl}(\mathbf{E}) = \frac{1}{r^2\sin(\phi)} \det \begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\phi & r\sin(\phi)\mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ m & rp & r\sin(\phi)q \end{bmatrix} = \frac{1}{r^2\sin(\phi)} \det \begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\phi & r\sin(\phi)\mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ -Ar^{-2} & 0 & 0 \end{bmatrix}.$$

As $\frac{\partial}{\partial \phi}(-Ar^{-2}) = \frac{\partial}{\partial \theta}(-Ar^{-2}) = 0$, we see that all terms vanish so $\operatorname{curl}(\mathbf{E}) = 0$ as well.

8. Curves

Often we need to deal with curves in three-dimensional space. For example:

(a) A wire in an electrical machine is a curve. To calculate the magnetic field created by a current in the wire, or the force exerted on the wire by an externally applied magnetic field, we need equations for the curve.

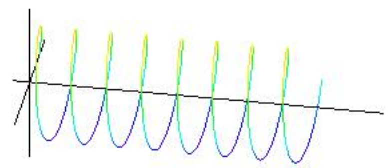
(b) The path of a moving particle over time defines a curve. If the particle is charged then it will feel a force from any electric or magnetic fields; to understand the effect of this, we need various equations relating the position, velocity, force and acceleration to the fields.

We can describe a curve by giving the x, y and z coordinates (or equivalently, the position vector $\mathbf{r} = (x, y, z)$) in terms of another parameter t. (In the case of a moving particle we often take t to be time, but that is not compulsory.)

Example 8.1. The equation

$$\mathbf{r} = (x, y, z) = (at,\ b\cos(t),\ b\sin(t))$$

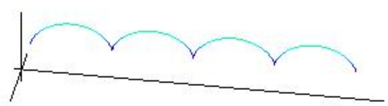
describes a helix winding around the x-axis.



This is the path followed by an electron moving in a uniform magnetic field. It could also describe a wire wound round a cylinder.

Example 8.2. Suppose that a car with axles of length a and wheels of radius b drives at constant speed c along the x-axis. A pebble stuck in the front left tyre will move along the curve with equation

$$\mathbf{r} = (ct, a/2, b) - b(\sin(ct/b), 0, \cos(ct/b)) = (ct - b\sin(ct/b), a/2, b - b\cos(ct/b)).$$



The first term $(ct, a/2, b)$ reflects the overall motion of the car, and the second term comes from the rotation of the wheel.

Example 8.3. A thrown ball will follow a parabolic path like

$$\mathbf{r} = (at, bt, ct - dt^2)$$

for some constants $a, \ldots, d$.

Often we need to integrate along a curve C. The general idea should by now be familiar. We first divide C into many small pieces, each running from some position $\mathbf{r}$ to a nearby position $\mathbf{r} + \delta\mathbf{r}$. Each such piece will give a contribution to the integral, and we add up the contributions to get an approximation to the required value. For the exact value, we pass to the limit where the length of the small pieces tends to zero.

(a) The length of the curve is approximately the sum of the lengths $|\delta\mathbf{r}|$ over all the small pieces. The exact length is denoted by $\int_C |d\mathbf{r}|$.
(b) The vector from the beginning to the end of the curve is the sum of the vectors $\delta\mathbf{r}$ over all the small pieces. In the limit we denote this by $\int_C d\mathbf{r}$.
(c) If a particle moves along a curve C through a force field $\mathbf{F}$, then the work done against the force is $-\int_C \mathbf{F}.d\mathbf{r}$.

For integrals of type (b) and (c), it makes a difference which direction we follow when traversing the curve: the answer we get when traversing the curve backwards will be the negative of the answer we get when traversing the curve forwards.

In practice, we calculate these integrals as follows. We parametrise the curve as $\mathbf{r} = (x(t), y(t), z(t))$ for some range of values of t (say $a \leq t \leq b$), and we write $\dot{x} = dx/dt$ and so on. We then have

$$
\begin{aligned}
d\mathbf{r} &= \frac{d\mathbf{r}}{dt}dt = \dot{\mathbf{r}}dt = (\dot{x}\,dt,\ \dot{y}\,dt,\ \dot{z}\,dt) \\
|d\mathbf{r}| &= \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,dt,
\end{aligned}
$$

so

$$
\begin{aligned}
\text{length}(C) &= \int_C |d\mathbf{r}| = \int_{t=a}^{b} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,dt \\
\text{work} &= \int_C \mathbf{F}.d\mathbf{r} = \int_{t=a}^{b} \mathbf{F}.\dot{\mathbf{r}}\,dt
\end{aligned}
$$

and so on.

Example 8.4. Let C be the curve from $(0,0,0)$ to $(6,12,8)$ given by

$$\mathbf{r} = (x, y, z) = (6t, 3\sqrt{2}t^2, 2t^3)$$

for $0 \leq t \leq 1$. We will calculate the length of this curve. We have

$$
\begin{aligned}
d\mathbf{r} &= (6, 6\sqrt{2}t, 6t^2)\,dt \\
|d\mathbf{r}| &= \sqrt{36 + 72t^2 + 36t^4}\,dt = 6\sqrt{1 + 2t^2 + t^4}\,dt = 6(1 + t^2)\,dt,
\end{aligned}
$$

so

$$\text{length} = \int_C |d\mathbf{r}| = \int_{t=0}^{1} 6(1+t^2)\,dt = \Big[6t + 2t^3\Big]_{t=0}^{1} = 8.$$

Example 8.5. Consider a particle moving along a path $\mathbf{r} = (x, y, z) = (t, 0, t/2)$ (for $0 \leq t \leq 1$) against a force field $\mathbf{F} = (y^2 + z^2 - 1, 0, 0)$. (This could reasonably model the wind force in a wind tunnel of radius one centred on the x-axis.) Note that

$$
\begin{aligned}
d\mathbf{r} &= (1, 0, 1/2)\,dt \\
\mathbf{F} &= (y^2 + z^2 - 1, 0, 0) = (t^2/4 - 1, 0, 0) \\
\mathbf{F}.d\mathbf{r} &= (t^2/4 - 1)\,dt.
\end{aligned}
$$

The work done against the force is therefore

$$\text{work} = -\int_C \mathbf{F}.d\mathbf{r} = \int_{t=0}^{1} (1 - t^2/4)\,dt = \Big[t - t^3/12\Big]_{t=0}^{1} = \frac{11}{12}.$$

If f is a function of one variable, it is a basic fact that

$$\int_{x=a}^{b} f'(x)\,dx = f(b) - f(a).$$

(This is known as the *Fundamental Theorem of Calculus.*) There is an analogous principle for path integrals, as follows:

Fact 8.6. For any curve C from a point $\mathbf{a}$ to a point $\mathbf{b}$, and any scalar field p, we have

$$\int_C \nabla(p).d\mathbf{r} = p(\mathbf{b}) - p(\mathbf{a}).$$

The reason is simple: for any short piece of the curve, the quantity

$$\nabla(p).\delta\mathbf{r} = p_x\delta x + p_y\delta y + p_z\delta z$$

is (to a good approximation) the change in p along that piece, and when we add up all these small changes, we get the overall change in p from $\mathbf{a}$ to $\mathbf{b}$.

This gives us the following method.

Method 8.7. Suppose we have a curve C from $\mathbf{a}$ to $\mathbf{b}$, and we want to calculate the integral $I = \int_C \mathbf{F}.d\mathbf{r}$ for some vector field $\mathbf{F}$. Suppose that $\mathbf{F}$ is conservative. We can then find a potential function p with $\nabla(p) = \mathbf{F}$, and it will follow that $\int_C \mathbf{F}.d\mathbf{r} = p(\mathbf{b}) - p(\mathbf{a})$.

Note that in this method, *we do not need to know anything about C except where it starts and ends*. This often makes calculations much easier.

Example 8.8. Let C be given by $(x, y, z) = (1 - 2t^2, 1, 2t^3)$ for $0 \leq t \leq 1$, and consider the vector field $\mathbf{F} = (-y/(x^2 + y^2), x/(x^2 + y^2), 0)$. It would be very unpleasant to calculate $\int_C \mathbf{F}.d\mathbf{r}$ directly. However, we know from Example 6.18 that $\mathbf{F}$ is conservative, with the polar coordinate function θ as a potential, so $\int_C \mathbf{F}.d\mathbf{r}$ is just the change in θ from the start of C to the end of C. At the start of C we have $t = 0$ so $(x, y, z) = (1, 1, 0)$, so $\theta = \pi/4$. At the end we have $t = 1$, so $(x, y, z) = (-1, 1, 2)$, which lies above the point $(-1, 1, 0)$ in the xy-plane; this means that $\theta = 3\pi/4$. This means that

$$\int_C \mathbf{F}.d\mathbf{r} = \frac{3\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2}.$$

We have cheated a little bit here (although our answer is in fact correct) by ignoring the multi-valued nature of θ. This becomes important if we need to deal with curves that wind several times around the z-axis. However, we will not explore this further at the moment.

If we have trouble finding a potential function, it may be better to use the following approach:

Method 8.9. Suppose we have a curve C from $\mathbf{a}$ to $\mathbf{b}$, and we want to calculate the integral $I = \int_C \mathbf{F}.d\mathbf{r}$ for some vector field $\mathbf{F}$. Suppose that $\mathbf{F}$ is conservative. We can then find a different curve C' from $\mathbf{a}$ to $\mathbf{b}$ for which the calculation is easier, and then I will be equal to $\int_{C'} \mathbf{F}.d\mathbf{r}$.

The reason why this method works is that both $\int_C \mathbf{F}.d\mathbf{r}$ and $\int_{C'} \mathbf{F}.d\mathbf{r}$ are equal to $p(\mathbf{b}) - p(\mathbf{a})$, where p is the potential function. For this to be valid, we need to know that p exists (so we must check that $\mathbf{F}$ is conservative) but we do not actually need to find p.

Example 8.10. Let C be the helical path given by $\mathbf{r} = (t, \cos(10\pi t), \sin(10\pi t))$ for $0 \leq t \leq 1$, which runs from $\mathbf{a} = (0, 1, 0)$ to $\mathbf{b} = (1, 1, 0)$. Let $\mathbf{F}$ be the vector field (yz, xz, xy). We would like to calculate $\int_C \mathbf{F}.d\mathbf{r}$. We first check whether $\mathbf{F}$ is conservative, by finding the curl:

$$\nabla \times \mathbf{F} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{bmatrix} = (x - x, y - y, z - z) = 0.$$

As $\mathbf{F}$ is conservative, we can replace C by a simpler path without changing the integral. In particular, we can use the straight line L given by $\mathbf{r} = (x, y, z) = (t, 1, 0)$. On L we have $\mathbf{F} = (0, 0, t)$ and $d\mathbf{r} = (1, 0, 0)dt$ so $\mathbf{F}.d\mathbf{r} = 0$, so we conclude that $\int_C \mathbf{F}.d\mathbf{r} = \int_L \mathbf{F}.d\mathbf{r} = 0$.

Example 8.11. Let C and L be as in the last example, and consider the vector field $\mathbf{G} = (0, -z, y)$. This one has

$$\nabla \times \mathbf{G} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -z & y \end{bmatrix} = (1 - (-1), 0 - 0, 0 - 0) = (2, 0, 0) \neq 0.$$

As $\mathbf{G}$ is not conservative, the integrals $\int_C \mathbf{G}.d\mathbf{r}$ and $\int_L \mathbf{G}.d\mathbf{r}$ need not be the same. (They could be the same by coincidence, but that would be unlikely.) On C we have

$$\begin{aligned} \mathbf{r} &= (x, y, z) = (t, \cos(10\pi t), \sin(10\pi t)) \\ d\mathbf{r} &= (\dot{x}, \dot{y}, \dot{z})\, dt = (1, -10\pi \sin(10\pi t), 10\pi \cos(10\pi t))\, dt \\ \mathbf{G} &= (0, -z, y) = (0, -\sin(10\pi t), \cos(10\pi t)) \\ \mathbf{G}.d\mathbf{r} &= 10\pi(\sin^2(10\pi t) + \cos^2(10\pi t))\, dt = 10\pi\, dt \end{aligned}$$

$$\int_C \mathbf{G}.d\mathbf{r} = \int_{t=0}^{1} 10\pi\, dt = 10\pi.$$

On L we have

$$\begin{aligned}
\mathbf{r} &= (x,y,z) = (t,1,0) \\
d\mathbf{r} &= (\dot{x},\dot{y},\dot{z})\,dt = (1,0,0)\,dt \\
\mathbf{G} &= (0,-z,y) = (0,0,-1) \\
\mathbf{G}.d\mathbf{r} &= 0 \\
\int_L \mathbf{G}.d\mathbf{r} &= 0.
\end{aligned}$$

Thus, the integrals over C and L are different, as expected.

We can also use Method 8.7 in reverse, as follows.

Method 8.12. Let $\mathbf{F}$ be an conservative vector field. We can then define a potential function for $\mathbf{F}$ by the rule

$$p(a,b,c) = \text{ the integral } \int_C \mathbf{F}.d\mathbf{r}, \text{ for any curve } C \text{ from } (0,0,0) \text{ to } (a,b,c)\ .$$

The answer will not depend on the choice of curve, so we can choose whichever curve makes the integral easiest. A straight line is often good, but sometimes a broken line (from $(0,0,0)$ to $(a,0,0)$ to $(a,b,0)$ to (a,b,c), for example) is better.

We should emphasis again that this is only valid for conservative fields. Fields that are not conservative do not have a potential function.

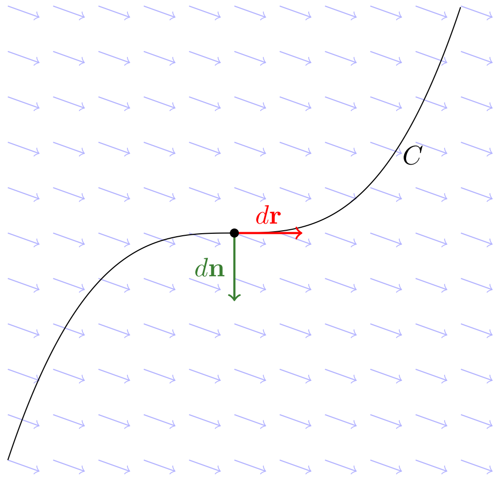
If you consider carefully the logical relationship between the various statements made above, you will see that we have not really justified the above method. It is possible to close this gap, but we will not do so here.

Example 8.13. In Example 8.10, we checked that the vector field $\mathbf{F} = (yz,xz,xy)$ is conservative, so it has a potential function p. To find $p(a,b,c)$, we evaluate $\int_L \mathbf{F}.d\mathbf{r}$, where L is the straight line from $(0,0,0)$ to (a,b,c). This can be parametrised by $\mathbf{r} = (x,y,z) = (ta,tb,tc)$ for $0 \leq t \leq 1$, which gives

$$\begin{aligned}
d\mathbf{r} &= (a,b,c)dt \\
\mathbf{F} &= (t^2bc, t^2ac, t^2ab) \\
\mathbf{F}.d\mathbf{r} &= 3t^2abc\,dt \\
p(a,b,c) &= \int_L \mathbf{F}.d\mathbf{r} = \int_{t=0}^1 3t^2abc\,dt = \Big[t^3abc\Big]_{t=0}^1 = abc.
\end{aligned}$$

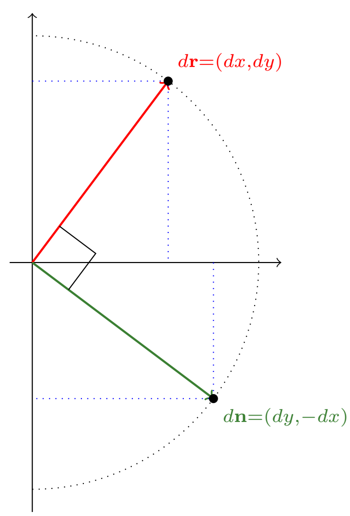
It is convenient to write this calculation in terms of a, b and c, to avoid confusion between the end of the path (where $(x,y,z) = (a,b,c)$) and the points along the path (where $(x,y,z) = (ta,tb,tc)$). However, we can restate the final answer as $p(x,y,z) = xyz$, which is more convenient for later use.

Everything that we have done so far works equally well in two dimensions or three dimensions. Now, however, we will consider a different kind of integral that only makes sense in the plane. The picture below shows a vector field $\mathbf{F}$ and a curve C, with the vector $d\mathbf{r}$ pointing along the curve, and another vector $d\mathbf{n}$ of the same length perpendicular to $d\mathbf{r}$.

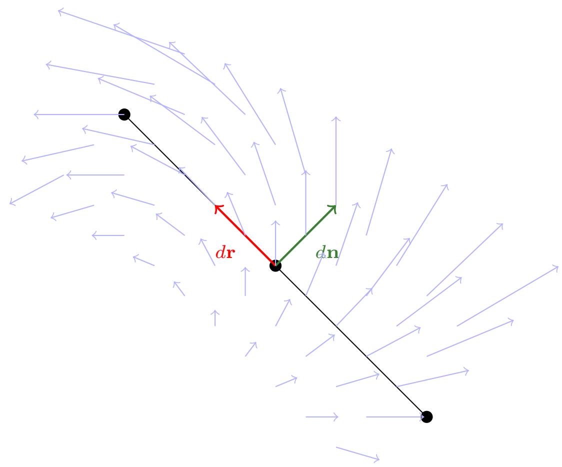


The integral $\int_C \mathbf{F}.d\mathbf{r}$ measures the extent to which $\mathbf{F}$ points along the curve. For some purposes, however, we want to measure the flow of $\mathbf{F}$ *across* the curve, in which case we want to evaluate $\int_C \mathbf{F}.d\mathbf{n}$ rather than $\int_C \mathbf{F}.d\mathbf{r}$.

Note that $d\mathbf{r} = (dx, dy) = (\dot{x}, \dot{y})dt$, and $d\mathbf{n}$ is obtained by rotating this through a quarter turn clockwise, so $d\mathbf{n} = (dy, -dx) = (\dot{y}, -\dot{x})dt$.



Example 8.14. Consider the vector field $\mathbf{F} = (x^2 - y^2, 2xy)$ and the straight line L from $(1, 0)$ to $(0, 1)$, given by $\mathbf{r} = (1 - t, t)$ for $0 \leq t \leq 1$.



This gives $d\mathbf{r} = (-1, 1)\,dt$ and

$$\mathbf{F} = ((1-t)^2 - t^2,\ 2(1-t)t) = (1-2t, 2t-2t^2),$$

so

$$\mathbf{F}.d\mathbf{r} = (1-2t,\ 2t-2t^2).(-1,\ 1)dt = (4t-1-2t^2)dt.$$

Integrating this, we get

$$\int_C \mathbf{F}.d\mathbf{r} = \int_{t=0}^{1} (4t-1-2t^2)dt = \left[2t^2 - t - \tfrac{2}{3}t^3\right]_{t=0}^{1} = \tfrac{1}{3}.$$

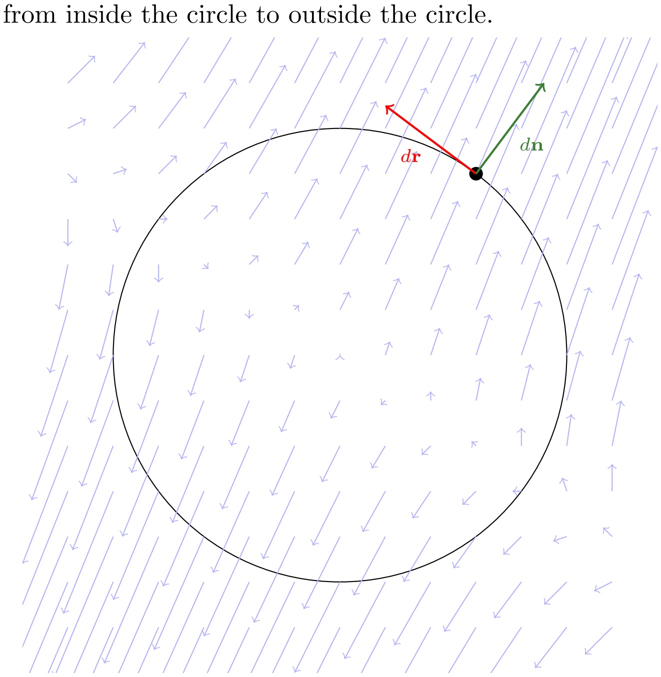
On the other hand, we have $d\mathbf{n} = (dy, -dx) = (1, 1)\,dt$ so

$$\mathbf{F}.d\mathbf{n} = (1-2t^2)\,dt,$$

so

$$\int_C \mathbf{F}.d\mathbf{n} = \int_{t=0}^{1} (1-2t^2)\,dt = \left[t - \tfrac{2}{3}t^3\right]_{t=0}^{1} = \tfrac{1}{3}.$$

Example 8.15. Let C be the unit circle, and take $\mathbf{F} = (x+2y, 3x+4y)$. We will calculate $\int_C \mathbf{F}.d\mathbf{n}$, which is the flow of $\mathbf{F}$ crossing from inside the circle to outside the circle.



We parametrise C as $\mathbf{r} = (x, y) = (\cos(t), \sin(t))$ for $0 \leq t \leq 2\pi$. This gives

$$\begin{aligned}
d\mathbf{r} &= (\dot{x}, \dot{y})dt = (-\sin(t), \cos(t))\, dt \\
d\mathbf{n} &= (\dot{y}, -\dot{x})dt = (\cos(t), \sin(t))\, dt \\
\mathbf{F} &= (\cos(t) + 2\sin(t), 3\cos(t) + 4\sin(t)) \\
\mathbf{F}.d\mathbf{n} &= (\cos^2(t) + 5\sin(t)\cos(t) + 4\sin^2(t))dt.
\end{aligned}$$

Now

$$\begin{aligned}
\int_0^{2\pi} \sin(t)\cos(t)\, dt &= \tfrac{1}{2}\int_0^{2\pi} \sin(2t)\, dt = 0 \\
\int_0^{2\pi} \sin^2(t)\, dt &= \int_0^{2\pi} \cos^2(t)\, dt = \pi
\end{aligned}$$

so

$$\int_C \mathbf{F}.d\mathbf{n} = \int_0^{2\pi} (\cos^2(t) + 5\sin(t)\cos(t) + 4\sin^2(t))dt = \pi + 0 + 4\pi = 5\pi.$$

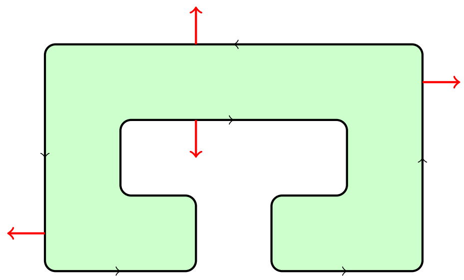
9. Integral theorems

Some important facts about electromagnetism are as follows:

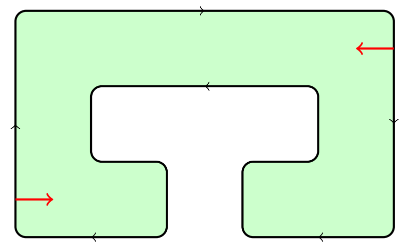
(a) For any three-dimensional region, the total electric field crossing the boundary of the region is ϵ_0^{-1} times the total charge in the region.
(b) On the other hand, the magnetic field crossing the boundary always cancels out to give a total of zero.
(c) Now suppose we have a surface S in three-dimensional space. Suppose that has a boundary that is a closed curve C (so the surface could be a disk or a hemispherical bowl, but not a complete sphere). Then the circulation of $\mathbf{E}$ around C is minus the rate of change of the total magnetic field passing through S.
(d) Similarly, the circulation of $\mathbf{B}$ around C is μ_0 times the rate of change of the current passing through S (including the "displacement current" $\epsilon_0\dot{\mathbf{E}}$).

These are not really new physical facts; they are mathematically equivalent to Maxwell's equations, which we discussed earlier. Maxwell's equations told us about the values of scalar and vector fields and their derivatives at every point in space. The above statements are about various kinds of integrals of such scalar and vector fields over curves, surfaces and three-dimensional regions. The main point of this final section of the course is to understand why these integral statements are the same as the earlier differential statements.

9.1. **The two-dimensional divergence theorem.** Let D be a region in the plane. The edge of the region will be a curve, which we call C. For any vector field $\mathbf{u}$, we can consider the integral $\int_C \mathbf{u}.d\mathbf{n}$ measuring the flux of $\mathbf{u}$ across C. This kind of integral depends on the direction in which we traverse the curve. We will always traverse in the direction which keeps the region D on our left. This means that we are basically going anticlockwise, although it may not always seem that way if C has a complicated shape. Recall also that $d\mathbf{n}$ is defined to be the vector obtained by turning $d\mathbf{r}$ through a quarter turn clockwise, so it points to the right of the direction of travel, and thus away from the region on the left.



In this direction
we keep the region on the left
so $d\mathbf{n}$ points outwards



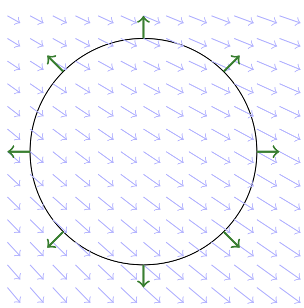
In this direction
we keep the region on the right
so $d\mathbf{n}$ points inwards

The two-dimensional divergence theorem says that for any vector field $\mathbf{u}$ that is well-behaved everywhere in D, we have

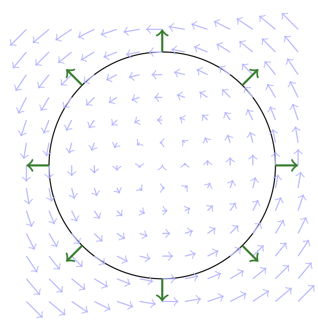
$$\iint_D \nabla.\mathbf{u}\, dA = \int_C \mathbf{u}.d\mathbf{n}.$$

Here "well-behaved" means that there are no discontinuous jumps (as with a square wave) or kinks (as with a sawtooth). Functions like $1/(x^2+y^2)$ (which blows up to infinity at the origin) are allowed if the origin lies outside D, but disallowed if the origin is inside D.

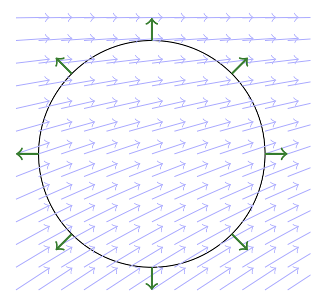
It is not too hard to see roughly why the theorem should be true. We can imagine that the vector field is describing the flow of some kind of particles. In regions where $\nabla.\mathbf{u} > 0$ the flow lines are spreading apart which means that new particles must be being created. In regions where $\nabla.\mathbf{u} < 0$ the flow lines are coming together which means that particles are being destroyed. The net rate of creation of particles is given by the integral $\iint_D \nabla.\mathbf{u}\, dA$. This rate of creation must be balanced the net flow of particles crossing the boundary curve C to leave the region, which is given by $\int_C \mathbf{u}.d\mathbf{n}$.



$\mathrm{div}(\mathbf{u}) > 0$



$\mathrm{div}(\mathbf{u}) = 0$



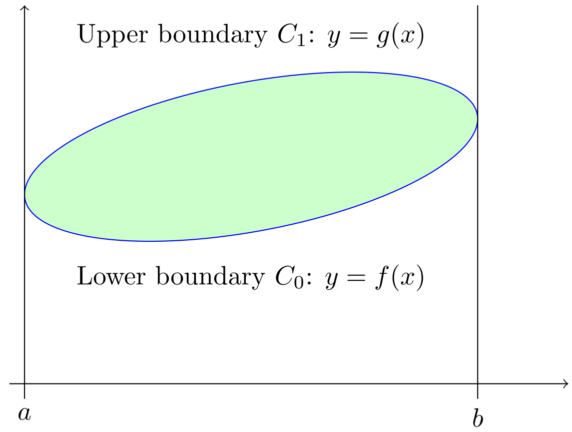
$\mathrm{div}(\mathbf{u}) < 0$

We can give a slightly more formal argument as follows. We can write $\mathbf{u} = (p, q)$ for some scalar functions p and q, so $\nabla.\mathbf{u} = p_x + q_y$. We will show that

$$\iint_D q_y\, dA = \int_C (0,q).(dy,-dx) = -\int_C q\, dx \tag{A}$$

$$\iint_D p_x\, dA = \int_C (p,0).(dy,-dx) = \int_C p\, dy. \tag{B}$$

Adding these two gives $\iint_D \nabla.\mathbf{u}\, dA = \int_C \mathbf{u}.d\mathbf{n}$ as claimed. For the first of these equations, we will assume for simplicity that the curve region D has the following general shape:



Recall that for any function $m(y)$, we have

$$\int_{y=y_0}^{y_1} m'(y)dy = m(y_1) - m(y_0).$$

We can fix x and regard $q(x, y)$ as a function of y alone, and we get

$$\int_{y=y_0}^{y_1} q_y(x, y)\, dy = q(x, y_1) - q(x, y_0).$$

This works for any y_0 and y_1. In particular, it works for $y_0 = f(x)$ and $y_1 = g(x)$, so we have

$$\int_{y=f(x)}^{g(x)} q_y(x, y)\, dy = q(x, g(x)) - q(x, f(x)).$$

Note that the right hand side is the change in q from the bottom boundary of D to the top boundary. We now integrate both sides:

$$\int_{x=a}^{b} \int_{y=f(x)}^{g(x)} q_y(x, y)\, dy\, dx = \int_{x=a}^{b} q(x, g(x))\, dx - \int_{x=a}^{b} q(x, f(x))\, dx. \tag{C}$$

The left hand side is $\iint_D q_y\, dA$.

We can parametrise the lower boundary C_0 as $y = f(x)$ for $a \leq x \leq b$. This gives

$$\int_{C_0} q(x, y)\, dx = \int_{x=a}^{b} q(x, f(x))dx. \tag{D}$$

We could also parametrise the upper boundary C_1 as $y = g(x)$ for $a \leq x \leq b$. However, this runs along C_1 keeping D on the right, but we are supposed to go in the opposite direction, keeping D on the left. The integral in the wrong direction is minus the integral in the right direction, we we see that

$$\int_{C_1} q(x, y)\, dx = -\int_{x=a}^{b} q(x, g(x))dx. \tag{E}$$

After substituting (D) and (E) in (C) we see that

$$\iint_D q_y\, dA = -\left(\int_{C_0} q_y\, dy + \int_{C_1} q_y\, dy \right) = -\int_C q_y\, dy,$$

which verifies equation (A). Equation (B) can be checked in the same way, except that we need to do the outer integral with respect to y and the inner integral with respect to x.

Example 9.1. Let D be the disc where $x^2 + y^2 \leq m^2$, so C is a circle of radius m. Take $\mathbf{u} = (ax + by, cx + dy)$ for some constants a, b, c and d. Then $\operatorname{div}(\mathbf{u}) = (ax + by)_x + (cx + dy)_y = a + d$, so $\iint_D \operatorname{div}(\mathbf{u})dA =$

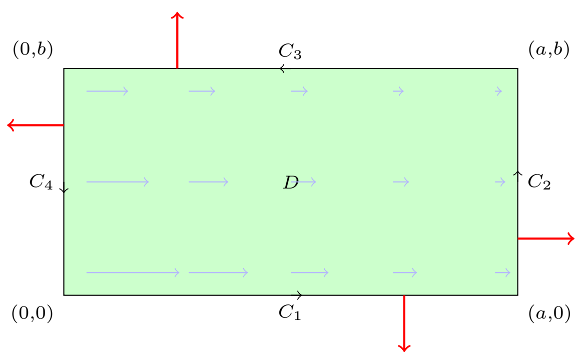
$(a+d)$area(D)= $\pi m^2(a+d)$. On the other hand, we can parametrise C by $\mathbf{r} = (x,y) = (m\cos(t), m\sin(t))$, so $d\mathbf{n} = (\dot{y}, -\dot{x})dt = (m\cos(t), m\sin(t))\,dt$. On C we also have

$$\mathbf{u} = (ax+by, cx+dy) = (am\cos(t) + bm\sin(t), cm\cos(t) + dm\sin(t))$$

so

$$\begin{aligned}
\mathbf{u}.d\mathbf{n} &= (am\cos(t) + bm\sin(t))(m\cos(t))dt + \\
&\qquad (cm\cos(t) + dm\sin(t))(m\sin(t))dt \\
&= m^2(a\cos^2(t) + (b+c)\sin(t)\cos(t) + d\sin^2(t))dt \\
&= \frac{m^2}{2}(a + a\cos(2t) + (b+c)\sin(2t) + d - d\cos(2t)) \\
&= \frac{m^2}{2}((a+d) + (a-d)\cos(2t) + (b+c)\sin(2t)) \\
\int_C \mathbf{u}.d\mathbf{n} &= \frac{m^2}{2}\left[(a+d)t + \tfrac{1}{2}(a-d)\sin(2t) - \tfrac{1}{2}(b+c)\cos(2t)\right]_{t=0}^{2\pi} \\
&= \frac{m^2}{2}2\pi(a+d) = \pi m^2(a+d).
\end{aligned}$$

Example 9.2. Let D be the rectangle as shown below. The boundary curve C consists of the four segments C_1, C_2, C_3 and C_4.



Consider the horizontal vector field $\mathbf{u} = (e^{-x-y}, 0)$. This has div$(\mathbf{u}) = \frac{\partial}{\partial x}(e^{-x-y}) = -e^{-x-y} = -e^{-x}e^{-y}$, so

$$\begin{aligned}
\iint_D \operatorname{div}(\mathbf{u})dA &= -\int_{x=0}^{a} e^{-x}\,dx \int_{y=0}^{b} e^{-y}\,dy = -\left[-e^{-x}\right]_{x=0}^{a}\left[-e^{-y}\right]_{y=0}^{b} \\
&= -(1-e^{-a})(1-e^{-b}) = e^{-a} + e^{-b} - e^{-a-b} - 1.
\end{aligned}$$

On C_1 and C_3 the normal $d\mathbf{n}$ is vertical but $\mathbf{u}$ is horizontal so $\mathbf{u}.d\mathbf{n} = 0$. On C_2 we have $d\mathbf{n} = (1,0)dy$ and $x = a$ so $\mathbf{u} = (e^{-a-y}, 0)$ so $\mathbf{u}.d\mathbf{n} = e^{-a-y}dy$ so

$$\int_{C_2} \mathbf{u}.d\mathbf{n} = \int_{y=0}^{b} e^{-a-y}dy = \left[-e^{-a-y}\right]_{y=0}^{b} = e^{-a} - e^{-a-b}$$

We can parametrise C_4 in the right direction by $(x,y) = (0, b-t)$ for $0 \leq t \leq b$. This gives $d\mathbf{n} = (\dot{y}, -\dot{x})dt = (-1,0)dt$ and $\mathbf{u} = (e^{-x-y}, 0) = (e^{t-b}, 0)$ so $\mathbf{u}.d\mathbf{n} = -e^{t-b}$ so

$$\int_{C_4} \mathbf{u}.d\mathbf{n} = \int_{t=0}^{b} -e^{t-b}\,dt = \left[-e^{t-b}\right]_{t=0}^{b} = -1 + e^{-b}$$

This gives

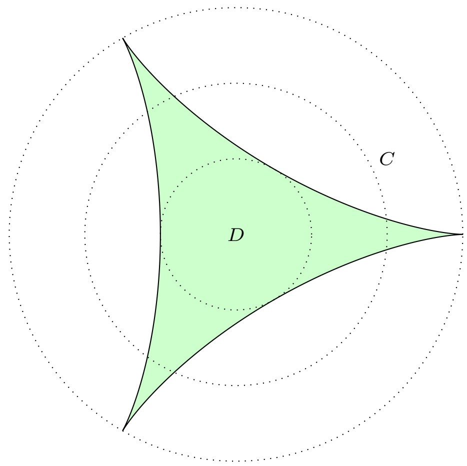
$$\int_C \mathbf{u}.d\mathbf{n} = (e^{-a} - e^{-a-b}) + (-1 + e^{-b}) = e^{-a} + e^{-b} - e^{-a-b} - 1 = \iint_D \operatorname{div}(\mathbf{u})dA$$

as expected.

Example 9.3. The picture shows the curve C with equations

$$x = 2\cos(t) + \cos(2t) \qquad\qquad y = 2\sin(t) - \sin(2t)$$

(for $0 \leq t \leq 2\pi$). This is called the *deltoid curve*.



This curve encloses a region D. It is hard to find the area of D directly. However, we can evaluate it by a trick using the divergence theorem. Consider the vector field $\mathbf{F} = (x, 0)$, so $\operatorname{div}(\mathbf{F}) = \partial x/\partial x + \partial 0/\partial y = 1$, so $\iint_D \operatorname{div}(\mathbf{F}) dA = \operatorname{area}(D)$. The Divergence Theorem tells us that this is the same as $\int_C \mathbf{F}.d\mathbf{n}$. Here

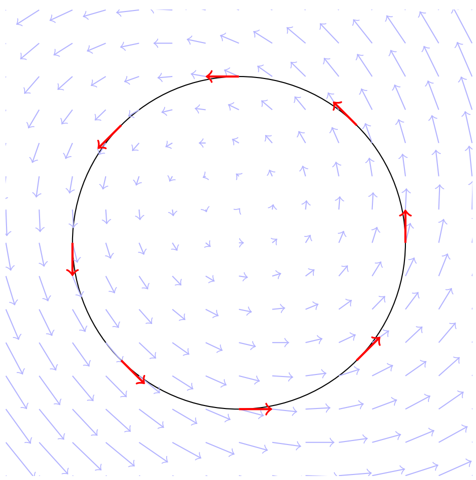
$$\begin{aligned}
d\mathbf{n} &= (\dot{y}, -\dot{x})\,dt = (2\cos(t) - 2\cos(2t),\ 2\sin(t) + 2\sin(2t))\,dt \\
\mathbf{F} &= (x, 0) = (2\cos(t) + \cos(2t),\ 0) \\
\mathbf{F}.d\mathbf{n} &= (2\cos(t) - 2\cos(2t))(2\cos(t) + \cos(2t)) \\
&= 4\cos^2(t) - 2\cos(t)\cos(2t) - 2\cos^2(2t) \\
&= (2 + 2\cos(2t)) - (\cos(3t) + \cos(t)) - (1 + \cos(4t)) \\
&= 1 - \cos(t) + 2\cos(2t) - \cos(3t) - \cos(4t) \\
\int_{t=0}^{2\pi} \mathbf{F}.d\mathbf{n} &= 2\pi
\end{aligned}$$

so we conclude that the area of D is 2π.

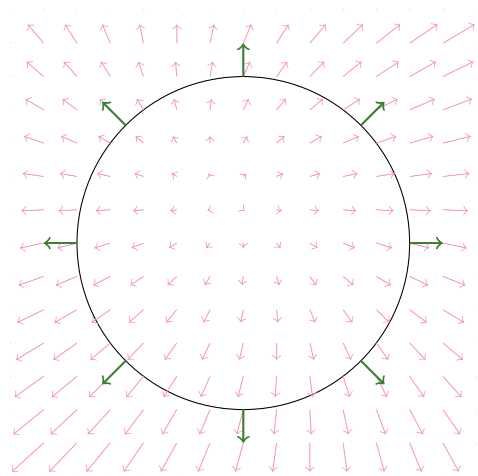
9.2. **Green's theorem.** Let D be a region in the plane whose boundary is a closed curve C. Green's theorem says that for any vector field $\mathbf{u}$ that is well-behaved everywhere in D, we have

$$\iint_D \operatorname{curl}(\mathbf{u})\,dA = \int_C \mathbf{u}.d\mathbf{r}.$$

To see this, let $\mathbf{v}$ be the field obtained by turning $\mathbf{u}$ clockwise by $\pi/2$.



u and $d\mathbf{r}$



v and $d\mathbf{n}$

We can apply the divergence theorem to $\mathbf{v}$ to get $\iint_D \operatorname{div}(\mathbf{v})dA = \int_C \mathbf{v}.d\mathbf{n}$. If $\mathbf{u} = (p, q)$ then $\mathbf{v} = (q, -p)$, so $\operatorname{div}(\mathbf{v}) = q_x - p_y = \operatorname{curl}(\mathbf{u})$ and

$$\mathbf{v}.d\mathbf{n} = (q, -p).(dy, -dx) = p\,dx + q\,dy = (p, q).(dx, dy) = \mathbf{u}.d\mathbf{r}$$

so

$$\iint_D \operatorname{curl}(\mathbf{u})\,dA = \int_C \mathbf{u}.d\mathbf{r}.$$

as claimed

Example 9.4. Let D be the unit disc, so the boundary curve C is the unit circle. Let $\mathbf{u}$ be the vector field (x^3, x^3). Green's Theorem tells us that $\iint_D \operatorname{curl}(\mathbf{u})\,dA = \int_C \mathbf{u}.d\mathbf{r}$. We will check this by evaluating both sides. First, we have

$$\operatorname{curl}(\mathbf{u}) = \det \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^3 & x^3 \end{bmatrix} = \frac{\partial}{\partial x}(x^3) - \frac{\partial}{\partial y}(x^3) = 3x^2 - 0 = 3x^2.$$

We will evaluate $\iint_D \operatorname{curl}(\mathbf{u})\,dA$ using polar coordinates, so $x = r\cos(\theta)$ and $dA = r\,dr\,d\theta$. This gives

$$\begin{aligned}\iint_D \operatorname{curl}(\mathbf{u})\,dA &= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} 3(r\cos(\theta))^2\,r\,dr\,d\theta \\ &= 3\left(\int_{\theta=0}^{2\pi}\cos^2(\theta)d\theta\right)\left(\int_{r=0}^{1} r^3\,dr\right) \\ &= 3\times\pi\times(1/4) = 3\pi/4.\end{aligned}$$

On the other side, we can parametrise C as $(x, y) = (\cos(\theta), \sin(\theta))$ (for $-\pi \leq \theta \leq \pi$), which gives

$$\begin{aligned} d\mathbf{r} &= (-\sin(\theta), \cos(\theta))\;d\theta \\ \mathbf{u} &= (x^3, x^3) = (\cos^3(\theta), \cos^3(\theta)) \\ \mathbf{u}.d\mathbf{r} &= (\cos^4(\theta) - \sin(\theta)\cos^3(\theta))\,d\theta.\end{aligned}$$

We will need to rewrite this in a form that is easier to integrate. We can square the relation $\cos^2(\theta) = (1 + \cos(2\theta))/2$ to get

$$\cos^4(\theta) = (1 + 2\cos(2\theta) + \cos^2(2\theta))/4.$$

We also have $\cos^2(2\theta) = (1 + \cos(4\theta))/2$, so after some rearrangement we get

$$\cos^4(\theta) = (3 + 4\cos(2\theta) + \cos(4\theta))/8.$$

On the other hand, we have

$$\begin{aligned}\sin(\theta)\cos^3(\theta) = (\sin(\theta)\cos(\theta))\ \cos^2(\theta) &= \tfrac{1}{2}\sin(2\theta) \times \tfrac{1}{2}(1+\cos(2\theta)) \\ &= \tfrac{1}{4}(\sin(2\theta) + \sin(2\theta)\cos(2\theta)) \\ &= \tfrac{1}{4}\sin(2\theta) + \tfrac{1}{8}\sin(4\theta).\end{aligned}$$

Putting these together, we get

$$\mathbf{u}.d\mathbf{r} = \frac{1}{8}\left(3 + 4\cos(2\theta) + \cos(4\theta) - 2\sin(2\theta) - \sin(4\theta)\right).$$

It is standard that

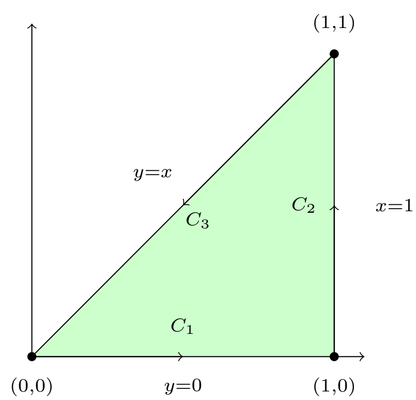
$$\int_{\theta=0}^{2\pi} \sin(2\theta)d\theta = \int_{\theta=0}^{2\pi} \sin(4\theta)d\theta = \int_{\theta=0}^{2\pi} \cos(2\theta)d\theta = \int_{\theta=0}^{2\pi} \cos(4\theta)d\theta = 0,$$

so

$$\begin{aligned}\int_C \mathbf{u}.d\mathbf{r} &= \frac{1}{8}\int_{\theta=0}^{2\pi} 3 + 4\cos(2\theta) + \cos(4\theta) - 2\sin(2\theta) - \sin(4\theta)d\theta \\ &= \frac{1}{8}\int_{\theta=0}^{2\pi} 3d\theta = \frac{1}{8}\times 3 \times 2\pi = 3\pi/4.\end{aligned}$$

As expected, this is the same as $\iint_D \operatorname{curl}(\mathbf{u})\,dA$.

Example 9.5. Let D be the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.



Let $\mathbf{u}$ be the vector field $(-y^2, x^2 - xy + y^2)$. This has

$$\operatorname{curl}(\mathbf{u}) = \det\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ -y^2 & x^2 - xy + y^2 \end{bmatrix} = (2x - y) - (-2y) = 2x + y,$$

so

$$\begin{aligned}\iint_D \operatorname{curl}(\mathbf{u})\,dA &= \int_{x=0}^{1}\int_{y=0}^{x} 2x + y\,dy\,dx = \int_{x=0}^{1}\left[2xy + \tfrac{1}{2}y^2\right]_{y=0}^{x} dx \\ &= \int_{x=0}^{1} 2x^2 + \tfrac{1}{2}x^2\,dx = \int_{x=0}^{1}\tfrac{5}{2}x^2\,dx = \left[\tfrac{5}{6}x^3\right]_{x=0}^{1} \\ &= 5/6.\end{aligned}$$

We will check that this is the same as $\int_C \mathbf{u}.d\mathbf{r}$ as predicted by Green's Theorem, where C is the boundary curve of the region D. This consists of three segments C_1, C_2 and C_3 as shown in the diagram. On C_1 we have $y = 0$ so $\mathbf{u} = (-y^2, x^2 - xy + y^2) = (0, x^2)$, but also $dy = 0$ so $d\mathbf{r} = (dx, 0)$ so $\mathbf{u}.d\mathbf{r} = 0$, so $\int_{C_1}\mathbf{u}.d\mathbf{r} = 0$. On C_2 we have $x = 1$ (so $dx = 0$) and

$$\mathbf{u} = (-y^2, x^2 - xy + y^2) = (-y^2, 1 - y + y^2),$$

so

$$\mathbf{u}.d\mathbf{r} = (-y^2, 1 - y + y^2).(0, dy) = (1 - y + y^2)\,dy,$$

so

$$\int_{C_2} \mathbf{u}.d\mathbf{r} = \int_{y=0}^{1} (1 - y + y^2)\, dy = \left[y - \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 \right]_{y=0}^{1} = 1 - \tfrac{1}{2} + \tfrac{1}{3} = 5/6.$$

On C_3 we have $y = x$ so $d\mathbf{r} = (dx, dx)$ but $\mathbf{u} = (-y^2, x^2 - xy + y^2) = (-y^2, y^2)$. From this it is clear that $\mathbf{u}.d\mathbf{r} = 0$ on C_3, so $\int_{C_3} \mathbf{u}.d\mathbf{r} = 0$. Putting this together, we get

$$\int_C \mathbf{u}.d\mathbf{r} = \int_{C_1} \mathbf{u}.d\mathbf{r} + \int_{C_2} \mathbf{u}.d\mathbf{r} + \int_{C_3} \mathbf{u}.d\mathbf{r} = 0 + 5/6 + 0 = 5/6.$$

As expected, this is the same as $\iint_D \mathrm{curl}(\mathbf{u})\, dA$.

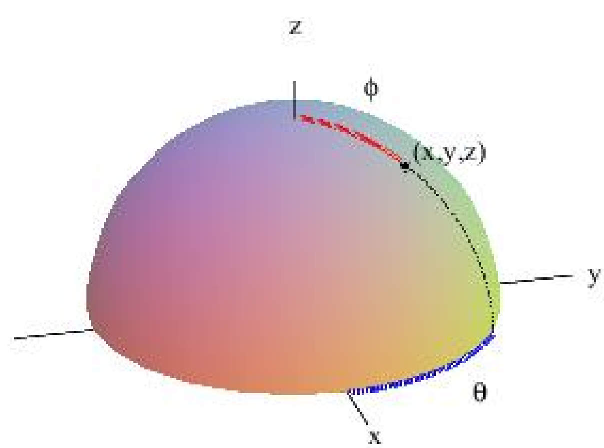
10. Surfaces

As well as considering curved paths, we also need to consider curved surfaces in three-dimensional space. Such a surface can be parametrised as $\mathbf{r} = (x(s,t), y(s,t), z(s,t))$ for some pair of parameters s and t.

Example 10.1. (a) The upper half of a spherical shell of radius 2 can be described in terms of parameters ϕ and θ by

$$(x, y, z) = (2\sin(\phi)\cos(\theta),\ 2\sin(\phi)\sin(\theta),\ 2\cos(\phi))$$

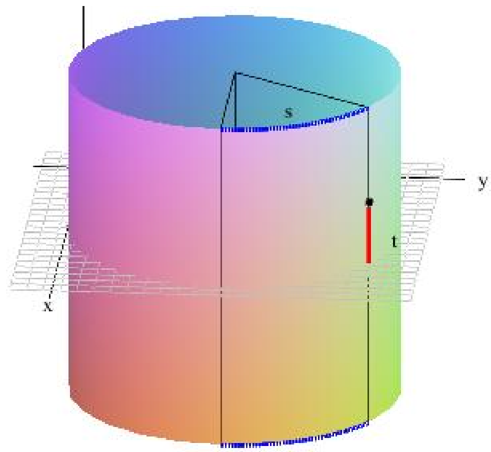
(for $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi/2$).



(b) Let S be a cylindrical surface of radius 1, centred on the line joining $(1, 1, -1)$ to $(1, 1, 1)$. Then S can be described in terms of parameters s and t by

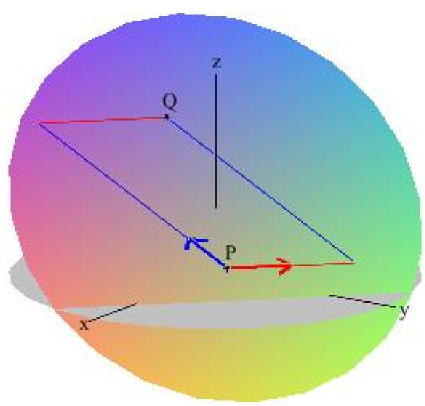
$$(x, y, z) = (1 + \cos(s), 1 + \sin(s), t)$$

(for $0 \leq s \leq 2\pi$ and $-1 \leq t \leq 1$).



(c) Let S be the plane where $x+y+z=3$. This can be parametrised in many different ways, one of which is

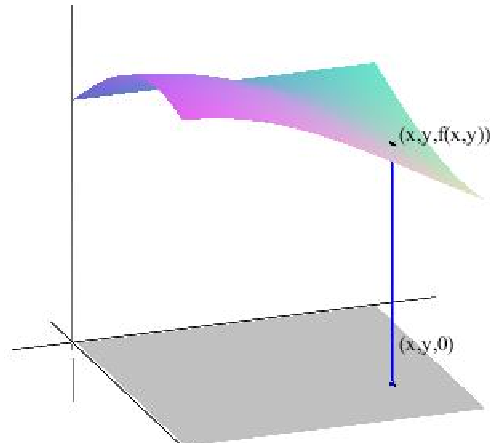
$$(x,y,z) = (1-s, 1+s-t, 1+t) = (1,1,1) + s(-1,1,0) + t(0,-1,1).$$



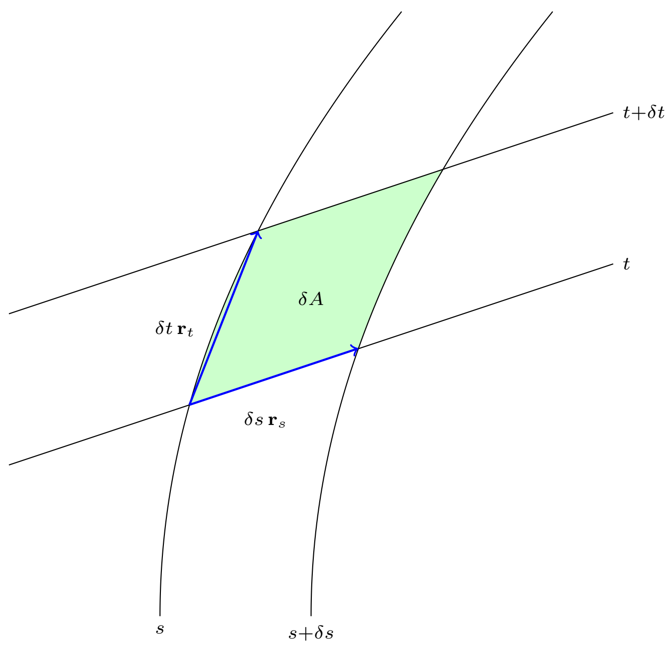
The picture shows the point $P=(1,1,1)$, which lies on S. Any other point on S (such as Q) can be reached from P by adding a multiple of the red vector $(-1,1,0)$ and a multiple of the blue vector $(0,-1,1)$.

(d) For any function $f(x,y)$, the equation $z=f(x,y)$ defines a surface. We can use the variables x and y themselves as parameters, and then the full parametrisation is

$$(x,y,z) = (x,y,f(x,y)).$$



There are two different kinds of integrals over a surface S that we need to be able to calculate. Firstly, given a scalar function f we can consider the integral with respect to area, written $\iint_S f\,dA$, defined in the same way as the integrals over plane regions that we studied in Section 4.1. To evaluate this kind of integral, we first need a parametrisation $\mathbf{r} = (x(s,t), y(s,t), z(s,t))$, then we need an expression for dA in terms of s and t. If we let s and t vary by small amounts δs and δt, then the corresponding part of the surface will be a small parallelogram spanned by the vectors $\mathbf{r}_s\,\delta s = (x_s \delta s,\ y_s\,\delta s)$ and $\mathbf{r}_t\,\delta t = (x_t \delta t,\ y_t\,\delta t)$.



We write δA for the area of this parallelogram. We also write $\delta\mathbf{A}$ for the vector $(\mathbf{r}_s \times \mathbf{r}_t)\delta s\,\delta t$; as explained in Section 5, this is perpendicular to $\mathbf{r}_s$ and $\mathbf{r}_t$ (which means that it is normal to the surface), and $|\delta\mathbf{A}| = \delta A$. In the limit we get $d\mathbf{A} = (\mathbf{r}_s \times \mathbf{r}_t) ds\,dt$ and $dA = |d\mathbf{A}| = |\mathbf{r}_s \times \mathbf{r}_t| ds\,dt$.

Example 10.2. Consider again a hemispherical shell of radius a as in Example 10.1(a). We have

$$\begin{aligned}
\mathbf{r} &= (a\sin(\phi)\cos(\theta),\ a\sin(\phi)\sin(\theta),\ a\cos(\phi)) \\
\mathbf{r}_\phi &= (a\cos(\phi)\cos(\theta),\ a\cos(\phi)\sin(\theta),\ -a\sin(\phi)) \\
\mathbf{r}_\theta &= (-a\sin(\phi)\sin(\theta),\ a\sin(\phi)\cos(\theta),\ 0) \\
\mathbf{r}_\phi \times \mathbf{r}_\theta &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos(\phi)\cos(\theta) & a\cos(\phi)\sin(\theta) & -a\sin(\phi) \\ -a\sin(\phi)\sin(\theta) & a\sin(\phi)\cos(\theta) & 0 \end{bmatrix} \\
&= (a^2\sin^2(\phi)\cos(\theta),\ a^2\sin^2(\phi)\sin(\theta),\ a^2\sin(\phi)\cos(\phi)) \\
&= a^2\sin(\phi)\mathbf{e}_r \\
d\mathbf{A} &= a^2\sin(\phi)\mathbf{e}_r\, d\phi\, d\theta \\
dA &= |d\mathbf{A}| = a^2\sin(\phi)d\theta\, d\phi.
\end{aligned}$$

It follows that the area of the surface is

$$\begin{aligned}
A = \iint_S 1\, dA &= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\frac{\pi}{2}} a^2\sin(\phi)d\theta\, d\phi \\
&= 2a^2\pi\int_{\phi=0}^{\frac{\pi}{2}}\sin(\phi)\, d\phi = 2a^2\pi\Big[-\cos(\phi)\Big]_{\phi=0}^{\frac{\pi}{2}} = 2a^2\pi.
\end{aligned}$$

Example 10.3. Consider a cylindrical surface as in Example 10.1(b). We have

$$\begin{aligned}
\mathbf{r} &= (1+\cos(s), 1+\sin(s), t) \\
\mathbf{r}_s &= (-\sin(s), \cos(s), 0) \\
\mathbf{r}_t &= (0,0,1) \\
\mathbf{r}_s \times \mathbf{r}_t &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin(s) & \cos(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} = (\cos(s), \sin(s), 0) \\
|\mathbf{r}_s \times \mathbf{r}_t| &= |(\cos(s),\sin(s),0)| = \sqrt{\cos^2(s)+\sin^2(s)} = 1 \\
dA &= |\mathbf{r}_s \times \mathbf{r}_t|\, ds\, dt = ds\, dt.
\end{aligned}$$

It follows that the area of the surface is

$$\iint_S 1\, dA = \int_{s=0}^{2\pi}\int_{t=-1}^{1} 1\, ds\, dt = 2\pi(1-(-1)) = 4\pi.$$

Example 10.4. Consider a surface of the form $z = f(x,y)$ as described in Example 10.1(d). We have

$$\begin{aligned}
\mathbf{r} &= (x, y, f(x,y)) \\
\mathbf{r}_x &= (1, 0, f_x) \\
\mathbf{r}_y &= (0, 1, f_y) \\
\mathbf{r}_x \times \mathbf{r}_y &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{bmatrix} = (-f_x, -f_y, 1) \\
d\mathbf{A} &= (\mathbf{r}_x \times \mathbf{r}_y)\, dx\, dy = (-f_x, -f_y, 1)\, dx\, dy \\
|\mathbf{r}_x \times \mathbf{r}_y| &= \sqrt{f_x^2 + f_y^2 + 1} \\
dA &= \sqrt{f_x^2 + f_y^2 + 1}dx\, dy.
\end{aligned}$$

Example 10.5. Now consider the surface S given by

$$z = \cosh(x+y)/\sqrt{2} = (e^{x+y} + e^{-x-y})/\sqrt{8}$$

for $0 \leq x, y \leq 1$. This is of the type considered above, with $f(x,y) = \cosh(x+y)/\sqrt{2}$. We have

$$\begin{aligned}
f_x &= \sinh(x+y)/\sqrt{2} \\
f_y &= \sinh(x+y)/\sqrt{2} \\
\sqrt{1+f_x^2+f_y^2} &= \sqrt{1+\tfrac{1}{2}\sinh^2(x+y)+\tfrac{1}{2}\sinh^2(x+y)} = \sqrt{1+\sinh^2(x+y)} \\
&= \sqrt{\cosh^2(x+y)} = \cosh(x+y) \\
dA &= \sqrt{1+f_x^2+f_y^2}\,dx\,dy = \cosh(x+y)\,dx\,dy.
\end{aligned}$$

It follows that the area of the surface is

$$\begin{aligned}
A &= \iint_S 1\,dA = \int_{x=0}^{1}\int_{y=0}^{1} \cosh(x+y)dy\,dx \\
&= \int_{x=0}^{1} \Big[\sinh(x+y)\Big]_{y=0}^{1} dx = \int_{x=0}^{1} \sinh(x+1)-\sinh(x)\,dx \\
&= \Big[\cosh(x+1)-\cosh(x)\Big]_{x=0}^{1} \\
&= (\cosh(2)-\cosh(1)) - (\cosh(1)-\cosh(0)) = \cosh(2)-2\cosh(1)+1 \\
&= \frac{e^2+e^{-2}}{2} - 2\frac{e+e^{-1}}{2} + 1 = \tfrac{1}{2}e^2 - e + 1 - e^{-1} + \tfrac{1}{2}e^{-2}.
\end{aligned}$$

The second kind of integral over surfaces is analogous to the integral $\int_C \mathbf{F}.d\mathbf{n}$ that measures the flux of a two-dimensional vector field $\mathbf{F}$ across a curve C. In the three-dimensional case, the corresponding thing is the flux of a vector field across a surface. As we explained above, the vector $d\mathbf{A}$ is normal to the surface, and the length is the area element dA. This means that $d\mathbf{A}$ is a natural analog of the vector $d\mathbf{n}$ in the two-dimensional case, so the total flux of a vector field $\mathbf{F}$ across a surface S is $\iint_S \mathbf{F}.d\mathbf{A}$.

Example 10.6. Let S be the surface given by $z = xy$ for $0 \leq x, y \leq 1$, and let $\mathbf{F}$ be the vector field $(x+y+z, x+y+z, x+y+z)$. This gives an instance of Example 10.4 with $f(x,y) = xy$, so

$$\begin{aligned}
\mathbf{r} &= (x, y, xy) \\
\mathbf{r}_x \times \mathbf{r}_y &= (-y, -x, 1) \\
d\mathbf{A} &= (-y, -x, 1)\,dx\,dy \\
\mathbf{F} &= (x+y+xy, x+y+xy, x+y+xy) \\
\mathbf{F}.d\mathbf{A} &= (-y(x+y+xy) - x(x+y+xy) + (x+y+xy))\,dx\,dy \\
&= \left(x+y-x^2-y^2-xy-xy^2-x^2y\right)\,dx\,dy \\
\iint_S \mathbf{F}.d\mathbf{A} &= \int_{x=0}^{1}\int_{y=0}^{1} \left(x+y-x^2-y^2-xy-xy^2-x^2y\right)\,dx\,dy \\
&= \int_{x=0}^{1} \left(x+\tfrac{1}{2}-x^2-\tfrac{1}{3}-\tfrac{1}{2}x-\tfrac{1}{3}x-\tfrac{1}{2}x^2\right)dx \\
&= \int_{x=0}^{1} \left(\tfrac{1}{6}+\tfrac{1}{6}x-\tfrac{3}{2}x^2\right)\,dx \\
&= \tfrac{1}{6}+\tfrac{1}{6}.\tfrac{1}{2}-\tfrac{3}{2}.\tfrac{1}{3} = \tfrac{2}{12}+\tfrac{1}{12}-\tfrac{6}{12} \\
&= -\tfrac{1}{4}.
\end{aligned}$$

11. Further integral theorems

11.1. **The (three-dimensional) Divergence Theorem.** Let E be the three-dimensional solid region enclosed by a surface S, and let $\mathbf{u}$ be a vector field that is well-behaved everywhere in E. The *Divergence*

Theorem says that

$$\iiint_E \operatorname{div}(\mathbf{u})\, dV = \iint_S \mathbf{u}.d\mathbf{A}$$

This can be proved by an argument similar to that used for the two-dimensional version. The physical interpretation is also similar: in a steady state, the rate of flow of particles escaping through S must balance the rate of creation of particles in E.

Example 11.1. Let S be the unit sphere, and let E be the solid ball enclosed by S. Consider the vector field $\mathbf{u} = (x, 0, 0)$. This has $\operatorname{div}(\mathbf{u}) = \partial x/\partial x + \partial 0/\partial y + \partial 0/\partial z = 1$, so

$$\iiint_E \operatorname{div}(\mathbf{u}) dV = \iiint_E dV = \text{ volume of } E = 4\pi/3.$$

On S we have

$$\begin{aligned}\mathbf{r} &= (x, y, z) = (\sin(\phi)\cos(\theta),\ \sin(\phi)\sin(\theta),\ \cos(\phi))\\ \mathbf{u} &= (x, 0, 0) = (\sin(\phi)\cos(\theta),\ 0,\ 0).\end{aligned}$$

The unit normal vector is $\mathbf{n} = \mathbf{e}_r = \mathbf{r}$, so $\mathbf{u}.\mathbf{n} = x^2 = \sin^2(\phi)\cos^2(\theta)$.
We have also seen before that $dA = \sin(\phi)\, d\phi\, d\theta$, so

$$\begin{aligned}\iint_S \mathbf{u}.d\mathbf{A} = \iint_S \mathbf{u}.\mathbf{n}\, dA &= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \sin^3(\phi)\cos^2(\theta)\, d\phi\, d\theta\\ &= \left(\int_{\theta=0}^{2\pi}\cos^2(\theta)\, d\theta\right)\left(\int_{\phi=0}^{\pi}\sin^3(\phi)\, d\phi\right) = \pi\int_{\phi=0}^{\pi}\frac{1}{4}(3\sin(\phi) - \sin(3\phi))d\phi\\ &= \frac{\pi}{4}\left[-3\cos(\phi) + \tfrac{1}{3}\cos(3\phi)\right]_{\phi=0}^{\pi} = \frac{\pi}{4}((3 - \tfrac{1}{3}) - (-3 + \tfrac{1}{3})) = \frac{4\pi}{3}\end{aligned}$$

As expected, this is the same as $\iiint_E \operatorname{div}(\mathbf{u})dV$.

Example 11.2. Let E be the solid vertical cylinder of radius a and height $2b$ centred at the origin, and let S be the surface of E. Consider the vector field $\mathbf{u} = (-y, x, z^3)$. We have

$$\operatorname{div}(\mathbf{u}) = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z^3) = 0 + 0 + 3z^2 = 3z^2.$$

The region E can be described in cylindrical polar coordinates by $0 \leq r \leq a$ and $-b \leq z \leq b$ (with $0 \leq \theta \leq 2\pi$ as usual). Moreover, the volume element in those coordinates is $dV = r\, dr\, d\theta\, dz$. It follows that

$$\begin{aligned}\iiint_E \operatorname{div}(\mathbf{u})\, dV &= \int_{z=-b}^{b}\int_{\theta=0}^{2\pi}\int_{r=0}^{a} 3z^2 r\, dr\, d\theta\, dz\\ &= 2\pi\left(\int_{r=0}^{a} r\, dr\right)\left(\int_{z=-b}^{b} 3z^2\, dz\right)\\ &= 2\pi\left[\tfrac{1}{2}r^2\right]_{r=0}^{a}\left[z^3\right]_{z=-b}^{b} = 2\pi a^2 b^3.\end{aligned}$$

Now consider instead $\iint_S \mathbf{u}.d\mathbf{A} = \iint_S \mathbf{u}.\mathbf{n}\, dA$. Let S_1 be the bottom end of E, where $z = -b$. Let S_2 be the top end, where $z = b$. Let S_3 be the curved outer surface, where $r = a$. We then have

$$\iint_S \mathbf{u}.\mathbf{n}\, dA = \iint_{S_1} \mathbf{u}.\mathbf{n}\, dA + \iint_{S_2} \mathbf{u}.\mathbf{n}\, dA + \iint_{S_3} \mathbf{u}.\mathbf{n}\, dA.$$

On S_1, the outward unit normal is clearly $\mathbf{n} = -\mathbf{k} = (0, 0, -1)$. We also have $z = -b$, so $\mathbf{u} = (-y, x, -b^3)$, so $\mathbf{u}.\mathbf{n} = b^3$. As this is constant, it follows that

$$\iint_{S_1} \mathbf{u}.\mathbf{n}\, dA = \iint_{S_1} b^3\, dA = b^3 \times (\text{ area of } S_1) = \pi a^2 b^3.$$

On S_2 we have $\mathbf{n} = (0, 0, 1)$ and $\mathbf{u} = (-y, x, b^3)$, and it follows easily that $\iint_{S_2} \mathbf{u}.\mathbf{n}\, dA$ is also equal to $\pi a^2 b^3$.

For S_3 it is convenient to work in cylindrical polar coordinates again. The outward unit normal is $\mathbf{n} = \mathbf{e}_r = (\cos(\theta), \sin(\theta), 0)$, and the vector field is

$$\mathbf{u} = (-y, x, z^3) = (-r\sin(\theta), r\cos(\theta), z^3).$$

From this it is clear that $\mathbf{u}.\mathbf{n} = 0$, so $\iint_{S_3} \mathbf{u}.\mathbf{n}\, dA = 0$. Putting this together, we get

$$\iint_S \mathbf{u}.\mathbf{n}\, dA = \pi a^2 b^3 + \pi a^2 b^3 + 0 = 2\pi a^2 b^3,$$

which is the same as $\iiint_E \operatorname{div}(\mathbf{u})\, dV$, as expected.

Example 11.3. Let E be the solid region where $-1 \leq x, y \leq 1$ and $0 \leq z \leq (1-x^2)(1-y^2)$. Let S be the surface of E, and let $\mathbf{u}$ be the vector field $(x, y, 0)$. This has

$$\operatorname{div}(\mathbf{u}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(0) = 1 + 1 + 0 = 2,$$

so

$$\begin{aligned}
\iiint_E \operatorname{div}(\mathbf{u})\, dV &= \int_{x=-1}^{1} \int_{y=-1}^{1} \int_{z=0}^{(1-x^2)(1-y^2)} 2\, dz\, dy\, dx \\
&= \int_{x=-1}^{1} \int_{y=-1}^{1} 2(1-x^2)(1-y^2)\, dy\, dx \\
&= 2\left(\int_{x=-1}^{1} 1 - x^2\, dx\right)\left(\int_{y=-1}^{1} 1 - y^2\, dy\right) \\
&= 2\left[x - \tfrac{1}{3}x^3\right]_{x=-1}^{1} \left[y - \tfrac{1}{3}y^3\right]_{y=-1}^{1}.
\end{aligned}$$

Here

$$\left[x - \tfrac{1}{3}x^3\right]_{x=-1}^{1} = (1 - \tfrac{1}{3}) - (-1 + \tfrac{1}{3}) = \tfrac{2}{3} - (-\tfrac{2}{3}) = \tfrac{4}{3},$$

and $\left[y - \frac{1}{3}y^3\right]_{y=-1}^{1}$ is the same, so

$$\iiint_E \operatorname{div}(\mathbf{u})\, dV = 2 \times \tfrac{4}{3} \times \tfrac{4}{3} = \tfrac{32}{9}.$$

Now consider instead $\iint_S \mathbf{u}.d\mathbf{A} = \iint_S \mathbf{u}.\mathbf{n}\, dA$. Let S_1 be the bottom surface of E (where $z = 0$) and let S_2 be the top surface (where $z = (1-x^2)(1-y^2)$. (Note that there is no side surface, because $(1-x^2)(1-y^2) = 0$ whenever $x = \pm 1$ or $y = \pm 1$.) On S_1 the unit normal vector is $\mathbf{n} = (0, 0, -1)$ but $\mathbf{u} = (x, y, 0)$ so $\mathbf{u}.\mathbf{n} = 0$ so $\iint_{S_1} \mathbf{u}.d\mathbf{A} = 0$. The surface S_2 is given by $z = f(x, y)$, where

$$f(x, y) = (1-x^2)(1-y^2) = 1 - x^2 - y^2 + x^2y^2.$$

It follows by Example 10.4 that on S_2 we have

$$d\mathbf{A} = (-f_x, -f_y, 1)\, dx\, dy = (2x - 2xy^2, 2y - 2x^2y, 1)\, dx\, dy,$$

so

$$\mathbf{u}.d\mathbf{A} = (x, y, 0).(2x - 2xy^2, 2y - 2x^2y, 1)\, dx\, dy = (2x^2 + 2y^2 - 4x^2y^2)\, dx\, dy.$$

This gives

$$\iint_{S_2} \mathbf{u}.d\mathbf{A} = \int_{y=-1}^{1} \int_{x=-1}^{1} (2x^2 + 2y^2 - 4x^2y^2)\, dx\, dy.$$

For the inner integral we have

$$\begin{aligned}
\int_{x=-1}^{1} (2x^2 + 2y^2 - 4x^2y^2)\, dx &= \left[\tfrac{2}{3}x^3 + 2xy^2 - \tfrac{4}{3}x^3y^2\right]_{x=-1}^{1} \\
&= \left(\tfrac{2}{3} + 2y^2 - \tfrac{4}{3}y^2\right) - \left(-\tfrac{2}{3} - 2y^2 + \tfrac{4}{3}y^2\right) \\
&= \tfrac{4}{3}(1 + y^2).
\end{aligned}$$

Feeding this into the outer integral gives

$$
\begin{aligned}
\iint_{S_2} \mathbf{u}.d\mathbf{A} &= \frac{4}{3}\int_{y=-1}^{1} 1+y^2\,dy = \frac{4}{3}\left[y+\tfrac{1}{3}y^3\right]_{y=-1}^{1} \\
&= \frac{4}{3}\left((1+\tfrac{1}{3})-(-1-\tfrac{1}{3})\right) \\
&= \tfrac{4}{3}\times\tfrac{8}{3} = \tfrac{32}{9}.
\end{aligned}
$$

This is the same as $\iiint_E \mathrm{div}(\mathbf{u})\,dV$, as expected.

11.2. **Stokes's Theorem.** Stokes's Theorem is analogous to Green's Theorem, but it applies to curved surfaces as well as to flat regions in the plane. The statement is as follows. Suppose we have a surface S whose boundary is a closed curve C, and a well-behaved vector field $\mathbf{u}$. Then

$$\iint_S \mathrm{curl}(\mathbf{u}).d\mathbf{A} = \pm\int_C \mathbf{u}.d\mathbf{r}.$$

We need a little more discussion to eliminate the ambiguity in the sign. To make sense of the right hand side, we need to specify the direction in which we move around C. The integral in one direction will be the negative of the integral in the opposite direction. Similarly, on the left hand side we have the integral of $\mathrm{curl}(\mathbf{u}).\mathbf{n}\,dA$, where $\mathbf{n}$ is a unit vector normal to the surface. There are two possible directions for $\mathbf{n}$ (each opposite to the other) and there is no natural rule to choose between them. However, the choice of $\mathbf{n}$ can be linked to the choice of direction around the curve as follows: if you walk in the specified direction with your feet on C and your head pointing in the direction of $\mathbf{n}$, then the surface S should be on your left. Provided that we follow this convention, we will have

$$\iint_S \mathrm{curl}(\mathbf{u}).d\mathbf{A} = \iint_S \mathrm{curl}(\mathbf{u}).\mathbf{n}\,dA = +\int_C \mathbf{u}.d\mathbf{r}.$$

Example 11.4. Consider the surface S given by $z = x^2 - y^2$ with $x^2 + y^2 \leq 1$. We will check Stokes's Theorem for the vector field $(-y, x, 0)$.

We can parametrise S as

$$\mathbf{r} = (x,y,z) = (r\cos(s),\ r\sin(s),\ r^2\cos^2(s) - r^2\sin^2(s))$$

with $0 \leq r \leq 1$ and $0 \leq s \leq 2\pi$. Using the standard identity $\cos^2(s) - \sin^2(s) = \cos(2s)$, we can also rewrite the parametrisation as

$$\mathbf{r} = (x,y,z) = (r\cos(s), r\sin(s), r^2\cos(2s)).$$

This gives

$$
\begin{aligned}
\mathbf{r}_r &= (\cos(s), \sin(s), 2r\cos(2s)) \\
\mathbf{r}_s &= (-r\sin(s),\ r\cos(s),\ -2r^2\sin(2s)) \\
\mathbf{r}_r\times\mathbf{r}_s &= \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(s) & \sin(s) & 2r\cos(2s)) \\ -r\sin(s) & r\cos(s) & -2r^2\sin(2s) \end{bmatrix} \\
&= (-2r^2\sin(s)\sin(2s) - 2r^2\cos(s)\cos(2s),\ 2r^2\cos(s)\sin(2s) - 2r^2\sin(s)\cos(2s),\ r\cos^2(s) + r\sin^2(s))
\end{aligned}
$$

Using the standard identities

$$
\begin{aligned}
\sin(a)\sin(b) + \cos(a)\cos(b) &= \cos(a-b) = \cos(b-a) \\
\sin(a)\cos(b) - \cos(a)\sin(b) &= \sin(a-b) = -\sin(b-a)
\end{aligned}
$$

this becomes

$$
\begin{aligned}
\mathbf{r}_r\times\mathbf{r}_s &= (-2r^2\cos(s),\ 2r^2\sin(s),\ r) \\
d\mathbf{A} &= (\mathbf{r}_r\times\mathbf{r}_s)\,dr\,ds \qquad = (-2r^2\cos(s),\ 2r^2\sin(s),\ r)\,dr\,ds.
\end{aligned}
$$

Next, we have

$$\begin{aligned}
\operatorname{curl}(\mathbf{u}) &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{bmatrix} \\
&= (0-0)\mathbf{i} - (0-0)\mathbf{j} + (1-(-1))\mathbf{k} = (0,0,2) \\
\operatorname{curl}(\mathbf{u}).d\mathbf{A} &= 2r\,dr\,ds \\
\iint_S \operatorname{curl}(\mathbf{u}).d\mathbf{A} &= \int_{s=0}^{2\pi} \int_{r=0}^{1} 2r\,dr\,ds \\
&= \int_{s=0}^{2\pi} 1\,ds = 2\pi.
\end{aligned}$$

On the other hand, we can parametrise the boundary curve C (where $r = 1$) as

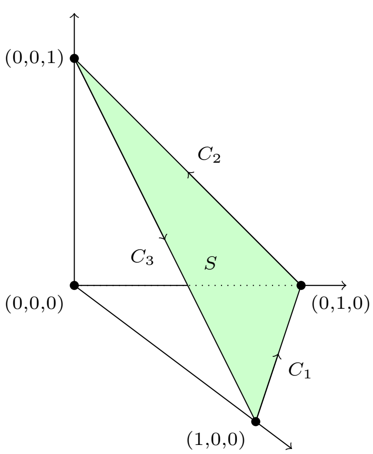
$$\mathbf{r} = (x,y,z) = (\cos(s), \sin(s), \cos(2s)).$$

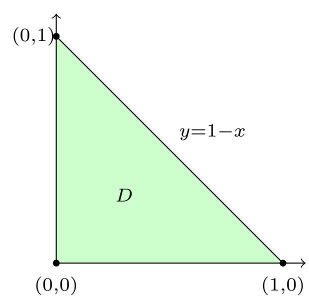
On this curve we have

$$\begin{aligned}
\mathbf{u} &= (-y,\ x,\ 0) = (-\sin(s),\ \cos(s),\ 0) \\
d\mathbf{r} &= (-\sin(s), \cos(s), -2\sin(2s))\,ds \\
\mathbf{u}.d\mathbf{r} &= (\sin^2(s) + \cos^2(s))\,ds = ds \\
\int_C \mathbf{u}.d\mathbf{r} &= \int_{s=0}^{2\pi} ds = 2\pi.
\end{aligned}$$

As expected, this is the same as $\iint_S \operatorname{curl}(\mathbf{u}).d\mathbf{A}$.

Example 11.5. Let S be the triangular surface shown on the left below, given by $x+y+z=1$ with $x,y,z \geq 0$. Let $\mathbf{u}$ be the vector field (z,x,y).





The boundary consists of the edges C_1, C_2 and C_3. We can parametrise C_1 by $\mathbf{r} = (x,y,z) = (1-t,t,0)$ for $0 \leq t \leq 1$. This gives $d\mathbf{r} = (-1,1,0)dt$. We can also substitute $x = 1-t$ and $y = t$ and $z = 0$ in the definition $\mathbf{u} = (z,x,y)$ to get $\mathbf{u} = (0, 1-t, t)$. This gives $\mathbf{u}.d\mathbf{r} = (1-t)dt$, so

$$\int_{C_1} \mathbf{u}.d\mathbf{r} = \int_{t=0}^{1} (1-t)\,dt = \left[t - \tfrac{1}{2}t^2\right]_{t=0}^{1} = 1/2.$$

The other edges work in the same way, as in the following table:

edge	C_1	C_2	C_3
$\mathbf{r}$	$(1-t,t,0)$	$(0,1-t,t)$	$(t,0,1-t)$
$d\mathbf{r}$	$(-1,1,0)dt$	$(0,-1,1)dt$	$(1,0,-1)dt$
$\mathbf{u}$	$(0,1-t,t)$	$(t,0,1-t)$	$(1-t,t,0)$
$\mathbf{u}.d\mathbf{r}$	$(1-t)dt$	$(1-t)dt$	$(1-t)dt$
$\int \mathbf{u}.d\mathbf{r}$	$1/2$	$1/2$	$1/2$

Altogether, we have $\int_C \mathbf{u}.d\mathbf{r} = 3/2$.

On the other hand, we have

$$\operatorname{curl}(\mathbf{u}) = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{bmatrix} = (1,1,1).$$

The shadow of S in the xy-plane is the triangle D shown on the right. The surface has the form $z = f(x,y)$, where $f(x,y) = 1-x-y$ and (x,y) lies in D. By Example 10.4 we have

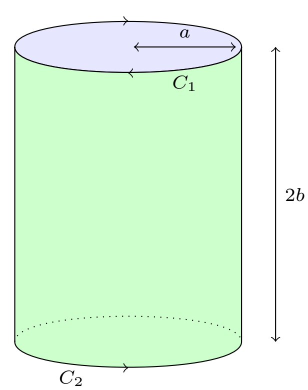
$$d\mathbf{A} = (-f_x, -f_y, 1)\, dx\, dy = (1,1,1)\, dx\, dy.$$

This gives

$$\begin{aligned} \iint_S \operatorname{curl}(\mathbf{u}).d\mathbf{A} &= \int_D (1,1,1).(1,1,1)\, dx\, dy = 3 \int_{x=0}^{1} \int_{y=0}^{1-x} dy dx \\ &= 3 \int_{x=0}^{1} (1-x)\, dx = 3 \left[x - \tfrac{1}{2}x^2 \right]_{x=0}^{1} = 3/2. \end{aligned}$$

As expected, this is the same as $\int_C \mathbf{u}.d\mathbf{r}$.

Example 11.6. Let S be the surface given in cylindrical polar coordinates by $r = a$ with $-b \le z \le b$ and $0 \le \theta \le 2\pi$.



We will check Stokes's Theorem for the vector field $\mathbf{u} = (-zy, zx, z^2)$.

We parametrise S as

$$\begin{aligned} \mathbf{r} &= (x,y,z) = (a\cos(\theta),\ a\sin(\theta),\ z) \\ \mathbf{r}_\theta &= (-a\sin(\theta),\ a\cos(\theta),\ 0) \\ \mathbf{r}_z &= (0,0,1) \\ \mathbf{r}_\theta \times \mathbf{r}_z &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin(\theta) & a\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = (a\cos(\theta),\ a\sin(\theta),\ 0) \\ d\mathbf{A} &= (\mathbf{r}_\theta \times \mathbf{r}_z)\, d\theta\, dz = a(\cos(\theta), \sin(\theta), 0)\, d\theta\, dz. \end{aligned}$$

Note that with this parametrisation, $d\mathbf{A}$ points outwards, away from the z-axis. We also have

$$\operatorname{curl}(\mathbf{u}) = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -zy & zx & z^2 \end{bmatrix} = (0 - x,\ -y - 0,\ z - (-z)) = (-x, -y, 2z).$$

On the surface S this becomes

$$\begin{aligned} \operatorname{curl}(\mathbf{u}) &= (-a\cos(\theta),\ -a\sin(\theta),\ 2z) \\ \operatorname{curl}(\mathbf{u}).d\mathbf{A} &= (-a^2\cos^2(\theta) - a^2\sin^2(\theta))\, d\theta\, dz = -a^2\, d\theta\, dz \\ \iint_S \operatorname{curl}(\mathbf{u}).d\mathbf{A} &= -a^2 \int_{\theta=0}^{2\pi} \int_{z=-b}^{b} d\theta\, dz = -a^2 \times 2\pi \times 2b = -4\pi a^2 b. \end{aligned}$$

The boundary of S consists of the two circles C_1 and C_2 shown in the diagram. To understand the direction of these curves, recall that when you walk with your head in the direction of $d\mathbf{A}$ (outwards from the z-axis), you need to keep the surface on your left. This means we must follow the arrows as shown in the diagram, so we traverse C_1 clockwise as seen from above, and C_2 anticlockwise. Thus, a suitable parametrisation for C_1 is $(x, y, z) = (a\cos(t), -a\sin(t), b)$, and a suitable parametrisation for C_2 is $(x, y, z) = (a\cos(t), a\sin(t), -b)$, with $0 \leq t \leq 2\pi$ in both cases. On C_1 we have

$$\begin{aligned} d\mathbf{r} &= (-a\sin(t), -a\cos(t), 0)\, dt \\ \mathbf{u} &= (-zy, zx, z^2) = (ab\sin(t), ab\cos(t), b^2) \\ \mathbf{u}.d\mathbf{r} &= -a^2b\sin^2(t) - a^2b\cos^2(t) = -a^2b \\ \int_{C_1} \mathbf{u}.d\mathbf{r} &= \int_{t=0}^{2\pi} -a^2b\, dt = -2\pi a^2 b. \end{aligned}$$

On C_2 we have

$$\begin{aligned} d\mathbf{r} &= (-a\sin(t), a\cos(t), 0)\, dt \\ \mathbf{u} &= (-zy, zx, z^2) = (ab\sin(t), -ab\cos(t), b^2) \\ \mathbf{u}.d\mathbf{r} &= -a^2b\sin^2(t) - a^2b\cos^2(t) = -a^2b \\ \int_{C_2} \mathbf{u}.d\mathbf{r} &= \int_{t=0}^{2\pi} -a^2b\, dt = -2\pi a^2 b. \end{aligned}$$

Putting these together, we get

$$\int_C \mathbf{u}.d\mathbf{r} = -4\pi a^2 b,$$

which is the same as $\iint_S \operatorname{curl}(\mathbf{u}).d\mathbf{A}$, as expected.