MAS243 PROBLEMS

Week 1 — Optimisation

Exercise 1. Find the maximum and minimum values of the function $f(x) = x^3 - 9x^2 + 15x + 35$ with $0 \leq x \leq 8$.

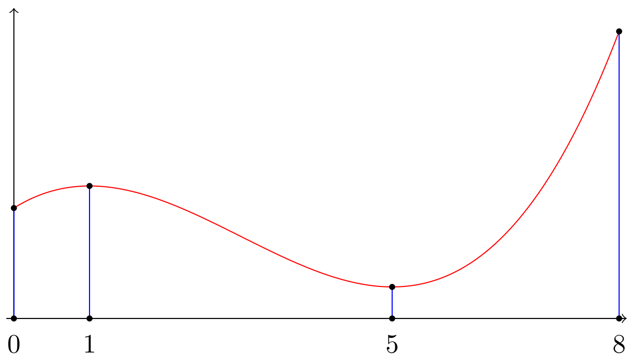
Solution: We have

$$f'(x) = 3x^2 - 18x + 15 = 3(x^2 - 6x + 5) = 3(x - 1)(x - 5),$$

so the critical points are $x = 1$ and $x = 5$. We need to check the values of $f(x)$ at the endpoints and the critical points:

$$\begin{aligned} f(0) &= 35 \\ f(1) &= 42 \\ f(5) &= 125 - 9 \times 25 + 15 \times 5 + 35 = 10 \\ f(8) &= 512 - 9 \times 64 + 15 \times 8 + 35 = 91. \end{aligned}$$

Thus, the smallest value is 10 (at the critical point $x = 5$) and the largest value is 91 (at the endpoint $x = 8$).



Note that this is very similar to Example (d) on pages 3-4 of the course notes.

Exercise 2. Find and classify the critical points of the function $f(x) = \cos(x) - \cos^3(x)/3$. What is the unusual feature of this example?

Solution: The derivatives are

$$\begin{aligned} f'(x) &= -\sin(x) + \cos^2(x)\sin(x) = -\sin(x)(1 - \cos^2(x)) = -\sin(x)\sin^2(x) \\ &= -\sin^3(x) \\ f''(x) &= -3\sin^2(x)\cos(x). \end{aligned}$$

The critical points are the points x where $\sin^3(x) = 0$ or equivalently $\sin(x) = 0$ or $x = n\pi$ for integers n. For such points we have $f''(x) = 3\sin^2(x)\cos(x) = 0$, so the second derivative does not help us to decide whether we have a local maximum or a local minimum (this is the unusual feature). However, when x lies between $2m\pi$ and $(2m + 1)\pi$ we have $\sin(x) > 0$ so $f'(x) = -\sin^3(x) < 0$ so $f(x)$ is decreasing. On the other hand, when x lies between $(2m + 1)\pi$ and $(2m + 2)\pi$ we have $\sin(x) < 0$ so $f'(x) = -\sin^3(x) > 0$ so $f(x)$ is increasing. It follows that the points $x = 2m\pi$ are maxima and the points $x = (2m + 1)\pi$ are minima, with

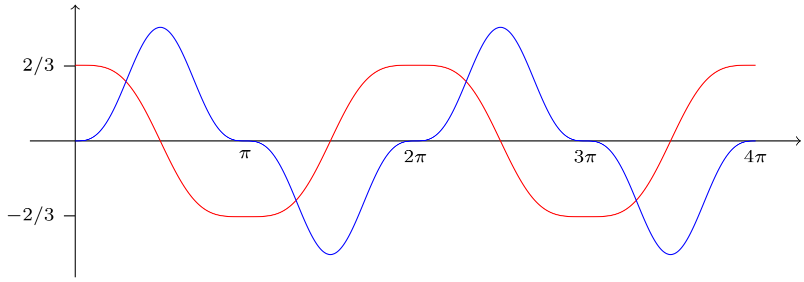
$$\begin{aligned} f(2m\pi) &= \cos(2m\pi) - \cos^3(2m\pi)/3 = 1 - 1/3 = 2/3 \\ f((2m + 1)\pi) &= \cos((2m + 1)\pi) - \cos^3((2m + 1)\pi)/3 = -1 + 1/3 = -2/3. \end{aligned}$$

If you did not notice the trick of rewriting $f'(x)$ as $-\sin^3(x)$, you would get

$$f'(x) = -\sin(x)(1-\cos^2(x))$$
$$f''(x) = -\cos(x) - 2\cos(x)\sin^2(x) + cos^3(x).$$

You would then say that there are critical points where $\sin(x) = 0$ or $\cos(x) = \pm 1$, but that again gives $x = n\pi$. After remembering that $\cos(n\pi) = (-1)^n$ you would then get $f''(n\pi) = 0$ again. This is less tidy but essentially the same.

In the following picture, the red graph is $f(x)$ and the blue graph is $f'(x)$.



You should see that the graph of $f(x)$ is unusually flat at the top and the bottom, much flatter than the graph of $\sin(x)$ or $\cos(x)$. This is a reflection of the fact that $f''(x) = 0$ at the critical points.

Exercise 3. Show that $\sqrt{2}\sin(x - \pi/4) = \sin(x) - \cos(x)$. Using this, find the maximum value of $e^{-x}\sin(x)$ for $x \geq 0$.

Solution: We start with the standard addition formula

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y).$$

Take $y = -\pi/4$, recalling that $\sin(-\pi/4) = -\sin(\pi/4) = -1/\sqrt{2}$ and $\cos(-\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$. This gives $\sin(x - \pi/4) = (\sin(x) - \cos(x))/\sqrt{2}$, so $\sqrt{2}\sin(x - \pi/4) = \sin(x) - \cos(x)$ as claimed.

Now consider the function $f(x) = e^{-x}\sin(x)$. This has

$$f'(x) = -e^{-x}\sin(x) + e^{-x}\cos(x) = e^{-x}(\cos(x) - \sin(x)) = -\sqrt{2}e^{-x}\sin(x - \pi/4).$$

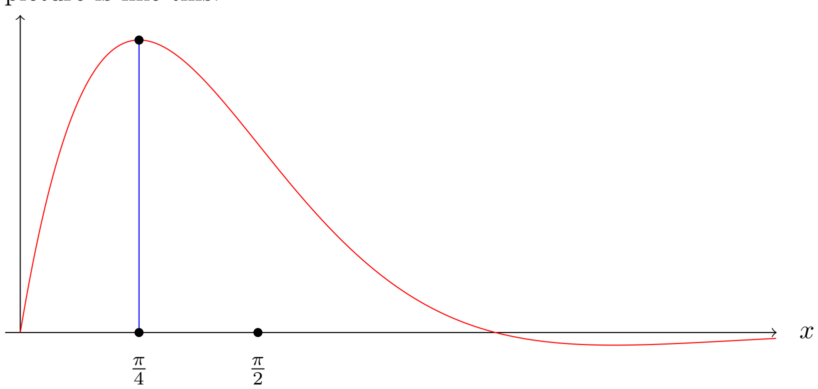
For a critical point we must have $f'(x) = 0$. As e^{-x} is never zero, this gives $\sin(x - \pi/4) = 0$, so $x - \pi/4$ must be a multiple of π, so $x = n\pi + \pi/4$ for some integer n. We are only interested in the case $x \geq 0$, so we must have $n \geq 0$. Note also that $\sin(\pi + x) = -\sin(x)$, so

$$\sin(n\pi + \pi/4) = (-1)^n \sin(\pi/4) = (-1)^n/\sqrt{2}.$$

Using this we get

$$f(n\pi + \pi/4) = (-1)^n e^{-n\pi} e^{-\pi/4}/\sqrt{2}.$$

These numbers alternate in sign, and the absolute values get smaller as n increases. It follows that we have the largest value when $n = 0$, namely $f(\pi/2) = e^{-\pi/4}/\sqrt{2} \simeq 0.32$. We also need to check what happens at the endpoints but $f(0) = 0$ and $f(x)$ decays rapidly to zero as $x \to \infty$, so our conclusionis unaffected. The picture is like this:



Exercise 4. For the following functions, calculate the partial derivatives f_x and f_y, and verify that $f_{xy} = f_{yx}$.

(a) $f(x, y) = x^3 + 3x^2y + xy^2 + 4y^3$

(b) $f(x,y) = xy^2\ln(x^2+y^2)$.

Solution:

(a)

$$f_x(x,y) = 3x^2+6xy+y^2 \qquad f_y(x,y) = 3x^2+2xy+12y^2$$
$$f_{xy}(x,y) = 6x+2y \qquad f_{yx}(x,y) = 6x+2y.$$

(b) First note that the online test asked you to enter f_x rather than f_{xy} here, in an attempt to save you some pain. Unfortunately many students misread the question.

To calculate the relevant derivatives, the chain rule gives

$$\frac{\partial}{\partial x}\ln(x^2+y^2) = \frac{2x}{x^2+y^2}.$$

To explain this in more detail, put $u = x^2+y^2$ and $v = \ln(u)$. We then have $dv/du = 1/u = 1/(x^2+y^2)$ and $\partial u/\partial x = 2x$ so the chain rule gives

$$\frac{\partial v}{\partial x} = \frac{dv}{du}\,\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}$$

as claimed. Similarly, we have $\partial v/\partial y = 2y/(x^2+y^2)$. Using this together with the product rule we get

$$f_x(x,y) = y^2\ln(x^2+y^2) + 2x^2y^2/(x^2+y^2)$$
$$f_y(x,y) = 2xy\ln(x^2+y^2) + 2xy^3/(x^2+y^2).$$

We now want to calculate $f_{xy}(x,y)$. We can differentiate $y^2\ln(x^2+y^2)$ by the same method that we just used for $xy^2\ln(x^2+y^2)$. For the second term we use the quotient rule $(u/v)_x = (u_xv-uv_x)/v^2$ (with $u = 2x^2y^2$ and $v = x^2+y^2$). This gives

$$\frac{\partial}{\partial y}\left(\frac{2x^2y^2}{x^2+y^2}\right) = \frac{4x^2y(x^2+y^2)-2x^2y^2.2y}{(x^2+y^2)^2} = \frac{4x^4y}{(x^2+y^2)^2},$$

so

$$\begin{aligned} f_{xy}(x,y) &= 2y\ln(x^2+y^2) + \frac{y^2.2y}{x^2+y^2} + \frac{4x^4y}{(x^2+y^2)^2} \\ &= 2y\ln(x^2+y^2) + \frac{2y^3(x^2+y^2)+4x^4y}{(x^2+y^2)^2} = 2y\ln(x^2+y^2) + \frac{4x^4y+2x^2y^3+2y^5}{(x^2+y^2)^2}. \end{aligned}$$

In the same way, we have

$$\frac{\partial}{\partial x}\left(\frac{2xy^3}{x^2+y^2}\right) = \frac{2y^3(x^2+y^2)-2xy^3.2x}{(x^2+y^2)^2} = \frac{2y^5-2x^2y^3}{(x^2+y^2)^2},$$

so

$$\begin{aligned} f_{yx}(x,y) &= 2y\ln(x^2+y^2) + \frac{2xy.2x}{x^2+y^2} + \frac{2y^5-2x^2y^3}{(x^2+y^2)^2} \\ &= 2y\ln(x^2+y^2) + \frac{4x^2y(x^2+y^2)+2y^5-2x^2y^3}{(x^2+y^2)^2} = 2y\ln(x^2+y^2) + \frac{4x^4y+2x^2y^3+2y^5}{(x^2+y^2)^2}. \end{aligned}$$

This is the same as $f_{xy}(x,y)$, as expected.

Exercise 5. Suppose we have a function $f(x,y)$ of two variables. We say that f is biharmonic if it satisfies the equation

$$f_{xxxx} + 2f_{xxyy} + f_{yyyy} = 0.$$

(This comes up in the theory of small elastic deformations of nearly rigid bodies.) Show that the function $f(x,y) = xy^2(x^2-y^2)$ is biharmonic but the function $g(x,y) = e^{x+y}$ is not.

Solution: We have $f = x^3y^2 - xy^4$ so

$$f_x = 3x^2y^2 - y^4 \qquad f_{xx} = 6xy^2 \qquad f_{xxx} = 6y^2 \qquad f_{xxxx} = 0$$
$$f_y = 2x^3y - 4xy^3 \qquad f_{yy} = 2x^3 - 12xy^2 \qquad f_{yyy} = -24xy \qquad f_{yyyy} = -24x$$
$$f_{xxy} = 12xy \qquad f_{xxyy} = 12x$$

so

$$f_{xxxx} + 2f_{xxyy} + f_{yyyy} = 0 + 2\times 12x - 24x = 0.$$

This means that f is biharmonic.

On the other hand we have $g_x = e^{x+y} = g$ and $g_y = e^{x+y} = g$, and it follows in turn that $g_{xx} = g$ and $g_{xxx} = g$ and $g_{xxxx} = g$. Similarly we have $g_{xxyy} = g$ and $g_{yyyy} = g$ so

$$g_{xxxx} + 2g_{xxyy} + g_{yyyy} = 4g = 4e^{x+y} \neq 0,$$

so g is not biharmonic.

Exercise 6. Show that the function $f(x,t) = t^{-1/2}e^{-x^2/t}$ satisfies the equation $4f_t = f_{xx}$. (This is relevant to the equations of heat flow, and also to the pricing of financial derivatives.)

Solution: To find f_t we break f into pieces, putting $u = t^{-1/2}$ and $v = e^{-x^2/t}$ and $w = -x^2/t$. It is straightforward that

$$\begin{aligned} u_t &= \frac{\partial u}{\partial t} = -\tfrac{1}{2}t^{-3/2} \\ w_t &= \frac{\partial w}{\partial t} = -x^2\frac{\partial}{\partial t}(t^{-1}) = -x^2 \times (-t^{-2}) = x^2t^{-2}. \end{aligned}$$

Next, we have $v = e^w$ so the chain rule gives

$$v_t = \frac{\partial v}{\partial t} = \frac{dv}{dw}\,\frac{\partial w}{\partial t} = e^w x^2 t^{-2} = x^2t^{-2}e^{-x^2/t}.$$

We now note that $f = uv$ and use the product rule:

$$\begin{aligned} f_t &= (uv)_t = u_t v + u\,v_t \\ &= -\tfrac{1}{2}t^{-3/2}e^{-x^2/t} + t^{-1/2}x^2t^{-2}e^{-x^2/t} \\ &= (-t^{-3/2}/2 + x^2t^{-5/2})e^{-x^2/t}. \end{aligned}$$

On the other hand, using the chain rule and te product rule we have

$$\begin{aligned} f_x &= (-2x/t)t^{-1/2}e^{-x^2/t} = -2xt^{-3/2}e^{-x^2/t} \\ f_{xx} &= -2t^{-3/2}e^{-x^2/t} + (2x/t)2xt^{-3/2}e^{-x^2/t} = (-2t^{-3/2} + 4x^2t^{-5/2})e^{-x^2/t}. \end{aligned}$$

It is now clear that $4f_t = f_{xx}$ as claimed.

Exercise 7. Find the critical points of the function $f(x,y) = (x+y+2)e^{-(x^2+y^2)/2}$.

Solution: The derivatives are

$$\begin{aligned} f_x &= e^{-(x^2+y^2)/2} - x(x+y+2)e^{-(x^2+y^2)/2} = (1 - x(x+y+2))e^{-(x^2+y^2)/2} \\ f_y &= e^{-(x^2+y^2)/2} - y(x+y+2)e^{-(x^2+y^2)/2} = (1 - y(x+y+2))e^{-(x^2+y^2)/2}. \end{aligned}$$

These must both be zero. As exponentials are never zero, this implies that $1 - x(x+y+2) = 1 - y(x+y+2) = 0$. This gives $x = 1/(x+y+2)$ and also $y = 1/(x+y+2)$ so $y = x$. Putting $y = x$ in the equation $1 - x(x+y+2) = 0$ gives $1 - 2x^2 - 2x = 0$ so $x^2 + x - 1/2 = 0$ which gives $x = (-1 \pm \sqrt{3})/2$. It follows that the critical points are $(x,y) = ((-1+\sqrt{3})/2, (-1+\sqrt{3})/2) \simeq (0.366, 0.366)$ and $(x,y) = ((-1-\sqrt{3})/2, (-1-\sqrt{3})/2) \simeq (-1.366, -1.366)$.

Exercise 8. Let ϕ be a constant. The function

$$f(x,y) = (x^2+y^2)^2 - x\cos(\phi)/2 - y\sin(\phi)/2$$

has only one critical point. Find it.

Solution: The derivatives are

$$\begin{aligned} f_x &= 2(x^2+y^2) \times 2x - \cos(\phi)/2 \\ f_y &= 2(x^2+y^2) \times 2y - \sin(\phi)/2. \end{aligned}$$

These must both be zero, which means that

$$\begin{aligned} \cos(\phi) &= 8x(x^2+y^2) \\ \sin(\phi) &= 8y(x^2+y^2). \end{aligned}$$

Squaring these equations and adding them together, we get

$$\begin{aligned}1 &= \cos^2(\phi) + \sin^2(\phi) = 64x^2(x^2+y^2)^2 + 64y^2(x^2+y^2)^2 \\ &= 64(x^2+y^2)(x^2+y^2)^2 = 64(x^2+y^2)^3.\end{aligned}$$

This gives $x^2 + y^2 = (1/64)^{1/3} = 1/4$. Feeding this back into the equation $\cos(\phi) = 8x(x^2+y^2)$ gives $\cos(\phi) = 2x$ and so $x = \cos(\phi)/2$. Similarly, the equation $\sin(\phi) = 8y(x^2+y^2)$ gives $\sin(\phi) = 2y$ and so $y = \sin(\phi)/2$. Thus, the unique critical point is $(x,y) = (\cos(\phi)/2, \sin(\phi)/2)$.

Week 2 — Constrained optimisation

Exercise 9. Show that the function $u = x^3 + x^2y - y^2 - 2x^2$ has critical points at $(0,0)$, $(1,1/2)$ and $(-4,8)$, and determine their nature.

Solution: The partial derivatives are

$$\begin{aligned} u_x &= 3x^2 + 2xy - 4x & & & u_y &= x^2 - 2y \\ u_{xx} &= 6x + 2y - 4 & u_{xy} &= 2x & u_{yy} &= -2. \end{aligned}$$

For a critical point, we must have $3x^2 + 2xy - 4x = 0$ and $x^2 = 2y$. The second of these gives $y = x^2/2$, which we substitute in the first to get $3x^2 + x^3 - 4x = 0$. This factorises as $x(x-1)(x+4) = 0$, so we have $x = 0$, $x = 1$ or $x = -4$. As $y = x^2/2$ the corresponding values of y are 0, $1/2$ and 8. This means that the critical points are $(0,0)$, $(1,1/2)$ and $(-4,8)$, as claimed. The Hessian matrix is $H = \begin{bmatrix} 6x+2y-4 & 2x \\ 2x & -2 \end{bmatrix}$, so $A_1 = 6x + 2y - 4$ and

$$A_2 = (6x + 2y - 4).(-2) - (2x)^2 = 8 - 12x - 4y - 4x^2.$$

At $(0,0)$ we have $A_1 = -4 < 0$ and $A_2 = 8 > 0$ so Method 3.8 tells us that this is a local maximum. At $(1,1/2)$ we have $A_2 = -10 < 0$ so this is a saddle point. At $(-4,8)$ we have $A_2 = -40 < 0$ so this is another saddle point.

Exercise 10. Find all the critical points of the function

$$u = x^3 - 3xy^2 + 4y^3 - 18y$$

and determine their nature.

Solution: The partial derivatives are

$$\begin{aligned} u_x &= 3x^2 - 3y^2 & & & u_y &= 12y^2 - 6xy - 18 \\ u_{xx} &= 6x & u_{xy} &= -6y & u_{yy} &= 24y - 6x. \end{aligned}$$

For a critical point we must have $u_x = 0$, so $x^2 = y^2$, which means that $y = \pm x$. If $y = x$ then the equation $u_y = 0$ becomes $6x^2 = 18$ so $x = \pm\sqrt{3}$. We thus have critical points $(\sqrt{3}, \sqrt{3})$ and $(-\sqrt{3}, -\sqrt{3})$. On the other hand, if $y = -x$ the equation $u_y = 0$ becomes $18x^2 = 18$ so $x = \pm 1$. We thus have two more critical points at $(1,-1)$ and $(-1,1)$, and this completes the list of critical points. The Hessian is $H = \begin{bmatrix} 6x & -6y \\ -6y & 24y - 6x \end{bmatrix}$, so $A_1 = 6x$ and

$$A_2 = 6x(24y - 6x) - (-6y)^2 = 144xy - 36x^2 - 36y^2 = 36(4xy - x^2 - y^2).$$

This gives the following table:

(x,y)	A_1	A_2	type
$(\sqrt{3}, \sqrt{3})$	$6\sqrt{3} > 0$	$216 > 0$	local minimum
$(-\sqrt{3}, -\sqrt{3})$	$-6\sqrt{3} < 0$	$216 > 0$	local maximum
$(1,-1)$	$6 > 0$	$-216 < 0$	saddle point
$(-1,1)$	$-6 < 0$	$-216 < 0$	saddle point

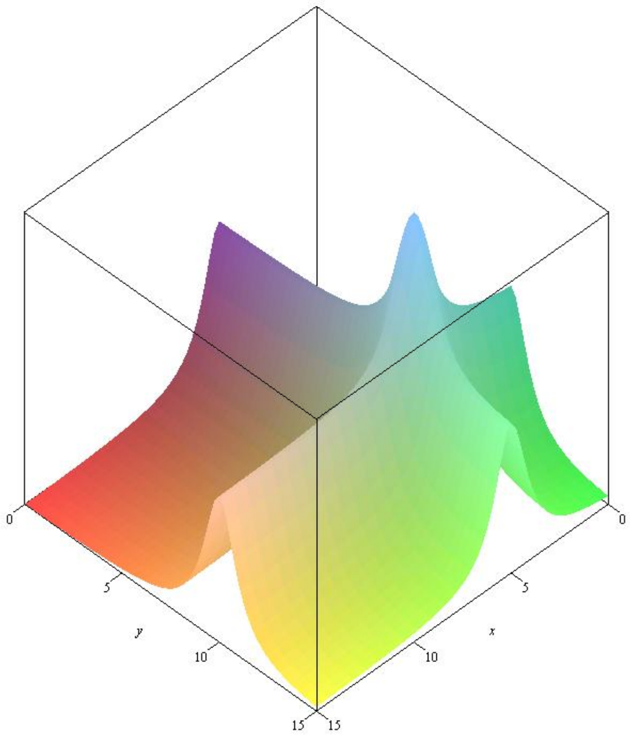
(Here we have again used Method 3.8: if $A_2 < 0$ we have a saddle point; if $A_2 > 0$ and $A_1 > 0$ we have a local minimum; if $A_2 > 0$ and $A_1 < 0$ we have a local maximum.)

Exercise 11. Find the maximum and minimum values of the function

$$f(x,y) = \frac{1}{1+(x-5)^2} + \frac{1}{1+(y-10)^2}.$$

Do this by thinking intelligently about the problem, not by grinding through the general method.

Solution: As the first term depends only on x, and the second term depends only on y, we can deal with them separately. To make the first term as large as possible we need to make $1+(x-5)^2$ as small as possible, which we do by taking $x = 5$. To make the second term as large as possible we need to make $1+(y-10)^2$ as small as possible, which we do by taking $y = 10$. Thus, the maximum is $f(5,10) = 2$. It is also clear that $f(x,y)$ is always positive, but we can make it as small as we like by taking x and y to be very large. Thus, the minimum is zero, but this minimum is never attained.



If we just followed the standard method, the calculation would go as follows. The partial derivatives are

$$f_x(x,y) = \frac{10-2x}{(1+(x-5)^2)^2} \qquad\qquad f_y(x,y) = \frac{20-2y}{(1+(y-10)^2)^2}$$

$$f_{xx}(x,y) = \frac{6x^2-60x+148}{(1+(x-5)^2)^3} \qquad f_{xy}(x,y) = 0 \qquad f_{yy}(x,y) = \frac{6y^2-120y+598}{(1+(y-10)^2)^3}.$$

At a critical point both f_x and f_y must be zero, so $10-2x = 20-2y = 0$, so $x = 5$ and $y = 10$ as before. At this point we have $f_{xx} = 6\times 25 - 60\times 5 + 148 = -2$ and $f_{yy} = 600-1200+598 = -2$, so the Hessian matrix is $H = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$, so $A_1 = -2 < 0$ and $A_2 = 4 > 0$, so we see that this is a local maximum. As the minimum is never attained, it cannot be found by looking for critical points.

Exercise 12. Find and classify the critical points of the function

$$f(x,y,z) = 1 - x^2 - y^2 - z^2 + 2xyz.$$

Solution: The first-order derivatives are

$$\begin{aligned} f_x(x,y,z) &= -2x + 2yz = 2(yz - x) \\ f_y(x,y,z) &= -2y + 2xz = 2(xz - y) \\ f_z(x,y,z) &= -2z + 2xy = 2(xy - z). \end{aligned}$$

Thus, for a critical point we must have $x = yz$ and $y = xz$ and $z = xy$. Multiplying these three equations together gives $xyz = x^2y^2z^2$, so $xyz(1 - xyz) = 0$, so either $x = 0$ or $y = 0$ or $z = 0$ or $xyz = 1$. If $x = 0$ then the equations $y = xz$ and $z = xy$ give $y = z = 0$. Similarly, if $y = 0$ then the equations $x = yz$ and $z = xy$ give $x = z = 0$, and if $z = 0$ then the equations $x = yz$ and $y = xz$ give $x = y = 0$. Thus, if any of x, y and z is zero, they all are.

Now consider the other case, where x, y and z are nonzero. In this case we can divide the equation $xyz(1 - xyz) = 0$ by xyz to get $xyz = 1$. We can then multiply $x = yz$ by x to get $x^2 = xyz = 1$, so $x = \pm 1$. Similarly, we can multiply $y = xz$ by y to get $y^2 = xyz = 1$ so $y = \pm 1$, and we can multiply $z = xy$ by z to get $z^2 = 1$ so $z = \pm 1$. We now see that each of x, y and z is ± 1. We can choose x and y freely, but then z has to be $1/(xy)$ because $xyz = 1$. This gives the following list of critical points:

$$(0, 0, 0), (1, 1, 1), (-1, 1, -1), (1, -1, -1), (-1, -1, 1).$$

The Hessian matrix is

$$H = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} -2 & 2z & 2y \\ 2z & -2 & 2x \\ 2y & 2x & -2 \end{bmatrix}$$

so

$$\begin{aligned}
A_1 &= -2 \\
A_2 &= \det \begin{bmatrix} -2 & 2z \\ 2z & -2 \end{bmatrix} = 4(1 - z^2) \\
A_3 &= \det(H) = -2 \det \begin{bmatrix} -2 & 2x \\ 2x & -2 \end{bmatrix} - 2z \det \begin{bmatrix} 2z & 2x \\ 2y & -2 \end{bmatrix} + 2y \det \begin{bmatrix} 2z & -2 \\ 2y & 2x \end{bmatrix} \\
&= -2(4 - 4x^2) - 2z(-4z - 4xy) + 2y(4xz + 4y) \\
&= 8(2xyz + x^2 + y^2 + z^2 - 1).
\end{aligned}$$

At the point $(0, 0, 0)$ we have $(A_1, A_2, A_3) = (-2, 4, -8)$. The signs alternate, starting with a negative, which means that we have a local maximum. At any of the other critical points, we have $x^2 = y^2 = z^2 = xyz = 1$ so $(A_1, A_2, A_3) = (-2, 0, 32)$. As $A_2 \neq 0$ but the signs are neither all positive nor alternating, we must have a saddle.

Exercise 13. Locate the critical points of $f(x, y) = x^2y$ subject to the constraint $x^2 + xy = 1$.

Solution: We need to find the unconstrained critical points of the function

$$L(\lambda, x, y) = x^2y - \lambda(x^2 + xy - 1).$$

These are the points where the following equations hold:

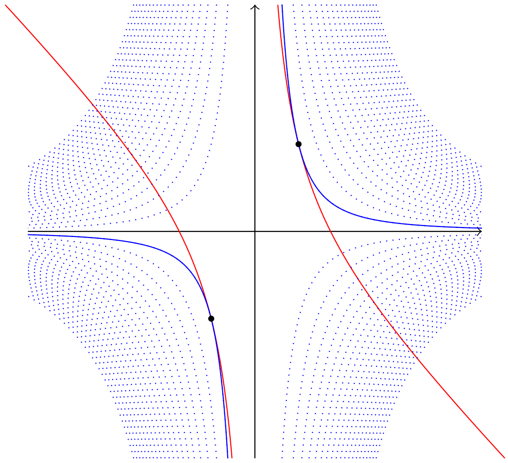
$$\text{(A)} \qquad L_\lambda(\lambda, x, y) = 1 - x^2 - xy = 0$$

$$\text{(B)} \qquad L_x(\lambda, x, y) = 2xy - 2x\lambda - y\lambda = 0$$

$$\text{(C)} \qquad L_y(\lambda, x, y) = x^2 - x\lambda = 0.$$

Equation (C) can be written as $x(x - \lambda) = 0$, so $x = 0$ or $\lambda = x$. On the other hand, (A) can be written as $x(x + y) = 1$, which clearly means that x cannot be 0. We therefore have $\lambda = x \neq 0$. Substituting this into (B) gives $2xy - 2x^2 - xy = 0$ or equivalently $x(y - 2x) = 0$. As $x \neq 0$ it follows that $y = 2x$. Substituting this in (A) gives $3x^2 = 1$, so $x = \pm 1/\sqrt{3}$. As $y = 2x$ we see that the critical points are $p = (1/\sqrt{3}, 2/\sqrt{3})$ and $q = (-1/\sqrt{3}, -2/\sqrt{3})$. At p we have $f(x, y) = x^2y = 2/3\sqrt{3}$ and at q we have $f(x, y) = -2/3\sqrt{3}$.

These equations can be visualised as shown below. The constraint curve (where $x^2 + xy = 1$) is shown in red, and the contours of f are shown in blue. Most of them are dotted, but the contours where $f(x, y) = \pm 2/3\sqrt{3}$ are shown as solid curves. These are the contours that touch the constraint curve at p and q.



Exercise 14. Locate the critical points of $f(x,y) = x^2+y^2$ subject to the condition $3x^2+4xy+6y^2 = 140$.

Solution: We need to find the unconstrained critical points of the function

$$L(\lambda, x, y) = x^2 + y^2 - \lambda(3x^2 + 4xy + 6y^2 - 140).$$

These are the points where the following equations hold:

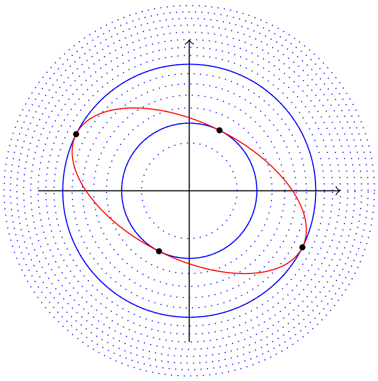
$$\text{(A)} \qquad L_\lambda(\lambda, x, y) = 3x^2 + 4xy + 6y^2 - 140 = 0$$
$$\text{(B)} \qquad L_x(\lambda, x, y) = 2x - (6x + 4y)\lambda = 0$$
$$\text{(C)} \qquad L_y(\lambda, x, y) = 2y - (4x + 12y)\lambda = 0.$$

The best way to start solving these is to eliminate λ from (B) and (C). We multiply (B) by $4x + 12y$ and (C) by $6x + 4y$ and then subtract. The terms involving λ cancel out and we are left with $2x(4x + 12y) - 2y(6x + 4y) = 0$. We can expand this out to get $8x^2 + 12xy - 8y^2 = 0$, and then factorise to get $4(2x - y)(x + 2y) = 0$. This means that either $y = 2x$ or $x = -2y$.

(1) Consider the case where $y = 2x$. Equation (A) becomes $3x^2 + 4x(2x) + 6(2x)^2 = 140$, which gives $35x^2 = 140$, so $x = \pm 2$. As $y = 2x$ we see that this gives critical points (for the constrained problem) at $(2, 4)$ and $(-2, -4)$. At these points we have $f(x, y) = x^2 + y^2 = 20$.
(2) Now consider the other case where $x = -2y$. Equation (A) becomes $3(-2y)^2 + 4(-2y)y + 6y^2 = 140$, which gives $10y^2 = 140$, so $y = \pm\sqrt{14}$. As $x = -2y$ we see that this gives critical points (for the constrained problem) at $(-2\sqrt{14}, \sqrt{14})$ and $(2\sqrt{14}, -\sqrt{14})$. At these points we have $f(x, y) = x^2 + y^2 = 70$.

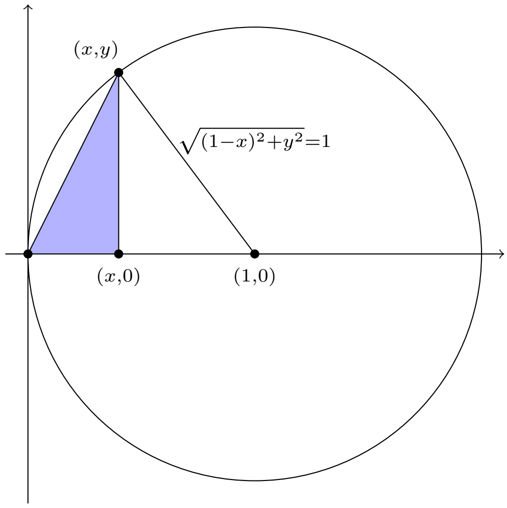
For an alternative approach, we can rearrange equation (B) to get $(2 - 6\lambda)x = 4y\lambda$ and so $x/y = 2\lambda/(1-3\lambda)$. Similarly, we can rearrange (C) to get $(2-12\lambda)y = 4x\lambda$ so $x/y = (1-6\lambda)/(2\lambda)$. Comparing these expressions gives $2\lambda/(1 - 3\lambda) = (1 - 6\lambda)/(2\lambda)$, or $4\lambda^2 = (1 - 3\lambda)(1 - 6\lambda) = 1 - 9\lambda + 18\lambda^2$, or $14\lambda^2 - 9\lambda + 1 = 0$. Using the quadratic formula we obtain $\lambda = (9 \pm \sqrt{25})/28$, so $\lambda = 1/2$ or $\lambda = 1/7$. If $\lambda = 1/2$ then the equation $x/y = 2\lambda/(1 - 3\lambda)$ gives $x = -2y$, and the rest of the calculation is as in case (2) above. If $\lambda = 1/7$ then the equation $x/y = 2\lambda/(1 - 3\lambda)$ gives $y = 2x$, and the rest of the calculation is as in case (1) above.

These equations can be visualised as shown below. The constraint curve (where $3x^2+4xy+6y^2 = 140$) is shown in red, and the contours of f are shown in blue. Most of them are dotted, but the contours where $f(x, y) = 20$ and $f(x, y) = 70$ are shown as solid curves. These are the contours that touch the constraint curve at the critical points.



Exercise 15. A triangle in the xy-plane has vertices at $(0,0)$, $(x,0)$ and (x,y), with $x,y \geq 0$. The point (x,y) lies on the circle of radius 1 with centre at $(1,0)$. Use the method of Lagrange multipliers to show that the maximum possible area for the triangle is $3\sqrt{3}/8$.

Solution: The geometry is as follows:



The area of the triangle is $S = xy/2$. Pythagoras's theorem tells us that the distance from (x,y) to $(1,0)$ is $\sqrt{(1-x)^2+y^2}$. As the point is supposed to lie on the circle of radius 1 centred at $(1,0)$, we must have $(1-x)^2+y^2 = 1$. Our problem is thus to maximise $xy/2$ subject to $(1-x)^2+y^2-1 = 0$, so we put

$$L = xy/2 - \lambda((1-x)^2+y^2-1) = xy/2 - \lambda(x^2+y^2-2x).$$

The equations for a critical point are

(A) $$L_\lambda = 2x - x^2 - y^2 = 0$$

(B) $$L_x = y/2 - 2\lambda x + 2\lambda = 0$$

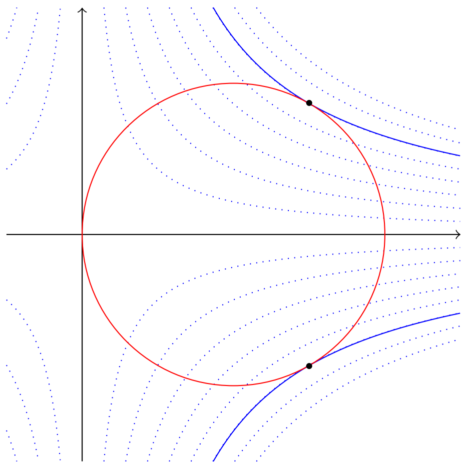
(C) $$L_y = x/2 - 2\lambda y = 0.$$

Equation (C) gives $\lambda = x/(4y)$. (To get this we had to divide by y. That is acceptable because it is clear that the maximum area does not occur when $y = 0$.) We can substitute this in (B) we get $y/2 - x^2/(2y) + x/(2y) = 0$. After multiplying by $2y$ this becomes

(D) $$y^2 - x^2 + x = 0.$$

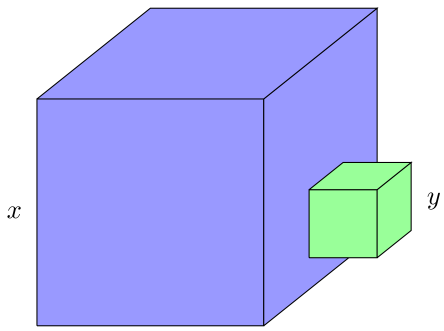
We now add (A) and (D) to get $3x - 2x^2 = 0$. Again, it is clear that x is not zero when we have maximum area, so it is acceptable to divide this by $2x$ to get $x = 3/2$. Equation (A) then becomes $y^2 = 3/4$, and $y \geq 0$ so $y = \sqrt{3}/2$. Thus, the maximum area is $xy/2 = 3\sqrt{3}/8$ and occurs for $(x,y) = (3/2, \sqrt{3}/2)$. There are further critical points at $(0,0)$ and $(3/2, -\sqrt{3}/2)$ but these are not relevant for the original problem.

The equations can be visualised as shown below. The constraint curve (where $(1-x)^2+y^2 = 1$) is the red circle, and the contours of $xy/2$ are shown in blue. Most of them are dotted, but the contours where $xy/2 = 3\sqrt{3}/8$ or $xy/2 = -2\sqrt{3}/8$ are shown as solid curves. These are the contours that touch the constraint curve at the critical points.



Exercise 16. A solid body of volume V and surface S is formed by joining together two cubes of different sizes so that every point of a face of the smaller cube is in contact with the larger cube. If $S = 7m^2$, use the method of Lagrange multipliers to find the critical value of V for which both cubes have non-zero volumes.

Solution: The solid body looks like this:



We let x be the length of the sides of the larger cube, and y the length of the sides of the smaller cube, so $0 < y < x$. The surface consists of 5 green faces of area y^2, plus five ordinary blue faces of order x^2, plus the face along which the two cubes are joined. This face originally had area x^2 but a square of area y^2 was removed when the cubes were joined, leaving an area of $x^2 - y^2$. Thus, the total area is

$$S = 5y^2 + 5x^2 + (x^2 - y^2) = 6x^2 + 4y^2.$$

On the other hand, we have $V = x^3 + y^3$. Thus, we are trying to optimise $x^3 + y^3$ subject to $6x^2 + 4y^2 - 7 = 0$, so we need to take

$$L = x^3 + y^3 - \lambda(6x^2 + 4y^2 - 7) = 0.$$

The equations for a critical point are

(A) $$L_\lambda = 7 - 6x^2 - 4y^2 = 0$$

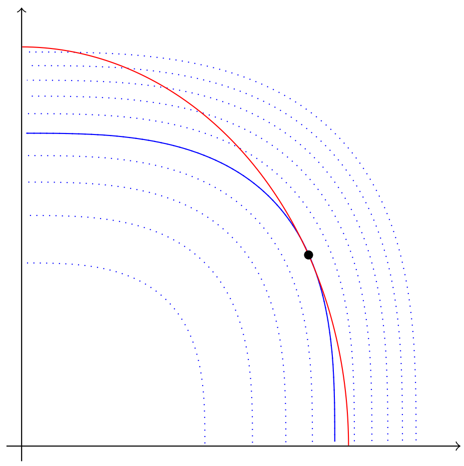
(B) $$L_x = 3x^2 - 12\lambda x = 3x(x - 4\lambda) = 0$$

(C) $$L_y = 3y^2 - 8\lambda y = 3y(y - 8\lambda/3) = 0.$$

We are asked to find a critical point where both cubes have nonzero size, so $x, y > 0$. It is therefore legitimate to divide (B) and (C) by $3x$ and $3y$, giving $x = 4\lambda$ and $y = 8\lambda/3$ (and so $\lambda > 0$). Substituting these into (A) gives $96\lambda^2 + 256\lambda^2/9 = 7$, so $1120\lambda^2 = 63$, so

$$\lambda = \sqrt{63/1120} = \frac{3}{4\sqrt{10}} \simeq 0.237.$$

From this we get $x = 4\lambda \simeq 0.949$ and $y = 8\lambda/3 \simeq 0.632$ and $V = x^3 + y^3 \simeq 1.107$.



Week 3 — Integrals over plane regions

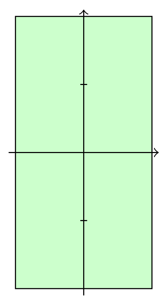
Exercise 17. Evaluate the following integrals, and sketch the corresponding regions in the (x, y)-plane.

(a) $\int_{x=-1}^{1} \int_{y=-2}^{2} (2x^2 + y^2)\, dy\, dx$
(b) $\int_{x=1}^{2} \int_{y=0}^{1} x\, e^y\, dy\, dx$
(c) $\int_{x=2/a}^{4/a} \int_{y=1/x}^{a} y^2\, dy\, dx$
(d) $\int_{y=0}^{\pi} \int_{x=0}^{\sin(y)} 1\, dx\, dy$.

Solution:

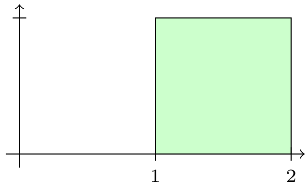
(a)

$$\begin{aligned}
&\int_{x=-1}^{1} \int_{y=-2}^{2} (2x^2 + y^2)\, dy\, dx \\
=&\int_{x=-1}^{1} \left[2x^2 y + y^3/3\right]_{y=-2}^{2} dx \\
=&\int_{x=-1}^{1} (8x^2 + 16/3) dx \\
=&\left[8x^3/3 + 16x/3\right]_{x=-1}^{1} = 24/3 - (-24/3) = 16.
\end{aligned}$$

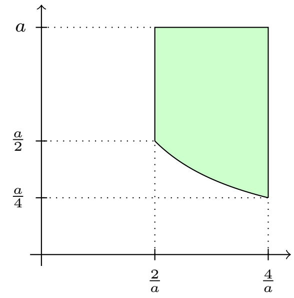


(b)

$$\begin{aligned}
&\int_{x=1}^{2} \int_{y=0}^{1} x\, e^y\, dy\, dx \\
=&\int_{x=1}^{2} \left[x\, e^y\right]_{y=0}^{1} dx \\
=&\int_{x=1}^{2} (e-1)x\, dx = \left[(e-1)x^2/2\right]_{x=1}^{2} \\
=&(e-1)(4/2 - 1/2) = 3(e-1)/2.
\end{aligned}$$

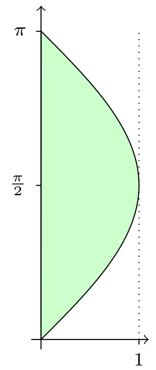


(c)



$$\begin{aligned}
&\int_{x=2/a}^{4/a}\int_{y=1/x}^{a} y^2\,dy\,dx\\
=&\int_{x=2/a}^{4/a}\left[\frac{y^3}{3}\right]_{y=1/x}^{a}dx\\
=&\frac{1}{3}\int_{x=2/a}^{4/a} a^3 - x^{-3}\,dx\\
=&\frac{1}{3}\left[a^3x+\frac{1}{2}x^{-2}\right]_{x=2/a}^{4/a}\\
=&\frac{1}{3}((4a^2+a^2/32)-(2a^2+a^2/8)) = 61a^2/96
\end{aligned}$$

(d)



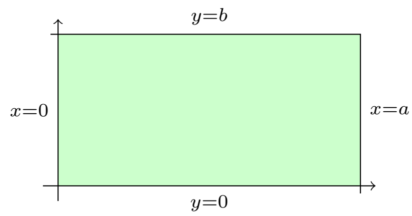
$$\begin{aligned}
&\int_{y=0}^{\pi}\int_{x=0}^{\sin(y)} 1\,dx\,dy\\
=&\int_{y=0}^{\pi}\sin(y)\,dx\,dy\\
=&\Big[-\cos(y)\Big]_{y=0}^{\pi}\\
=&1-(-1)=2.
\end{aligned}$$

Exercise 18. Express the following double integrals as repeated integrals and evaluate them:

(a) $\iint_D xy\,dA$, where D is the rectangle bounded by the lines $x=0$, $x=a$, $y=0$ and $y=b$.
(b) $\iint_D e^{x+y}\,dA$, where D is the region bounded by the lines $x=0$, $y=0$ and $x+y=1$.
(c) $\iint_D e^{y^2}\,dA$, where D is the triangle with vertices $(0,0)$, $(-1,1)$ and $(1,1)$.
(d) $\iint_D x^2\,dA$, where D is the trapezium with vertices $(-1,0)$, $(1,0)$, $(0,1)$ and $(1,1)$.

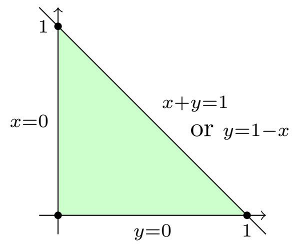
Solution:

(a)



$$\begin{aligned}
\iint_D xy\,dA &= \int_{x=0}^{a}\int_{y=0}^{b} xy\,dy\,dx\\
&=\int_{x=0}^{a}\left[\tfrac{1}{2}xy^2\right]_{y=0}^{b}dx = \int_{x=0}^{a}\tfrac{1}{2}xb^2\,dx\\
&=\left[\tfrac{1}{4}x^2b^2\right]_{x=0}^{a} = \frac{a^2b^2}{4}.
\end{aligned}$$

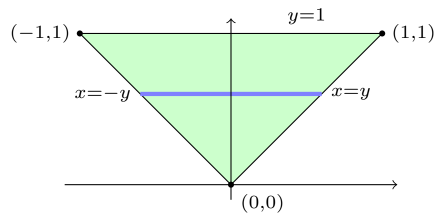
(b)



$$\begin{aligned}
\iint_D e^{x+y}\,dA &= \int_{x=0}^{1}\int_{y=0}^{1-x} e^{x+y}dy\,dx\\
&=\int_{x=0}^{1}\left[e^{x+y}\right]_{y=0}^{1-x}dx = \int_{x=0}^{1} e - e^x\,dx\\
&=\left[ex-e^x\right]_{x=0}^{1} = (e-e)-(0-1)=1.
\end{aligned}$$

(c)

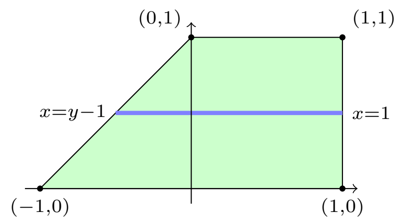
$$\iint_D e^{y^2}\,dA = \int_{y=0}^{1}\int_{x=-y}^{y} e^{y^2}\,dx\,dy = \int_{y=0}^{1} e^{y^2}.2y\,dy$$
$$= \int_{u=0}^{1} e^u\,du \qquad\qquad (u = y^2,\ du = 2y\,dy)$$
$$= \Big[e^u\Big]_{u=0}^{1} = e - 1.$$



We could alternatively try to do this using vertical strips, but then we would need to do the left half and the right half separately and add them together. The left half would be $\int_{x=-1}^{0}\int_{y=-x}^{1} e^{y^2}\,dy\,dx$, and the right half would be $\int_{x=0}^{1}\int_{y=x}^{1} e^{y^2}\,dy\,dx$. The inner integral $\int e^{y^2}\,dy$ cannot be expressed in terms of familiar functions, so this is not a useful approach.

(d) First, we have

$$\iint_D x^2\,dA = \int_{y=0}^{1}\int_{x=y-1}^{1} x^2\,dx\,dy.$$



The inner integral is

$$\int_{x=y-1}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{x=y-1}^{1} = \tfrac{1}{3}(1-(y-1)^3)$$
$$= \tfrac{1}{3}(1 - y^3 + 3y^2 - 3y + 1)$$
$$= \tfrac{2}{3} - \tfrac{1}{3}y^3 + y^2 - y.$$

Putting this into the outer integral gives

$$\iint_D x^2\,dA = \int_{y=0}^{1}(\tfrac{2}{3} - \tfrac{1}{3}y^3 + y^2 - y)\,dy$$
$$= \left[\tfrac{2}{3}y - \tfrac{1}{12}y^4 + \tfrac{1}{3}y^3 - \tfrac{1}{2}y^2\right]_0^1 = \tfrac{2}{3} - \tfrac{1}{12} + \tfrac{1}{3} - \tfrac{1}{2}$$
$$= 5/12.$$

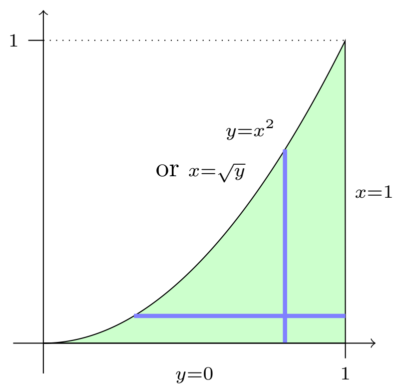
We could alternatively do this using vertical strips, but then we would need to do the left half and the right half separately and add them together. The left half would be $\int_{x=-1}^{0}\int_{y=0}^{1+x} x^2\,dy\,dx = \int_{x=-1}^{0}(x^2 + x^3)\,dx = \frac{1}{12}$, and the right half would be $\int_{x=0}^{1}\int_{y=0}^{1} x^2\,dy\,dx = \frac{1}{3}$ giving $\frac{1}{12} + \frac{1}{3} = \frac{5}{12}$ overall as before.

Exercise 19. By sketching the region of integration, show that

$$\int_{y=0}^{1}\int_{x=\sqrt{y}}^{1} 1\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{x^2} 1\,dy\,dx.$$

Evaluate both integrals and check that they are the same.

Solution: The region is as follows:



The horizontal slice at height y runs from $x = \sqrt{y}$ to $x = 1$, in accordance with the limits in the left hand integral. The vertical slice at position x runs from $y = 0$ to $y = x^2$, in accordance with the limits in the right hand integral. Thus, the two integrals should be the same. We can evaluate them as follows:

$$\int_{y=0}^{1}\int_{x=\sqrt{y}}^{1} 1\,dx\,dy = \int_{y=0}^{1} 1-\sqrt{y}\,dy = \int_{y=0}^{1} 1-y^{\frac{1}{2}}\,dy$$
$$= \left[y - \tfrac{2}{3}y^{\frac{3}{2}}\right]_{y=0}^{1} = 1 - \tfrac{2}{3} = \tfrac{1}{3}$$
$$\int_{x=0}^{1}\int_{y=0}^{x^2} 1\,dy\,dx = \int_{x=0}^{1} x^2\,dx = \left[\tfrac{1}{3}x^3\right]_{x=0}^{1} = \tfrac{1}{3}.$$

Exercise 20. Sketch the region of the (x,y)-plane over which the integral

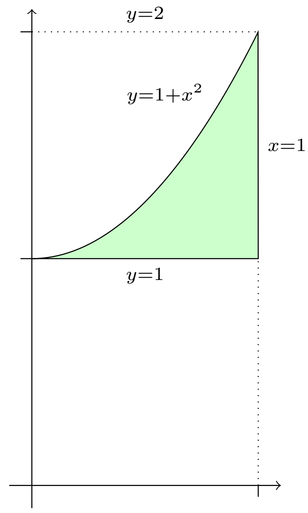
$$I = \int_{x=0}^{1}\int_{y=1}^{x^2+1} f(x,y)\,dy\,dx$$

is taken. Obtain a similar expression for I by reversing the order of integration.

Solution:

The picture as shown on the right. The equation $y = x^2+1$ for the curved edge can be rewritten as $x = \sqrt{y-1}$. Thus, we can divide the region into horizontal stripes running from $x = \sqrt{y-1}$ to $x = 1$ for $1 \leq y \leq 2$. This gives

$$I = \int_{y=1}^{2}\int_{x=\sqrt{y-1}}^{1} f(x,y)\,dx\,dy.$$



Exercise 21. Change the order of integration in the following integrals, and hence evaluate them:

(a) $\displaystyle\int_{y=0}^{\infty}\int_{x=y}^{\infty} \frac{e^{-x}}{x}\,dx\,dy$

(b) $\displaystyle\int_{x=0}^{a}\int_{y=x}^{a} \frac{y^2}{(x^2+y^2)^{1/2}}\,dy\,dx$

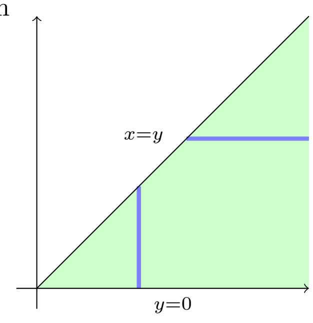
You may find the following integral useful:

$$\int \frac{1}{(x^2+y^2)^{1/2}}\,dx = \ln(x+\sqrt{x^2+y^2}).$$

Solution:

(a) Horizontal slices run from $x = y$ to $x = \infty$. Vertical slices run from $y = 0$ to $y = x$. The integral can therefore be rewritten as

$$\int_{x=0}^{\infty}\int_{y=0}^{x} \frac{e^{-x}}{x}\,dy\,dx = \int_{x=0}^{\infty}\left[y\frac{e^{-x}}{x}\right]_{y=0}^{x} dx = \int_{x=0}^{\infty} e^{-x}\,dx$$
$$= \left[-e^{-x}\right]_{x=0}^{\infty} = ((-0)-(-1)) = 1.$$

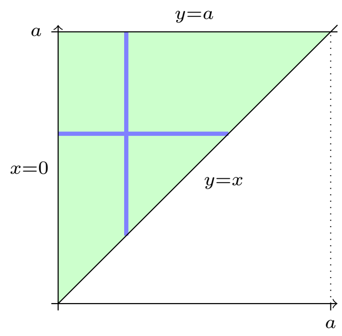


(b) Vertical slices run from $y = x$ to $y = a$. Horizontal slices run from $x = 0$ to $x = y$. The integral can therefore be rewritten as

$$I = \int_{y=0}^{a} \int_{x=0}^{y} \frac{y^2}{(x^2+y^2)^{1/2}} dx\, dy$$

Using the hint, the inner integral gives

$$\begin{aligned}
\int_{x=0}^{y} \frac{y^2}{(x^2+y^2)^{1/2}} dx &= \left[y^2 \ln(x + \sqrt{x^2+y^2}) \right]_{x=0}^{y} \\
&= y^2 \ln(y + \sqrt{2y^2}) - y^2 \ln(0 + \sqrt{y^2}) \\
&= y^2 (\ln((1+\sqrt{2})y) - \ln(y)) \\
&= y^2 (\ln(1+\sqrt{2}) + \ln(y) - \ln(y)) \\
&= y^2 \ln(1+\sqrt{2}).
\end{aligned}$$



Note here that $\ln(1+\sqrt{2})$ is just a constant, approximately 0.881. The outer integral is now easy:

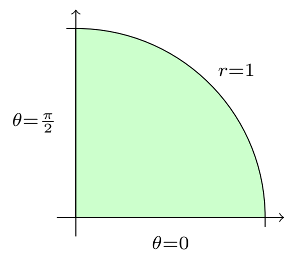
$$I = \int_{y=0}^{a} y^2 \ln(1+\sqrt{2})\, dy = \left[\ln(1+\sqrt{2}) y^3/3 \right]_{y=0}^{a} = \frac{\ln(1+\sqrt{2})a^3}{3}.$$

Week 4 — Plane polar integrals and volume integrals

Exercise 22. Consider the integral given in polar coordinates by $I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} r^2 \sin(\theta)\, dr\, d\theta$. Sketch the corresponding region in the (x, y)-plane, and evaluate the integral.

Solution:

$$\begin{aligned}
I &= \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} r^2 \sin(\theta)\, dr\, d\theta \\
&= \int_{\theta=0}^{\pi/2} \frac{1}{3} \sin(\theta)\, d\theta \\
&= \left[-\frac{1}{3} \cos(\theta) \right]_{\theta=0}^{\pi/2} \\
&= 0 - (-\tfrac{1}{3}) = 1/3.
\end{aligned}$$



Exercise 23. Evaluate $\iint_D xy\, dA$, where D is the quadrant of the disk $x^2 + y^2 \leq a^2$ where $x \geq 0$ and $y \geq 0$. (Hint: use polar coordinates.)

Solution: The region is given in polar coordinates by the limits $0 \leq \theta \leq \pi/2$ and $0 \leq r \leq a$. We also have $x = r\cos(\theta)$ and $y = r\sin(\theta)$ and $dA = r\, dr\, d\theta$ so

$$xy\, dA = r^3 \sin(\theta)\cos(\theta)\, dr\, d\theta = \tfrac{1}{2} r^3 \sin(2\theta)\, dr\, d\theta.$$

This gives

$$\begin{aligned}
\iint_D xy\, dA &= \frac{1}{2} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} r^3 \sin(2\theta)\, dr\, d\theta = \frac{a^4}{8} \int_{\theta=0}^{\frac{\pi}{2}} \sin(2\theta)\, d\theta \\
&= \frac{a^4}{8} \left[-\frac{\cos(2\theta)}{2} \right]_{\theta=0}^{\frac{\pi}{2}} = \frac{a^4}{16}(1 - (-1)) = \frac{a^4}{8}.
\end{aligned}$$

Exercise 24. Evaluate the following integrals, where D is the region given by $x^2 + y^2 \leq a^2$.

(a) $\iint_D (x^2+y^2)^{\frac{1}{2}}\, dA$

(b) $\iint_D e^{-(x^2+y^2)}\, dA$.

Solution: We will use polar coordinates, so $x^2 + y^2 = r^2$ and $dA = r\,dr\,d\theta$. The relevant limits are $0 \le r \le a$ and $0 \le \theta \le 2\pi$. For part (a) we have

$$\iint_D (x^2+y^2)^{\frac{1}{2}}\,dA = \int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^2\,dr\,d\theta = 2\pi\int_{r=0}^{a} r^2\,dr = \frac{2\pi a^3}{3}$$

Similarly, for part (b) we have

$$\iint_D e^{-(x^2+y^2)}\,dA = \int_{r=0}^{a}\int_{\theta=0}^{2\pi} e^{-r^2} r\,dr\,d\theta. = \int_{r=0}^{a} e^{-r^2} 2\pi r\,dr\,d\theta.$$

We now substitute $u = r^2$, so $du = 2r\,dr$ and the limits $0 \le r \le a$ become $0 \le u \le a^2$. The integral becomes

$$\int_{u=0}^{a^2} e^{-u}\pi\,du = \Big[-\pi e^{-u}\Big]_{u=0}^{a^2} = \pi(1-e^{-a^2}).$$

Exercise 25. Evaluate $\iint_D x^2\,dA$, where D is the ring-shaped region given by $4 \le x^2 + y^2 \le 9$.

Solution: In polar coordinates the region is described by $4 \le r^2 \le 9$ or equivalently $2 \le r \le 3$, with θ running from 0 to 2π as usual. We can write the integrand x^2 as $r^2\cos^2(\theta)$ and we also have $dA = r\,dr\,d\theta$. This gives

$$\iint_D x^2\,dA = \int_{\theta=0}^{2\pi}\int_{r=2}^{3} r^3\cos^2(\theta)\,dr\,d\theta.$$

The integrand is just a function of r times a function of θ and the limits are constants so the integral breaks apart giving

$$\begin{aligned}\iint_D x^2\,dA &= \left(\int_{\theta=0}^{2\pi}\cos^2(\theta)d\theta\right)\left(\int_{r=2}^{3} r^3\,dr\right)\\ &= \left[\tfrac{1}{4}\cos(2\theta)+\tfrac{1}{2}\theta\right]_{\theta=0}^{2\pi}\left[\tfrac{1}{4}r^4\right]_{r=2}^{3}\\ &= \pi\times(3^4-2^4)/4 = 65\pi/4.\end{aligned}$$

Exercise 26. Use polar coordinates to evaluate $\displaystyle\iint_D e^{-\sqrt{x^2+y^2}}\,dA$, where D is the region given by $x \ge 0$.

Solution: The region is given in polar coordinates by $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $0 \le r < \infty$. We therefore have

$$\iint_D e^{-\sqrt{x^2+y^2}}\,dA = \int_{r=0}^{\infty}\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-r}\,r\,d\theta\,dr = \pi\int_{r=0}^{\infty} e^{-r}\,r\,dr.$$

We will evaluate this by parts. In more detail, we put $\frac{dv}{dr} = e^{-r}$ and $u = r$, so $v = -e^{-r}$ and $\frac{du}{dr} = 1$. This gives

$$\int e^{-r}\,r\,dr = -re^{-r} + \int e^{-r}\,dr = -re^{-r} - e^{-r} = -(r+1)e^{-r}.$$

Feeding this back into the original integral, we get

$$\iint_D e^{-\sqrt{x^2+y^2}}\,dA = \pi\Big[-(r+1)e^{-r}\Big]_{r=0}^{\infty} = \pi(0-(-1)) = \pi.$$

Exercise 27. Evaluate $\displaystyle\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} xyz\,dz\,dy\,dx.$

Solution:

$$\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} xyz\,dz\,dy\,dx = \left(\int_0^1 x\,dx\right)\left(\int_0^1 y\,dy\right)\left(\int_0^1 z\,dz\right) = \left[\frac{x^2}{2}\right]_0^1\left[\frac{y^2}{2}\right]_0^1\left[\frac{z^2}{2}\right]_0^1 = \frac{1}{2}.\frac{1}{2}.\frac{1}{2} = \frac{1}{8}$$

Exercise 28. Evaluate $\displaystyle\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=\sqrt{x^2+y^2}}^{2} xyz\,dz\,dy\,dx.$

Solution: For the innermost integral, we have

$$\int_{z=\sqrt{x^2+y^2}}^{2} xyz\,dz = \left[\tfrac{1}{2}xyz^2\right]_{z=\sqrt{x^2+y^2}}^{2} = 2xy - \tfrac{1}{2}xy(x^2+y^2) = 2xy - \tfrac{1}{2}x^3y - \tfrac{1}{2}xy^3.$$

The middle integral is thus

$$\int_{y=0}^{1} 2xy - \tfrac{1}{2}x^3y - \tfrac{1}{2}xy^3\,dy = \left[xy^2 - \tfrac{1}{4}x^3y^2 - \tfrac{1}{8}xy^4\right]_{y=0}^{1} = x - \tfrac{1}{4}x^3 - \tfrac{1}{8}x = \tfrac{7}{8}x - \tfrac{1}{4}x^3.$$

Finally, the outermost integral is

$$\int_{x=0}^{1} \tfrac{7}{8}x - \tfrac{1}{4}x^3\,dx = \left[\tfrac{7}{16}x^2 - \tfrac{1}{16}x^4\right]_{x=0}^{1} = \tfrac{7}{16} - \tfrac{1}{16} = \tfrac{3}{8}.$$

Exercise 29. Evaluate $\displaystyle\int_{x=0}^{1}\int_{y=x}^{1}\int_{z=y}^{1} x\,dz\,dy\,dx$.

Solution: For the innermost integral, we have

$$\int_{z=y}^{1} x\,dz = \Big[xz\Big]_{z=y}^{1} = x - xy.$$

The middle integral is thus

$$\int_{y=x}^{1} x - xy\,dy = \Big[xy - xy^2/2\Big]_{y=x}^{1} = (x - x/2) - (x^2 - x^3/2) = x/2 - x^2 + x^3/2.$$

Finally, the outermost integral is

$$\int_{x=0}^{1} x/2 - x^2 + x^3/2\,dx = \Big[x^2/4 - x^3/3 + x^4/8\Big]_{x=0}^{1} = 1/4 - 1/3 + 1/8 = 1/24.$$

Exercise 30. The region D in the (x,y)-plane is given by $|x| \leq 1$ and $|y| \leq 1$, and the surface S consists of the points (x,y,z) where (x,y) lies in D and $z = x^2 + xy$. Let E be the three-dimensional region between D and S. What is the volume of E?

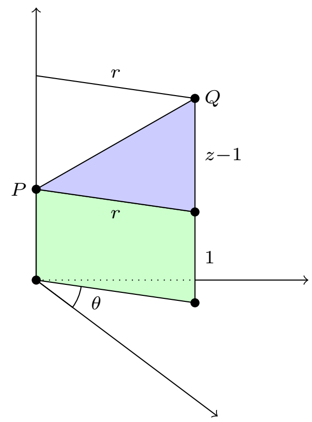
Solution: The volume is

$$\begin{aligned}
V &= \int_{x=-1}^{1}\int_{y=-1}^{1}\int_{z=0}^{x^2+xy} 1\,dz\,dy\,dx = \int_{x=-1}^{1}\int_{y=-1}^{1} x^2 + xy\,dy\,dx \\
&= \int_{x=-1}^{1} \left[x^2y + \tfrac{1}{2}xy^2\right]_{y=-1}^{1} dx = \int_{x=-1}^{1} 2x^2\,dx \\
&= \left[\frac{2}{3}x^3\right]_{x=-1}^{1} = \frac{4}{3}.
\end{aligned}$$

Week 5 — Cylindrical and spherical integrals

Exercise 31. The point P has rectangular coordinates $x = y = 0$ and $z = 1$, and the point Q has cylindrical coordinates r, θ and z. What is the distance from P to Q?

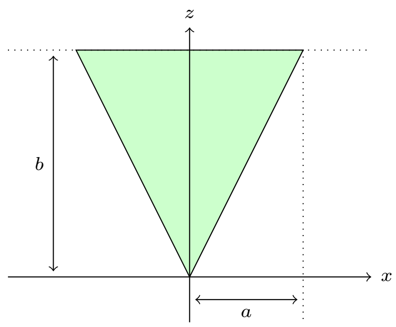
Solution: To go from P to Q we travel a distance $z - 1$ along the z-axis, and a distance r horizontally (perpendicular to the z-axis). By Pythagoras's theorem the straight-line distance is $\sqrt{(z-1)^2 + r^2}$.



More algebraically, we can say that the rectangular coordinates of Q are $(r\cos(\theta), r\sin(\theta), z)$, whereas the rectangular coordinates of P are $(0, 0, 1)$. This means that the vector from P to Q is $(r\cos(\theta), r\sin(\theta), z-1)$, which has length

$$\sqrt{r^2\cos^2(\theta) + r^2\sin^2(\theta) + (z-1)^2} = \sqrt{r^2 + (z-1)^2}.$$

Exercise 32. Let D be a flat-topped circular cone with cross-section as shown below:



Fint the centre of mass and the moment of inertia about the z-axis (assuming that the density is 1).

Solution: It is clear that θ runs from 0 to 2π, and z runs from 0 to b. Any horizontal slice through the cone is a disk, whose radius depends on z. At the top of the cone (where $z = b$) the radius is a, and at the bottom of the cone (where $z = 0$) the radius is 0. Moreover, the radius increases uniformly in z, so we just have $r = za/b$. This formula describes the outer wall of the cone. For the solid interior of the cone, we have $0 \leq r \leq za/b$. We can now calculate the moment of inertia about the z-axis. This is the integral of the square of the distance from the z-axis, and that distance is just r, so

$$\begin{aligned}
I &= \iiint_D r^2\, dV = \int_{\theta=0}^{2\pi}\int_{z=0}^{b}\int_{r=0}^{az/b} r^2\, r\, dr\, dz\, d\theta \\
&= 2\pi \int_{z=0}^{b}\int_{r=0}^{az/b} r^3\, dr\, dz = 2\pi\int_{z=0}^{b}\left[\frac{r^4}{4}\right]_{r=0}^{az/b} dz = 2\pi\int_{z=0}^{b}\frac{a^4z^4}{4b^4}\, dz \\
&= \frac{\pi a^4}{2b^4}\left[\frac{z^5}{5}\right]_{z=0}^{b} = \frac{\pi a^4 b}{10}.
\end{aligned}$$

Next, it is clear by symmetry that the centre of mass must lie on the z-axis, say at $(0, 0, \overline{z})$. The general theory tells us that $\overline{z} = Z/V$, where $Z = \iiint_D z\, dV$ and $V = \iiint_D 1\, dV$. These can be evaluated as follows:

$$
\begin{aligned}
Z &= \int_{\theta=0}^{2\pi}\int_{z=0}^{b}\int_{r=0}^{az/b} z\,r\,dr\,dz\,d\theta = 2\pi\int_{z=0}^{b}\left[\frac{zr^2}{2}\right]_{r=0}^{az/b} dz \\
&= 2\pi\int_{z=0}^{b}\frac{z}{2}\left(\frac{az}{b}\right)^2 dz = \frac{\pi a^2}{b^2}\int_{z=0}^{b} z^3\,dz \\
&= \frac{\pi a^2}{b^2}\,\frac{b^4}{4} = \frac{\pi a^2 b^2}{4} \\
V &= \int_{\theta=0}^{2\pi}\int_{z=0}^{b}\int_{r=0}^{az/b} r\,dr\,dz\,d\theta = 2\pi\int_{z=0}^{b}\left[\frac{r^2}{2}\right]_{r=0}^{az/b} dz \\
&= \frac{\pi a^2}{b^2}\int_{z=0}^{b} z^2\,dz \\
&= \frac{\pi a^2}{b^2}\,\frac{b^3}{3} = \frac{\pi a^2 b}{3} \\
\overline{z} &= \frac{Z}{V} = \frac{\pi a^2 b^2}{4}\,\frac{3}{\pi a^2 b} = \frac{3b}{4}.
\end{aligned}
$$

Exercise 33. Let D be the solid given by $x, y, z \geq 0$ and $x^2+y^2 \leq 1$ and $z \leq 2$. By rewriting everything in cylindrical coordinates, evaluate the integral

$$I = \iiint_D x + y + z\,dV.$$

Solution: The conditions $x, y \geq 0$ give $0 \leq \theta \leq \pi/2$, and the condition $x^2 + y^2 \leq 1$ gives $0 \leq r \leq 1$. We are also given that $0 \leq z \leq 2$. (Geometrically, we have a quarter of a cylinder with radius one and height two.) For the integrand, we have

$$x + y + z = r(\cos(\theta) + \sin(\theta)) + z,$$

and $dV = r\,dr\,d\theta\,dz$. We thus have

$$I = \int_{z=0}^{2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^2(\cos(\theta)+\sin(\theta)) + zr\,dr\,d\theta\,dz.$$

For the innermost integral we have

$$\int_{r=0}^{1} r^2(\cos(\theta)+\sin(\theta)) + zr\,dr = \left[\tfrac{1}{3}r^3(\cos(\theta)+\sin(\theta)) + \tfrac{1}{2}zr^2\right]_{r=0}^{1} = (\cos(\theta)+\sin(\theta))/3 + z/2.$$

The middle integral now becomes

$$\int_{\theta=0}^{\pi/2}(\cos(\theta)+\sin(\theta))/3+z/2\,d\theta = \Big[(\sin(\theta)-\cos(\theta))/3 + z\theta/2\Big]_{\theta=0}^{\pi/2} = (1/3+\pi z/4)-(-1/3) = 2/3+\pi z/4.$$

Finally, the outer integral is

$$I = \int_{z=0}^{2} 2/3 + \pi z/4\,dz = \Big[2z/3 + \pi z^2/8\Big]_{z=0}^{2} = 4/3 + \pi/2.$$

Exercise 34. Let D be a solid cylinder, with base of radius a centred at the origin in the (x, y)-plane, and height h. The charge density on D is given by $\rho(x, y, z) = x^2\sin(\pi z/h)$. Find the total charge.

Solution: We use cylindrical polar coordinates. The relevant limits are $0 \leq r \leq a$ and $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq h$. The volume element dV is $r\,dr\,d\theta\,dz$, and the charge density is

$$\rho = x^2\sin(\pi z/h) = r^2\cos^2(\theta)\sin(\pi z/h) = \tfrac{1}{2}r^2\sin(\pi z/h)(1+\cos(2\theta)).$$

The total charge is

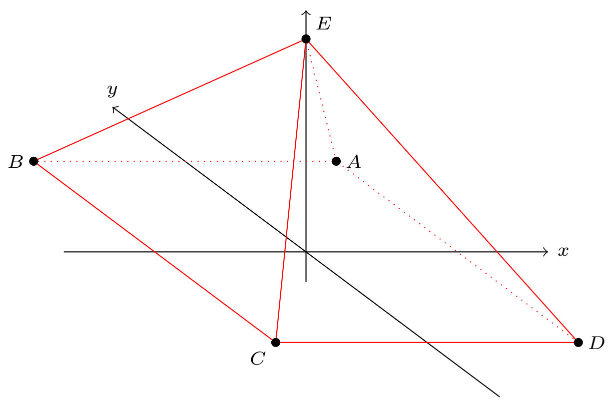
$$\begin{aligned}
Q = \iiint_D \rho dV &= \int_{z=0}^{h} \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \tfrac{1}{2} r^3 \sin(\pi z/h)(1+\cos(2\theta))\, dr\, d\theta\, dz \\
&= \frac{1}{2}\left(\int_{z=0}^{h} \sin(\pi z/h)\, dz\right)\left(\int_{\theta=0}^{2\pi} (1+\cos(2\theta))\, d\theta\right)\left(\int_{r=0}^{a} r^3\, dr\right) \\
&= \frac{1}{2}\left[-\frac{h}{\pi}\cos\left(\frac{\pi z}{h}\right)\right]_{z=0}^{h} \left[\theta + \tfrac{1}{2}\sin(2\theta)\right]_{\theta=0}^{2\pi} \left[\frac{r^4}{4}\right]_{r=0}^{a} \\
&= \frac{1}{2}.\frac{2h}{\pi}.2\pi.\frac{a^4}{4} = \frac{a^4 h}{2}.
\end{aligned}$$

Exercise 35. Let P be the shape obtained by joining together points $A, \ldots, E$ with spherical coordinates as listed below:

	A	B	C	D	E
r	1	1	1	1	1
θ	$\pi/4$	$3\pi/4$	$5\pi/4$	$7\pi/4$	0
ϕ	$\pi/2$	$\pi/2$	$\pi/2$	$\pi/2$	0

Sketch P and describe it geometrically.

Solution: The points $A, \ldots, D$ have $\phi = \pi/2$, so they are at an angle of $\pi/2$ to the z-axis, so they lie in the xy-plane. They all have $r = 1$, which means that they have distance one from the origin. Point A is at angle $\theta = \pi/4$ to the x-axis, and B, C and D are obtained by moving round an extra $\pi/2$ each time, so they form a square in the xy-plane. On the other hand, the point E has $\phi = 0$ so it is on the z-axis. The picture is as follows:



Thus, P is a pyramid with a square base.

Exercise 36. Let Q be the surface given in spherical coordinates by the equation $\tan(\phi)\sin(\theta) = 1$. Explain why Q is a plane and give an equation for Q in terms of rectangular coordinates.

Solution: Recall that rectangular and spherical coordinates are related as follows:

$$\begin{aligned}
x &= r\ \sin(\phi)\ \cos(\theta) \\
y &= r\ \sin(\phi)\ \sin(\theta) \\
z &= r\ \cos(\phi).
\end{aligned}$$

From this we get

$$\frac{y}{z} = \frac{r\sin(\phi)\sin(\theta)}{r\cos(\phi)} = \tan(\phi)\sin(\theta).$$

Thus, the equation $\tan(\phi)\sin(\theta) = 1$ is equivalent to $y/z = 1$ or $y = z$, which clearly describes a plane.

Exercise 37. Suppose that $b > a > 0$, and let D be the three-dimensional region between the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$. Evaluate the integral

$$I = \iiint_D \frac{1}{(x^2+y^2+z^2)^{3/2}} dV.$$

Solution: We use spherical polar coordinates, noting that $x^2+y^2+z^2=r^2$ and $dV=r^2\sin(\phi)\,dr\,d\theta\,d\phi$. The integral becomes

$$\begin{aligned} I &= \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi}\int_{r=a}^{b} r^{-3}.r^2\sin(\phi)\,dr\,d\theta\,d\phi \\ &= 2\pi\left(\int_{\phi=0}^{\pi}\sin(\phi)\,d\phi\right)\left(\int_{r=a}^{b} r^{-1}\,dr\right) = 2\pi\Big[-\cos(\phi)\Big]_{\phi=0}^{\pi}\Big[\ln(r)\Big]_{r=a}^{b} \\ &= 4\pi(\ln(b)-\ln(a)) = 4\pi\ln(b/a). \end{aligned}$$

Exercise 38. By rewriting everything in spherical polar coordinates, evaluate the integral

$$I=\int_{x=0}^{\infty}\int_{y=0}^{\infty}\int_{z=0}^{\infty}\exp\left(-(x^2+y^2+z^2)^{3/2}\right)dz\,dy\,dx.$$

Solution: The region of integration has $x,y\geq 0$ (which means that $0\leq\theta\leq\pi/2$ and $z\geq 0$ (which means that $0\leq\phi\leq\pi/2$). The distance r from the origin can be any nonnegative number. For the integrand, we have

$$\exp\left(-(x^2+y^2+z^2)^{3/2}\right)=e^{-r^3}.$$

We also have

$$dz\,dy\,dx=dV=r^2\sin(\phi)dr\,d\phi\,d\theta.$$

Thus, the integral is

$$I=\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2}\int_{r=0}^{\infty}r^2e^{-r^3}\sin(\phi)dr\,d\phi\,d\theta=\frac{\pi}{2}\left(\int_{\phi=0}^{\pi/2}\sin(\phi)\,d\phi\right)\left(\int_{r=0}^{\infty}r^2e^{-r^3}\,dr\right).$$

For the ϕ integral we have

$$\int_{\phi=0}^{\pi/2}\sin(\phi)\,d\phi=\Big[-\cos(\phi)\Big]_{\phi=0}^{\pi/2}=0-(-1)=1.$$

For the r integral we use the substitution $u=r^3$, so $du=3r^2\,dr$, so $dr=du/(3r^2)$. The limits $r=0$ and $r=\infty$ correspond to $u=0$ and $u=\infty$. We thus get

$$\int_{r=0}^{\infty}r^2e^{-r^3}\,dr=\int_{u=0}^{\infty}r^2e^{-u}\frac{du}{3r^2}=\frac{1}{3}\int_{u=0}^{\infty}e^{-u}\,du=\frac{1}{3}\Big[-e^{-u}\Big]_{u=0}^{\infty}=1/3.$$

Putting this together we get

$$I=\frac{\pi}{2}\times 1\times\frac{1}{3}=\frac{\pi}{6}.$$

Week 6 — Vector algebra and gradients

Unless otherwise specified, $\mathbf{r}$ refers to the position vector $\mathbf{r}=(x,y,z)$, and $r=|\mathbf{r}|=\sqrt{x^2+y^2+z^2}$.

Exercise 39. Consider the vectors $\mathbf{p}=2\mathbf{i}-\mathbf{j}$, $\mathbf{q}=4\mathbf{i}-2\mathbf{j}$ and $\mathbf{r}=2\mathbf{i}+4\mathbf{j}$. Which of them are parallel to each other, and which of them are perpendicular to each other?

Solution: Recall that vectors are perpendicular when their dot product is zero, and parallel (or antiparallel) when their cross product is zero. We have

$$\begin{aligned} \mathbf{p}.\mathbf{q} &= 2\times 4+(-1)\times(-2)=10 \\ \mathbf{p}.\mathbf{r} &= 2\times 2+(-1)\times 4=0 \\ \mathbf{q}.\mathbf{r} &= 4\times 2+(-2)\times 4=0 \end{aligned}$$

so $\mathbf{p}$ and $\mathbf{q}$ are perpendicular to $\mathbf{r}$ but not to each other. For the cross products, we have

$$\begin{aligned} \mathbf{p}\times\mathbf{q} &= \det\begin{bmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-1&0\\4&-2&0\end{bmatrix}=(2\times(-2)-(-1)\times 4)\mathbf{k}=0 \\ \mathbf{p}\times\mathbf{r} &= \det\begin{bmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-1&0\\2&4&0\end{bmatrix}=(2\times 4-(-1)\times 2)\mathbf{k}=10\mathbf{k} \\ \mathbf{q}\times\mathbf{r} &= \det\begin{bmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\4&-2&0\\2&4&0\end{bmatrix}=(4\times 4-(-2)\times 2)\mathbf{k}=20\mathbf{k}, \end{aligned}$$

so $\mathbf{p}$ and $\mathbf{q}$ are parallel to each other, but not to $\mathbf{r}$. More obviously, we can just observe that $\mathbf{q} = 2\mathbf{p}$, so $\mathbf{p}$ and $\mathbf{q}$ are parallel.

Exercise 40. If $\mathbf{a} = (2, -1, 2)$, $\mathbf{b} = (-1, 2, 1)$ and $\mathbf{c} = (1, -2, 1)$, find the following quantities:

(a) $|\mathbf{a}|$, $|\mathbf{b}|$ and $|\mathbf{c}|$.
(b) $\mathbf{a}.\mathbf{b}$, $\mathbf{a}.\mathbf{c}$ and $\mathbf{b}.\mathbf{c}$.
(c) $\mathbf{a} \times \mathbf{b}$, $\mathbf{a} \times \mathbf{c}$ and $\mathbf{b} \times \mathbf{c}$.
(d) The unit vector $\widehat{\mathbf{a}}$.
(e) The angle between $\mathbf{b}$ and $\mathbf{c}$.
(f) The area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{c}$.
(g) The component of $\mathbf{a}$ parallel to $\mathbf{b}$.
(h) The component of $\mathbf{a}$ perpendicular to $\mathbf{b}$.

Be sure to type-check your answers: do not give a vector where a scalar is required, or *vice-versa*.

Solution:

(a) $|\mathbf{a}| = \sqrt{4+1+4} = 3$; $|\mathbf{b}| = \sqrt{1+4+1} = \sqrt{6}$; $|\mathbf{c}| = \sqrt{1+4+1} = \sqrt{6}$.
(b) $\mathbf{a}.\mathbf{b} = 2 \times (-1) + (-1) \times 2 + 2 \times 1 = -2$; $\mathbf{a}.\mathbf{c} = 2 \times 1 + (-1) \times (-2) + 2 \times 1 = 6$; $\mathbf{b}.\mathbf{c} = (-1) \times 1 + 2 \times (-2) + 1 \times 1 = -4$.
(c)

$$\mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ -1 & 2 & 1 \end{bmatrix} = (-5, -4, 3)$$
$$\mathbf{a} \times \mathbf{c} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ 1 & -2 & 1 \end{bmatrix} = (3, 0, -3)$$
$$\mathbf{b} \times \mathbf{c} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 1 \\ 1 & -2 & 1 \end{bmatrix} = (4, 2, 0).$$

(d) $\widehat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}| = \mathbf{a}/3 = (2/3, -1/3, 2/3)$.
(e) We have

$$\cos(\theta) = \frac{\mathbf{b}.\mathbf{c}}{|\mathbf{b}||\mathbf{c}|} = \frac{-4}{\sqrt{6}\sqrt{6}} = -\tfrac{2}{3},$$

so $\theta = \arccos(-2/3)$, which is 2.3 radians or 132 degrees.
(f) The area is $|\mathbf{a} \times \mathbf{c}| = \sqrt{3^2 + 0^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2} \simeq 4.24$.
(g) The parallel component is

$$\mathbf{a}_{||} = \frac{\mathbf{a}.\mathbf{b}}{\mathbf{b}.\mathbf{b}}\mathbf{b} = \frac{-2}{6}(-1, 2, 1) = (\tfrac{1}{3}, -\tfrac{2}{3}, -\tfrac{1}{3}).$$

(h) The perpendicular component is

$$\mathbf{a}_{\perp} = \mathbf{a} - \mathbf{a}_{||} = (2, -1, 2) - (\tfrac{1}{3}, -\tfrac{2}{3}, -\tfrac{1}{3}) = (\tfrac{5}{3}, -\tfrac{1}{3}, \tfrac{7}{3}).$$

Exercise 41. Consider vectors $\mathbf{a} = (u, v, w)$ and $\mathbf{b} = (x, y, z)$. Give formulae for $\mathbf{a}.\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ and $|\mathbf{a} \times \mathbf{b}|^2$. Verify by direct expansion that

$$(\mathbf{a}.\mathbf{b})^2 + |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2.$$

Solution:

$$\begin{aligned}
\mathbf{a}.\mathbf{b} &= ux + vy + wz \\
(\mathbf{a}.\mathbf{b})^2 &= u^2x^2 + v^2y^2 + w^2z^2 + 2uvxy + 2uwxz + 2vwyz \\
\mathbf{a} \times \mathbf{b} &= (vz - wy,\ wx - uz,\ uy - vx) \\
|\mathbf{a} \times \mathbf{b}|^2 &= (vz - wy)^2 + (wx - uz)^2 + (uy - vx)^2 \\
&= v^2z^2 - 2vwyz + w^2y^2 + w^2x^2 - 2uwxz + u^2z^2 + u^2y^2 - 2uvxy + v^2x^2. \\
(\mathbf{a}.\mathbf{b})^2 + |\mathbf{a} \times \mathbf{b}|^2 &= u^2x^2 + v^2y^2 + w^2z^2 + v^2z^2 + w^2y^2 + w^2x^2 + u^2z^2 + u^2y^2 + v^2x^2 \\
&= (u^2 + v^2 + w^2)(x^2 + y^2 + z^2) \\
&= |\mathbf{a}|^2|\mathbf{b}|^2.
\end{aligned}$$

Note that if we believe the formulae $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$ and $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\theta)$ then we have a simpler argument:

$$(\mathbf{a}.\mathbf{b})^2 + |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2\cos^2(\theta) + |\mathbf{a}|^2|\mathbf{b}|^2\sin^2(\theta) = |\mathbf{a}|^2|\mathbf{b}|^2.$$

However, this is not really a satisfactory argument, because the more complicated calculation given above is required in order to prove the formula $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\theta)$ in the first place.

Exercise 42. Find $\operatorname{grad}(f)$ for the following functions

(a) $f = x^2y + y^2z + z^2x$
(b) $f = \sin(r)/r$
(c) $f = e^{-x^2-y^2} + z$.

Solution:

(a) $\operatorname{grad}(f) = (f_x, f_y, f_z) = (2xy + z^2, 2yz + x^2, 2zx + y^2)$.
(b) Recall that

$$r_x = \frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{\frac{1}{2}} = \tfrac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}.2x = x/r,$$

and similarly $r_y = y/r$ and $r_z = z/r$. We also have

$$\frac{df}{dr} = \frac{\cos(r)r - \sin(r)}{r^2},$$

and

$$f_x = \frac{\partial f}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x} = \frac{\cos(r)r - \sin(r)}{r^2}\,\frac{x}{r} = \frac{\cos(r)r - \sin(r)}{r^3}\,x.$$

We can find f_y and f_z in the same way, giving

$$\operatorname{grad}(f) = (f_x, f_y, f_z) = \frac{\cos(r)r - \sin(r)}{r^3}(x, y, z).$$

(c) In this case we have $\operatorname{grad}(f) = (-2xe^{-x^2-y^2}, -2ye^{-x^2-y^2}, 1)$.

Exercise 43. If $f = x^2yz^3$ and $\mathbf{n} = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, find the directional derivative $\mathbf{n}.\nabla(f)$.

Solution: $\nabla(f) = (2xyz^3, x^2z^3, 3x^2yz^2)$, so

$$\mathbf{n}.\nabla(f) = \frac{2}{3}xyz^3 + \frac{2}{3}x^2z^3 + 2x^2yz^2.$$

Exercise 44. The scalar field f is given by $f = x\sin(xy) + z\cos(xy)$. Find the component of $\operatorname{grad}(f)$ in parallel to $(-1, 1, -1)$ at the point $(\pi/2, 2, 0)$.

Solution: Here

$$\nabla(f) = (\sin(xy) + xy\cos(xy) - yz\sin(xy), x^2\cos(xy) - xz\sin(xy), \cos(xy)).$$

At the point $(x, y, z) = (\pi/2, 2, 0)$ we have $xy = \pi$ so $\sin(xy) = 0$ and $\cos(xy) = -1$ so

$$\nabla(f) = (-\pi, -\pi^2/4, -1).$$

Now put $\mathbf{m} = (-1, 1, -1)$, so $\nabla(f).\mathbf{m} = \pi - \pi^2/4 + 1 = \frac{1}{4}(4 + 4\pi - \pi^2)$ and $\mathbf{m}.\mathbf{m} = 3$. The component of $\nabla(f)$ parallel to the $\mathbf{m}$ is given by

$$\nabla(f)_{||} = \frac{\nabla(f).\mathbf{m}}{\mathbf{m}.\mathbf{m}}\mathbf{m} = \frac{4 + 4\pi - \pi^2}{12}(-1, 1, -1).$$

Exercise 45. Put $f = x^2 - z^2$ and $g = 2xz + y^2$. Show that $\nabla(f)$ is always perpendicular to $\nabla(g)$.

Solution:

$$\begin{aligned}\nabla(f) &= (2x, 0, -2z)\\ \nabla(g) &= (2z, 2y, 2x)\\ \nabla(f).\nabla(g) &= (2x)\times(2z) + 0\times 2y + (-2z)\times(2x) = 4xz - 4xz = 0.\end{aligned}$$

As $\nabla(f).\nabla(g) = 0$ for all x, y and z, we see that $\nabla(f)$ and $\nabla(g)$ are perpendicular at all points.

Exercise 46. Find $\nabla.\mathbf{u}$ and $\nabla\times\mathbf{u}$ for the following vector fields:

$$\text{(a):}\qquad \mathbf{u}=(xy,yz,0)\qquad\qquad \text{(b):}\qquad \mathbf{u}=(z,x,y).$$

Solution:

(a) $\nabla.\mathbf{u}=\frac{\partial}{\partial x}(xy)+\frac{\partial}{\partial y}(yz)+\frac{\partial}{\partial z}(0)=y+z$ and

$$\nabla\times\mathbf{u}=\det\begin{bmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ xy&yz&0\end{bmatrix}=\left(\frac{\partial}{\partial y}(0)-\frac{\partial}{\partial z}(yz),\frac{\partial}{\partial z}(xy)-\frac{\partial}{\partial x}(yz),\frac{\partial}{\partial x}(yz)-\frac{\partial}{\partial y}(xy)\right)=(-y,0,-x).$$

(b) $\nabla.\mathbf{u}=\frac{\partial}{\partial x}(z)+\frac{\partial}{\partial y}(x)+\frac{\partial}{\partial z}(y)=0+0+0=0$ and

$$\nabla\times\mathbf{u}=\det\begin{bmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ z&x&y\end{bmatrix}=\left(\frac{\partial}{\partial y}(y)-\frac{\partial}{\partial z}(x),\frac{\partial}{\partial z}(z)-\frac{\partial}{\partial x}(y),\frac{\partial}{\partial x}(x)-\frac{\partial}{\partial y}(z)\right)=(1,1,1).$$

Exercise 47. Consider a vector field of the form $\mathbf{u}=f(r)\mathbf{r}$, where f is a function of r only. Show that $\nabla.\mathbf{u}=3\,f(r)+r\,f'(r)$. Show that if $\nabla.\mathbf{u}=0$, then $f(r)=c/r^3$ for some constant c.

[**Hint:** remember the chain rule $\frac{\partial}{\partial x}f(r)=f'(r)\frac{\partial r}{\partial x}$.]

Solution: We have $\mathbf{u}=(f(r)x,f(r)y,f(r)z)$, so

$$\nabla.\mathbf{u}=\frac{\partial}{\partial x}(f(r)x)+\frac{\partial}{\partial y}(f(r)y)+\frac{\partial}{\partial z}(f(r)z).$$

Using the product rule, the chain rule, and the the standard fact that $\partial r/\partial x=x/r$, we get

$$\frac{\partial}{\partial x}(f(r)x)=f(r)+x\frac{\partial}{\partial x}f(r)=f(r)+x\,f'(r)\frac{\partial r}{\partial x}=f(r)+x^2\,f'(r)/r.$$

We can treat the other two terms in the same way to get

$$\begin{aligned}\nabla.\mathbf{u}&=\frac{\partial}{\partial x}(f(r)x)+\frac{\partial}{\partial y}(f(r)y)+\frac{\partial}{\partial z}(f(r)z)\\&=f(r)+\frac{x^2}{r}f'(r)+f(r)+\frac{y^2}{r}f'(r)+f(r)+\frac{z^2}{r}f'(r)=3\,f(r)+\frac{x^2+y^2+z^2}{r}\,f'(r)=3\,f(r)+\frac{r^2}{r}\,f'(r)\\&=3\,f(r)+r\,f'(r).\end{aligned}$$

Now suppose that $\nabla.\mathbf{u}=0$, so $3f(r)+r\,f'(r)=0$. Put $g(r)=r^3\,f(r)$, so

$$g'(r)=3r^2\,f(r)+r^3\,f'(r)=r^2(3f(r)+r\,f'(r))=0.$$

This means that $g(r)$ is a constant, say c. As $c=g(r)=r^3\,f(r)$, it follows that $f(r)=c/r^3$.

Exercise 48. Find constants a, b and c such that the vector field

$$\mathbf{v}=(x+2y+az,\ bx-3y-z,\ 4x+cy+2z)$$

satisfies $\mathrm{curl}(\mathbf{v})=0$. For these values of a, b and c, find a potential function f with $\mathrm{grad}(f)=\mathbf{v}$.

Solution: We have

$$\mathrm{curl}(\mathbf{v})=\det\begin{bmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ x+2y+az&bx-3y-z&4x+cy+2z\end{bmatrix}=(c-(-1),a-4,b-2).$$

For this to be zero we must have $a=4$ and $b=2$ and $c=-1$, so

$$\mathbf{v}=(x+2y+4z,\ 2x-3y-z,\ 4x-y+2z).$$

We want this to be equal to $\mathrm{grad}(f)$, which means that

(A) $$f_x=x+2y+4z$$

(B) $$f_y=2x-3y-z$$

(C) $$f_z=4x-y+2z$$

By integrating (A), we see that

(D) $$f=\int x+2y+4z\,dx=\tfrac{1}{2}x^2+2xy+4xz+p,$$

where p is constant with respect to x, so p depends only on y and z. We now differentiate (D) with respect to y to get

(E) $$f_y = 2x + p_y.$$

On the other hand, equation (B) says that $f_y = 2x - 3y - z$. By comparing (B) and (E) we see that $p_y = -3y - z$. We can now integrate with respect to y to get

(F) $$p = -\tfrac{3}{2}y^2 - yz + q,$$

where q is constant with respect to both x and y, so it depends only on z. We can substitute (F) in (D) to get

(G) $$f = \tfrac{1}{2}x^2 + 2xy + 4xz - \tfrac{3}{2}y^2 - yz + q.$$

We can differentiate (G) with respect to z to get

(H) $$f_z = 4x - y + q_z.$$

On the other hand, equation (C) says that $f_z = 4x - y + 2z$. By comparing (C) and (H) we see that $q_z = 2z$, so $q = z^2$ (plus a constant, which we can take to be zero). We can substitute this in (G) to get

$$f = \tfrac{1}{2}x^2 + 2xy + 4xz - \tfrac{3}{2}y^2 - yz + z^2.$$

As a final check, we can calculate directly that

$$\begin{aligned} f_x &= x + 2y + 4z \\ f_y &= 2x - 3y - z \\ f_z &= 4x - y + 2z \end{aligned}$$

as required.

Exercise 49. If $r = \sqrt{x^2 + y^2 + z^2}$, show that $\nabla^2(r^n) = n(n+1)r^{n-2}$.

Solution: We have seen before that $r_x = x/r$, which implies that

(A) $$(r^n)_x = n\,r^{n-1}r_x = n\,r^{n-1}x/r = n\,r^{n-2}x.$$

In the same way, we have

$$(r^{n-2})_x = (n-2)r^{n-4}x.$$

We can differentiate (A) once more using the product rule to get

$$\begin{aligned} (r^n)_{xx} &= (n\,r^{n-2}x)_x = n\,(r^{n-2})_x\,x + n\,r^{n-2} \\ &= n((n-2)r^{n-4}\,x)\,x + nr^{n-2} = (n^2-2n)r^{n-4}\,x^2 + nr^{n-2}. \end{aligned}$$

In the same way, we have

$$\begin{aligned} (r^n)_{yy} &= (n^2-2n)r^{n-4}\,y^2 + nr^{n-2} \\ (r^n)_{zz} &= (n^2-2n)r^{n-4}\,z^2 + nr^{n-2}. \end{aligned}$$

Adding these together, we get

$$\begin{aligned} \nabla^2(r^n) &= (r^n)_{xx} + (r^n)_{yy} + (r^n)_{zz} \\ &= (n^2-2n)r^{n-4}\,x^2 + nr^{n-2} + (n^2-2n)r^{n-4}\,y^2 + nr^{n-2} + (n^2-2n)r^{n-4}\,z^2 + nr^{n-2} \\ &= (n^2-2n)r^{n-4}(x^2+y^2+z^2) + 3nr^{n-2} = (n^2-2n)r^{n-4}\,r^2 + 3nr^{n-2} \\ &= (n^2+n)r^{n-2} = n(n+1)r^{n-2} \end{aligned}$$

as claimed.

Exercise 50. Let Ω be a scalar field, and let $\mathbf{F}$ be a vector field. Show that

(a) $\mathrm{curl}(\Omega\mathbf{F}) = \Omega\,\mathrm{curl}(\mathbf{F}) - \mathbf{F} \times \mathrm{grad}(\Omega)$
(b) $\mathrm{curl}(\mathrm{grad}(\Omega)) = 0$.

Rewrite these identities in ∇ notation.

Solution: Write $\mathbf{F}=(P,Q,R)$, so $\Omega\mathbf{F}=(\Omega P,\Omega Q,\Omega R)$. We have

$$\operatorname{curl}(\Omega\mathbf{F})=\det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \Omega P & \Omega Q & \Omega R\end{bmatrix}=((\Omega R)_y-(\Omega Q)_z,\ (\Omega P)_z-(\Omega R)_x,\ (\Omega Q)_x-(\Omega P)_y)$$

$$=(\Omega_y R+\Omega R_y-\Omega_z Q-\Omega Q_z,\ \Omega_z P+\Omega P_z-\Omega_x R-\Omega R_x,\ \Omega_x Q+\Omega Q_x-\Omega_y P-\Omega P_y)$$

(A) $$=\Omega(R_y-Q_z,\ P_z-R_x,\ Q_x-P_y)+(\Omega_y R-\Omega_z Q,\ \Omega_z P-\Omega_x R,\ \Omega_x Q-\Omega_y P)$$

(B) $$\Omega\operatorname{curl}(\mathbf{F})=\Omega\det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{bmatrix}=\Omega(R_y-Q_z,\ P_z-R_x,\ Q_x-P_y)$$

(C)

$$\mathbf{F}\times\operatorname{grad}(\Omega)=\det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ P & Q & R\\ \Omega_x & \Omega_y & \Omega_z\end{bmatrix}=(\Omega_z Q-\Omega_y R,\ \Omega_x R-\Omega_z P,\ \Omega_y P-\Omega_x Q)$$

Combining (A), (B) and (C) makes it clear that $\operatorname{curl}(\Omega\mathbf{F})=\Omega\operatorname{curl}(\mathbf{F})-\mathbf{F}\times\operatorname{grad}(\Omega)$. In ∇ notation, this becomes $\nabla\times(\Omega\mathbf{F})=\Omega\,\nabla\times\mathbf{F}-\mathbf{F}\times\nabla(\Omega)$.

Next, we have

$$\operatorname{grad}(\Omega)=(\Omega_x,\ \Omega_y,\ \Omega_z)$$

$$\operatorname{curl}(\operatorname{grad}(\Omega))=\det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \Omega_x & \Omega_y & \Omega_z\end{bmatrix}=(\Omega_{zy}-\Omega_{yz},\Omega_{xz}-\Omega_{zx},\Omega_{yx}-\Omega_{xy})=(0,0,0).$$

In ∇ notation, this says that $\nabla\times(\nabla(\Omega))=0$.

Exercise 51. Let $\mathbf{H}$ be a vector field that can be expressed as $\mathbf{H}=f\operatorname{grad}(g)$ for some scalar fields f and g. Show that $\mathbf{H}$ is perpendicular to $\operatorname{curl}(\mathbf{H})$ at every point.
[**Hint:** use the previous question.]

Now consider the vector field $\mathbf{H}=x^2y\,\mathbf{r}$ (where $\mathbf{r}=(x,y,z)$ as usual). Find scalar fields f and g such that $\mathbf{H}=f\operatorname{grad}(g)$. Calculate $\operatorname{curl}(\mathbf{H})$ and check directly that it is perpendicular to $\mathbf{H}$.

Solution: Using the previous question (with $\Omega=f$ and $\mathbf{F}=\operatorname{grad}(g)$) we see that

$$\operatorname{curl}(\mathbf{H})=\operatorname{curl}(f\operatorname{grad}(g))=f\operatorname{curl}(\operatorname{grad}(g))-\operatorname{grad}(g)\times\operatorname{grad}(f).$$

As $\operatorname{curl}(\operatorname{grad}(g))=0$, this simplifies to $\operatorname{curl}(\mathbf{H})=-\operatorname{grad}(g)\times\operatorname{grad}(f)=\operatorname{grad}(f)\times\operatorname{grad}(g)$. It is a general rule that $\mathbf{a}\times\mathbf{b}$ is always perpendicular to $\mathbf{a}$ and to $\mathbf{b}$, so $\operatorname{curl}(\mathbf{H})$ is perpendicular to $\operatorname{grad}(g)$. Moreover, $\mathbf{H}$ is just the scalar f times the vector $\operatorname{grad}(g)$, so it has the same direction as $\operatorname{grad}(g)$, so it is also perpendicular to $\mathbf{H}$.

Now consider the case

$$\mathbf{H}=x^2y\mathbf{r}=(x^3y,\ x^2y^2,\ x^2yz).$$

We know that $\operatorname{grad}(r)=(x/r,y/r,z/r)=\mathbf{r}/r$, so

$$\mathbf{H}=x^2yr\ \mathbf{r}/r=x^2yr\operatorname{grad}(r).$$

Thus, if we take $f=x^2yr$ and $g=r$ then $\mathbf{H}=f\operatorname{grad}(g)$. A different, but equally valid, solution is to take $f=x^2y$ and $g=r^2/2=(x^2+y^2+z^2)/2$.

The first part of the question now tells us that $\operatorname{curl}(\mathbf{H})$ should be perpendicular to $\mathbf{H}$. We can check this directly as follows. We have

$$\operatorname{curl}(\mathbf{H})=\det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^3y & x^2y^2 & x^2yz\end{bmatrix}=(x^2z-0,\ 0-2xyz,\ 2xy^2-x^3)=(x^2z,\ -2xyz,\ 2xy^2-x^3),$$

so

$$\mathbf{H}.\operatorname{curl}(\mathbf{H})=(x^3y,\ x^2y^2,\ x^2yz).(x^2z,\ -2xyz,\ 2xy^2-x^3)=x^5yz-2x^3y^3z+2x^3y^3z-x^5yz=0,$$

which means that $\mathbf{H}$ and $\operatorname{curl}(\mathbf{H})$ are perpendicular.

Exercise 52. For the vector field $\mathbf{A}=(x^2y,\ y^2z,\ z^2x)$, calculate

(a) $\nabla.\mathbf{A}$
(b) $\nabla(\nabla.\mathbf{A})$

(c) $\nabla \times \mathbf{A}$
(d) $\nabla \times (\nabla \times \mathbf{A})$
(e) $\nabla^2(\mathbf{A})$.

Verify the identity $\nabla^2(\mathbf{A}) = \nabla(\nabla.\mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$ in this case.

Solution:

(a) $\nabla.\mathbf{A} = \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(y^2 z) + \frac{\partial}{\partial z}(z^2 x) = 2xy + 2yz + 2zx$.
(b) $\nabla(\nabla.\mathbf{A}) = \nabla(2xy + 2yz + 2zx) = (2y + 2z,\ 2z + 2x,\ 2x + 2y)$.
(c) $\nabla \times \mathbf{A} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & y^2 z & z^2 x \end{bmatrix} = (0 - y^2,\ 0 - z^2,\ 0 - x^2) = (-y^2, -z^2, -x^2)$.
(d) $\nabla \times (\nabla \times \mathbf{A}) = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & -z^2 & -x^2 \end{bmatrix} = (0 - (-2z),\ 0 - (-2x),\ 0 - (-2y)) = (2z, 2x, 2y)$.
(e) Note that $\nabla(x^2 y) = (2xy, x^2, 0)$, so

$$\nabla^2(x^2 y) = \nabla.(2xy, x^2, 0) = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial z}(0) = 2y.$$

In the same way, we have $\nabla^2(y^2 z) = 2z$ and $\nabla^2(z^2 x) = 2x$, so

$$\nabla^2(\mathbf{A}) = (\nabla^2(x^2 y), \nabla^2(y^2 z), \nabla^2(z^2 x))) = (2y, 2z, 2x).$$

We now see that

$$\nabla(\nabla.\mathbf{A}) - \nabla \times (\nabla \times (\mathbf{A})) = (2y + 2z,\ 2z + 2x,\ 2x + 2y) - (2z, 2x, 2y) = (2y, 2z, 2x) = \nabla^2(\mathbf{A})$$

as claimed.

Exercise 53. Show that for any vector fields $\mathbf{u} = (p,\ q,\ r)$ and $\mathbf{v} = (f,\ g,\ h)$ we have

$$\nabla.(\mathbf{u} \times \mathbf{v}) = (\nabla \times \mathbf{u}).\mathbf{v} - (\nabla \times \mathbf{v}).\mathbf{u}.$$

Solution:

$$\begin{aligned}
\nabla \times \mathbf{u} &= (r_y - q_z,\ p_z - r_x,\ q_x - p_y) \\
\nabla \times \mathbf{v} &= (h_y - g_z,\ f_z - h_x,\ g_x - f_y) \\
\mathbf{u} \times \mathbf{v} &= (qh - rg,\ rf - ph,\ pg - qf) \\
\nabla.(\mathbf{u} \times \mathbf{v}) &= (qh - rg)_x \ + \ (rf - ph)_y \ + \ (pg - qf)_z \\
\\
&= q_x h + q h_x - r_x g - r g_x + \\
&\quad r_y f + r f_y - p_y h - p h_y + \\
&\quad p_z g + p g_z - q_z f - q f_z \\
&= (r_y - q_z) f + (p_z - r_x) g + (q_x - p_y) h + \\
&\quad p(g_z - h_y) + q(h_x - f_z) + r(f_y - g_x) \\
&= (r_y - q_z,\ p_z - r_x,\ q_x - p_y).(f,\ g,\ h) - \\
&\quad (p,\ q,\ r).(h_y - g_z,\ f_z - h_x,\ g_x - f_y) \\
&= (\nabla \times \mathbf{u}).\mathbf{v} - (\nabla \times \mathbf{v}).\mathbf{u}.
\end{aligned}$$

Week 8 — Polar fields and line integrals

There are formulae for div, grad and curl in polar coordinates on the back of this sheet.

Exercise 54. Let $\mathbf{u}$ be the vector field given in spherical polar coordinates by

$$\mathbf{u} = r^2 \cos(\phi) \mathbf{e}_r + r^{-1} \mathbf{e}_\phi + (r \sin(\phi))^{-1} \mathbf{e}_\theta$$

Find $\operatorname{div}(\mathbf{u})$ and $\operatorname{curl}(\mathbf{u})$.

Solution: The general formulae are

$$\operatorname{div}(m\mathbf{e}_r + p\mathbf{e}_\phi + q\mathbf{e}_\theta) = r^{-2}(r^2 m)_r + (r\sin(\phi))^{-1}(\sin(\phi)p)_\phi + (r\sin(\phi))^{-1}q_\theta$$

$$\operatorname{curl}(m\mathbf{e}_r + p\mathbf{e}_\phi + q\mathbf{e}_\theta) = \frac{1}{r^2\sin(\phi)}\det\begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\phi & r\sin(\phi)\mathbf{e}_\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ m & rp & r\sin(\phi)q \end{bmatrix}.$$

In the present case we have $m = r^2\cos(\phi)$ and $p = r^{-1}$ and $q = (r\ \sin(\phi))^{-1}$. This gives

$$\begin{aligned}
r^{-2}(r^2 m)_r &= r^{-2}(r^4\cos(\phi))_r = r^{-2}\times 4r^3\cos(\phi) = 4r\cos(\phi) \\
(r\sin(\phi))^{-1}(\sin(\phi)p)_\phi &= (r\sin(\phi))^{-1}(r^{-1}\sin(\phi))_\phi = \frac{r^{-1}\cos(\phi)}{r\sin(\phi)} = r^{-2}\cot(\phi) \\
(r\sin(\phi))^{-1}q_\theta &= 0 \\
\operatorname{div}(\mathbf{u}) &= 4r\cos(\phi) + r^{-2}\cot(\phi)
\end{aligned}$$

Next, we have

$$\operatorname{curl}(\mathbf{u}) = \frac{1}{r^2\sin(\phi)}\det\begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\phi & r\sin(\phi)\mathbf{e}_\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ r^2\cos(\phi) & 1 & 1 \end{bmatrix}.$$

The relevant 2×2 determinants are

$$\begin{aligned}
\det\begin{bmatrix} \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ 1 & 1 \end{bmatrix} &= 0 - 0 = 0 \\
\det\begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} \\ r^2\cos(\phi) & 1 \end{bmatrix} &= 0 - 0 = 0 \\
\det\begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} \\ r^2\cos(\phi) & 1 \end{bmatrix} &= 0 - (-r^2\sin(\phi)) = r^2\sin(\phi).
\end{aligned}$$

This gives

$$\operatorname{curl}(\mathbf{u}) = \frac{1}{r^2\sin(\phi)}\left(0\mathbf{e}_r - 0(r\mathbf{e}_\phi) + r^2\sin(\phi)(r\sin(\phi)\mathbf{e}_\theta)\right) = r\ \sin(\phi)\mathbf{e}_\theta.$$

Exercise 55. Let $\mathbf{u}$ be the vector field given in cylindrical polar coordinates by

$$\mathbf{u} = r\cos(\theta)\mathbf{e}_r + r\sin(\theta)\mathbf{e}_\theta + \mathbf{e}_z.$$

Find $\operatorname{div}(\mathbf{u})$ and $\operatorname{curl}(\mathbf{u})$.

Solution: The general formulae are

$$\operatorname{div}(m\mathbf{e}_r + p\mathbf{e}_\theta + q\mathbf{e}_z) = r^{-1}m + m_r + r^{-1}p_\theta + q_z = r^{-1}(rm)_r + r^{-1}p_\theta + q_z$$

$$\operatorname{curl}(m\mathbf{e}_r + p\mathbf{e}_\theta + q\mathbf{e}_z) = \frac{1}{r}\det\begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ m & rp & q \end{bmatrix}.$$

In the present case we have $m = r\cos(\theta)$ and $p = r\sin(\theta)$ and $q = 1$. This gives

$$\begin{aligned}
r^{-1}m &= \cos(\theta) \\
m_r &= \cos(\theta) \\
r^{-1}p_\theta &= \cos(\theta) \\
q_z &= 0 \\
\operatorname{div}(\mathbf{u}) &= 3\cos(\theta)
\end{aligned}$$

Next, we have

$$\operatorname{curl}(\mathbf{u}) = \frac{1}{r}\det\begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ r\cos(\theta) & r^2\sin(\theta) & 1 \end{bmatrix}.$$

The relevant 2×2 determinants are

$$\det \begin{bmatrix} \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ r^2 \sin(\theta) & 1 \end{bmatrix} = 0 - 0 = 0$$

$$\det \begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial z} \\ r\cos(\theta) & 1 \end{bmatrix} = 0 - 0 = 0$$

$$\det \begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} \\ r\cos(\theta) & r^2 \sin(\theta) \end{bmatrix} = 2r\sin(\theta) - (-r\sin(\theta)) = 3r\sin(\theta)$$

so

$$\mathrm{curl}(\mathbf{u}) = \frac{1}{r}\left(0\mathbf{e}_r - 0(r\mathbf{e}_\theta) + 3r\sin(\theta)\mathbf{e}_z\right) = 3\sin(\theta)\mathbf{e}_z.$$

Exercise 56. Consider the vector field $\mathbf{u} = r^{-2}\mathbf{r}$, where $\mathbf{r} = (x, y, z)$ and $r = |\mathbf{r}|$. Show that $\mathrm{curl}(\mathbf{u}) = 0$ and $\mathrm{div}(\mathbf{u}) = r^{-2}$.

Solution: One approach is to use spherical polar coordinates. Recall that $\mathbf{r} = r\,\mathbf{e}_r$, so we can write $\mathbf{u} = r^{-2}r\mathbf{e}_r = r^{-1}\mathbf{e}_r$. The general formulae are

$$\mathrm{div}(m\mathbf{e}_r + p\mathbf{e}_\phi + q\mathbf{e}_\theta) = r^{-2}(r^2 m)_r + (r\sin(\phi))^{-1}(\sin(\phi)p)_\phi + (r\sin(\phi))^{-1}q_\theta$$

$$\mathrm{curl}(m\mathbf{e}_r + p\mathbf{e}_\phi + q\mathbf{e}_\theta) = \frac{1}{r^2\sin(\phi)} \det \begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\phi & r\sin(\phi)\mathbf{e}_\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ m & rp & r\sin(\phi)q \end{bmatrix}.$$

In the present case we have $m = r^{-1}$ and $p = q = 0$, so

$$\mathrm{div}(\mathbf{u}) = r^{-2}(r^2 m)_r = r^{-2}(r)_r = r^{-2}.$$

We also have

$$\mathrm{curl}(\mathbf{u}) = \frac{1}{r^2\sin(\phi)} \det \begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\phi & r\sin(\phi)\mathbf{e}_\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ r^{-1} & 0 & 0 \end{bmatrix} = 0 - 0 + 0 = 0.$$

Alternatively, we could use rectangular coordinates. We have

$$\mathbf{u} = \mathbf{r}/r^2 = \left(\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2}\right)$$

The first term in the divergence is

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right) = \frac{1.(x^2+y^2+z^2) - x.2x}{(x^2+y^2+z^2)^2} = r^{-2} - 2x^2 r^{-4}.$$

Similarly, the second term is $r^{-2} - 2y^2r^{-4}$ and the third term is $r^{-2} - 2z^2r^{-4}$. Adding these together, we get

$$\mathrm{div}(\mathbf{u}) = 3r^{-2} - 2(x^2+y^2+z^2)r^{-4} = 3r^{-2} - 2r^2r^{-4} = r^{-2}.$$

For the curl we note that

$$\frac{\partial}{\partial y}\left(\frac{z}{r^2}\right) = \frac{\partial}{\partial y}\left(\frac{z}{x^2+y^2+z^2}\right) = \frac{-2yz}{(x^2+y^2+z^2)^4} = -2yz/r^4$$

and similarly $\frac{\partial}{\partial x}(z/r^2) = -2xz/r^4$ and so on. This means that

$$\mathrm{curl}(\mathbf{u}) = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x/r^2 & y/r^2 & z/r^2 \end{bmatrix} = (-yz/r^2 + yz/r^2,\ -xz/r^2 + xz/r^2,\ -xy/r^2 + xy/r^2) = (0, 0, 0).$$

Exercise 57. Let r denote $\sqrt{x^2+y^2}$ as in cylindrical polar coordinates. Use the formula for ∇^2 in those coordinates to show that $\nabla^2(r^2+z^2) = 6$.

Solution: The general rule is

$$\nabla^2(f) = r^{-1}f_r + f_{rr} + r^{-2}f_{\theta\theta} + f_{zz}.$$

In the present case we have $f = r^2 + z^2$ so

$$\begin{aligned} f_r &= 2r & f_\theta &= 0 & f_z &= 2z \\ f_{rr} &= 2 & f_{\theta\theta} &= 0 & f_{zz} &= 2 \end{aligned}$$

so

$$\nabla^2(f) = r^{-1}.(2r) + 2 + 0 + 2 = 6.$$

Exercise 58. Evaluate $\int_C x^2\,|d\mathbf{r}|$, where C is the circle of radius a centred at the origin.

Solution: We can parametrise C as $\mathbf{r} = (a\cos(t), a\sin(t))$ (for $0 \leq t \leq 2\pi$). This gives

$$\begin{aligned}
d\mathbf{r} &= (-a\sin(t), a\cos(t))\,dt \\
|d\mathbf{r}| &= \sqrt{(a\sin(t))^2 + (a\cos(t))^2}dt = a\sqrt{\sin^2(t) + \cos^2(t)}\,dt = a\,dt \\
x^2 &= (a\cos(t))^2 = a^2\,\cos^2(t) \\
\int_C x^2\,|d\mathbf{r}| &= \int_{t=0}^{2\pi} a^2\cos^2(t)\,a\,dt = a^3\int_{t=0}^{2\pi}\cos^2(t)\,dt \\
&= a^3\int_{t=0}^{2\pi}(\tfrac{1}{2} + \tfrac{1}{2}\cos(2t))\,dt = a^3\left[\tfrac{1}{2}t + \tfrac{1}{4}\sin(2t)\right]_{t=0}^{2\pi} \\
&= a^3\pi.
\end{aligned}$$

Exercise 59. Consider the vector field $\mathbf{u} = (-z, 0, x)$ and the following three paths from $(0,0,0)$ to $(1,1,1)$.

- C_1 is just a straight line.
- C_2 is given by $(x,y,z) = (t, t^2, t^3)$ for $0 \leq t \leq 1$.
- C_3 is given by $(x,y,z) = (\sin(\theta), 2\theta/\pi, 1-\cos(\theta))$ for $0 \leq \theta \leq \pi/2$.

Write $I_1 = \int_{C_1}\mathbf{u}.d\mathbf{r}$ and $I_2 = \int_{C_2}\mathbf{u}.d\mathbf{r}$ and $I_3 = \int_{C_3}\mathbf{u}.d\mathbf{r}$.

(a) Would you expect I_1, I_2 and I_3 to be the same? Why?
(b) Calculate I_1, I_2 and I_3, and check your answer to (a).

Solution:

(a) For a conservative field $\mathbf{u}$, the integral $\int_C \mathbf{u}.d\mathbf{r}$ depends only on the endpoints of C, not on the precise path. Thus, if our field $\mathbf{u}$ were conservative, we would have $I_1 = I_2 = I_3$. To see whether this is the case, we must calculate the curl of $\mathbf{u}$:

$$\nabla\times\mathbf{u} = \det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z & 0 & x\end{bmatrix} = (0-0,\ -1-1,\ 0-0) = -2\mathbf{j} \neq 0.$$

As this is nonzero, we see that $\mathbf{u}$ is not conservative, so there is no good reason for I_1, I_2 and I_3 to be the same. (They might still be the same by coincidence.)

(b) The curve C_1 can be parametrised as $\mathbf{r} = (x,y,z) = (t,t,t)$ for $0 \leq t \leq 1$. This gives $d\mathbf{r} = (1,1,1)dt$. Also, on C_1 we have $\mathbf{u} = (-z,0,x)$ but $x = z = t$ so $\mathbf{u} = (-t,0,t)$. We thus have

$$I_1 = \int_{C_1}\mathbf{u}.d\mathbf{r} = \int_{t=0}^{1}(-t,0,t).(1,1,1)dt = \int_{t=0}^{1}0\,dt = 0.$$

Next, for C_2 we have $\mathbf{r} = (x,y,z) = (t, t^2, t^3)$ so

$$\begin{aligned}
\mathbf{u} &= (-z,0,x) = (-t^3, 0, t) \\
d\mathbf{r} &= \dot{\mathbf{r}}\,dt = (1, 2t, 3t^2)\,dt \\
\mathbf{u}.d\mathbf{r} &= (-t^3\times 1 + t\times 3t^2)dt = 2t^3\,dt \\
I_2 &= \int_{C_2}\mathbf{u}.d\mathbf{r} = \int_{t=0}^{1}2t^3\,dt = \left[\frac{2t^4}{4}\right]_{t=0}^{1} = \tfrac{1}{2}.
\end{aligned}$$

Finally, for C_3 we have $\mathbf{r} = (x, y, z) = (\sin(\theta), 2\theta/\pi, 1 - \cos(\theta))$ so

$$\begin{aligned}
\mathbf{u} &= (-z, 0, x) = (\cos(\theta) - 1, 0, \sin(\theta)) \\
d\mathbf{r} &= \frac{d\mathbf{r}}{d\theta}\, d\theta = (\cos(\theta), 2/\pi, \sin(\theta))\, d\theta \\
\mathbf{u}.d\mathbf{r} &= (\cos^2(\theta) - \cos(\theta) + \sin^2(\theta))d\theta = (1 - \cos(\theta))d\theta \\
I_3 &= \int_{C_3} \mathbf{u}.d\mathbf{r} = \int_{\theta=0}^{\pi/2} (1 - \cos(\theta))d\theta = \Big[\theta - \sin(\theta)\Big]_{\theta=0}^{\pi/2} \\
&= (\pi/2 - 1) - (0 - 0) = \pi/2 - 1.
\end{aligned}$$

Exercise 60. Let C be the curve given by

$$\mathbf{r} = (2^t \cos(10\pi t^2), 2^t \sin(10\pi t^2), 2\pi)$$

for $0 \leq t \leq 1$, and let $\mathbf{u}$ be the vector field

$$\mathbf{u} = (e^x \cos(y)\cos(z),\ -e^x \sin(y)\cos(z),\ -e^x \cos(y)\sin(z)).$$

Calculate $\int_C \mathbf{u}.d\mathbf{r}$. Think carefully about the most efficient method before launching into calculations.

Solution: First, we check whether $\mathbf{u}$ is conservative. We have

$$\begin{aligned}
\operatorname{curl}(\mathbf{u}) = &\det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \cos(y)\cos(z) & -e^x \sin(y)\cos(z) & -e^x \cos(y)\sin(z) \end{bmatrix} \\
=&(e^x \sin(y)\sin(z) - e^x \sin(y)\sin(z), \\
&- e^x \cos(y)\sin(z) + e^x \cos(y)\sin(z), \\
&- e^x \sin(y)\cos(z) + e^x \sin(y)\cos(z)) \\
=&(0, 0, 0),
\end{aligned}$$

so $\mathbf{u}$ is indeed conservative. This means that we can replace C by any other curve with the same endpoints, and we will get the same integral. At $t = 0$ we have $\mathbf{r} = (2^0 \cos(0), 2^0 \sin(0), 2\pi) = (1, 0, 2\pi)$. At $t = 1$ we have $\mathbf{r} = (2^1 \cos(10\pi), 2^1 \sin(10\pi), 2\pi) = (2, 0, 2\pi)$. The obvious replacement curve is just the straight line L given by $\mathbf{r} = (x, y, z) = (1 + t, 0, 2\pi)$ for $0 \leq t \leq 1$. For this curve we have

$$\begin{aligned}
\mathbf{u} &= (e^{1+t} \cos(0)\cos(2\pi),\ -e^{1+t} \sin(0)\cos(2\pi),\ -e^{1+t}\cos(0)\sin(2\pi)) = (e^{1+t}, 0, 0) \\
d\mathbf{r} &= (1, 0, 0) \\
\mathbf{u}.d\mathbf{r} &= e^{1+t} \\
\int_L \mathbf{u}.d\mathbf{r} &= \int_{t=0}^{1} e^{1+t}\, dt = \Big[e^{1+t}\Big]_{t=0}^{1} = e^2 - e.
\end{aligned}$$

We conclude that $\int_C \mathbf{u}.d\mathbf{r}$ is also $e^2 - e$.

As an alternative approach, after we checked that $\mathbf{u}$ is conservative, we could look for a potential function f, which must satisfy $\operatorname{grad}(f) = \mathbf{u}$ or equivalently

(A) $$f_x = e^x \cos(y)\cos(z)$$

(B) $$f_y = -e^x \sin(y)\cos(z)$$

(C) $$f_z = -e^x \cos(y)\sin(z).$$

Integrating (A) with respect to x gives $f = e^x \cos(y)\cos(z) + g$, where g depends only on y and z. Substituting this into (B) gives

$$-e^x \sin(y)\cos(z) + g_y = -e^x \sin(y)\cos(z),$$

so $g_y = 0$. Similarly, we can substitute $f = e^x \cos(y)\cos(z) + g$ in (C) to get

$$-e^x \cos(y)\sin(z) + g_z = -e^x \cos(y)\sin(z),$$

so $g_z = 0$. As g depends only on y and z but $g_y = g_z = 0$ we see that g is constant, so we can take it to be zero, so $f = e^x \cos(y)\cos(z)$. We now have

$$\int_C \mathbf{u}.d\mathbf{r} = \int_C \operatorname{grad}(f).d\mathbf{r} = f(2, 0, 2\pi) - f(1, 0, 2\pi) = e^2 - e,$$

just as before.

Two-dimensional polar coordinates

(a) For any two-dimensional scalar field f (expressed as a function of r and θ) we have

$$\nabla(f) = \operatorname{grad}(f) = f_r\,\mathbf{e}_r + r^{-1} f_\theta\,\mathbf{e}_\theta.$$

(b) For any 2-dimensional vector field $\mathbf{u} = m\,\mathbf{e}_r + p\,\mathbf{e}_\theta$ (where m and p are expressed as functions of r and θ) we have

$$\begin{aligned} \operatorname{div}(\mathbf{u}) &= r^{-1}m + m_r + r^{-1}p_\theta \\ \operatorname{curl}(\mathbf{u}) &= r^{-1}p + p_r - r^{-1}m_\theta. \end{aligned}$$

Note that the product rule gives $(rm)_r = m + r\,m_r$ and $(rp)_r = p + r\,p_r$. Using this, we can rewrite the above equations as

$$\begin{aligned} \operatorname{div}(\mathbf{u}) &= r^{-1}\left((rm)_r + p_\theta\right) \\ \operatorname{curl}(\mathbf{u}) &= r^{-1}\left((rp)_r - m_\theta\right) = \frac{1}{r}\det\begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} \\ m & rp \end{bmatrix}. \end{aligned}$$

(c) For any two-dimensional scalar field f we have

$$\nabla^2(f) = r^{-1}f_r + f_{rr} + r^{-2}f_{\theta\theta} = r^{-1}(rf_r)_r + r^{-2}f_{\theta\theta}$$

Cylindrical polar coordinates

(a) For any three-dimensional scalar field f (expressed as a function of r, θ and z) we have

$$\nabla(f) = \operatorname{grad}(f) = f_r\,\mathbf{e}_r + r^{-1}f_\theta\,\mathbf{e}_\theta + f_z\mathbf{e}_z.$$

(b) For any three-dimensional vector field $\mathbf{u} = m\,\mathbf{e}_r + p\,\mathbf{e}_\theta + q\,e_z$ (where m, p and q are expressed as functions of r, θ and z) we have

$$\begin{aligned} \operatorname{div}(\mathbf{u}) &= r^{-1}m + m_r + r^{-1}p_\theta + q_z = r^{-1}(rm)_r + r^{-1}p_\theta + q_z \\ \operatorname{curl}(\mathbf{u}) &= \frac{1}{r}\det\begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ m & rp & q \end{bmatrix}. \end{aligned}$$

(c) For any three-dimensional scalar field f we have

$$\nabla^2(f) = r^{-1}f_r + f_{rr} + r^{-2}f_{\theta\theta} + f_{zz} = r^{-1}(rf_r)_r + r^{-2}f_{\theta\theta} + f_{zz}.$$

Spherical polar coordinates

(a) For any three-dimensional scalar field f (expressed as a function of r, ϕ and θ) we have

$$\nabla(f) = \operatorname{grad}(f) = f_r\,\mathbf{e}_r + r^{-1}f_\phi\,\mathbf{e}_\phi + (r\,\sin(\phi))^{-1}f_\theta\mathbf{e}_\theta.$$

(b) For any three-dimensional vector field $\mathbf{u} = m\,\mathbf{e}_r + p\,\mathbf{e}_\phi + q\,e_\theta$ (where m, p and q are expressed as functions of r, ϕ and θ) we have

$$\begin{aligned} \operatorname{div}(\mathbf{u}) &= r^{-2}(r^2m)_r + (r\sin(\phi))^{-1}(\sin(\phi)p)_\phi + (r\sin(\phi))^{-1}q_\theta \\ \operatorname{curl}(\mathbf{u}) &= \frac{1}{r^2\sin(\phi)}\det\begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\phi & r\sin(\phi)\mathbf{e}_\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ m & rp & r\sin(\phi)q \end{bmatrix}. \end{aligned}$$

(c) For any three-dimensional scalar field f we have

$$\nabla^2(f) = r^{-2}(r^2f_r)_r + (r^2\sin(\phi))^{-1}(\sin(\phi)f_\phi)_\phi + (r^2\sin^2(\phi))^{-1}f_{\theta\theta}.$$

Exercise 61. Let D be the disc of radius a centred at $(0,0)$, and let $\mathbf{u}$ be the vector field $(xy^2,0)$. Let C be the boundary curve of D. Verify the divergence theorem $\iint_D \operatorname{div}(\mathbf{u})\,dA = \int_C \mathbf{u}.d\mathbf{n}$ in this case.

Solution: First, we have

$$\operatorname{div}(\mathbf{u}) = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(0) = y^2.$$

We will evaluate $\iint_D \operatorname{div}(\mathbf{u})\,dA$ using polar coordinates. This means that $\operatorname{div}(\mathbf{u}) = y^2$ becomes $r^2\sin^2(\theta)$, whereas $dA = r\,dr\,d\theta$. The limits for D are $0 \le r \le a$ and $0 \le \theta \le 2\pi$. This gives

$$\begin{aligned}\iint_D \operatorname{div}(\mathbf{u})\,dA &= \int_{\theta=0}^{2\pi}\int_{r=0}^{a} r^2\sin^2(\theta)\,r\,dr\,d\theta = \left(\int_{\theta=0}^{2\pi}\sin^2(\theta)\,d\theta\right)\left(\int_{r=0}^{a} r^3\,dr\right)\\ &= \left[\frac{2\theta-\sin(2\theta)}{4}\right]_{\theta=0}^{2\pi}\left[\frac{r^4}{4}\right]_{r=0}^{a} = \frac{\pi a^4}{4}.\end{aligned}$$

On the other hand, the boundary curve C is just the circle of radius a centred at the origin, so it can be parametrised as $(x,y) = (a\cos(\theta), a\sin(\theta))$ with $0 \le \theta \le 2\pi$. Along C we therefore have

$$\mathbf{u} = (xy^2, 0) = (a^3\cos(\theta)\sin^2(\theta), 0).$$

We can also differentiate the formula $(x,y) = (a\cos(\theta),\ a\sin(\theta))$ to get $d\mathbf{r} = (dx, dy) = (-a\sin(\theta),\ a\cos(\theta))\,d\theta$ and so

$$d\mathbf{n} = (dy, -dx) = (a\cos(\theta),\ a\sin(\theta)))\,d\theta.$$

This gives $\mathbf{u}.d\mathbf{n} = a^4\cos^2(\theta)\sin^2(\theta)d\theta$. To simplify this, we recall that $\sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta)$, so

$$\cos^2(\theta)\sin^2(\theta) = \tfrac{1}{4}\sin^2(2\theta) = \frac{1}{4}\,\frac{1-\cos(4\theta)}{2} = \frac{1-\cos(4\theta)}{8}.$$

We now have

$$\int_C \mathbf{u}.d\mathbf{n} = \frac{a^4}{8}\int_{\theta=0}^{2\pi} 1-\cos(4\theta)d\theta = \frac{a^4}{8}\Big[\theta - \sin(4\theta)/4\Big]_{\theta=0}^{2\pi} = \frac{a^4}{8}\times 2\pi = \frac{\pi a^4}{4}.$$

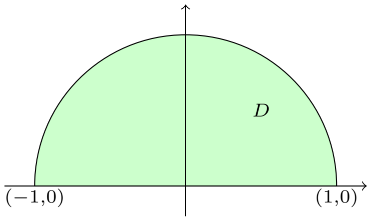
As expected, this is the same as $\iint_D \operatorname{div}(\mathbf{u})\,dA$.

Exercise 62. Show that $\cos^2(\theta)\sin(\theta) = (\sin(3\theta)+\sin(\theta))/4$.

Solution: One approach is to use complex exponentials:

$$\begin{aligned}\cos(\theta) &= (e^{j\theta}+e^{-j\theta})/2\\ \sin(\theta) &= (e^{j\theta}-e^{-j\theta})/(2j)\\ \cos^2(\theta)\sin(\theta) &= \frac{1}{8j}(e^{j\theta}+e^{-j\theta})^2(e^{j\theta}-e^{-j\theta}) = \frac{1}{8j}(e^{2j\theta}+2+e^{-2j\theta})(e^{j\theta}-e^{-j\theta})\\ &= \frac{1}{8j}(e^{3j\theta}+2e^{j\theta}+e^{-j\theta}-e^{j\theta}-2e^{-2j\theta}-e^{-3j\theta})\\ &= \frac{1}{8j}(e^{3j\theta}-e^{-3j\theta}+e^{j\theta}-e^{-j\theta}) = \frac{1}{4}\left(\frac{e^{3j\theta}-e^{-3j\theta}}{2j}+\frac{e^{j\theta}-e^{-j\theta}}{2j}\right)\\ &= \frac{1}{4}(\sin(3\theta)+\sin(\theta)).\end{aligned}$$

Exercise 63. Consider the region D as shown, and the vector field $\mathbf{u} = (0, x^4+x^2y^2-x^2)$.



Check the divergence theorem in this case. (The previous exercise will be helpful.)

Solution: First, we have

$$\operatorname{div}(\mathbf{u}) = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(x^4 + x^2y^2 - x^2) = 2x^2y.$$

We will evaluate $\iint_D \operatorname{div}(\mathbf{u})\, dA$ using polar coordinates. This means that $\operatorname{div}(\mathbf{u}) = 2x^2y$ becomes $2r^3\cos^2(\theta)\sin(\theta)$, whereas $dA = r\,dr\,d\theta$. The limits for D are $0 \le r \le 1$ and $0 \le \theta \le \pi$ (not 2π, because D lies above the x-axis). This gives

$$\iint_D \operatorname{div}(\mathbf{u})\, dA = \int_{\theta=0}^{\pi}\int_{r=0}^{1} 2r^3\cos^2(\theta)\sin(\theta)\, r\, dr\, d\theta = 2\left(\int_{r=0}^{1} r^4\, dr\right)\left(\int_{\theta=0}^{2\pi} \cos^2(\theta)\sin(\theta)\, d\theta\right)$$

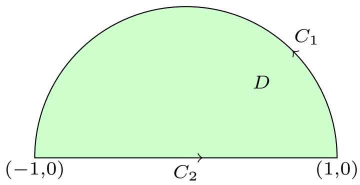
The first integral is just $\left[r^5/5\right]_{r=0}^{1} = 1/5$. The second can be evaluated using the previous question:

$$\begin{aligned}\int_{\theta=0}^{\pi} \cos^2(\theta)\sin(\theta)\, d\theta &= \frac{1}{4}\int_{\theta=0}^{\pi} \sin(3\theta) + \sin(\theta)\, d\theta \\ &= \frac{1}{4}\Big[-\cos(3\theta)/3 - \cos(\theta)\Big]_{\theta=0}^{\pi} = \frac{1}{4}\left((1/3+1) - (-1/3-1)\right) \\ &= 2/3.\end{aligned}$$

Putting this together, we get

$$\iint_D \operatorname{div}(\mathbf{u})\, dA = 2 \times \frac{1}{5} \times \frac{2}{3} = \frac{4}{15}.$$

On the other hand, the boundary curve can be divided into a circular section C_1 and a straight line C_2:



We can parametrise C_1 as $(x,y) = (\cos(t), \sin(t))$ for $0 \le t \le \pi$. On C_1 we then have

$$\mathbf{u} = (0, x^4 + x^2y^2 - x^2) = (0, \cos^4(t) + \cos^2(t)\sin^2(t) - \cos^2(t)).$$

If we rewrite $\sin^2(t)$ as $1 - \cos^2(t)$ then everything cancels out and we get $\mathbf{u} = (0,0)$ on C_1, so $\int_{C_1} \mathbf{u}.d\mathbf{n} = 0$. On the other hand, we can parametrise C_2 as $(x,y) = (t,0)$ for $-1 \le t \le 1$. On C_2 we then have

$$\mathbf{u} = (0, x^4 + x^2y^2 - x^2) = (0, t^4 - t^2).$$

We also have $dx = dt$ and $dy = 0$ so $d\mathbf{n} = (dy, -dx) = (0, -dt)$. This gives

$$\int_{C_1} \mathbf{u}.d\mathbf{n} = \int_{t=-1}^{1} (0, t^4 - t^2).(0, -dt) = \int_{t=-1}^{1} t^2 - t^4\, dt = \left[\frac{t^3}{3} - \frac{t^5}{5}\right]_{t=-1}^{1} = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}.$$

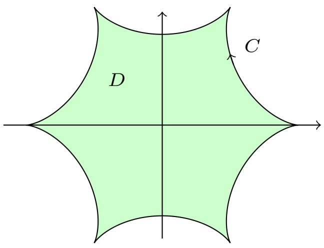
Putting this together, we get

$$\int_C \mathbf{u}.d\mathbf{n} = \int_{C_1} \mathbf{u}.d\mathbf{n} + \int_{C_2} \mathbf{u}.d\mathbf{n} = 0 + 4/15 = 4/15,$$

which is the same as $\iint_D \operatorname{div}(\mathbf{u})\, dA$, as expected.

Exercise 64. The following picture shows a hypocycloid curve C, which can be parametrised as

$$(x,y) = (5\cos(t) + \cos(5t),\ 5\sin(t) - \sin(5t)).$$



Use the divergence theorem to find the area of the region D enclosed by C. It may help to recall the identity

$$\cos(\alpha)\cos(\beta) = \tfrac{1}{2}\cos(\beta+\alpha) + \tfrac{1}{2}\cos(\beta-\alpha).$$

Solution: We use same the method as was used for the deltoid in the notes and lectures. We put $\mathbf{u} = (x, 0)$, so $\operatorname{div}(\mathbf{u}) = 1$ and $\iint_D \operatorname{div}(\mathbf{u})\, dA = \operatorname{area}(D)$. On the other hand, the divergence theorem tells us that this is the same as $\int_C \mathbf{u}.d\mathbf{n}$. Using the given parametrisation of C we have

$$d\mathbf{n} = (dy, -dx) = (5\cos(t) - 5\cos(5t),\ -5\sin(t) - 5\sin(5t))\ dt.$$

Moreover, on C we have $\mathbf{u} = (x, 0) = (5\cos(t) + \cos(5t), 0)$. This gives

$$\begin{aligned} \mathbf{u}.d\mathbf{n} &= (5\cos(t) + \cos(5t))(5\cos(t) - 5\cos(5t)) \\ &= 25\cos^2(t) - 20\cos(t)\cos(5t) - 5\cos^2(5t). \end{aligned}$$

We now use the identities

$$\begin{aligned} \cos^2(t) &= \tfrac{1}{2}(1 + \cos(2t)) \\ \cos^2(5t) &= \tfrac{1}{2}(1 + \cos(10t)) \\ \cos(t)\cos(5t) &= \tfrac{1}{2}(\cos(6t) + \cos(4t)) \end{aligned}$$

to get

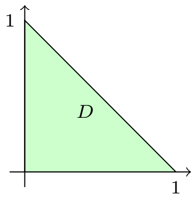
$$\mathbf{u}.d\mathbf{n} = \frac{25}{2} + \frac{25}{2}\cos(2t) - 10\cos(6t) - 10\cos(4t) - \frac{5}{2} - \frac{5}{2}\cos(10t).$$

We now integrate this from $t = 0$ to $t = 2\pi$. It is standard that $\int_{t=0}^{2\pi} \cos(kt)\, dt = 0$ for any $k > 0$, so most of the terms do not contribute anything, and we are left with

$$\int_C \mathbf{u}.d\mathbf{n} = \int_{t=0}^{2\pi} \frac{25}{2} - \frac{5}{2}\, dt = \int_{t=0}^{2\pi} 10\, dt = 20\pi.$$

Thus, the area of D is 20π.

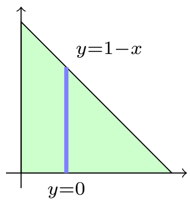
Exercise 65. Consider the following region D, and the vector field $\mathbf{u} = (-y^2, x^2)$. Verify Green's theorem for this case.



Solution: First, we have

$$\operatorname{curl}(\mathbf{u}) = \det \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ -y^2 & x^2 \end{bmatrix} = \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2) = 2x + 2y.$$

We now need to evaluate $\iint_D \operatorname{curl}(\mathbf{u})\, dA$. The overall limits of x in the region are $0 \leq x \leq 1$. For a fixed value of x (corresponding to a vertical strip as shown), the limits are from $y = 0$ (at the bottom) to $y = 1 - x$ (at the top).



We thus have

$$\iint_D \operatorname{curl}(\mathbf{u})\, dA = \int_{x=0}^{1} \int_{y=0}^{1-x} 2x + 2y\, dy\, dx.$$

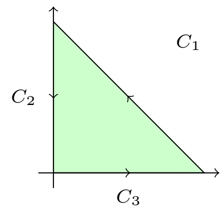
The inner integral is

$$\int_{y=0}^{1-x} 2x + 2y\, dy = \left[2xy + y^2\right]_{y=0}^{1-x} = 2x(1-x) + (1-x)^2 = 2x - 2x^2 + 1 - 2x + x^2 = 1 - x^2.$$

The outer integral is therefore

$$\iint_D \operatorname{curl}(\mathbf{u})\, dA = \int_{x=0}^{1} 1 - x^2\, dx = \left[x - x^3/3\right]_{x=0}^{1} = 2/3.$$

Green's theorem tells us that this is the same as $\int_C \mathbf{u}.d\mathbf{r}$, where C is the boundary curve of C. This can be broken into three pieces as follows:



We can parametrise C_1 as $(x, y) = (1 - t, t)$ for $0 \leq t \leq 1$. This gives $d\mathbf{r} = (dx, dy) = (-dt, dt)$ and $\mathbf{u} = (-y^2, x^2) = (-t^2, t^2)$, so $\mathbf{u}.d\mathbf{r} = 2t^2\, dt$. It follows that

$$\int_{C_1} \mathbf{u}.d\mathbf{r} = \int_{t=0}^{1} 2t^2\, dt = \left[\frac{2}{3}t^3\right]_{t=0}^{1} = \frac{2}{3}.$$

Next, we can parametrise C_2 as $(x, y) = (0, 1 - t)$ so $\mathbf{u} = (-y^2, x^2 = (-(1-t)^2, 0)$ and $d\mathbf{r} = (dx, dy) = (0, -dt)$. This gives $\mathbf{u}.d\mathbf{r} = 0$ and so $\int_{C_2} \mathbf{u}.d\mathbf{r} = 0$. Similarly, we can parametrise C_3 as $(x, y) = (t, 0)$ giving $\mathbf{u} = (0, t^2)$ and $d\mathbf{r} = (dt, 0)$ so $\mathbf{u}.d\mathbf{r} = 0$ and $\int_{C_3} \mathbf{u}.d\mathbf{r} = 0$. Putting this together, we get

$$\int_C \mathbf{u}.d\mathbf{r} = \int_{C_1} \mathbf{u}.d\mathbf{r} + \int_{C_2} \mathbf{u}.d\mathbf{r} + \int_{C_3} \mathbf{u}.d\mathbf{r} = 2/3 + 0 + 0 = 2/3.$$

As expected, this is the same as $\iint_D \operatorname{curl}(\mathbf{u})\, dA$.

Exercise 66. Let f be any well-behaved function of two variables, and let C be the curve where $f(x, y) = 1$. Suppose that this is a finite, closed curve like a circle, not branching or extending to infinity. Explain three different reasons why $\int_C \operatorname{grad}(f).d\mathbf{r} = 0$.

Solution: The most basic reason is just the fundamental theorem of calculus for path integrals, which says that the integral along C of $\operatorname{grad}(f).d\mathbf{r}$ is the change in f from the beginning of C to the end. Here we are assuming that C is a closed curve like a circle, so the end is the same as the beginning, so there is no change in f and $\int_C \operatorname{grad}(f).d\mathbf{r} = 0$. Note that we do not even need the fact that C is given by $f(x, y) = 1$ here.

Next, we can recall that $\operatorname{grad}(f)$ points in the direction of fastest possible increase of f, and is perpendicular to the curves where f is constant. As we move around C the vector $d\mathbf{r}$ will point along C so $\operatorname{grad}(f)$ will be perpendicular to $d\mathbf{r}$ and $\operatorname{grad}(f).d\mathbf{r} = 0$, so $\int_C \operatorname{grad}(f).d\mathbf{r} = 0$ again.

Finally, we can use Green's theorem, which says that $\int_C \operatorname{grad}(f).d\mathbf{r} = \iint_D \operatorname{curl}(\operatorname{grad}(f))\, dA$, where D is the region enclosed by C. However, we have the standard identity

$$\operatorname{curl}(\operatorname{grad}(f)) = \operatorname{curl}(f_x, f_y) = f_{yx} - f_{xy} = 0,$$

so this integral over D is again zero.

Week 10 — Surface integrals and the divergence theorem

Exercise 67. Show that

$$\begin{aligned} \sin(\alpha)\cos^2(\alpha) &= \tfrac{1}{4}(\sin(3\alpha) + \sin(\alpha)) \\ \cos^3(\alpha) &= \tfrac{1}{4}\cos(3\alpha) + \tfrac{3}{4}\cos(\alpha). \end{aligned}$$

(You will need these identities in the questions below.)

Solution: We just expand everything out using the identities $\cos(\alpha) = (e^{j\alpha} + e^{-j\alpha})/2$ and $\sin(\alpha) = (e^{j\alpha} - e^{-j\alpha})/(2j)$. We get

$$\begin{aligned}
\sin(\alpha)\cos^2(\alpha) &= \tfrac{1}{8j}(e^{j\alpha} - e^{-j\alpha})(e^{j\alpha} + e^{-j\alpha})^2 \\
&= \tfrac{1}{8j}(e^{j\alpha} - e^{-j\alpha})(e^{2j\alpha} + 2 + e^{-2j\alpha}) \\
&= \tfrac{1}{8j}(e^{3j\alpha} + 2e^{j\alpha} + e^{-j\alpha} - e^{j\alpha} - 2e^{-j\alpha} - e^{-3j\alpha}) \\
&= \tfrac{1}{8j}(e^{3j\alpha} - e^{-3j\alpha} + e^{j\alpha} - e^{-j\alpha}) \\
&= \tfrac{1}{4}\left(\frac{e^{3j\alpha} - e^{-3j\alpha}}{2j} + \frac{e^{j\alpha} - e^{-j\alpha}}{2j}\right) \\
&= \tfrac{1}{4}(\sin(3\alpha) + \sin(\alpha)) \\
\cos^3(\alpha) &= \tfrac{1}{8}(e^{j\alpha} + e^{-j\alpha})^3 \\
&= \tfrac{1}{8}(e^{3j\alpha} + 3e^{2j\alpha}e^{-j\alpha} + 3e^{j\alpha}e^{-2j\alpha} + e^{-3j\alpha}) \\
&= \tfrac{1}{8}(e^{3j\alpha} + e^{-3j\alpha} + 3e^{j\alpha} + 3e^{-j\alpha}) \\
&= \tfrac{1}{4}\left(\frac{e^{3j\alpha} + e^{-3j\alpha}}{2} + \frac{e^{j\alpha} + e^{-j\alpha}}{2}\right)
\end{aligned}$$

Exercise 68. Evaluate $\iint_S \mathbf{F}.d\mathbf{A}$, where $\mathbf{F} = x^2\mathbf{i} + y^3\mathbf{j} + z^4\mathbf{k}$ and S is the surface of the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$.

Solution: We need to consider the six faces of the cube separately. Let S_1 be the face where $x = 0$, and let S_2 be the face where $x = 1$; in both cases y and z vary from 0 to 1. On S_1 the outward unit normal vector $\mathbf{n}$ is $-\mathbf{i}$, but on S_2 we have $\mathbf{n} = +\mathbf{i}$. On S_1 we therefore have

$$\mathbf{F}.\mathbf{n} = -\mathbf{i}.(x^2\mathbf{i} + y^3\mathbf{j} + z^4\mathbf{k}) = -x^2,$$

but $x = 0$ on S_1 so $\mathbf{F}.\mathbf{n} = 0$, so

$$\int_{S_1} \mathbf{F}.d\mathbf{A} = \int_{S_1} \mathbf{F}.\mathbf{n}dA = \int_{S_1} 0dA = 0.$$

Similarly, on S_2 we have $x = 1$ and $\mathbf{n} = \mathbf{i}$ so $\mathbf{F}.d\mathbf{A} = \mathbf{F}.\mathbf{n}dA = x^2\,dA = dA$, which means that

$$\int_{S_2} \mathbf{F}.d\mathbf{A} = \int_{S_2} dA = \text{ area of } S_2 = 1.$$

The other faces work in essentially the same way. We have the following table:

face	equation	$\mathbf{n}$	$\mathbf{F}.\mathbf{n}$	$\int \mathbf{F}.d\mathbf{A}$
S_1	$x = 0$	$-\mathbf{i}$	0	0
S_2	$x = 1$	$\mathbf{i}$	1	1
S_3	$y = 0$	$-\mathbf{j}$	0	0
S_4	$y = 1$	$\mathbf{j}$	1	1
S_5	$z = 0$	$-\mathbf{k}$	0	0
S_6	$z = 1$	$\mathbf{k}$	1	1

By adding the numbers in the last column we get $\int_S \mathbf{F}.d\mathbf{A} = 3$.

Exercise 69. Evaluate $\iint_S z^2\,dA$, where S is the hemispherical shell given by $x^2 + y^2 + z^2 = a^2$ with $z \geq 0$.

Solution: This surface is given in terms of the spherical coordinates ϕ and θ by

$$\mathbf{r} = (x, y, z) = (a\sin(\phi)\cos(\theta),\ a\sin(\phi)\sin(\theta),\ a\cos(\phi))$$

with $0 \leq \phi \leq \pi/2$ (because $\phi = \pi/2$ on the xy-plane) and $0 \leq \theta \leq 2\pi$. This gives

$$\begin{aligned}
d\mathbf{A} =& \mathbf{r}_\phi \times \mathbf{r}_\theta \, d\phi \, d\theta \\
=& (a\cos(\phi)\cos(\theta),\ a\cos(\phi)\sin(\theta),\ -a\sin(\phi)) \times \\
& (-a\sin(\phi)\sin(\theta),\ a\sin(\phi)\cos(\theta),\ 0) \, d\phi \, d\theta \\
=& (a^2\sin^2(\phi)\cos(\theta), a^2\sin^2(\phi)\sin(\theta), a^2\sin(\phi)\cos(\phi)(\cos^2(\theta)+\sin^2(\theta))) \, d\phi \, d\theta \\
=& (a^2\sin^2(\phi)\cos(\theta), a^2\sin^2(\phi)\sin(\theta), a^2\sin(\phi)\cos(\phi)) \, d\phi \, d\theta \\
=& a^2\sin(\phi)\ (\sin(\phi)\cos(\theta),\ \sin(\phi)\sin(\theta),\ \cos(\phi)) \, d\phi \, d\theta \\
=& a^2\sin(\phi)\mathbf{e}_r \, d\phi \, d\theta \\
dA =& |d\mathbf{A}| = a^2\sin(\phi) \, d\phi \, d\theta \\
z^2 \, dA =& (a\cos(\phi))^2 \ a^2\sin(\phi) \, d\phi \, d\theta = a^4 \ \cos^2(\phi)\sin(\phi) \, d\phi \, d\theta.
\end{aligned}$$

Using Exercise 1, this can be rewritten as

$$\begin{aligned}
z^2 \, dA =& \frac{a^4}{4}(\sin(3\phi)+\sin(\phi)) \, d\phi \, d\theta \\
\int_S z^2 \, dA =& \frac{a^4}{4}\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{2\pi} \sin(3\phi)+\sin(\phi) \, d\theta \, d\phi \\
=& \frac{a^4\pi}{2}\int_{\phi=0}^{\pi/2} \sin(3\phi)+\sin(\phi) \, d\phi \\
=& \frac{a^4\pi}{2}\left[-\tfrac{1}{3}\cos(3\phi) - \cos(\phi) \right]_{\phi=0}^{\pi/2} \\
=& \frac{a^4\pi}{2}(0-(-4/3)) = 2\pi a^4/3.
\end{aligned}$$

Exercise 70. Let S be the hemispherical shell given by $x^2+y^2+z^2 = 1$ with $x \geq 0$ (not $z \geq 0$). Evaluate $\iint_S \mathbf{F}.d\mathbf{A}$, where $\mathbf{F} = (1,-z,y)$.

Solution: This surface is given in terms of the spherical coordinates ϕ and θ by

$$\mathbf{r} = (x,y,z) = (\sin(\phi)\cos(\theta),\ \sin(\phi)\sin(\theta),\ \cos(\phi))$$

with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ (to ensure that $x \geq 0$) and $0 \leq \phi \leq \pi$. We can substitute these values for x, y and z into the equation $\mathbf{F} = (1,\ -z,\ y)$ to get

$$\mathbf{F} = (1,\ -\cos(\phi),\ \sin(\phi)\sin(\theta)).$$

Moreover, just as in the previous exercise (with $a = 1$) we have

$$d\mathbf{A} = \sin(\phi)\mathbf{e}_r = (\sin^2(\phi)\cos(\theta),\ \sin^2(\phi)\sin(\theta),\ \sin(\phi)\cos(\phi)) \, d\phi \, d\theta,$$

so

$$\mathbf{F}.d\mathbf{A} = \big(\sin^2(\phi)\cos(\theta) + (-\cos(\phi))\sin^2(\phi)\sin(\theta) + \sin(\phi)\sin(\theta)\sin(\phi)\cos(\phi)\big) \ d\phi \, d\theta.$$

Here the last two terms cancel, leaving

$$\mathbf{F}.d\mathbf{A} = \sin^2(\phi)\cos(\theta) \, d\phi \, d\theta.$$

Integrating this gives

$$\begin{aligned}
\iint_S \mathbf{F}.d\mathbf{A} &= \int_{\theta=-\pi/2}^{\pi/2}\int_{\phi=0}^{\pi} \sin^2(\phi)\cos(\theta) \, d\phi \, d\theta = \int_{\theta=-\pi/2}^{\pi/2} \cos(\theta) \, d\theta \int_{\phi=0}^{\pi} \sin^2(\phi) \, d\phi \\
&= \left[\sin(\theta) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\tfrac{1}{2}\phi - \tfrac{1}{4}\cos(2\phi) \right]_{\phi=0}^{\pi} \\
&= (1-(-1))((\tfrac{\pi}{2}-\tfrac{1}{4}) - (0-\tfrac{1}{4})) = \pi.
\end{aligned}$$

Exercise 71.

(a) Let S be the cylindrical surface given parametrically by $x = a\cos(\theta)$ and $y = a\sin(\theta)$ with $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq b$. Evaluate $\iint_S \mathbf{F}.d\mathbf{A}$, where $\mathbf{F} = (x^2, y, 0)$.

(b) Now let E be the solid region bounded by S together with the planes $z = 0$ and $z = b$. Evaluate $\iiint_E \mathrm{div}(\mathbf{F}) dV$.

Solution:

(a) For this surface we have

$$\begin{aligned}
\mathbf{r} &= (a\cos(\theta),\ a\sin(\theta),\ z) \\
\mathbf{r}_\theta &= (-a\sin(\theta),\ a\cos(\theta),\ 0) \\
\mathbf{r}_z &= (0,0,1) \\
d\mathbf{A} &= (\mathbf{r}_\theta \times \mathbf{r}_z) d\theta\, dz = (a\cos(\theta),\ a\sin(\theta),\ 0) d\theta\, dz \\
\mathbf{F} &= (x^2, y, 0) = (a^2\cos^2(\theta),\ a\sin(\theta),\ 0) \\
\mathbf{F}.d\mathbf{A} &= (a^3\cos^3(\theta) + a^2\sin^2(\theta)) d\theta\, dz.
\end{aligned}$$

This can be rewritten using Exercise 1 as

$$\mathbf{F}.d\mathbf{A} = (\frac{a^3}{4}(\cos(3\theta) + 3\cos(\theta)) + \frac{a^2}{2}(1 - \cos(2\theta))) d\theta\, dz.$$

It is standard that

$$\int_{\theta=0}^{2\pi} \cos(\theta)\, d\theta = \int_{\theta=0}^{2\pi} \cos(2\theta)\, d\theta = \int_{\theta=0}^{2\pi} \cos(3\theta)\, d\theta = 0.$$

Using this, we find that

$$\begin{aligned}
\int_S \mathbf{F}.d\mathbf{A} &= \int_{z=0}^{b} \int_{\theta=0}^{2\pi} (\frac{a^3}{4}(\cos(3\theta) + 3\cos(\theta)) + \frac{a^2}{2}(1 - \cos(2\theta))) d\theta\, dz \\
&= \int_{z=0}^{b} \int_{\theta=0}^{2\pi} \frac{a^2}{2} d\theta\, dz \\
&= \int_{z=0}^{b} a^2\pi = a^2 b\pi.
\end{aligned}$$

(b) First, we have

$$\mathrm{div}(\mathbf{F}) = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(0) = 2x + 1.$$

If we work in cylindrical polar coordinates then this becomes $\mathrm{div}(\mathbf{F}) = 2r\cos(\theta) + 1$. On the other hand, the volume element is $dV = r\, dr\, d\theta\, dz$, so

$$\begin{aligned}
\int_E \mathrm{div}(\mathbf{F})\, dV &= \int_{z=0}^{b} \int_{\theta=0}^{2\pi} \int_{r=0}^{a} (2r\cos(\theta) + 1)\, r\, dr\, d\theta\, dz \\
&= \int_{z=0}^{b} \int_{\theta=0}^{2\pi} \int_{r=0}^{a} (2r^2\cos(\theta) + r)\, dr\, d\theta\, dz \\
&= \int_{z=0}^{b} \int_{\theta=0}^{2\pi} \left[\tfrac{2}{3}r^3\cos(\theta) + \tfrac{1}{2}r^2\right]_{r=0}^{a} d\theta\, dz \\
&= \int_{z=0}^{b} \int_{\theta=0}^{2\pi} \tfrac{2}{3}a^3\cos(\theta) + \tfrac{1}{2}a^2 d\theta\, dz \\
&= \int_{z=0}^{b} \left[\tfrac{2}{3}a^3\sin(\theta) + \tfrac{1}{2}a^2\theta\right]_{\theta=0}^{2\pi} dz \\
&= \int_{z=0}^{b} a^2\pi\, dz = a^2 b\pi.
\end{aligned}$$

Exercise 72. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F}.d\mathbf{A}$, where $\mathbf{F} = (y^2x, z^2y, x^2z)$ and S is the surface of the sphere of radius a centred at the origin.

Solution: First, we have

$$\mathrm{div}(\mathbf{F}) = \frac{\partial}{\partial x}(y^2x) + \frac{\partial}{\partial y}(z^2y) + \frac{\partial}{\partial z}(x^2z) = y^2 + z^2 + x^2.$$

If we use spherical polar coordinates, this can be written as $\mathrm{div}(\mathbf{F}) = r^2$.

Now let E be the unit ball enclosed by S. After recalling that the volume element in spherical polar coordinates is $dV = r^2\sin(\phi)dr\,d\phi\,d\theta$, we find that

$$\begin{aligned}\iiint_E \operatorname{div}(\mathbf{F})\,dV &= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{r=0}^{a} r^4\sin(\phi)\,dr\,d\phi\,d\theta \\ &= 2\pi\left(\int_{\phi=0}^{\pi}\sin(\phi)\,d\phi\right)\left(\int_{r=0}^{a} r^4\,dr\right) \\ &= 2\pi\Big[-\cos(\phi)\Big]_{\phi=0}^{\pi}\left[\frac{r^5}{5}\right]_{r=0}^{a} = 2\pi\times 2\times\tfrac{a^5}{5} = 4\pi a^5/5.\end{aligned}$$

By the Divergence Theorem, we must have $\iint_S \mathbf{F}.d\mathbf{A} = 4\pi a^5/5$ as well.

Exercise 73. Let E be the solid cylinder with equations $0 \leq x^2+y^2 \leq a^2$ and $0 \leq z \leq b$. Let S be the surface of E, and let $\mathbf{F}$ be the vector field (x^3, y^3, z^3). Use the Divergence Theorem to evaluate $\iint_S \mathbf{F}.d\mathbf{A}$.

Solution: First, we have

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3(x^2+y^2+z^2).$$

If we use cylindrical polar coordinates, this can be written as $\operatorname{div}(\mathbf{F}) = 3r^2+3z^2$. In those coordinates the volume element is $dV = r\,dr\,d\theta\,dz$, so

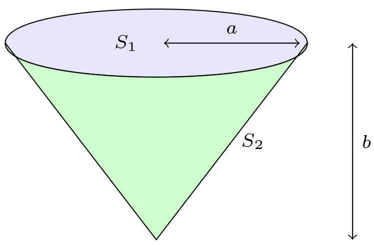
$$\begin{aligned}\iiint_E \operatorname{div}(\mathbf{F})dV &= \int_{z=0}^{b}\int_{\theta=0}^{2\pi}\int_{r=0}^{a}(3r^3+3rz^2)dr\,d\theta\,dz \\ &= \int_{z=0}^{b}\int_{\theta=0}^{2\pi}\left(\frac{3a^4}{4}+\frac{3a^2z^2}{2}\right)d\theta\,dz = \int_{z=0}^{b}(\tfrac{3}{2}\pi a^4 + 3\pi a^2 z^2)\,dz \\ &= \tfrac{3}{2}\pi a^4 b + \pi a^2 b^3 = \pi a^2 b(3a^2/2+b^2).\end{aligned}$$

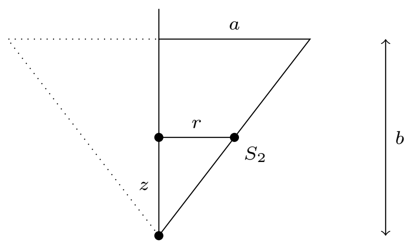
By the Divergence Theorem, we must have $\iint_S \mathbf{F}.d\mathbf{A} = \pi a^2 b(3a^2/2+b^2)$ as well.

Exercise 74. Let E be the cone whose top is a flat disc of radius a centred on the z-axis at height b, and whose point is at the origin. Let S_1 be the flat top of E, and let S_2 be the lower curved surface (so S_1 and S_2 together form the whole boundary of E).

(a) Give equations for S_1, S_2 and E in cylindrical polar coordinates.
(b) Put $\mathbf{F} = \operatorname{grad}(f)$, where $f = x^2+y^2+z^2$. Show that $\int_{S_2}\mathbf{F}.d\mathbf{A} = 0$, and calculate $\int_{S_1}\mathbf{F}.d\mathbf{A}$.
(c) Use the Divergence Theorem to deduce the volume of E.

Solution: The picture on the left below shows the situation in three dimensions. The picture on the right is a vertical cross-section.





(a) The surface S_1 is given by $z = b$ with $0 \leq r \leq a$ and $0 \leq \theta \leq 2\pi$. By inspecting the right-hand picture, we see that the surface S_2 is given by $r/z = a/b$, or equivalently $r = az/b$, with $0 \leq z \leq b$ and $0 \leq \theta \leq 2\pi$. The solid cylinder E is given by $0 \leq r \leq az/b$, again with $0 \leq z \leq b$ and $0 \leq \theta \leq 2\pi$.
(b) If $f = x^2+y^2+z^2$ and $\mathbf{F} = \operatorname{grad}(f)$ then

$$\mathbf{F} = (2x, 2y, 2z) = (2r\cos(\theta),\ 2r\sin(\theta),\ 2z).$$

In other words, $\mathbf{F}$ points directly away from the origin. This means that on the surface S_1, the vector field $\mathbf{F}$ points along the surface, whereas $d\mathbf{A}$ is perpendicular to the surface, so $\mathbf{F}.d\mathbf{A} = 0$, so $\iint_{S_1} \mathbf{F}.d\mathbf{A} = 0$. This can be seen more algebraically as follows. On S_2, we have $r = az/b$, so

$$\begin{aligned}
\mathbf{r} &= (az\cos(\theta)/b, az\sin(\theta)/b, z) \\
\mathbf{F} &= (2az\cos(\theta)/b, 2az\sin(\theta)/b, 2z) \\
\mathbf{r}_\theta &= (-az\sin(\theta)/b, az\cos(\theta)/b, 0) \\
\mathbf{r}_z &= (a\cos(\theta)/b,\ a\sin(\theta)/b,\ 1) \\
\mathbf{F}.d\mathbf{A} &= \mathbf{F}.(\mathbf{r}_\theta \times \mathbf{r}_z)d\theta\, dz \\
&= \det \begin{bmatrix} 2az\cos(\theta)/b & 2az\sin(\theta)/b & 2z \\ -az\sin(\theta)/b & az\cos(\theta)/b & 0 \\ a\cos(\theta)/b & a\sin(\theta)/b & 1 \end{bmatrix}.
\end{aligned}$$

In this matrix the top row is $2z$ times the bottom row, and it follows by standard properties of determinants that the determinant is zero. Even more explicitly, the relevant 2×2 subdeterminants are

$$\begin{aligned}
\det \begin{bmatrix} az\cos(\theta)/b & 0 \\ a\sin(\theta)/b & 1 \end{bmatrix} &= az\cos(\theta)/b \\
\det \begin{bmatrix} -az\sin(\theta)/b & 0 \\ a\cos(\theta)/b & 1 \end{bmatrix} &= -az\sin(\theta)/b \\
\det \begin{bmatrix} -az\sin(\theta)/b & az\cos(\theta)/b \\ a\cos(\theta)/b & a\sin(\theta)/b \end{bmatrix} &= -a^2z/b^2
\end{aligned}$$

so the full 3×3 determinant is

$$2az\cos(\theta)/b \times az\cos(\theta)/b - 2az\sin(\theta)/b \times (-az\sin(\theta)/b) + 2z \times (-a^2z/b^2) = 2a^2z^2/b^2(\cos^2(\theta)+\sin^2(\theta)-1) = 0.$$

Now consider instead the surface S_2. Here we have $\mathbf{n} = \mathbf{k}$ and $dA = r\,dr\,d\theta$ so $d\mathbf{A} = \mathbf{n}\,dA = (0,0,r)dr\,d\theta$. We also have $z = b$ so

$$\begin{aligned}
\mathbf{F} &= (2x, 2y, 2z) = (2r\cos(\theta),\ 2r\sin(\theta),\ 2b) \\
\mathbf{F}.d\mathbf{A} &= 2br\,dr\,d\theta \\
\iint_{S_2} \mathbf{F}.d\mathbf{A} &= \int_{\theta=0}^{2\pi}\int_{r=0}^{a} 2br\,dr\,d\theta = \int_{\theta=0}^{2\pi} a^2b\,d\theta = 2\pi a^2 b.
\end{aligned}$$

(c) Now note that

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(2z) = 6,$$

so

$$\iiint_E \operatorname{div}(\mathbf{F})\,dV = 6 \times \operatorname{volume}(E).$$

Using the Divergence Theorem we also see that

$$\iiint_E \operatorname{div}(\mathbf{F})\,dV = \iint_{S_1} \mathbf{F}.d\mathbf{A} + \iint_{S_2} \mathbf{F}.d\mathbf{A} = 0 + 2\pi a^2 b = 2\pi a^2 b.$$

Rearranging this gives $\operatorname{volume}(E) = (2\pi a^2 b)/6 = \pi a^2 b/3$.

Week 11 — Stokes's theorem

Exercise 75. Let C be the vertical circle given by $y = a\sin(t)$ and $z = a\cos(t)$ with $x = 0$. Use Stokes's Theorem to evaluate $\int_C (x^2y, z, 0).d\mathbf{r}$. Check your answer by calculating the integral directly.

Solution: Let D be the vertical disc whose boundary is C, so D can be parametrised by

$$(x, y, z) = (0, s\sin(t), s\cos(t))$$

with $0 \leq s \leq a$ and $0 \leq t \leq 2\pi$. We are asked to calculate $\int_C \mathbf{u}.d\mathbf{r}$, where $\mathbf{u} = (x^2y, z, 0)$. Stokes's Theorem tells us that this is the same as $\iint_S \operatorname{curl}(\mathbf{u}).d\mathbf{A}$. Here

$$\operatorname{curl}(\mathbf{u}) = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & z & 0 \end{bmatrix} = (-1, 0, -x^2).$$

It is clear that the unit normal vector to D is $\mathbf{n} = \pm\mathbf{i}$. By inspecting the diagram we see that the $\mathbf{n}$ must be $-\mathbf{i}$ to ensure that S stays on the left as we walk around C in the direction of increasing t. We also have $dA = dy\,dz = s\,ds\,dt$, so

$$\mathrm{curl}(\mathbf{u}).d\mathbf{A} = (-1, 0, -x^2).(-\mathbf{i})s\,ds\,dt = s\,ds\,dt$$

$$\iint_S \mathrm{curl}(\mathbf{u}).d\mathbf{A} = \int_{t=0}^{2\pi}\int_{s=0}^{a} s\,ds\,dt = \int_{t=0}^{2\pi} \tfrac{1}{2}a^2\,dt = \pi a^2.$$

Alternatively, we can calculate the line integral directly. On C we have

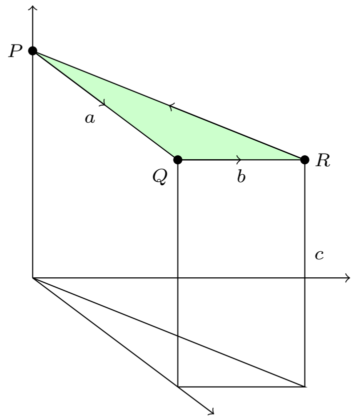
$$\begin{aligned}
\mathbf{r} &= (0, a\sin(t), a\cos(t)) \\
d\mathbf{r} &= (0, a\cos(t), -a\sin(t))\,dt \\
\mathbf{u} &= (x^2 y, z, 0) = (0, a\cos(t), 0) \\
\mathbf{u}.d\mathbf{r} &= a^2\cos^2(t)\,dt \\
\int_C \mathbf{u}.d\mathbf{r} &= \int_{t=0}^{2\pi} a^2\cos^2(t)\,dt = a^2\int_{t=0}^{2\pi} \tfrac{1}{2}(1+\cos(2t))\,dt \\
&= a^2\left[\tfrac{1}{2}t + \tfrac{1}{4}\sin(2t)\right]_{t=0}^{2\pi} = \pi a^2.
\end{aligned}$$

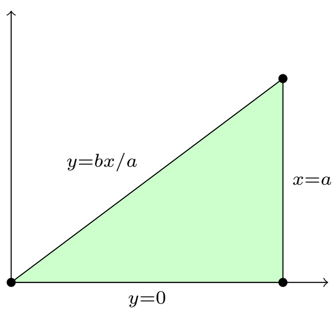
Exercise 76. Consider points

$$P = (0, 0, c) \qquad Q = (a, 0, c) \qquad R = (a, b, c).$$

Let C be the triangular path that goes from P to Q to R and back to P. Use Stokes's Theorem to evaluate $\int_C (yz^2, x^3, xy^2).d\mathbf{r}$.

Solution: The path C encloses a triangular region S as shown on the left below. The shadow in the xy-plane is shown on the right.





Consider the vector field $\mathbf{u} = (yz^2, x^3, xy^2)$. Stokes's Theorem tells us that $\int_C \mathbf{u}.d\mathbf{r} = \iint_S \mathrm{curl}(\mathbf{u}).\mathbf{n}dA$. Here

$$\mathrm{curl}(\mathbf{u}) = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & x^3 & xy^2 \end{bmatrix} = (2xy, 2yz - y^2, 3x^2 - z^2).$$

Next, $\mathbf{n}$ is clearly $(0, 0, \pm 1)$. If you walk around C in the indicated direction with your head pointing upwards, then S is on the left. This means that the correct choice for $\mathbf{n}$ is $(0, 0, 1)$, so $\mathrm{curl}(\mathbf{u}).\mathbf{n} = 3x^2 - z^2$, but $z = c$ on S, so $\mathbf{u}.\mathbf{n} = 3x^2 - c^2$. As S is flat we just have $dA = dx\,dy$, and the right hand diagram gives us the limits, so

$$\begin{aligned}
\iint_S \mathrm{curl}(\mathbf{u}).\mathbf{n}\,dA &= \int_{x=0}^{a}\int_{y=0}^{bx/a} 3x^2 - c^2\,dy\,dx = \int_{x=0}^{a}\left[3x^2 y - c^2 y\right]_{y=0}^{bx/a} dx \\
&= \int_{x=0}^{a} \tfrac{3b}{a}x^3 - \tfrac{bc^2}{a}x\,dx = \left[\tfrac{3b}{4a}x^4 - \tfrac{bc^2}{2a}x^2\right]_{x=0}^{a} \\
&= \tfrac{3}{4}a^3 b - \tfrac{1}{2}abc^2
\end{aligned}$$