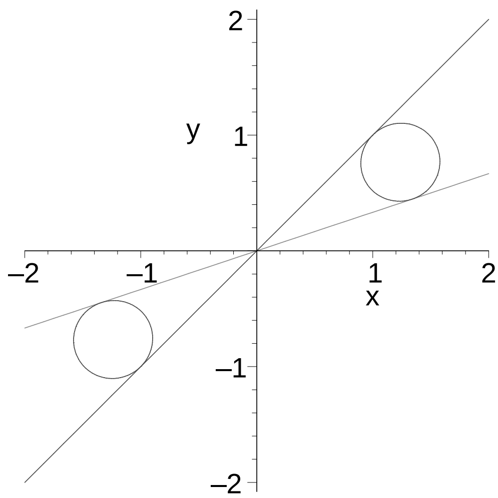
# Maths with Maple — Exam

## 1 Maple questions

### 1.1 Plotting

**(1)** [**0910**] Put $f(x,y) = (x^2 - y^2 - 1)^2 + 4(xy - 1)^2$. The picture below shows the curve $C$ with equation $f(x,y) = 1$, together with the line $L$ passing through the origin with slope 1, and the line $M$ passing through the origin with slope 1/3.




(a) Give Maple commands to produce the above picture. Include options to ensure that Maple uses the same scale on both axes, and that it plots enough points to give a smooth picture of $C$. **(7 marks)**

(b) Find (by hand) the coordinates of the two points where $M$ meets $C$. **(5 marks)**

**Solution:**

(a)
```
with(plots):
f := (x,y) -> (x^2-y^2-1)^2+4*(x*y-1)^2;
display(
 implicitplot(f(x,y)=1,x=-2..2,y=-2..2,grid=[200,200],scaling=constrained),
 plot([x,x/3],x=-2..2)
);
```

([**1**]for `display`, [**1**]for `implicitplot`, [**1**]for ranges, [**1**]for `grid` option, [**1**]for `scaling` option, [**2**] for `plot` command)

(b) We need to find the points where $y = x/3$ and $f(x,y) = 1$. We have

$$\begin{aligned} f(x, x/3) - 1 &= (x^2 - x^2/9 - 1)^2 + 4(x^2/3 - 1)^2 - 1 \\ &= (8x^2/9 - 1)^2 + 4(x^2/3 - 1)^2 - 1 \\ &= 64x^4/81 - 16x^2/9 + 1 + 4x^4/9 - 8x^2/3 + 4 - 1 \\ &= 100x^4/81 - 40x^2/9 + 4 \\ &= 4(5x^2/9 - 1)^2.\mathbf{[3]} \end{aligned}$$

This vanishes when $5x^2/9 = 1$, so $x = \pm\sqrt{9/5} = \pm 3/\sqrt{5}$. [**1**]We must also have $y = x/3$, so the intersection points are $(-3/\sqrt{5}, -1/\sqrt{5})$ and $(3/\sqrt{5}, 1/\sqrt{5})$. [**1**]

**(2)** [**0910**] Give Maple commands to plot the list of values $\frac{10!}{k!(10-k)!}$ for $k = 1, 2, \ldots, 10$, together with the graph of the function $252e^{-(x-5)^2/5}$ for $0 \le x \le 10$. **(6 marks)**

**Solution:**

```
with(plots):
display(
 listplot([seq(10!/(k!*(10-k)!),k=1..10)]),
 plot(252*exp(-(x-5)^2/5),x=0..10)
);
```

([**1**]for `display`, [**2**] for `listplot`, [**1**]for `seq`, [**1**]for square brackets, [**1**]for `plot` command.)

**(3)** What would you type to get Maple to draw the curve given parametrically by $x = \cos(t)/r$ and $y = \sin(t)/r$, where $r = (\cos(t)^8 + \sin(t)^8)^{1/8}$ and $t$ runs from 0 to $2\pi$.

**Solution:**

```
r := (cos(t)^8+sin(t)^8)^(1/8);
plot([cos(t)/r,sin(t)/r,t=0..2*Pi]);
```

**(4)** [**0708**] What would you type to get Maple to draw the curve $y = \sin(10x) + \sin(11x)$ together with the curve $y = 2|\cos(x/2)|$, for $-4\pi \le x \le 4\pi$? **(4 marks)**

**Solution:** `plot([sin(10*x)+sin(11*x),2*abs(cos(x/2))],x=-4*Pi..4*Pi);` [**4**]

**(5)** [**0506; 0708R**]

(a) What would you enter to plot the curve given by $x = t - \sin(t)$ and $y = 2 - \cos(t)$, for $0 \le t \le 6\pi$? You should include an option to make Maple use the same scales on the two axes. **(3 marks)**

(b) What would you enter to plot the curves $y = t - \sin(t)$ and $y = 2 - \cos(t)$ together on the same graph, for $0 \le t \le 6\pi$? **(2 marks)**

**Solution:**

(a) `plot([t-sin(t),2-cos(t),t=0..6*Pi],scaling=constrained);` [**3**]

(b) `plot([t-sin(t),2-cos(t)],t=0..6*Pi);` [**2**]

**(6)** [**Mock2**] How would you ask Maple to plot the curve $x^6 + y^6 = 1$ for $|x| \le 3$ and $|y| \le 3$? If you found that the picture was jagged or inaccurate, how would you improve it? **(4 marks)**

**Solution:** Enter `implicitplot(x^6+y^6=1,x=-3..3,y=-3..3);` to plot the curve [**2**]. If it is inaccurate, include the option `grid=[100,100]` [**2**].

**(7)** [**0405R**] Consider the function

$$f(x, y) = (x - y)^4 + (x + y)^4 - x^6 - y^6.$$

How would you ask Maple to plot the curve $f = 0$ for $|x| \le 3$ and $|y| \le 3$? If you found that the picture was jagged or inaccurate, how would you improve it? **(5 marks)**

**Solution:** Enter `f:=(x,y)->(x-y)^4+(x+y)^4 - x^6 - y^6;` to define the function, then

```
implicitplot(f(x,y)=0,x=-3..3,y=-3..3);
```

to plot the curve [**3**]. (It is perfectly acceptable to include the formula in the plot command rather than defining a function.) If it is inaccurate, include the option `grid=[100,100]` [**2**].

**(8)** [**Mock1**] How would you ask Maple to plot the lines $y = x$ and $y = -x$ together with the curve $x^4 + y^4 = 1 + 9x^2y^2/5$, for $-3 \le x, y \le 3$? **(6 marks)**

**Solution:**

```
display(
 plot([x,-x],x=-2..2),
 implicitplot(x^4-9*x^2*y^2/5+y^4=1,
              x=-3..3,y=-3..3,
              grid=[100,100])
);
```

[**1**]for `display()`, [**2**] for $y = \pm x$, [**3**] for the curve.

**(9)** [**0506R**]

(a) What would you enter to plot the curves $y = t^2$ and $y = t(t^2 - 1)(t^2 - 2)$ together on the same graph? You should include an option to make Maple show only the part of the curve where $-2 \le t \le 2$ and $-5 \le y \le 5$. **(4 marks)**

(b) What would you enter to plot the curve given by $x = t^2$ and $y = t(t^2 - 1)(t^2 - 2)$, for $-2 \le t \le 2$? You should include an option to make Maple show only the part of the curve where $-1 \le x \le 4$ and $-5 \le y \le 5$. **(4 marks)**

**Solution:**

(a) `plot([t^2,t*(t^2-1)*(t^2-2)],t=-2..2,-5..5);` [**4**]

(b) `plot([t^2,t*(t^2-1)*(t^2-2),t=-2..2],view=[-1..4,-5..5]);` [**4**]

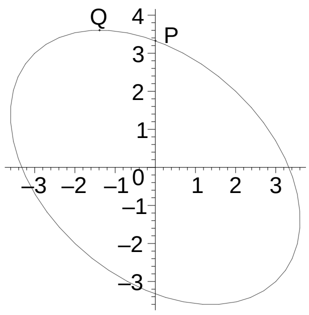
**(10)** [**0405**] How would you ask Maple to plot the curve $(x + \sin(2\pi x))^2 + y^2 = 4$, with the same scale on the $x$ and $y$ axes? You should decide for yourself what is an appropriate range for the axes. You should also include an option telling Maple to plot extra points, so as to give a picture of reasonable quality. **(5 marks)**

**Solution:**

```
plots[implicitplot](
  (x+sin(2*Pi*x))^2+y^2=4,
  x=-3..3,y=-3..3,
  scaling=constrained,
  grid=[200,200]
);
```

[**1**]for `implicitplot`, [**1**]for `x+sin(2*Pi*x))^2+y^2=4`, [**1**]for ranges, [**1**]for `scaling=constrained`, [**1**]for `grid=[..,..]`. (Any grid size is acceptable.)

**(11)** [**Mock2**] Consider the curve $C$ given parametrically by the equations $x = 2\cos(t) + 3\sin(t)$ and $y = 2\cos(t) - 3\sin(t)$:




(a) How would you generate the above picture? (You need not include commands to mark or label the points $P$ and $Q$.) **(3 marks)**

(b) How would you ask Maple to find the coordinates of the point $P$, where the curve crosses the $y$-axis? **(3 marks)**

(c) How would you ask Maple to find the coordinates of the point $Q$ (the highest point on the curve)? **(3 marks)**

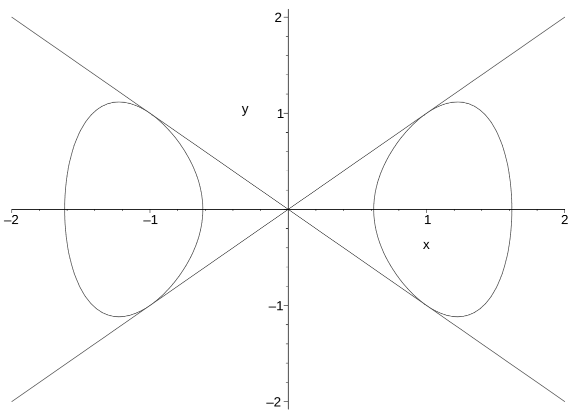
**Solution:**

(a)
```
x := 2*cos(t)+3*sin(t);
y := 2*cos(t)-3*sin(t);
plot([x,y,t=0..2*Pi]);
```
[**3**]

(b) `sol := solve(x=0,{t}); subs(sol[1],[x,y]);` [**3**]

(c) `sol := solve(diff(y,t)=0,{t}); subs(sol[1],[x,y]);` [**3**]

**(12)** [**0506; 0708R**] The following picture shows the curve $x^4 - 3x^2 + 1 = -y^2$, together with the lines $y = x$ and $y = -x$.




(a) How would you generate the above picture? (You should include an option to ensure that Maple plots enough points to give a smooth picture.) **(6 marks)**

(b) Let $P$ and $Q$ be the points where the curve meets the line $y = -x$. What would you enter in Maple to find the coordinates of $P$ and $Q$? **(2 marks)**

(c) Find the coordinates of $P$ and $Q$ by hand. **(4 marks)**

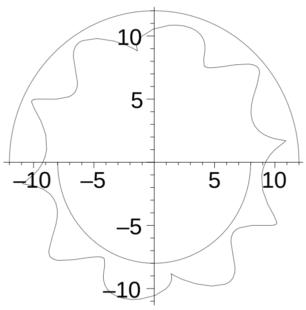
**Solution:**

(a)
```
with(plots):
display(
 implicitplot(x^4-3*x^2+1=-y^2,x=-2..2,y=-2..2,grid=[200,200]),
 plot(x,x=-2..2),
 plot(-x,x=-2..2)
);
```
[**1**]for `display()`, [**3**] for the curve, [**2**] for the lines. No penalty for omitting `with(plots)`.

(b) `solve(x^4-3*x^2+1=-(-x)^2,{x});`. [**2**]

(c) We have to solve $x^4 - 3x^2 + 1 = -x^2$ [**1**], or equivalently $x^4 - 2x^2 + 1 = 0$, or equivalently $(x^2 - 1)^2 = 0$ [**1**]. The solutions are $x = \pm 1$ [**1**], so the intersection points are $(1, -1)$ and $(-1, 1)$ [**1**].

**(13)** [**0405R**] Consider the following picture:



It shows the curve $C$ given parametrically by the equations

$$\begin{aligned} x &= 10\cos(t) + \sin(11t) \\ y &= 10\sin(t) + \cos(9t), \end{aligned}$$

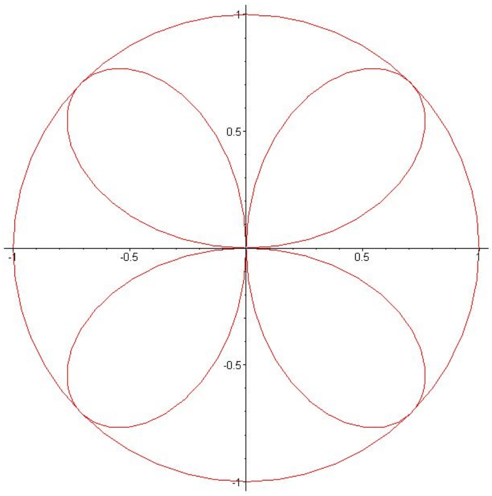
together with a semicircle of radius 12 in the upper half plane, and a semicircle of radius 8 in the lower half plane. What Maple command would you use to generate this picture? **(7 marks)**

**Solution:**

```
plots[display](
 plot([10*cos(t)+sin(11*t),10*sin(t)+cos(9*t),t=0..2*Pi]),
 plot([12*cos(t),12*sin(t),t=0..Pi]),
 plot([8*cos(t),8*sin(t),t=Pi..2*Pi])
);
```

[**1**]for `display`, then [**3**] for the curve, then [**2**] for the first semicircle, and [**1**]for the other one.

**(14)** [**0506R**] The following picture shows the curve $C$ with equation $x = \sin(2t)\sin(t)$ and $y = \sin(2t)\cos(t)$, together with the circle of radius one centred at the origin.




(a) How would you generate the above picture? **(6 marks)**

(b) What are the coordinates of the four points at which $C$ meets the circle? (You may guess the answer, but you should then give an argument to show that your guess is correct.) **(4 marks)**

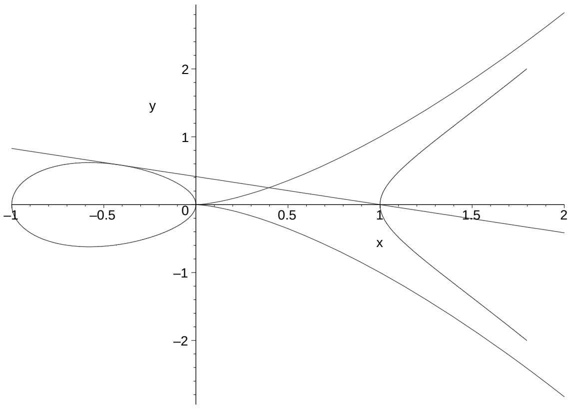
**Solution:**

(a)
```
with(plots):
display(
 plot([sin(2*t)*sin(t),sin(2*t)*cos(t),t=0..2*Pi]),
 plot([sin(t),cos(t),t=0..2*Pi])
);
```

[**1**]for `display()`, [**3**] for the curve, [**2**] for the circle. No penalty for omitting `with(plots)`.

(b) The intersection points are $(x, y) = (\pm 1, \pm 1)/\sqrt{2}$ [**2**]. Indeed, these points are certainly on the unit circle [**1**], and $C$ passes through them at $t = \pi/4$, $t = 3\pi/4$, $t = 5\pi/4$ and $t = 7\pi/4$ [**1**].

**(15)** [**0708**] The picture below shows the curve $C$ with equation $y^2 = x^3 - x$, the curve $C'$ given parametrically by $x = t^2$ and $y = t^3$ with $-\sqrt{2} \leq t \leq \sqrt{2}$, and the line $L$ of slope $1 - \sqrt{2}$ passing through the point $(1, 0)$.




(a) What is the equation of $L$? **(2 marks)**

(b) How would you generate the above picture? (You should include an option to ensure that Maple plots enough points to give a smooth picture. You may assume that the `plots` package has already been loaded.) **(9 marks)**

(c) The line $L$ meets $C$ at $(1, 0)$ and also at another point $P$. How would you ask Maple to find the coordinates of $P$? **(3 marks)**

**Solution:**

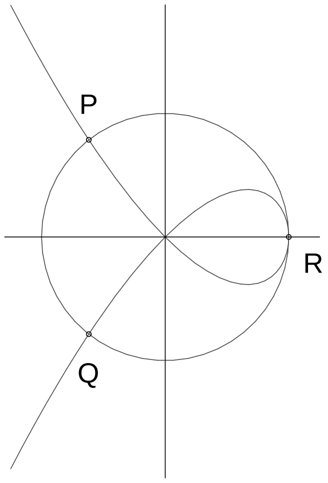
(a) The equation of $L$ is $y = (1 - \sqrt{2})(x - 1)$. [**2**]

(b)
```
display(
  implicitplot(y^2=x^3-x,x=-2..2,y=-2..2,grid=[200,200]),
  plot([t^2,t^3,t=-sqrt(2)..sqrt(2)]),
  plot((1-sqrt(2))*(x-1),x=-1..2)
);
```
[**1**]for `display`, [**3**] for the implicit plot, [**3**] for the parametric plot, [**2**] for the line.

(c) `solve({y^2=x^3-x,y=(1-sqrt(2))*(x-1)},{x,y});` [**3**] (`fsolve` would also be acceptable.)

**(16)** [**0405**] The picture below shows the unit circle, together with the curve $C$ given by $x = 1 - t^2$

and $y = t - t^3$, for $-3/2 \le t \le 3/2$.




(a) How would you generate the above picture? (You need not include commands to mark or label the points $P$, $Q$ and $R$.) **(5 marks)**

(b) How would you ask Maple to find the values of $t$ at the points $P$, $Q$ and $R$ where the curve meets the circle? (Do not worry about the possibility of spurious complex roots.) **(3 marks)**

(c) Given that $t = \sqrt{2+2\sqrt{5}}/2$ at $Q$, how would you ask Maple to find the coordinates of $Q$? **(2 marks)**

**Solution:**

(a)
```
plots[display](
  plot([cos(t),sin(t),t=0..2*Pi]),
  plot([1-t^2,t-t^3,t=-3/2..3/2]),
  scaling=constrained
 );
```

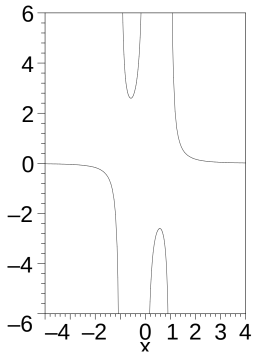
[**1**]for `display()`, [**2**] for the circle, [**2**] for the curve. No penalty for omitting `plots[...]` or `scaling=constrained`.

(b)
```
x := 1-t^2;
y := t-t^3;
sols := solve(x^2+y^2=1,{t});
```

(This will find two complex values of $t$ as well, but I will ignore that.) [**3**]

(c) `subs(t=sqrt(2+2*sqrt(5))/2,[x,y]);` [**2**]

**(17)** **[Mock1]** Here is a graph of the function $y = 1/(x^3 - x)$:




What Maple command would you use to generate this plot? Your command should reproduce the following features:

- The horizontal and vertical ranges are as shown.
- The distance from 0 to 2 is the same on the two axes.
- There are no vertical lines at the points of discontinuity.
- The axes appear on the outside of the plot, rather than passing through the origin.

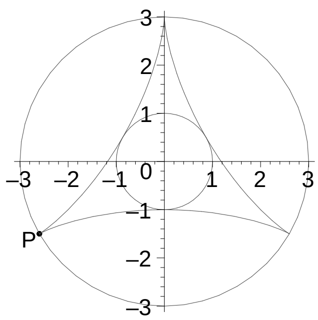
**(4 marks)**

**Solution:**

```
plot(1/(x^3-x),x=-4..4,-6..6,
     axes=boxed,
     scaling=constrained,
     discont=true):
```

[4]

**(18)** **[0809]**
Consider the curve $C$ given parametrically by the equations $x = 2\sin(t) + \sin(2t)$ and $y = -2\cos(t) + \cos(2t)$. The picture below shows $C$ together with two circles centred at the origin.



(a) How would you generate the above picture? (You need not include commands to mark the point $P$.) **(5 marks)**

(b) Write down the addition formula for $\cos(a+b)$.**(1 marks)**

(c) Show that $x^2 + y^2 = 5 - 4\cos(3t)$, and thus that $1 \leq x^2 + y^2 \leq 9$. **(5 marks)**

(d) Explain how (b) relates to the geometry of the picture. **(1 marks)**

(e) There is a point $P$ close to $t = 2$ where $y = 0$. What would you enter in Maple to find the approximate values of $t$ and $x$ at $P$, to 100 decimal places? **(5 marks)**

**Solution:**

(a)
```
x := 2*sin(t)+sin(2*t);
y := -2*cos(t)+cos(2*t);
plots[display](
 plot([x,y,t=0..2*Pi]),
 plot([cos(t),sin(t),t=0..2*Pi]),
 plot([3*cos(t),3*sin(t),t=0..2*Pi])
);
```

[**5**]

(b) $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$. [**1**]

(c) We have

$$\begin{aligned} x^2 + y^2 &= 4\sin(t)^2 + 4\sin(t)\sin(2t) + \sin(2t)^2 + 4\cos(t)^2 - 4\cos(t)\cos(2t) + \cos(2t)^2[\mathbf{1}] \\ &= 5 - 4(\cos(t)\cos(2t) - \sin(t)\sin(2t))[\mathbf{1}] \\ &= 5 - 4\cos(3t).[\mathbf{1}] \end{aligned}$$

(Here the last step used part (b).) Moreover, $\cos(3t)$ runs between $-1$ and $+1$, so $5-4\cos(3t)$ runs between $5-4=1$ and $5+4=9$. Thus $1 \leq x^2+y^2 \leq 9$. [**2**]

(d) Geometrically this means that the curve $C$ lies between the circles of radius $\sqrt{1} = 1$ and $\sqrt{9} = 3$ centred at the origin, which is also clear from the picture. [**1**]

(e)
```
Digits := 100;
t0 := fsolve(y=0, t=2);
x0 := evalf(subs(t=t0,x));
```

([**1**], [**2**], [**2**])

## 1.2 Numerical evaluation

**(19)** [**0405**] What would you enter in Maple to calculate $\log_2(\pi)$ to 50 decimal places? **(2 marks)**

**Solution:** `evalf[50](log[2](Pi));` [**2**] (1/2 mark each for `evalf`, `[50]`, `log[2]`, `Pi`; halves rounded down. It is acceptable (in fact, preferable) to use `Digits:=50;` rather than giving 50 as an index to `evalf`.)

**(20)** [**Mock2**] What would you enter in Maple to calculate $\ln(\ln(\ln(\pi)))$ to 40 decimal places? **(2 marks)**

**Solution:** `evalf[40](ln(ln(ln(Pi))));` [**2**]

**(21)** [**Mock1**] What would you enter in Maple to calculate $\sin(2^{3^6-3})$ to 1000 decimal places? **(2 marks)**

**Solution:** `evalf[1000](sin(2^(3^6-3)));` [**2**]

## 1.3 Fixing errors

**(22)** [**0910**] The following table shows some Maple commands, together with the output that the user expected to get. In each case, the command has one or more errors, so the output would not be as expected. Give a corrected version of each command. **(9 marks)**

| | Input | Expected output |
|---|---|---|
| (a) | `digits=15;eval(pi);` | 3.14159265358979 |
| (b) | `solve(ax+b=0);` | $\{x = -b/a\}$ |
| (c) | `dif(t^3);` | $3t^2$ |

**Solution:**

(a) `Digits:=15;evalf(Pi);` [**4**]

(b) `solve(a*x+b=0,{x});` [**3**]

(c) `diff(t^3,t);` [**2**]

**(23)** [**Mock2; 0708R**] The following table shows some Maple commands, together with the output that the user expected to get. In each case, the command has one or more errors, so the output would not be as expected. Give a corrected version of each command. **(6 marks)**

| | Input | Expected output |
|---|---|---|
| (a) | `simplify((xy+y)/(x+1));` | $y$ |
| (b) | `Expand(((a+b)*-c);` | $-ac - bc$ |
| (c) | `diff(x^2);` | $2x$ |
| (d) | `f(x)=x^2; f(3);` | 9 |

**Solution:**

(a) `simplify((x*y+y)/(x+1));` [**1**]

(b) `expand((a+b)*(-c));` [**2**]

(c) `diff(x^2,x);` [**1**]

(d) `f := (x)->x^2; f(3);` [**2**]

**(24)** [**0405; 0809**] The following table shows some Maple commands, together with the output that the user expected to get. In each case, the command has one or more errors, so the output would not be as expected. Give a corrected version of each command. **(8 marks)**

| | Input | Expected output |
|---|---|---|
| (a) | `solve(x+y:=2,x-y:=0,x,y);` | $\{x = 1, y = 1\}$ |
| (b) | `simplify(Sqrt(a^-8));` | $a^{-4}$ |
| (c) | `solve(cos(5*x)=0,x);` | $\{x = 0.3141592654\}$ |
| (d) | `simplify([x^2-y^2]/[x+y]);` | $x - y$ |

**Solution:**

(a) `solve({x+y=2,x-y=0},{x,y});` [**2**]

(b) `simplify(sqrt(a^(-8)),symbolic);` [**3**]

(c) `fsolve(cos(5*x)=0,{x});` [**2**]

(d) `simplify((x^2-y^2)/(x+y));` [**1**]

**(25)** [**0506R**] The following table shows some Maple commands, together with the output that the user expected to get. In each case, the command has one or more errors, so the output would not be as expected. Give a corrected version of each command. **(10 marks)**

| | Input | Expected output |
|---|---|---|
| (a) | `Simplify(a^2-b^2/a-b);` | $a + b$ |
| (b) | `solve(x+3=pi);` | $\{x = 0.1415926540\}$ |
| (c) | `xy^-3/xy;` | $1/y^4$ |
| (d) | `[seq(n^2,1..5)];` | $[1, 4, 9, 16, 25]$ |

**Solution:**

(a) `simplify((a^2-b^2)/(a-b));` [**2**]

(b) `fsolve(x+3=Pi,{x});` [**3**]

(c) `x*y^(-3)/(x*y);` [**4**]

(d) `[seq(n^2,n=1..5)];` [**1**]

**(26)** [**Mock1**] The following table shows some Maple commands, together with the output that the user expected to get. In each case, the command has one or more errors, so the output would not be as expected. Give a corrected version of each command. **(7 marks)**

| | Input | Expected output |
|---|---|---|
| (a) | `Sin(2 pi);` | 0 |
| (b) | `simplify((x^2-y^2)(X-Y)^-1);` | $x + y$ |
| (c) | `x := 1; unassign(x); expand((1+x)^2);` | $1 + 2x + x^2$ |
| (d) | `ln(e^2(x+y));` | $2(x + y)$ |

**Solution:**

(a) `sin(2*Pi);` [**2**]

(b) `simplify((x^2-y^2)*(x-y)^(-1));` [**2**]

(c) `x := 1; unassign('x'); expand((1+x)^2);` [**1**]

(d) `ln(exp(2*(x+y)));` [**2**]

(For part (d), you really need `simplify(ln(exp(2*(x+y))),symbolic);` to get $2(x + y)$ as the answer, but the answer given above would be accepted as correct.)

**(27)** [**0405R**] The following table shows some Maple commands, together with the output that the user expected to get. In each case, the command has one or more errors, so the output would not be as expected. Give a corrected version of each command. **(7 marks)**

| | Input | Expected output |
|---|---|---|
| (a) | `simplify((xy+y)/(x+1));` | $y$ |
| (b) | `Expand((a+b)(a-b)*-1);` | $b^2 - a^2$ |
| (c) | `diff(x^4);` | $4x^3$ |
| (d) | `e^2log(x);` | $x^2$ |

**Solution:**

(a) `simplify((x*y+y)/(x+1));` [**1**]

(b) `expand((a+b)*(a-b)*(-1));` [**3**]

(c) `diff(x^4,x);` [**1**]

(d) `exp(2*log(x));` [**2**]

**(28)** [**0506**] The following table shows some Maple commands, together with the output that the user expected to get. In each case, the command has one or more errors, so the output would not be as expected. Give a corrected version of each command. **(10 marks)**

| | Input | Expected output |
|---|---|---|
| (a) | `Expand((a+b)(a-b));` | $a^2 - b^2$ |
| (b) | `pi^-2;` | 0.1013211836 |
| (c) | `solve(tan(x)+1=a);` | $\{x = \arctan(a-1)\}$ |
| (d) | `seq(ln(10..13));` | $[\ln(10), \ln(11), \ln(12), \ln(13)]$ |

**Solution:**

(a) `expand((a+b)*(a-b));` [**2**]

(b) `evalf(Pi^(-2));` [**3**]

(c) `solve(tan(x)+1=a,{x});` [**2**]

(d) `[seq(ln(n),n=10..13)];` [**3**]

**(29)** [**0708**] The following table shows some Maple commands, together with the output that the user expected to get. In each case, the command has one or more errors, so the output would not be as expected. Give a corrected version of each command. **(9 marks)**

| | Input | Expected output |
|---|---|---|
| (a) | `Log(e^a);` | $a$ |
| (b) | `y=1/(1+t); solve(y:=3/4,t);` | 0.3333333333 |
| (c) | `factor([t^4-1]/[(t-1)(t+1)]);` | $t^2 + 1$ |
| (d) | `seq(x^n,5..7);` | $[x^5, x^6, x^7]$ |

**Solution:**

(a) `simplify(log(exp(a)),symbolic);` [**2**]
(No penalty for omitting `simplify(...,symbolic)`)

(b) `y:=1/(1+t): fsolve(y=3/4,t);` [**3**]
(No penalty for semicolon instead of colon)

(c) `factor((t^4-1)/((t-1)*(t+1)));` [**2**]
(`simplify()` will do instead of `factor()`)

(d) `[seq(x^n,n=5..7)];` [**2**]

## 1.4 Conversion and simplification

**(30)** [**0506R; 0809**] Suppose that the following definitions have been made:

$$\begin{aligned} P &= 1 + x + x^2 + O(x^3) \\ Q &= 6t^3 + 7t^2 - 1 \\ R &= (1 + 3X + 3X^2 + X^3)^{1/3} \\ S &= a^4 b + ab^2 + ab^3 + a^4 b^4 \end{aligned}$$

(a) What would you enter to convert $P$ to the form $1 + x + x^2$?

(b) What would you enter to convert $Q$ to the form $(3t - 1)(2t + 1)(t + 1)$?

(c) What would you enter to convert $R$ to the form $1 + X$?

(d) What would you enter to convert $S$ to the form $(b + b^4)a^4 + (b^2 + b^3)a$?

**(4 marks)**

**Solution:**

(a) `convert(P,polynom);` [**1**]

(b) `factor(Q);` [**1**]

(c) `simplify(R,symbolic);` [**1**]

(d) `collect(S,a);` [**1**]

**(31)** [**0506**] Suppose that the following definitions have been made:

$$\begin{aligned} P &= 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 \\ Q &= t + t^4 + t^9 + O(t^{16}) \\ R &= uv + uv^2 + u^2v^2 + u^2v^3 + u^3v^3 + u^3v^4 \\ S &= \sqrt{1 + 2u^4 + u^8} \end{aligned}$$

(a) What would you enter to convert $P$ to the form $(x + 1)(x^2 + 1)(x^4 + 1)$?

(b) What would you enter to convert $Q$ to the form $t + t^4 + t^9$?

(c) What would you enter to convert $R$ to the form $(v^3 + v^4)u^3 + (v^2 + v^3)u^2 + (v + v^2)u$?

(d) What would you enter to convert $S$ to the form $1 + u^4$?

**(4 marks)**

**Solution:**

(a) `factor(P);` [**1**]

(b) `convert(Q,polynom);` [**1**]

(c) `collect(R,u);` [**1**]

(d) `simplify(S,symbolic);` [**1**]

**(32)** [**0405**] The following expressions are all mathematically equivalent:

(a) $u^3q^3 - u^3q^2 - u^2q^2 + u^2q - uq^2 + uq + q - 1$

(b) $(q-1)(uq-1)(u^2q-1)$

(c) $(q^3 - q^2)u^3 + (-q^2 + q)u^2 + (-q^2 + q)u + q - 1$

(d) $u^3q^3 + (-u^3 - u^2 - u)q^2 + (u^2 + u + 1)q - 1$

Suppose that `A` has been set equal to expression (a). What would you enter in Maple to convert this to forms (b), (c) and (d)? (Do not worry about the precise ordering of terms.) **(3 marks)**

**Solution:**

(b) `factor(A);` [**1**]

(c) `collect(A,u);` [**1**]

(d) `collect(A,q);` [**1**]

**(33)** [**Mock1**] The following expressions are all mathematically equivalent:

(a) $y^2x^2 + yx^3 + x^4 + yx + 1/yx^3 + x^2 + 1/yx + 1/y^2x^2 + 1$

(b) $1 + x(y + 1/y) + x^2(y^2 + 1 + 1/(y^2)) + (y + 1/y)x^3 + x^4$

(c) $(y^2x^2 + 1 + yx)(y^2 + yx + x^2)/y^2$

(d) $x^2y^2 + (x + x^3)y + x^4 + x^2 + 1 + (x + x^3)y^{-1} + x^2y^{-2}$

Suppose that `A` has been set equal to expression (a). What would you enter in Maple to convert this to forms (b), (c) and (d)? (Do not worry about the precise ordering of terms.) **(3 marks)**

**Solution:**

(b) `collect(A,x);` [**1**]

(c) `factor(A);` [**1**]

(d) `collect(A,y);` [**1**]

**(34)** [**Mock2**] Suppose that Maple variables $a$, $b$ and $c$ have been given the following values:

$$a = 1 + x + xy + xy^2 + x^2y + x^2y^2 + x^2y^3 + x^3y^3$$
$$b = x^3y^3 + (y^3 + y^2 + y)x^2 + (y^2 + y + 1)x + 1$$
$$c = (1 + x)(1 + xy)(1 + xy^2).$$

All three expressions are mathematically equivalent.

(i) What Maple command would you use to convert form $a$ to form $b$? **(1 marks)**

(ii) What Maple command would you use to convert form $b$ to form $a$? **(1 marks)**

(iii) What Maple command would you use to convert form $a$ to form $c$? **(1 marks)**

**Solution:**

(i) `collect(a,x);` [**1**]

(ii) `expand(b);` **[1]**

(iii) `factor(a);` **[1]**

**(35)** [**Mock1**] What Maple command would you use to convert $(x^4)^{1/4}$ to $x$? Explain why this conversion is not always valid. **(3 marks)**

**Solution:** The relevant command is `simplify((x^4)^(1/4),symbolic);` [**1**]. The resulting simplification is not always valid because when $x$ is negative we have $(x^4)^{1/4} = |x| \neq x$ [**2**].

**(36)** [**0405**] What Maple command would you use to convert $\sqrt{p^2 + 2pq + q^2}$ to $p + q$? Explain why this conversion is not always valid. **(3 marks)**

**Solution:** The relevant command is `simplify(sqrt(p^2+2*p*q+q^2),symbolic);` [**1**]. The resulting simplification is not always valid because when $p+q$ is negative we have $\sqrt{p^2 + 2pq + q^2} = |p + q| \neq p + q$ [**2**].

**(37)** [**0405R**] What Maple command would you use to convert $\arcsin(\sin(x))$ to $x$? Give an example of a value of $x$ for which this simplification is not valid. **(4 marks)**

**Solution:** The relevant command is `simplify(arcsin(sin(x)),symbolic);`. [**1**]The resulting simplification is not valid when $x = 2\pi$ (for example): we have $\sin(2\pi) = 0$ and so

$$\arcsin(\sin(2\pi)) = \arcsin(0) = 0 \neq 2\pi.[\mathbf{3}]$$

**(38)** [**Mock2**] What Maple command would you use to convert $\arccos(\cos(x))$ to $x$? Give an example of $x$ for which this simplification is not valid. **(4 marks)**

**Solution:** The relevant command is `simplify(arccos(cos(x)),symbolic);`. [**1**]The resulting simplification is not valid when $x = 2\pi$ (for example): we have $\cos(2\pi) = 1$ and so

$$\arccos(\cos(2\pi)) = \arccos(1) = 0 \neq 2\pi.[\mathbf{3}]$$

**(39)** [**0405R**]

(i) What Maple command would you use to convert the expression $x^4 - y^4$ to the form $(x - y)(x + y)(x^2 + y^2)$? **(1 marks)**

(ii) What Maple command would you use to convert the expression $a^2 + ab + a^3 + a^2b + ab^2$ to the form $a^3 + (1 + b)a^2 + (b + b^2)a$? **(1 marks)**

(iii) What Maple command would you use to convert the expression $(x + 1/x)^4$ to the form $x^4 + 4x^2 + 6 + 4x^{-2} + x^{-4}$? **(1 marks)**

**Solution:**

(i) `factor(x^4-y^4);` **[1]**

(ii) `collect(a^2+a*b+a^3+a^2*b+a*b^2,a);` **[1]**

(iii) `expand((x+1/x)^4);` **[1]**

## 1.5 Sequences

**(40)** [**0405**] What would you enter in Maple to generate the sequence $1, x^2, x^4, x^6, \ldots, x^{100}$? **(2 marks)**

**Solution:** `seq(x^(2*k),k=0..50);` [**2**]

**(41)** [**Mock2; 0405R**] What would you enter in Maple to generate the sequence $dy/dx, d^2y/dx^2, \ldots, d^{10}y/dx^{10}$? **(3 marks)**

**Solution:** `seq(diff(y,x$k),k=1..10);` [**3**]

**(42)** [**Mock1**] What would you enter in Maple to generate the sequence

$$x, \frac{1}{x^2}, x^3, \frac{1}{x^4}, \ldots, x^{19}, \frac{1}{x^{20}}?$$ **(4 marks)**

**Solution:** `seq(x^(-(-1)^k * k),k=1..20);` [**4**]

**(43)** Suppose that $y$ has been set equal to some function of $x$. What would you enter in Maple to generate the sequence $dy/dx, d^2y/dx^2, \ldots, d^{10}y/dx^{10}$? **(3 marks)**

**Solution:** `seq(diff(y,x$k),k=1..10);` [**3**]

**(44)** [**0708R; 0809**] What would you enter in Maple to generate the sequence

$$\frac{1}{x+2}, \frac{2}{x+4}, \frac{3}{x+6}, \ldots, \frac{10}{x+20}$$ **(4 marks)**

**Solution:** `seq(k/(x+2*k),k=1..10);` [**4**]

**(45)** [**0809**] What would you enter in Maple to do the following?

(a) Define the function $g(n) = \dfrac{(2n)!\,\sqrt{n}}{2^{2n}\,n!^2}$. **(3 marks)**

(b) Find numerical approximations to $1/g(1)^2, 1/g(2)^2, \ldots, 1/g(100)^2$ (all in one go). **(3 marks)**

**Solution:**

(a) `g := (n) -> ((2*n)! * sqrt(n))/(2^(2*n) * n!^2);`
([**1**]for `g := (n)->` and [**2**] for the rest.)

(b) `seq(evalf(1/g(n)^2),n=1..100);` [**3**]

## 1.6 Coefficients

**(46)** [**0405R**] What would you enter in Maple to find the coefficient of $x$ in

$$x(x-1)(x-2)(x-3)(x-4)/24? \qquad \textbf{(3 marks)}$$

**Solution:** `coeff(x*(x-1)*(x-2)*(x-3)*(x-4)/24,x);`. [**3**]

**(47)** [**Mock1**] What would you enter in Maple to find the coefficient of $x^4$ in $(((x^2+a)^2+a)^2+a)^2+a$? **(2 marks)**

**Solution:** `coeff((((x^2+a)^2+a)^2+a)^2+a,x^4);` [**2**]

**(48)** [**Mock2**] What would you enter in Maple to find the coefficient of $a$ in

$$(a^{10}-b^{10})(a^5-b^5)^{-1}(a^2-b^2)^{-1}(a-b)? \qquad \textbf{(3 marks)}$$

**Solution:** `coeff(simplify((a^10-b^10)/(a^5-b^5)/(a^2-b^2)*(a-b)),a);` [**3**] (Note that the `simplify` command is necessary here; many students will miss that.)

**(49)** What would you enter in Maple to find the coefficient of $t^5$ in $(t\cos(\pi/8)+(1-t)\sin(\pi/8))^6$? What would you enter to find the constant term? **(2 marks)**

**Solution:**

```
a := (t*cos(Pi/8)+(1-t)*sin(Pi/8))^6;
coeff(a,t,5);
coeff(a,t,0);
```

For the coefficient of $t^5$ it will also work to enter `coeff(a,t^5);`, but `coeff(a,t^0);` will give an error. [**2**]

**(50)** [**0708R; 0809**] What would you enter in Maple to find the coefficient of $t^5$ in $(1+t)^{10}+(1+2t)^{10}$? What would you enter to find the constant term? **(2 marks)**

**Solution:**

```
a := (1+t)^10 + (1+2*t)^10;
coeff(a,t,5);
coeff(a,t,0);
```

For the coefficient of $t^5$ it will also work to enter `coeff(a,t^5);`, but `coeff(a,t^0);` will give an error. [**2**]

## 1.7 Series

**(51)** [**0506**] Write down commands to

(a) Find the Taylor series of $x/(e^x-1)$, discarding terms involving $x^{12}$ and higher; **(2 marks)**

(b) Convert the result to an ordinary polynomial (with no $O(x)$ term); **(2 marks)**

(c) Find the coefficient of $x^{10}$ in the result. **(1 marks)**

**Solution:**

(a) `A:=series(x/(exp(x)-1),x=0,12);` [**2**]

(b) `B:=convert(A,polynom);` [**2**]

(c) `C:=coeff(B,x^10);`. [**1**]

**(52)** [**0405; 0708R**] What would you enter in Maple to find the Taylor series of $\tan(x)$ about $x = \pi/4$, up to and including the term in $(x - \pi/4)^6$? **(2 marks)**

**Solution:** `series(tan(x),x=Pi/4,7);` [**2**]

**(53)** [**0405R**] What would you enter in Maple to find the Taylor series of $\arctan(x)$ about $x = 1$, up to and including the term in $(x - 1)^6$? **(2 marks)**

**Solution:** `series(arctan(x),x=1,7);` [**2**]

**(54)** What would you type to do the following?

(a) Define the functions $f(x) = x/(x^4 - a)$ and $g(x) = x/(x^4 + 1/a)$. **(4 marks)**

(b) Find the Taylor series for $f(g(x))$ near $x = 0$, keeping the term in $x^9$ but discarding higher terms. **(2 marks)**

(c) Find and simplify the coefficient of $x^5$ in this result. **(2 marks)**

**Solution:**

(a)
```
f := (x) -> x/(x^4-a);
g := (x) -> x/(x^4+1/a);
```
[**4**]

(b) `A := series(f(g(x)),x=0,10);` [**2**]

(c) `B := simplify(coeff(A,x,5));` [**2**]

**(55)** [**0506R**] Write down commands to

(a) Find the Taylor series at zero of $\tan(x - x^2)$, discarding terms involving $x^{20}$ and higher; **(2 marks)**

(b) Convert the result to an ordinary polynomial (with no $O(x)$ term); **(2 marks)**

(c) Find the coefficient of $x^6$ in the result. **(1 marks)**

**Solution:**

(a) `A:=series(tan(x-x^2),x=0,20);` [**2**]

(b) `B:=convert(A,polynom);` [**2**]

(c) `C:=coeff(B,x^6);`. [**1**]

## 1.8 Miscellaneous

**(56)** [**0910**] Consider the functions $y = e^{-x^2}$ and $z = \dfrac{d^6y}{dy^6} + 30\dfrac{d^4y}{dx^4} + 180\dfrac{d^2y}{dx^2} + 120y$.

(a) How would you enter these definitions in Maple? **(3 marks)**

(b) You may assume that $z$ works out to be $64x^6e^{-x^2}$. Find by hand the maximum value of $z$. **(4 marks)**

**Solution:**

(a)
```
y := exp(-x^2);
z := diff(y,x$6) + 30*diff(y,x$4) + 180*diff(y,x$2) + 120*y;
```
([**1**]for $y$, [**2**] for $z$)

(c) We have $dz/dx = 64(6x^5e^{-x^2} + x^6.(-2x)e^{-x^2}) = 128x^5e^{-x^2}(3-x^2)$, [**2**] which vanishes for $x = 0$ and $x = \pm\sqrt{3}$ [**1**]. When $x = 0$ we have $z = 0$, and when $x = \pm\sqrt{3}$ we have $z = 64 \times \sqrt{3}^6 e^{-3} = 1728e^{-3}$. It follows that the maximum value is $1728e^{-3}$. [**1**]

**(57)** [**0910**] Suppose that

$$w = t^2 - 1 \qquad x = w^2 - 2 \qquad y = x^2 - 3 \qquad z = y^2 - 4.$$

There are six real values of $t$ for which $z = 1$, say $t_1, \dots, t_6$. Give Maple commands to do the following:

(a) Enter all the definitions, and plot $y$ and $z$ together for $|t| \leq 1.8$, with the vertical range from $-10$ to 10. **(3 marks)**

(b) Find numerical approximations to the values $t_i$. Arrange your syntax so that Maple replies in the form

$$sols := \{t = -1.752372095\}, \dots, \{t = 1.752372095\}$$

**(2 marks)**

(c) Find the value of $x$ when $t = t_3$. **(1 marks)**

(d) Find the coefficient of $t^8$ in $z$. **(1 marks)**

**Solution:**

```
w := t^2-1; x := w^2-2; y := x^2-3; z := y^2-4;
plot([y,z],t=-1.8..1.8,-10..10);
sols := fsolve(z=1,{t});
subs(sols[3],x);
coeff(z,t,8);
```

[**7**] divided as in the question.

**(58)** [**0708**] Put $y = e^{ax-x^2}$ and $z = d^3y/dx^3$.

(a) How would you enter these definitions in Maple? **(3 marks)**

(b) How would you find the value of $z$ at the point $x = a/2$? **(1 marks)**

**Solution:**

(a) `y := exp(a*x-x^2);` [**1**]`z := diff(y,x,x,x);` [**2**]

(b) `subs(x=a/2,z);` [**1**]

**(59)** [**0506R**]

(a) How would you ask Maple to find a numerical solution to the following equations:

$$\begin{aligned} x^4 + y^2 + z^2 &= 21 \\ x^2 + y^4 + z^2 &= 35 \\ x^2 + y^2 + z^4 &= 57 \end{aligned}$$

You should give a command that will make Maple respond in the following form:

$$\text{sol} := \{x = -1.732050808, y = 2.236067977, z = -2.645751311\}$$

**(4 marks)**

(b) How would you then find the value of $x^2 + y^2 + z^2$, where $x$, $y$ and $z$ are as above? (You should give an answer that does not involve any retyping, cutting or pasting.) **(2 marks)**

**Solution:**

(a) `sol := fsolve({x^4+y^2+z^2=21,x^2+y^4+z^2=35,x^2+y^2+z^4=57},{x,y,z});` [**4**]

(b) `subs(sol,x^2+y^2+z^2);` [**2**]

**(60)** [**0506**]

(a) How would you ask Maple to find the numerical solution to the equations $y = \sin(x)$ and $x = \cos(y)$ ? You should give a command that will make Maple respond in the following form:

$$\text{sol} := \{x = .7681691567, y = .6948196907\}$$

**(2 marks)**

(b) How would you then find the value of $\tan(x)\tan(y)$, where $x$ and $y$ are as above? (You should give an answer that does not involve any retyping, cutting or pasting.) **(2 marks)**

**Solution:**

(a) `sol := fsolve({y=sin(x),x=cos(y)},{x,y});` [**2**]

(b) `evalf(subs(sol,tan(x)*tan(y)));` [**2**] (no penalty for omitting `evalf()`)

**(61)** [**0708; 0708R**] Write down commands to:

(a) Tell Maple to perform all numerical calculations to 20 significant figures. **(2 marks)**

(b) Define the function $f(x) = (x + x^4)/(1 + x^9)$; **(2 marks)**

(c) Find an approximate root of $f(x) = 1$ close to $x = 3/4$. **(3 marks)** You should arrange the details so that Maple responds in the form

$$sol := \{x = 0.75487766624669276005\}.$$

(d) Find the value of $\tan(\pi x)$ at this approximate root (without retyping). **(2 marks)**

**Solution:**

(a) `Digits := 20;` [**2**]

(b) `f := (x) -> (x+x^4)/(1+x^9);` [**2**]

(c) `sol := fsolve(f(x)=1,{x=3/4});` [**3**]

(d) `evalf(subs(sol,tan(Pi*x)));` [**2**] (no penalty for omitting `evalf`).

**(62)** [**0809**] Consider the function $y = (x+1)^{10} - (x+2)^{10} - (x+3)^{10} + (x+4)^{10}$. What would you enter in Maple to do the following?

(a) Find the coefficient of $x^8$ in $y$. **(1 marks)**

(b) Find the value of $y$ when $x = -5/2$. **(1 marks)**

(c) Find and factorise the third derivative $d^3y/dx^3$. **(2 marks)**

(d) Find the integral $\int_0^1 y\,dx$. **(1 marks)**

**Solution:**

(a) `coeff(y,x^8);` or `coeff(y,x,8);` [**1**]

(b) `subs(x=-5/2,y);` [**1**]

(c) `factor(diff(y,x,x,x));` [**2**]

(d) `int(y,x=0..1);` [**1**]

**(63)** [**0809**] Let $a$ and $b$ be constants. How would you ask Maple to find the points $(x, y)$ where $x^4 + x^2y^2 + y^4 = a$ and $3x^2y^2 = b$? **(2 marks)**

**Solution:** `solve({x^4+x^2*y^2+y^4=a,3*x^2*y^2=b},{x,y});` [**2**]

# 2 Mathematics questions

## 2.1 Special functions

**(64)** [**0910**] Prove that $\sin(\theta)(\sin(\theta) + \sin(3\theta) + \sin(5\theta)) = \sin(3\theta)^2$. **(5 marks)**

**Solution:** Write $e = e^{i\theta}$, so

$$\sin(\theta) = \frac{u - u^{-1}}{2i} \qquad \sin(3\theta) = \frac{u^3 - u^{-3}}{2i} \qquad \sin(5\theta) = \frac{u^5 - u^{-5}}{2i}.[\mathbf{1}]$$

Then

$$\begin{aligned}
&\sin(\theta)(\sin(\theta) + \sin(3\theta) + \sin(5\theta)) \\
=&\frac{u - u^{-1}}{2i}\frac{1}{2i}\left(u - u^{-1} + u^3 - u^{-3} + u^5 - u^{-5}\right)[\mathbf{1}] \\
=&\frac{1}{(2i)^2}((u^2 - 1 + u^4 - u^{-2} + u^6 - u^{-4}) - (1 - u^{-2} + u^2 - u^{-4} + u^4 - u^{-6})) \\
=&\frac{1}{(2i)^2}(u^6 - 2 - u^{-6})[\mathbf{1}] = \frac{(u^3 - u^{-3})^2}{(2i)^2}[\mathbf{1}] \\
=&\sin(3\theta)^2.[\mathbf{1}]
\end{aligned}$$

**(65) [Mock1; 0708R]** Show that $\dfrac{1}{\tanh(x)} - \tanh(x) = \dfrac{2}{\sinh(2x)}$. **(6 marks)**

**Solution:** Put $u = e^x$, so

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{u - u^{-1}}{u + u^{-1}}.\mathbf{[2]}$$

It follows that

$$\begin{aligned}
1/\tanh(x) - \tanh(x) &= \frac{u + u^{-1}}{u - u^{-1}} - \frac{u - u^{-1}}{u + u^{-1}} \\
&= \frac{(u + u^{-1})^2 - (u - u^{-1})^2}{(u + u^{-1})(u - u^{-1})} \\
&= \frac{u^2 + 2 + u^{-2} - u^2 + 2 - u^{-2}}{u^2 - u^{-2}} \\
&= \frac{4}{u^2 - u^{-2}} = 2/((u^2 - u^{-2})/2) \\
&= 2/\sinh(2x).\mathbf{[4]}
\end{aligned}$$

**(66) [Mock2]** Simplify the expression $\cosh(x)^4 - 2\cosh(x)^2\sinh(x)^2 + \sinh(x)^4$. (Some preliminary rearrangement will make this easier.) **(6 marks)**

**Solution:** First, we have

$$\begin{aligned}
\cosh(x)^2 - \sinh(x)^2 &= \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4} \\
&= (e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x})/4 = 4/4 = 1.\mathbf{[3]}
\end{aligned}$$

Squaring both sides gives

$$\cosh(x)^4 - 2\cosh(x)^2\sinh(x)^2 + \sinh(x)^4 = 1^2 = 1.\mathbf{[3]}$$

Alternatively:

$$\begin{aligned}
\cosh(x)^4 &= \frac{1}{16}(e^{4x} + 4e^{2x} + 6 + 4e^{-2x} + e^{-4x}) \\
\sinh(x)^4 &= \frac{1}{16}(e^{4x} - 4e^{2x} + 6 - 4e^{-2x} + e^{-4x}) \\
-2\cosh(x)^2\sinh(x)^2 &= \frac{-2}{16}(e^{2x} + 2 + e^{-2x})(e^{2x} - 2 + e^{-2x}) \\
&= \frac{-2}{16}(e^{4x} - 2 + e^{-4x}).
\end{aligned}$$

Adding these together gives $(6 + 6 + 4)/16 = 1$ again.

**(67) [0405; 0405R]** Simplify the expression $1 + 8(\cosh(x)^4 - \cosh(x)^2)$. Your answer should be in the form $\sinh(nx)$ or $\cosh(nx)$ or $\tanh(nx)$. **(5 marks)**

**Solution:** Put $u = e^x$. Then

$$\begin{aligned}
&1 + 8(\cosh(x)^4 - \cosh(x)^2) \\
=&1 + 8\frac{(u + u^{-1})^4}{2^4} - 8\frac{(u + u^{-1})^2}{2^2}\mathbf{[1]} \\
=&1 + \tfrac{1}{2}(u^4 + 4u^2 + 6 + 4u^{-2} + u^{-4}) - 2(u^2 + 2 + u^{-2})\mathbf{[2]} \\
=&\tfrac{1}{2}(2 + u^4 + 4u^2 + 6 + 4u^{-2} + u^{-4} - 4u^2 - 8 - 4u^{-2}) \\
=&(u^4 + u^{-4})/2\mathbf{[1]} = \cosh(4x)\mathbf{[1]}.
\end{aligned}$$

**(68)** **[0708]** Prove that $\sin(10x) + \sin(11x) = 2\cos(x/2)\sin(21x/2)$. **(5 marks)**

**Solution:**

$$\begin{aligned}
2\cos(x/2)\sin(21x/2) &= 2\frac{(e^{ix/2} + e^{-ix/2})}{2}\frac{(e^{21ix/2} - e^{-21ix/2})}{2i}[\mathbf{1}] \\
&= \tfrac{1}{2i}\left(e^{22ix/2} - e^{-20ix/2} + e^{20ix/2} - e^{-22ix/2}\right)[\mathbf{2}] \\
&= \frac{e^{11ix} - e^{-11ix}}{2i} + \frac{e^{10ix} - e^{-10ix}}{2i}[\mathbf{1}] \\
&= \sin(11x) + \sin(10x).[\mathbf{1}]
\end{aligned}$$

**(69)** **[0506]** Simplify $\cosh(x)^4 - \sinh(x)^4$, and thus find $\int \cosh(x)^4 - \sinh(x)^4\,dx$. **(7 marks)**

**Solution:** Put $u = e^x$. Then

$$\begin{aligned}
\cosh(u)^4 &= \left(\frac{u + u^{-1}}{2}\right)^4 = \frac{1}{16}(u^4 + 4u^2 + 6 + 4u^{-2} + u^{-4}) \\
\sinh(u)^4 &= \left(\frac{u - u^{-1}}{2}\right)^4 = \frac{1}{16}(u^4 - 4u^2 + 6 - 4u^{-2} + u^{-4})[\mathbf{3}] \\
\cosh(u)^4 - \sinh(u)^4 &= \frac{1}{16}(8u^2 + 8u^{-2}) = \frac{u^2 + u^{-2}}{2} = \cosh(2x).[\mathbf{2}]
\end{aligned}$$

It follows that

$$\int \cosh(x)^4 - \sinh(x)^4\,dx = \sinh(2x)/2.[\mathbf{2}]$$

**(70)** **[0708]** Write $\cos(x)^3$ as a trigonometric polynomial, and thus find $\int \cos(x)^3\,dx$. **(6 marks)**

**Solution:** Put $u = e^{ix}$, so

$$\begin{aligned}
\cos(x)^3 &= \left(\frac{u + u^{-1}}{2}\right)^3[\mathbf{1}] = \frac{1}{8}(u^3 + 3u + 3u^{-1} + u^{-3})[\mathbf{1}] = \frac{1}{4}\,\frac{(u^3 + u^{-3})}{2} + \frac{3}{4}\,\frac{(u + u^{-1})}{2}[\mathbf{1}] \\
&= \cos(3x)/4 + 3\cos(x)/4.[\mathbf{1}]
\end{aligned}$$

This gives

$$\int \cos(x)^3\,dx = \int \frac{\cos(3x)}{4} + \frac{3\cos(x)}{4}\,dx = \frac{\sin(3x)}{12} + \frac{3\sin(x)}{4} = \frac{\sin(3x) + 9\sin(x)}{12}.[\mathbf{2}]$$

**(71)** **[0506R; 0809]** Put $f(x) = 4\cosh(x)^3 + 4\sinh(x)^3 - 3\cosh(x) + 3\sinh(x)$. Simplify $f(x)$, and thus find $\int_{-1}^{1} f(x)\,dx$. **(7 marks)**

**Solution:** Put $u = e^x$. Then

$$\begin{aligned}
4\cosh(x)^3 &= 4\left(\frac{u+u^{-1}}{2}\right)^3 = \frac{1}{2}(u^3+3u+3u^{-1}+u^{-3})[\mathbf{1}] \\
4\sinh(x)^3 &= 4\left(\frac{u-u^{-1}}{2}\right)^3 = \frac{1}{2}(u^3-3u+3u^{-1}-u^{-3})[\mathbf{1}] \\
-3\cosh(x) &= \frac{1}{2}(-3u-3u^{-1})[\mathbf{1}] \\
3\sinh(x) &= \frac{1}{2}(3u-3u^{-1})[\mathbf{1}] \\
f(x) &= u^3 = e^{3x}[\mathbf{1}]
\end{aligned}$$

It follows that

$$\int_{-1}^{1} f(x)\,dx = [e^{3x}/3]_{-1}^{1} = (e^3 - e^{-3})/3.[\mathbf{2}]$$

**(72)** **[Mock1; 0405R]** Find $\int \sinh(x)^2\cosh(x)^2\,dx$. **(6 marks)**

**Solution:**

$$\begin{aligned}
\sinh(x)^2\cosh(x)^2 &= \left(\frac{e^x-e^{-x}}{2}\,\frac{e^x+e^{-x}}{2}\right)^2 = \tfrac{1}{16}(e^{2x}-e^{-2x})^2 \\
&= \tfrac{1}{16}(e^{4x}+e^{-4x}) - \tfrac{1}{8} = \tfrac{1}{8}\cosh(4x) - \tfrac{1}{8}[\mathbf{4}]
\end{aligned}$$

so

$$\int \sinh(x)^2\cosh(x)^2\,dx = \tfrac{1}{32}\sinh(4x) - \tfrac{1}{8}x[\mathbf{2}].$$

**(73)** **[Mock2; 0708R]** Find $\int \cosh(x)^2\,dx$. **(4 marks)**

**Solution:**

$$\begin{aligned}
\cosh(x)^2 &= \left(\frac{e^x+e^{-x}}{2}\right)^2 = \frac{1}{4}(e^{2x}+2+e^{-2x}) \\
&= \tfrac{1}{2}\cosh(2x) + \tfrac{1}{2}[\mathbf{2}]
\end{aligned}$$

so

$$\int \cosh(x)^2\,dx = \tfrac{1}{4}\sinh(2x) + \tfrac{1}{2}x.[\mathbf{2}]$$

**(74)** **[0405]** Find $\int \sinh(x)^3\,dx$, expressing your answer in terms of hyperbolic functions. **(5 marks)**

**Solution:** We have

$$\sinh(x)^3 = \left(\frac{e^x-e^{-x}}{2}\right)^3 = \frac{1}{8}(e^{3x}-3e^x+3e^{-x}-e^{-3x})[\mathbf{2}]$$

so

$$\int \sinh(x)^3\,dx = \frac{1}{8}\int e^{3x} - 3e^{x} + 3e^{-x} - e^{-3x}\,dx = \tfrac{1}{24}e^{3x} - \tfrac{3}{8}e^{x} - \tfrac{3}{8}e^{-x} + \tfrac{1}{24}e^{3x}$$
$$= \tfrac{1}{12}\cosh(3x) - \tfrac{3}{4}\cosh(x).\mathbf{[3]}$$

**(75)** Find $\int \sinh(x)^2\cosh(x)^2\,dx$. **(6 marks)**

**Solution:**

$$\sinh(x)^2\cosh(x)^2 = \left(\frac{e^{x}-e^{-x}}{2}\,\frac{e^{x}+e^{-x}}{2}\right)^2 = \tfrac{1}{16}(e^{2x}-e^{-2x})^2$$
$$= \tfrac{1}{16}(e^{4x}+e^{-4x}) - \tfrac{1}{8} = \tfrac{1}{8}\cosh(4x) - \tfrac{1}{8}\mathbf{[4]}$$

so

$$\int \sinh(x)^2\cosh(x)^2\,dx = \tfrac{1}{32}\sinh(4x) - \tfrac{1}{8}x\mathbf{[2]}.$$

**(76)** Show that $\dfrac{1+\tanh(x)^2}{1-\tanh(x)^2} = \cosh(2x)$. **(6 marks)**

**Solution:** Put $u = e^x$. Then

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{(u-u^{-1})/2}{(u+u^{-1})/2} = \frac{u-u^{-1}}{u+u^{-1}},\mathbf{[2]}$$

so

$$1+\tanh(x)^2 = 1 + \left(\frac{u-u^{-1}}{u+u^{-1}}\right)^2 = 1 + \frac{u^2-2+u^{-2}}{u^2+2+u^{-2}}$$
$$= \frac{(u^2+2+u^{-2})+(u^2-2+u^{-2})}{u^2+2+u^{-2}} = \frac{2u^2+2u^{-2}}{u^2+2+u^{-2}}\mathbf{[2]}$$
$$1-\tanh(x)^2 = 1 - \left(\frac{u-u^{-1}}{u+u^{-1}}\right)^2 = 1 - \frac{u^2-2+u^{-2}}{u^2+2+u^{-2}}$$
$$= \frac{(u^2+2+u^{-2})-(u^2-2+u^{-2})}{u^2+2+u^{-2}} = \frac{4}{u^2+2+u^{-2}}\mathbf{[1]}$$

so

$$\frac{1+\tanh(x)^2}{1-\tanh(x)^2} = \frac{2u^2+2u^{-2}}{4} = \frac{e^{2x}+e^{-2x}}{2} = \cosh(2x)\mathbf{[1]}.$$

## 2.2 Differentiation

**(77)** [**0910**] Find and simplify $\cos(x)^2\dfrac{d}{dx}\left(\tan(x)^2\right)$. **(3 marks)**

**Solution:** The power rule gives $\frac{d}{dx}(\tan(x)^2) = 2\tan(x)\tan'(x)$, [**1**]and it is standard that $\tan'(x) = \sec(x)^2 = \cos(x)^{-2}$, [**1**]so $\cos(x)^2\frac{d}{dx}(\tan(x)^2) = 2\tan(x)$. [**1**]Various other approaches are also possible.

**(78)** [**0910**] Find and simplify $\dfrac{d}{dx}\log\left(\dfrac{x^3-1}{x^3+1}\right)$. **(5 marks)**

**Solution:** Put $u = x^3$ and $v = (u-1)/(u+1)$ and $y = \log(v) = \log\left(\frac{x^3-1}{x^3+1}\right)$. We then have

$$\begin{aligned}
\frac{dy}{dv} &= 1/v = \frac{x^3+1}{x^3-1}[\mathbf{1}] \\
\frac{dv}{du} &= \frac{1.(u+1) - 1.(u-1)}{(u+1)^2} = \frac{2}{(x^3+1)^2}[\mathbf{1}] \\
\frac{du}{dx} &= 3x^2[\mathbf{1}] \\
\frac{dy}{dx} &= \frac{dy}{dv}\frac{dv}{du}\frac{du}{dx} = \frac{x^3+1}{x^3-1}\,\frac{2}{(x^3+1)^2}\,3x^2 \\
&= \frac{6x^2}{(x^3-1)(x^3+1)} = \frac{6x^2}{x^6-1}.[\mathbf{2}]
\end{aligned}$$

**(79)** [**0910**] There are numbers $a$ and $b$ such that the function $y = x^3(a\log(x) + b)$ satisfies $d^3y/dx^3 = \log(x)$. Find these numbers. **(5 marks)**

**Solution:** Using the product rule repeatedly, we have

$$\begin{aligned}
y &= x^3(a\log(x) + b) \\
dy/dx &= 3x^2(a\log(x) + b) + x^3(a/x) = x^2(3a\log(x) + (a+3b))[\mathbf{1}] \\
d^2y/dx^2 &= 2x(3a\log(x) + (a+3b)) + x^2(3a/x) = x(6a\log(x) + (5a+6b))[\mathbf{1}] \\
d^3y/dx^3 &= (6a\log(x) + (5a+6b)) + x(6a/x) = 6a\log(x) + (11a+6b).[\mathbf{1}]
\end{aligned}$$

We want this to be equal to $\log(x)$, so we must have $6a = 1$ and $11a + 6b = 0$, so $a = 1/6$ and $b = -11a/6 = -11/36$ [**2**].

**(80)** Put $f(x) = x/\sqrt{1+x^2}$. Simplify $\sqrt{1+x^2}f'(x)$, and hence find a constant $c$ such that $f'(x) = (1+x^2)^c$. **(5 marks)**

**Solution:**

**(81)** Let $a$, $b$ and $\omega$ be constants. Find $f'(x)$, where $f(x) = e^{-(x-a)^2/b}\sin(\omega x)$. **(4 marks)**

**Solution:** First put $u = -(x-a)^2/b$, so $du/dx = -2(x-a)/b$. Then put $v = \exp(u) = e^{-(x-a)^2/b}$, so the chain rule gives

$$\frac{dv}{dx} = -2(x-a)b^{-1}e^{-(x-a)^2/b}.[\mathbf{2}]$$

Finally, we apply the product rule:

$$\begin{aligned}
\frac{d}{dx}\left(e^{-(x-a)^2/b}\sin(\omega x)\right) &= -2(x-a)b^{-1}e^{-(x-a)^2/b}\sin(\omega x) + e^{-(x-a)^2/b}\omega\cos(\omega x) \\
&= e^{-(x-a)^2/b}(\omega\cos(\omega x) - 2(x-a)b^{-1}\sin(\omega x))[\mathbf{2}].
\end{aligned}$$

**(82)** Let $a$, $b$ and $n$ be constants. Find $f'(x)$, where $f(x) = \left(\dfrac{x-a}{x-b}\right)^n$. **(3 marks)**

**Solution:** Put $u = (x-a)/(x-b)$ and $y = f(x) = u^n$. Then

$$\frac{du}{dx} = \frac{1.(x-b)-(x-a).1}{(x-b)^2} = \frac{a-b}{(x-b)^2},$$

so

$$f'(x) = \frac{dy}{dx} = nu^{n-1}\frac{du}{dx} = n(a-b)\left(\frac{x-a}{x-b}\right)^{n-1}(x-b)^{-2} = n(a-b)(x-a)^{n-1}(x-b)^{-n-1}.$$

**(83)** Find $\dfrac{d}{dx}\cos\left(\left(\dfrac{x+1}{2}\right)^2\right)$. **(2 marks)**

**Solution:** By the chain rule, we have

$$\frac{d}{dx}\cos\left(\left(\frac{x+1}{2}\right)^2\right) = -\sin\left(\left(\frac{x+1}{2}\right)^2\right).\frac{d}{dx}\left(\frac{x+1}{2}\right)^2 = -\sin\left(\left(\frac{x+1}{2}\right)^2\right).\frac{x+1}{2}.$$

**(84)** Find $\dfrac{d}{dx}\log(\cos(x))$. **(2 marks)**

**Solution:** By the logarithmic rule, we have

$$\frac{d}{dx}\log(\cos(x)) = \frac{\cos'(x)}{\cos(x)} = -\frac{\sin(x)}{\cos(x)} = -\tan(x).$$

**(85)** Let $a$, $b$, $c$ and $d$ be constants. Find $\dfrac{d}{dx}\left(\dfrac{ax+bx^{-1}}{cx+dx^{-1}}\right)$. **(4 marks)**

**Solution:** Put $y = \sqrt{2\pi}x^{x-1/2}e^{-x}$, so

$$\log(y) = \log(\sqrt{2\pi}) + (x-1/2)\log(x) - x,$$

so

$$\begin{aligned}\frac{y'}{y} &= \log(y)' \\ &= 0 + 1.\log(x) + (x-1/2).\log'(x) - 1 \\ &= \log(x) + (x-1/2)/x - 1 = \log(x) + 1 - 1/(2x) - 1 \\ &= \log(x) - 1/(2x).\end{aligned}$$

**(86)** **[Mock2]** Find $f'(x)$, where $f(x) = \ln((x^3+1)^5)$. **(3 marks)**

**Solution:**

$$f'(x) = \frac{1}{(x^3+1)^5}.5(x^3+1)^4.3x^2 = \frac{15x^2}{x^3+1}.\textbf{[3]}$$

**(87)** Find $\dfrac{d}{dx}\left(\dfrac{x}{\log(x)}\right)$. **(3 marks)**

**Solution:** The quotient rule gives

$$\frac{d}{dx}\left(\frac{x}{\log(x)}\right) = \frac{1.\log(x) - x.\log'(x)}{\log(x)^2}[\mathbf{1}] = \frac{\log(x) - x.x^{-1}}{\log(x)^2}[\mathbf{1}]$$
$$= \frac{1}{\log(x)} - \frac{1}{\log(x)^2}[\mathbf{1}].$$

**(88)** $\frac{d}{dx}\left(\frac{3x+2}{4x+3}\right)$. **(2 marks)**

**Solution:**

$$\frac{d}{dx}\left(\frac{3x+2}{4x+3}\right) = \frac{3(4x+3) - 4(3x+2)}{(4x+3)^2}[\mathbf{1}]$$
$$= \frac{12x+9-12x-8}{(4x+3)^2} = (4x+3)^{-2}[\mathbf{1}]$$

**(89)** Find $\frac{d}{dx}\log(1+x+x^2+x^3)$. **(2 marks)**

**Solution:** Put $u = 1+x+x^2+x^3$ and $y = \log(u)$, so

$$y' = \frac{u'}{u} = \frac{1+2x+3x^2}{1+x+x^2+x^3}[\mathbf{2}].$$

**(90) [Mock2; 0405R]** Find $\frac{d}{dx}\left(\frac{x^4+x^2+1}{x^3-x}\right)$. **(4 marks)**

**Solution:**

$$\frac{d}{dx}\left(\frac{x^4+x^2+1}{x^3-x}\right) = \frac{(4x^3+2x)(x^3-x) - (x^4+x^2+1)(3x^2-1)}{(x^3-x)^2} \qquad [\mathbf{2}]$$
$$= \frac{4x^6 - 4x^4 + 2x^4 - 2x^2 - 3x^6 + x^4 - 3x^4 + x^2 - 3x^2 + 1}{(x^3-x)^2}$$
$$= \frac{x^6 - 4x^4 - 4x^2 + 1}{(x^3-x)^2} \qquad [\mathbf{2}]$$

**(91) [Mock2]** Find $\frac{d}{dx}\sin(2x)^4$. **(2 marks)**

**Solution:** $4\sin(2x)^3.2\cos(2x) = 8\sin(2x)^3\cos(2x)$. **[2]**

**(92) [0405; 0708R]** Find $dy/dx$, where $y = \tan(x^2+1)$. **(3 marks)**

**Solution:**

$$\frac{dy}{dx} = \sec(x^2+1)^2\frac{d}{dx}(x^2+1)[\mathbf{2}] = \frac{2x}{\cos(x^2+1)^2}[\mathbf{1}]$$

**(93) [0405]** Find $\frac{d}{dx}(\ln(x+x^{-1}))$. **(2 marks)**

**Solution:**

$$\frac{d}{dx}(\ln(x+x^{-1})) = \frac{1}{x+x^{-1}}\frac{d}{dx}(x+x^{-1})[\mathbf{1}] = \frac{1-x^{-2}}{x+x^{-1}} = \frac{x^2-1}{x^3+x}[\mathbf{1}].$$

**(94)** Find $\frac{d}{dx}\sin\left(\frac{1+x}{1-x}\right)$. **(3 marks)**

**Solution:**

$$\frac{d}{dx}\sin\left(\frac{1+x}{1-x}\right) = \cos\left(\frac{1+x}{1-x}\right).\frac{1.(1-x)-(-1).(1+x)}{(1-x)^2} = \frac{-2x}{(1-x)^2}\cos\left(\frac{1+x}{1-x}\right) \qquad [\mathbf{3}]$$

**(95) [0405]** Find $f'(x)$, where $f(x) = \frac{ax^4+b}{cx^4+d}$. Simplify your answer as much as possible. **(4 marks)**

**Solution:**

$$\begin{aligned} f'(x) &= \frac{4ax^3(cx^4+d)-4cx^3(ax^4+b)}{(cx^4+d)^2}[\mathbf{2}] \\ &= \frac{4acx^7+4adx^3-4acx^7-4bcx^3}{(cx^4+d)^2} \\ &= \frac{4x^3(ad-bc)}{(cx^4+d)^2}[\mathbf{2}] \end{aligned}$$

**(96) [Mock1]** Find $f'(x)$, where $f(x) = \cosh(\ln(x))$. Your answer should not involve sinh, cosh, exp or ln. **(4 marks)**

**Solution:**

$$f(x) = (e^{\ln(x)} + e^{-\ln(x)})/2 = (x+x^{-1})/2, [\mathbf{2}]$$

so

$$f'(x) = (1-x^{-2})/2[\mathbf{2}].$$

**(97) [Mock1]** Find $dy/dx$, where $y = \ln(\tan(x))$. **(3 marks)**

**Solution:**

$$\frac{d}{dx}\ln(\tan(x)) = \frac{\tan'(x)}{\tan(x)}[\mathbf{1}] = \frac{1+\tan(x)^2}{\tan(x)} = \cot(x)+\tan(x).[\mathbf{2}]$$

**(98) [Mock1]** Find $dy/dx$, where $y = \frac{x^3-6x-6}{x^3-3x-4}$. Simplify your answer as much as possible. **(4 marks)**

**Solution:**

$$\begin{aligned} \frac{dy}{dx} &= \frac{(3x^2-6)(x^3-3x-4)-(x^3-6x-6)(3x^2-3)}{(x^3-3x-4)^2}[\mathbf{2}] \\ &= \frac{3x^5-9x^3-12x^2-6x^3+18x+24-3x^5+3x^3+18x^3-18x+18x^2-18}{(x^3-3x-4)^2} \\ &= \frac{6x^3+6x^2+6}{(x^3-3x-4)^2} = 6\frac{x^3+x^2+1}{(x^3-3x-4)^2}[\mathbf{2}] \end{aligned}$$

**(99)** [**Mock1**] Find $f'(x)$, where $f(x) = (x^n + 1)^{-m}$. **(2 marks)**

**Solution:**

$$f'(x) = -nmx^{n-1}(x^n + 1)^{-m-1}.\mathbf{[2]}$$

**(100)** [**0405**] Find $\frac{d}{dx}\ln(x)^5$. **(2 marks)**

**Solution:** $5\ln(x)^4/x$ [**2**]

**(101)** [**0405R**] Find $\frac{d}{dx}(\log(x^2 + x^{-2}))$. Simplify your answer as much as possible. **(3 marks)**

**Solution:**

$$\frac{d}{dx}(\log(x^2 + x^{-2})) = \frac{1}{x^2 + x^{-2}}\frac{d}{dx}(x^2 + x^{-2})\mathbf{[1]} = \frac{2x - 2x^{-3}}{x^2 + x^{-2}}\mathbf{[1]} = \frac{2(x^4 - 1)}{x(x^4 + 1)}\mathbf{[1]}.$$

**(102)** [**0405R; 0708**] Simplify $\sin(2x)\frac{d}{dx}\log(\tan(x))$ **(5 marks)**

**Solution:**

$$\begin{aligned}\sin(2x)\frac{d}{dx}\log(\tan(x)) &= \sin(2x)\frac{\tan'(x)}{\tan(x)}\mathbf{[1]} = \sin(2x)\frac{\cos(x)^{-2}}{\sin(x)/\cos(x)}\mathbf{[1]} \\ &= \sin(2x)\frac{1}{\sin(x)\cos(x)}\mathbf{[1]} = \frac{2\sin(x)\cos(x)}{\sin(x)\cos(x)}\mathbf{[1]} = 2\mathbf{[1]}\end{aligned}$$

**(103)** Find $\frac{d}{dx}\left(\frac{x^4 + x^2 + 1}{x^3 - x}\right)$. **(4 marks)**

**Solution:**

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^4 + x^2 + 1}{x^3 - x}\right) &= \frac{(4x^3 + 2x)(x^3 - x) - (x^4 + x^2 + 1)(3x^2 - 1)}{(x^3 - x)^2} \quad \mathbf{[2]} \\ &= \frac{4x^6 - 4x^4 + 2x^4 - 2x^2 - 3x^6 + x^4 - 3x^4 + x^2 - 3x^2 + 1}{(x^3 - x)^2} \\ &= \frac{x^6 - 4x^4 - 4x^2 + 1}{(x^3 - x)^2} \quad \mathbf{[2]}\end{aligned}$$

**(104)** Simplify $\sin(2x)\frac{d}{dx}\log(\tan(x))$ **(5 marks)**

**Solution:**

$$\begin{aligned}\sin(2x)\frac{d}{dx}\log(\tan(x)) &= \sin(2x)\frac{\tan'(x)}{\tan(x)}\mathbf{[1]} = \sin(2x)\frac{\cos(x)^{-2}}{\sin(x)/\cos(x)}\mathbf{[1]} \\ &= \sin(2x)\frac{1}{\sin(x)\cos(x)}\mathbf{[1]} = \frac{2\sin(x)\cos(x)}{\sin(x)\cos(x)}\mathbf{[1]} = 2\mathbf{[1]}\end{aligned}$$

**(105)** [**0708**] Put $x = \sin(e^t)$ and $y = \cos(e^t)$.

(a) Simplify $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$. **(3 marks)**

(b) Show that $e^{2t}x + \frac{d^2x}{dt^2} = e^t y$. **(3 marks)**

**Solution:**

(a) By the chain rule, $dx/dt = e^t \cos(e^t)$ **[1]**and $dy/dt = -e^t \sin(e^t)$. **[1]**Thus

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t}\cos(e^t)^2 + e^{2t}\sin(e^t)^2 = e^{2t}(\cos(e^t)^2 + \sin(e^t)^2) = e^{2t}.\mathbf{[1]}$$

(b) We can differentiate the equation $dx/dt = e^t \cos(e^t)$ using the product rule and the chain rule to get

$$\frac{d^2x}{dt^2} = \cos(e^t)\frac{d}{dt}e^t + e^t\frac{d}{dt}\cos(e^t)\mathbf{[1]} = e^t\cos(e^t) - e^{2t}\sin(e^t)\mathbf{[1]} = e^t y - e^{2t}x.$$

This rearranges to give $e^{2t}x + \frac{d^2x}{dt^2} = e^t y$ **[1]**as claimed.

**(106) [0506R; 0708R]** Find $\frac{d}{dx}\left(\frac{x+\ln(x)}{x-\ln(x)}\right)$, simplifying your answer as much as possible. **(5 marks)**

**Solution:** Put $u = x + \ln(x)$ and $v = x - \ln(x)$, so $u' = 1 + 1/x$ **[1]**and $v' = 1 - 1/x$ **[1]**. Then

$$\begin{aligned}(u/v)' &= \frac{u'v - uv'}{v^2}\mathbf{[1]} = \frac{(1+1/x)(x-\ln(x)) - (x+\ln(x))(1-1/x)}{(x-\ln(x))^2}\mathbf{[1]} \\ &= \frac{x - \ln(x) + 1 - \ln(x)/x - x + 1 - \ln(x) + \ln(x)/x}{(1-\ln(x))^2} \\ &= \frac{2(1-\ln(x))}{(x-\ln(x))^2}\mathbf{[1]}\end{aligned}$$

**(107) [0506R]** Find $dy/dt$, where $y = \sin(\alpha t)^2\cos(\beta t)$. **(5 marks)**

**Solution:** Put $u = \sin(\alpha t)^2$ and $v = \cos(\beta t)$, so $y = uv$. Then $u' = 2\alpha\sin(\alpha t)\cos(\alpha t)$ **[2]** and $v' = -\beta\sin(\beta t)$ **[1]**, so

$$\begin{aligned}y' &= u'v + uv'\mathbf{[1]} \\ &= 2\alpha\sin(\alpha t)\cos(\alpha t)\cos(\beta t) - \beta\sin(\alpha t)^2\sin(\beta t) \\ &= \sin(\alpha t)(2\alpha\cos(\alpha t)\cos(\beta t) - \beta\sin(\alpha t)\sin(\beta t))\mathbf{[1]}\end{aligned}$$

**(108) [0506R]** Find $dy/dx$, where $y = \sin(x + x^3/6 + 3x^5/40)$. **(3 marks)**

**Solution:** Put $u = x + x^3/6 + 3x^5/40$, so $du/dx = 1 + x^2/2 + 3x^4/8$ **[1]**. Then $y = \sin(u)$, so $dy/du = \cos(u) = \cos(x + x^3/6 + 3x^5/40)$ **[1]**, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (1 + x^2/2 + 3x^4/8)\cos(x + x^3/6 + 3x^5/40).\mathbf{[1]}$$

**(109)** Simplify $f'(x)\,g'(x)$, where $f(x) = \log(x^3-1)$ and $g(x) = x^2 + 2/x$. **(5 marks)**

**Solution:** We have $f'(x) = (x^3-1)^{-1}\frac{d}{dx}(x^3-1) = \frac{3x^2}{x^3-1}$ **[2]** and $g'(x) = 2x - 2x^{-2} = 2(x^3-1)/x^2$ **[1]**, so

$$f'(x)\,g'(x) = \frac{3x^2}{x^3-1}\frac{2(x^3-1)}{x^2} = 6.\mathbf{[2]}$$

**(110)** [**0506**] Find $\frac{d}{dx}\left(\frac{1+\ln(x)}{1-\ln(x)}\right)$. **(3 marks)**

**Solution:** Put $u = 1+\ln(x)$ and $v = 1-\ln(x)$, so $u' = 1/x$ and $v' = -1/x$. Then

$$\begin{aligned}(u/v)' &= \frac{u'v - uv'}{v^2} = \frac{(1-\ln(x))/x + (1+\ln(x))/x}{(1-\ln(x))^2} \\ &= \frac{2}{x(1-\ln(x))^2}\mathbf{[3]}\end{aligned}$$

**(111)** [**0506**] Find $dy/dt$, where $y = e^{-\lambda t}\sin(\omega t)^2$. **(4 marks)**

**Solution:** Put $u = e^{-\lambda t}$ and $v = \sin(\omega t)^2$, so $y = uv$. Then $u' = -\lambda e^{-\lambda t}$ **[1]**and $v' = 2\omega\sin(\omega t)\cos(\omega t)$ **[1]**, so

$$\begin{aligned}y' &= u'v + uv'\mathbf{[1]} \\ &= -\lambda e^{-\lambda t}\sin(\omega t)^2 + 2\omega e^{-\lambda t}\sin(\omega t)\cos(\omega t) \\ &= e^{-\lambda t}\sin(\omega t)(2\omega\cos(\omega t) - \lambda\sin(\omega t)).\mathbf{[1]}\end{aligned}$$

**(112)** [**0506**] Find $dy/dx$, where $y = \tan(x - x^3/3 + x^5/5)$. **(3 marks)**

**Solution:** Put $u = x - x^3/3 + x^5/5$, so $du/dx = 1 - x^2 + x^4$ **[1]**. Then $y = \tan(u)$, so $dy/du = 1/\cos(u)^2$ **[1]**, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1-x^2+x^4}{\cos(x - x^3/3 + x^5/5)^2}.\mathbf{[1]}$$

**(113)**

(a) Using the identity $1 + \tan(x)^2 = \cos(x)^{-2}$ show that $\cos(\arctan(t)) = 1/\sqrt{1+t^2}$.

(b) Using the identity $\sin(x) = \tan(x)\cos(x)$, deduce that $\sin(\arctan(t)) = t/\sqrt{1+t^2}$.

(c) Use (a) and (b) and the addition formula for sin to expand out $\sin(x - \arctan(1/\lambda))$.

(d) Deduce that

$$\frac{d}{dx}\left[\frac{\sin(x - \arctan(1/\lambda))e^{\lambda x}}{\sqrt{1+\lambda^2}}\right] = \sin(x)e^{\lambda x}.$$

**Solution:**

(a) Put $x = \arctan(t)$, so $\tan(x) = t$. Then $1 + t^2 = 1 + \tan(x)^2 = \cos(x)^{-2}$, so $\sqrt{1+t^2} = \cos(x)^{-1}$, so $1/\sqrt{1+t^2} = \cos(x) = \cos(\arctan(t))$, as claimed.

(b) Now

$$\sin(\arctan(t)) = \tan(\arctan(t))\cos(\arctan(t)) = t\cos(\arctan(t)) = \frac{t}{\sqrt{1+t^2}}.$$

(c) The addition formula

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

gives

$$\begin{aligned}\sin(x - \arctan(1/\lambda)) &= \sin(x)\cos(-\arctan(1/\lambda)) + \cos(x)\sin(-\arctan(1/\lambda)) \\ &= \frac{\sin(x)}{\sqrt{1+\lambda^{-2}}} - \frac{\lambda^{-1}\cos(x)}{\sqrt{1+\lambda^{-2}}} \\ &= \frac{\lambda\sin(x) - \cos(x)}{\sqrt{\lambda^2+1}}\end{aligned}$$

(d) It follows that

$$\frac{\sin(x - \arctan(1/\lambda))e^{\lambda x}}{\sqrt{1+\lambda^2}} = \frac{\lambda\sin(x) - \cos(x)}{\sqrt{\lambda^2+1}}\frac{e^{\lambda x}}{\sqrt{\lambda^2+1}} = \frac{\lambda\sin(x) - \cos(x)}{\lambda^2+1}e^{\lambda x}.$$

We can differentiate this to get

$$\begin{aligned}\frac{d}{dx}\left[\frac{\sin(x - \arctan(1/\lambda))e^{\lambda x}}{\sqrt{1+\lambda^2}}\right] &= (\lambda^2+1)^{-1}((\lambda\cos(x) - \sin(x))e^{\lambda x} + (\lambda\sin(x) + \cos(x))\lambda e^{\lambda x}) \\ &= \frac{\lambda\cos(x) + \sin(x) + \lambda^2\sin(x) - \lambda\cos(x)}{\lambda^2+1}e^{\lambda x} \\ &= \sin(x)e^{\lambda x}\end{aligned}$$

**(114) [0809]** Simplify the expression $\left(\frac{d}{dx}\log(\sin(x))\right)\left(\frac{d}{dx}\log(\cos(x))\right)$. **(4 marks)**

**Solution:** Using the logarithmic rule, we have

$$\begin{aligned}\frac{d}{dx}\log(\sin(x)) &= \frac{\sin'(x)}{\sin(x)} = \frac{\cos(x)}{\sin(x)}\textbf{[2]} \\ \frac{d}{dx}\log(\cos(x)) &= \frac{\cos'(x)}{\cos(x)} = -\frac{\sin(x)}{\cos(x)}.\textbf{[1]}\end{aligned}$$

We can multiply these together to get

$$\left(\frac{d}{dx}\log(\sin(x))\right)\left(\frac{d}{dx}\log(\cos(x))\right) = -\frac{\cos(x)}{\sin(x)}\frac{\sin(x)}{\cos(x)} = -1.\textbf{[1]}$$

**(115) [0809]** Consider the function $f(x) = \frac{2x^4 - 3x^2 + 2}{x^3 - x}$. Find and simplify $f'(x)$, and evaluate $f'(\sqrt{2})$. **(4 marks)**

**Solution:** Put $u = 2x^4 - 3x^2 + 2$ and $v = x^3 - x$. Using the quotient rule, we have

$$\begin{aligned} f'(x) &= (u'v - uv')/v^2 \\ u' &= 8x^3 - 6x \\ v' &= 3x^2 - 1 \\ u'v - uv' &= (8x^3 - 6x)(x^3 - x) - (2x^4 - 3x^2 + 2)(3x^2 - 1) \\ &= 8x^6 - 8x^4 - 6x^4 + 6x^2 - 6x^6 + 2x^4 + 9x^4 - 3x^2 - 6x^2 + 2 \\ &= 2x^6 - 3x^4 - 3x^2 + 2 \\ f'(x) &= \frac{2x^6 - 3x^4 - 3x^2 + 2}{x^3 - x}.\mathbf{[2]} \end{aligned}$$

Now take $x = \sqrt{2}$, so $x^2 = 2$ and $x^4 = 4$ and $x^6 = 8$. We have

$$2x^6 - 3x^4 - 3x^2 + 2 = 2 \times 8 - 3 \times 4 - 3 \times 2 + 2 = 0,$$

so $f'(\sqrt{2}) = 0$. **[2]**

**(116)** [**0809**] Consider the function $g(x) = \exp(x - \frac{1}{2}x^2)$. Evaluate $g''(1)$. **(4 marks)**

**Solution:** The chain and product rules give

$$\begin{aligned} g'(x) &= \frac{d}{dx}(x - \tfrac{1}{2}x^2)\exp'(x - \tfrac{1}{2}x^2) = (1-x)\exp(x - \tfrac{1}{2}x) = (1-x)g(x)\mathbf{[1]} \\ g''(x) &= \left(\tfrac{d}{dx}(1-x)\right)g(x) + (1-x)g'(x) \\ &= -g(x) + (1-x)^2 g(x) = (x^2 - 2x)g(x)\mathbf{[2]} \\ g''(1) &= (1^2 - 2)g(1) = -\exp(1 - \tfrac{1}{2}) = -\sqrt{e}.\mathbf{[1]} \end{aligned}$$

## 2.3 Implicit differentiation

**(117)** [**Mock2**] Find $dy/dx$ in terms of $x$ and $y$, where $x$ and $y$ are related by the equation $e^{-x^2-y^2}\sin(\omega x) = a$. **(5 marks)**

**Solution:** Put $u = \sin(\omega x) - ae^{x^2+y^2}$, so the relation is $u = 0$. We have

$$\begin{aligned} \partial u/\partial x &= \omega\cos(\omega x) - 2axe^{x^2+y^2} \\ \partial u/\partial y &= -2aye^{x^2+y^2} \end{aligned}$$

so

$$\begin{aligned} \frac{dy}{dx} &= -\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\omega\cos(\omega x) - 2axe^{x^2+y^2}}{2aye^{x^2+y^2}} \\ &= \frac{\omega}{2a}\cos(\omega x)e^{-x^2-y^2} - \frac{x}{y} \end{aligned}$$

**(118)** [**0708R**] Find $dy/dx$ in terms of $x$ and $y$, where $x$ and $y$ are related by the equation $e^{-x^2-xy-y^2}\sin(x) = a$. **(5 marks)**

**Solution:** Differentiate the relation to get

$$(-2x-y)e^{-x^2-xy-y^2}\sin(x)+(-x-2y)e^{-x^2-xy-y^2}\sin(x)\frac{dy}{dx}+e^{-x^2-xy-y^2}\cos(x)=0.[\mathbf{3}]$$

We can now multiply by $e^{x^2+xy+y^2}$ and rearrange to get

$$\frac{dy}{dx}=\frac{\cos(x)-(2x+y)\sin(x)}{(x+2y)\sin(x)}.[\mathbf{2}]$$

**(119)** [**0405**] Find $dy/dx$ in terms of $x$ and $y$, where $x$ and $y$ are related by $x^2+y^2+\sin(ax+by)=1$. **(4 marks)**

**Solution:** Put $u=x^2+y^2+\sin(ax+by)-1$, so the relation is $u=0$. We have

$$\partial u/\partial x=2x+a\cos(ax+by)[\mathbf{1}]$$
$$\partial u/\partial y=2y+b\cos(ax+by)[\mathbf{1}]$$

so

$$\frac{dy}{dx}=-\frac{\partial u/\partial x}{\partial u/\partial y}[\mathbf{1}]=-\frac{2x+a\cos(ax+by)}{2y+b\cos(ax+by)}[\mathbf{1}]$$

(The obvious approach starting with

$$2x+2y\frac{dy}{dx}+\left(a+b\frac{dy}{dx}\right)\cos(ax+by)=0$$

is also acceptable. Indeed, I would have taught the students to do it that way if I had thought of it at the relevant time.)

**(120)** [**0405R**] Find $dy/dx$ in terms of $x$ and $y$, where $x$ and $y$ are related by $x^2+y^2+axy=1$. **(4 marks)**

**Solution:** Put $u=x^2+y^2+axy-1$, so the relation is $u=0$. We have

$$\partial u/\partial x = 2x+ay[\mathbf{1}]$$
$$\partial u/\partial y = 2y+ax[\mathbf{1}]$$

so

$$\frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y}[\mathbf{1}]=-\frac{2x+ay}{2y+ax}[\mathbf{1}]$$

**(121)** [**Mock1**] Find $dy/dx$ in terms of $x$ and $y$, where $x$ and $y$ are related by the equation $x+\ln(x)=y-\ln(y)$. **(5 marks)**

**Solution:** Put $u=\ln(x)+\ln(y)+x-y$, so the relation is $u=0$. We have $\partial u/\partial x=1/x+1$ [**1**]and $\partial u/\partial y=1/y-1$ [**1**], so the derivative is

$$\frac{dy}{dx}=-\frac{\partial u/\partial x}{\partial u/\partial y}[\mathbf{1}]=-\frac{1/x+1}{1/y-1}=\frac{1+1/x}{1-1/y}=\frac{(x+1)y}{x(y-1)}[\mathbf{2}]$$

**(122)** **[0708]** If $x$ and $y$ are related by $y^2 = x^3 - x$, find $dy/dx$ in terms of $x$ and $y$. **(3 marks)**

**Solution:** Differentiating the relation $y^2 = x^3 - x$ gives $2y\frac{dy}{dx} = 3x^2 - 1$ **[2]**, so $dy/dx = (3x^2 - 1)/(2y)$ **[1]**.

**(123)** **[0506R]** Find $du/dv$, where $u$ and $v$ are related by the equation $uv^2 + u^2v^3 = 1$. **(4 marks)**

**Solution:** Apply $d/du$ to the equation to get

$$v^2 + 2uv\frac{dv}{du} + 2uv^3 + 3u^2v^2\frac{dv}{du} = 0\textbf{[2]}$$

Rearrange this to get

$$(2uv + 3u^2v^2)\frac{dv}{du} = -v^2 - 2uv^3\textbf{[1]}$$

and so

$$\frac{dv}{du} = -\frac{v^2 + 2uv^3}{2uv + 3u^2v^2} = -\frac{v(1 + 2uv)}{u(2 + 3uv)}\textbf{[1]}$$

**(124)** **[0506]** Find $d\theta/d\phi$, where $\theta$ and $\phi$ are related by the equation $\cos(\theta + \phi) + 1/2 = \cos(\theta) + \cos(\phi)$. **(4 marks)**

**Solution:** Apply $d/d\phi$ to the equation to get

$$-\sin(\theta + \phi)\left(\frac{d\theta}{d\phi} + 1\right) = -\sin(\theta)\frac{d\theta}{d\phi} - \sin(\phi).\textbf{[2]}$$

Rearrange this to get

$$\frac{d\theta}{d\phi}(\sin(\theta) - \sin(\theta + \phi)) = \sin(\theta + \phi) - \sin(\phi), \textbf{[1]}$$

and so

$$\frac{d\theta}{d\phi} = \frac{\sin(\theta + \phi) - \sin(\phi)}{\sin(\theta) - \sin(\theta + \phi)}.\textbf{[1]}$$

## 2.4 Parametric differentiation

**(125)** **[0405]** Find $dy/dx$ in terms of $t$, where $x = \cos(nt + \alpha)$ and $y = \cos(mt + \beta)$. **(3 marks)**

**Solution:**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}\textbf{[1]} = \frac{-m\sin(mt + \beta)}{-n\sin(nt + \alpha)} = \frac{m\sin(mt + \beta)}{n\sin(nt + \alpha)}.\textbf{[2]}$$

**(126)** **[Mock2]** Find $dy/dx$ in terms of $t$, where $x = \omega t - \sin(\omega t)$ and $y = 1 - \cos(\omega t)$. **(2 marks)**

**Solution:**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\omega\sin(\omega t)}{\omega - \omega\cos(\omega t)} = \frac{\sin(\omega t)}{1 - \cos(\omega t)} \qquad \textbf{[2]}$$

**(127)** Find $dy/dx$ in terms of $t$, where $x = 1 + t^2 + t^4$ and $y = t + t^3$. **(2 marks)**

**Solution:**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}[\mathbf{1}] = \frac{3t^2+1}{2t+4t^3}[\mathbf{1}].$$

**(128)** **[0708]** Suppose we have

$$x = 7\cos(t) - \cos(7t) \qquad\qquad y = 7\sin(t) - \sin(7t).$$

Find $dy/dx$ in terms of $t$. **(4 marks)**

**Solution:**

$$\begin{aligned} dy/dt &= 7\cos(t) - 7\cos(7t)[\mathbf{1}] \\ dx/dt &= -7\sin(t) + 7\sin(7t)[\mathbf{1}] \\ \frac{dy}{dx} &= \frac{dy/dt}{dx/dt}[\mathbf{1}] = \frac{7\cos(t) - 7\cos(7t)}{-7\sin(t) + 7\sin(7t)} = \frac{\cos(t) - \cos(7t)}{\sin(7t) - \sin(t)}.[\mathbf{1}] \end{aligned}$$

**(129)** **[Mock1]** Find $dy/dx$ in terms of $t$, where $x = 1 + t^4$ and $y = t + t^2$. **(2 marks)**

**Solution:**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}[\mathbf{1}] = \frac{2t+1}{4t^3}[\mathbf{1}].$$

**(130)** **[0506R]** Find $dy/dx$ in terms of $t$, where $x = t - \sin(t)$ and $y = 1 - \cos(t)$. **(3 marks)**

**Solution:**

$$\begin{aligned} dx/dt &= 1 - \cos(t)[\mathbf{1}] \\ dy/dt &= \sin(t)[\mathbf{1}] \\ \frac{dy}{dx} &= \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{\sin(t)}{1-\cos(t)}.[\mathbf{1}] \end{aligned}$$

**(131)** **[0506; 0708R]** Find $dy/dx$ in terms of $t$, where $x = \ln(1 + t^2)$ and $y = \ln(1 - t^2)$. **(3 marks)**

**Solution:**

$$\begin{aligned} dx/dt &= 2t/(1+t^2)[\mathbf{1}] \\ dy/dt &= -2t/(1-t^2)[\mathbf{1}] \\ \frac{dy}{dx} &= \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{-2t}{1-t^2}\,\frac{1+t^2}{2t} = \frac{t^2+1}{t^2-1}.[\mathbf{1}] \end{aligned}$$

## 2.5 Integration by substitution

**(132)** **[0910]** Use a suitable substitution to evaluate the integral

$$\int_{\exp(\exp(1))}^{\exp(\exp(2))} \frac{dx}{x\ \log(x)\ \log(\log(x))^{n+1}}$$

**(5 marks)**

**Solution:** We put $u = \log(\log(x))$ **[1]**, so the endpoints $x = \exp(\exp(1))$ and $x = \exp(\exp(2))$ correspond to $u = 1$ and $u = 2$ **[1]**. The chain rule gives

$$\frac{du}{dx} = \frac{\log'(x)}{\log(x)} = \frac{1}{x\ \log(x)},$$

so $dx = x\ \log(x)\, du$ **[1]**. Our integral can thus be rewritten as $\int_{u=1}^{2} u^{-n-1} du = -n^{-1}\left[u^{-n}\right]_1^2 = (1-2^{-n})/n$. **[2]**

**(133)** By making a suitable substitution, find $\int \sin(x)\log(\cos(x))\, dx$. **(6 marks)**

**Solution:** Put $u = \cos(x)$, so $du = -\sin(x)\, dx$ **[2]**. Then

$$\begin{aligned}\int \sin(x)\log(\cos(x))\, dx &= -\int \log(u)\, du[\mathbf{1}] = -(u\log(u) - u)[\mathbf{2}] = u(1-\log(u)) \\ &= \cos(x)(1-\log(\cos(x)))[\mathbf{1}].\end{aligned}$$

**(134)** By putting $u = \log(x)$, find $\int \frac{(1+\log(x))^2}{x}\, dx$. **(4 marks)**

**Solution:** Put $u = \log(x)$, so $du = x^{-1}dx$. Then

$$\begin{aligned}\int \frac{(1+\log(x))^2}{x}\, dx &= \int (1+u)^2 du = (1+u)^3/3 \\ &= (1+\log(x))^3/3.\end{aligned}$$

**(135)** By substituting $u = x^n$, find $\int \frac{dx}{x\sqrt{x^{-2n}-1}}$. **(7 marks)**

**Solution:** Put $u = x^n$, so $du = nx^{n-1}\, dx$, so $dx = du/(nx^{n-1})$. The integral becomes

$$\begin{aligned}\int \frac{dx}{x\sqrt{x^{-2n}-1}} &= \int \frac{du}{nx^{n-1}.x\sqrt{x^{-2n}-1}} = \frac{1}{n}\int \frac{du}{x^n\sqrt{x^{-2n}-1}} \\ &= \frac{1}{n}\int \frac{du}{u\sqrt{u^{-2}-1}} = \frac{1}{n}\int \frac{du}{\sqrt{1-u^2}} = \arcsin(u)/n \\ &= \arcsin(x^n)/n.\end{aligned}$$

**(136)** **[Mock2]** Find $\int x^2\ \sin(x^3)\, dx$. **(3 marks)**

**Solution:** Put $u = x^3$, so $du = 3x^2\, dx$, so $x^2\, dx = du/3$. The integral becomes

$$\int \sin(u)\, du/3 = -\cos(u)/3 = -\cos(x^3)/3.[\mathbf{3}]$$

**(137)** **[0405R]** Find $\int x\,e^{-4x^2}\,dx$. **(5 marks)**

**Solution:** If we put $u = -4x^2$ [**1**]then $du = -8x\,dx$, so $dx = -du/(8x)$ [**1**]. The integral becomes

$$\int x\,e^u\,\frac{-du}{8x} = -\frac{1}{8}\int e^u\,du[\mathbf{2}] = -e^u/8 = -e^{-4x^2}/8.[\mathbf{1}]$$

**(138)** [**Mock1**] By substituting $u = 1/x$, find $\int \tan(1/x)/x^2\,dx$. **(4 marks)**

**Solution:** If we put $u = 1/x$ then $du = -x^{-2}\,dx$ [**1**], so the integral becomes $\int -\tan(u)\,du = \ln(\cos(u)) = \ln(\cos(1/x))$ [**3**].

**(139)** [**0405**] Find $\int x\,e^{-x^2}\,dx$. **(5 marks)**

**Solution:** If we put $u = -x^2$ [**1**]then $du = -2x\,dx$, so $dx = -du/(2x)$ [**1**]. The integral becomes

$$\int x\,e^u\,\frac{-du}{2x} = -\frac{1}{2}\int e^u\,du[\mathbf{2}] = -e^u/2 = -e^{-x^2}/2.[\mathbf{1}]$$

**(140)** [**0708; 0708R**] By substituting $u = 1+t^2$ or otherwise, find $\displaystyle\int \frac{2t\,dt}{2+2t^2+t^4}$. **(5 marks)**

**Solution:** With the suggested substitution we have $du = 2t\,dt$ [**1**]and $t^2 = u-1$ so $t^4 = (u-1)^2 = u^2-2u+1$ [**1**], so $2+2t^2+t^4 = 2+2u-2+u^2-2u+1 = u^2+1$ [**1**]. Thus, the integral becomes $\int \frac{du}{u^2+1} = \arctan(u) = \arctan(1+t^2)$ [**2**].

**(141)** [**0506R**] Use integration by substitution to find $\int x^{-2}e^{-2/x}\,dx$. **(5 marks)**

**Solution:** Put $u = -2/x$ [**1**], so $du/dx = 2/x^2$, so $dx = \frac{1}{2}x^2\,du$ [**1**]. This gives

$$\int x^{-2}e^{-2/x}\,dx = \int x^{-2}e^u.\tfrac{1}{2}x^2\,du[\mathbf{1}] = \tfrac{1}{2}\int e^u\,du = \tfrac{1}{2}e^u[\mathbf{1}] = \tfrac{1}{2}e^{-2/x}.[\mathbf{1}]$$

**(142)** [**0809**] By substituting $x = \tan(t)^2$, evaluate $\displaystyle\int_1^3 \frac{dx}{x^{1/2}+x^{3/2}}$ **(7 marks)**

**Solution:** We have

$$\begin{aligned}\frac{dx}{dt} &= 2\tan(t)\tan'(t) = 2\tan(t)\sec(t)^2[\mathbf{1}] = 2\tan(t)(1+\tan(t)^2)[\mathbf{1}]\\ &= 2\sqrt{x}(1+x^2) = 2(x^{1/2}+x^{3/2}),[\mathbf{1}]\end{aligned}$$

so $dx/(x^{1/2}+x^{3/2}) = 2\,dt$. Moreover, we have $\tan(t) = \sqrt{x}$ and so $t = \arctan(\sqrt{x})$ [**1**]. When $x = 1$ this gives $t = \arctan(1) = \pi/4$, [**1**]and when $x = 3$ we have $t = \arctan(\sqrt{3})$ it gives $t = \arctan(\sqrt{3}) = \pi/3$ [**1**]. Thus, our integral becomes

$$\int_{x=1}^3 \frac{dx}{x^{1/2}+x^{3/2}} = \int_{t=\pi/4}^{\pi/3} 2\,dt = 2(\pi/3-\pi/4) = \pi/6.[\mathbf{1}]$$

## 2.6 Integration by parts

**(143) [0910]**

(a) Find $\int (x+1)e^x \, dx$. **(4 marks)**

(b) Find $\int (x+1)e^x \log(x) \, dx$. **(4 marks)**
**Hint:** you should integrate by parts, using (a) to help.

**Solution:**

(a) We integrate by parts, taking $u = x+1$ and $dv/dx = e^x$ **[1]**, so $du/dx = 1$ and $v = \int e^x \, dx = e^x$ **[1]**. This gives

$$\int (x+1)e^x \, dx = \int u \frac{dv}{dx} \, dx uv - \int \frac{du}{dx} v \, dx = (x+1)e^x - \int e^x \, dx = x \, e^x. \qquad \textbf{[2]}$$

(b) We integrate by parts again, this time taking $u = \log(x)$ and $dv/dx = (x+1)e^x$ **[1]**. We then have $du/dx = 1/x$ and part (a) tells us that $v = x \, e^x$ **[1]**. We thus have

$$\int (x+1)e^x \log(x) \, dx = x \, e^x \log(x) - \int x \, e^x / x \, dx = x \, e^x \log(x) - e^x = (x \log(x) - 1)e^x.$$

**[2]**

**(144) [0708]** Find $\int_1^e x^2 \log(x) \, dx$. **(6 marks)**

**Solution:** We use integration by parts. We take $u = \log(x)$ and $dv/dx = x^2$, so $du/dx = 1/x$ and we can take $v$ to be $x^3/3$ **[2]**. This gives

$$\begin{aligned}\int x^2 \log(x) \, dx &= \int u \frac{dv}{dx} \, dx = uv - \int \frac{du}{dx} \, v \, dx \textbf{[1]} \\ &= \frac{x^3 \, \log(x)}{3} - \int \frac{x^2}{3} \, dx = \frac{x^3 \, \log(x)}{3} - \frac{x^3}{9} \textbf{[1]} \\ &= \frac{(3\log(x) - 1)x^3}{9}.\end{aligned}$$

This in turn gives

$$\int_1^e x^2 \log(x) \, dx = \left[ \frac{(3\log(x) - 1)x^3}{9} \right]_{x=1}^{e} \textbf{[1]} = \frac{(3\log(e) - 1)e^3}{9} - \frac{(3\log(1) - 1)1^3}{9} = \frac{2e^3 + 1}{9}. \textbf{[1]}$$

**(145) [0506R; 0708R]** Use integration by parts to find $\int x^n \ln(x) \, dx$. **(5 marks)**

**Solution:** Put $du/dx = x^n$ and $v = \ln(x)$ **[1]**, so $u = x^{n+1}/(n+1)$ **[1]**and $dv/dx = 1/x$ **[1]**. This gives

$$\int x^n \ln(x) \, dx = \int \frac{du}{dx} v \, dx = uv - \int u \frac{dv}{dx} \, dx \textbf{[1]} = \frac{x^{n+1}}{n+1} \ln(x) - \int \frac{x^n}{n+1} \, dx = \frac{x^{n+1}}{n+1} \ln(x) - \frac{x^{n+1}}{(n+1)^2} \textbf{[1]}$$

**(146)** [**0809**] Find $\int_0^1 x\sin(\pi x)\,dx$. **(5 marks)**

**Solution:** We integrate by parts [**1**], using $u = x$ and $dv/dx = \sin(\pi x)$ [**1**], so $du/dx = 1$ and $v = -\cos(\pi x)/\pi$ [**1**]. This gives

$$\begin{aligned}\int_0^1 x\sin(\pi x)\,dx &= \left[-\frac{x\cos(\pi x)}{\pi}\right]_0^1 + \int_0^1 \frac{\cos(\pi x)}{\pi}\,dx[\mathbf{1}] \\ &= \left[-\frac{x\cos(\pi x)}{\pi} + \frac{\sin(\pi x)}{\pi^2}\right]_0^1 = \left[\frac{\sin(\pi x) - \pi x\cos(\pi x)}{\pi^2}\right]_0^1 \\ &= ((\sin(\pi) - \pi\cos(\pi)) - (\sin(0) - 0))/\pi^2 = 1/\pi[\mathbf{1}]\end{aligned}$$

## 2.7 Integration by undetermined coefficients

**(147)** Find $\int e^{-x}\sin(x)^2\,dx$ **(9 marks)**

**Solution:**

**(148)** Find $\int(4x^2 + 2x + 1)e^{2x}\,dx$. **(4 marks)**

**Solution:** The general form is

$$\int(4x^2 + 2x + 1)e^{2x}\,dx = (Ax^2 + Bx + C)e^{2x}$$

for some constants $A$, $B$ and $C$. Differentiating, we find that

$$\begin{aligned}(4x^2 + 2x + 1)e^{2x} &= \frac{d}{dx}((Ax^2 + Bx + C)e^{2x}) \\ &= (2Ax + B)e^{2x} + (Ax^2 + Bx + C).2e^{2x} \\ &= (2Ax^2 + (2A + 2B)x + (B + 2C))e^{2x},\end{aligned}$$

so $2A = 4$ and $2A + 2B = 2$ and $B + 2C = 1$. It follows that $A = 2$ and $B = -1$ and $C = 1$, so

$$\int(4x^2 + 2x + 1)e^{2x}\,dx = (2x^2 - x + 1)e^{2x}.$$

**(149)** Find $\int 8x\sin(x)\cos(x)\,dx$ **(7 marks)**

**Solution:** First note that $8x\sin(x)\cos(x) = 4x\sin(2x)$, so

$$\begin{aligned}\int 8x\sin(x)\cos(x)\,dx &= \int 4x\sin(2x)\,dx \\ &= -2x\cos(2x) + \int 2\cos(2x)\,dx \\ &= -2x\cos(2x) + \sin(2x).\end{aligned}$$

**(150)** Find $\int x^2e^x\,dx$ **(5 marks)**

**Solution:** We know that

$$\int x^2 e^x \, dx = (ax^2 + bx + c)e^x$$

for some constants $a$, $b$ and $c$ **[2]**. To find these, we differentiate to get

$$\begin{aligned} x^2 e^x &= \frac{d}{dx}((ax^2 + bx + c)e^x) = (2ax + b)e^x + (ax^2 + bx + c)e^x\,[\mathbf{1}] \\ &= (ax^2 + (2a + b)x + (b + c))e^x. \end{aligned}$$

We equate coefficients to see that $a = 1$ and $2a + b = b + c = 0$ **[1]**, which gives $b = -2$ and $c = 2$. We conclude that

$$\int x^2 e^x \, dx = (x^2 - 2x + 2)e^x.[\mathbf{1}]$$

**(151)** Find $\int e^{3x} \sin(4x) \, dx$ **(5 marks)**

**Solution:** We know that

$$\int e^{3x} \sin(4x) \, dx = e^{3x}(A \cos(4x) + B \sin(4x))$$

for some $A$ and $B$ **[2]**. To find these, we differentiate and equate coefficients:

$$\begin{aligned} e^{3x} \sin(4x) &= \frac{d}{dx}\left(e^{3x}(A \cos(4x) + B \sin(4x))\right) \\ &= 3e^{3x}(A \cos(4x) + B \sin(4x)) + e^{3x}(-4A \sin(4x) + 4B \cos(4x)) \\ &= e^{3x}((3A + 4B) \cos(4x) + (3B - 4A) \sin(4x))[\mathbf{1}], \end{aligned}$$

so $3A + 4B = 0$ and $3B - 4A = 1$ **[1]**. This gives $A = -4B/3$ so $1 = 3B - 4A = 3B + 16B/3 = 25B/3$, so $B = 3/25$, so $A = -4B/3 = -4/25$. The conclusion is that

$$\int e^{3x} \sin(4x) \, dx = e^{3x}(3 \sin(4x) - 4 \cos(4x))/25.[\mathbf{1}]$$

**(152)** **[0809]** You may assume that $\int x^2 \log(x)^2 \, dx = x^3(a \log(x)^2 + b \log(x) + c)$ for some constants $a$, $b$ and $c$. Find these constants, and thus evaluate $\int_1^e x^2 \log(x)^2 \, dx$. **(7 marks)**

**Solution:** We first note that

$$\begin{aligned} \frac{d}{dx}\left(x^3(a \log(x)^2 + b \log(x) + c)\right) &= 3x^2(a \log(x)^2 + b \log(x) + c) + x^3(2a \log(x)/x + b/x)[\mathbf{1}] \\ &= x^2(3a \log(x)^2 + (3b + 2a) \log(x) + (3c + b)).[\mathbf{1}] \end{aligned}$$

This must also be equal to $x^2 \log(x)^2$ for all $x$, **[1]**so we must have

$$\begin{aligned} 3a &= 1 \\ 3b + 2a &= 0 \\ 3c + b &= 0, [\mathbf{1}] \end{aligned}$$

so $a = 1/3$ and $b = -2/9$ and $c = 2/27$, **[1]**giving

$$\int x^2 \log(x)^2 \, dx = x^3(\log(x)^2/3 - 2 \log(x)/9 + 2/27).$$

It follows that

$$\int_1^e x^2 \log(x)^2\,dx = \left[x^3(\log(x)^2/3 - 2\log(x)/9 + 2/27)\right]_1^e \,\mathbf{[1]}$$
$$= e^3(1/3 - 2/9 + 2/27) - 1^3(0/3 - 0/9 + 2/27)$$
$$= (5e^3 - 2)/27.\mathbf{[1]}$$

**(153)** [**Mock1**] Find $\int (x\,e^x)^3\,dx$ **(7 marks)**

**Solution:** For general reasons we know that the

$$\int (x\,e^x)^3\,dx = \int x^3 e^{3x}\,dx = (ax^3 + bx^2 + cx + d)e^{3x}\mathbf{[3]}$$

for some constants $a$, $b$, $c$ and $d$. Differentiating this gives

$$x^3e^{3x} = (3ax^2{+}2bx{+}c)e^{3x}{+}(ax^3{+}bx^2{+}cx{+}d).(3e^{3x}) = (3ax^3{+}(3a{+}3b)x^2{+}(2b{+}3c)x{+}(c{+}3d))e^{3x}.\mathbf{[2]}$$

By comparing coefficients, we see that $3a = 1$ and $3a + 3b = 2b + 3c = c + 3d = 0$, so $a = 1/3$, $b = -1/3$, $c = 2/9$ and $d = -2/27$. Thus

$$\int (x\,e^x)^3\,dx = \left(\tfrac{1}{3}x^3 - \tfrac{1}{3}x^2 + \tfrac{2}{9}x - \tfrac{2}{27}\right)e^{3x}.\mathbf{[2]}$$

**(154)** [**Mock1; 0405R**] Find $\int e^x(\sin(x) + \cos(x))\,dx$. **(6 marks)**

**Solution:** For general reasons we know that

$$\int e^x(\sin(x) + \cos(x))\,dx = e^x(a\sin(x) + b\cos(x))$$

for some constants $a$ and $b$ **[2]**. Differentiating this, we get

$$e^x(\sin(x) + \cos(x)) = e^x(a\sin(x) + b\cos(x)) + e^x(a\cos(x) - b\sin(x))$$
$$= e^x((a-b)\sin(x) + (a+b)\cos(x))\mathbf{[2]}.$$

By comparing coefficients, we see that $a + b = a - b = 1$, so $a = 1$ and $b = 0$. We thus have

$$\int e^x(\sin(x) + \cos(x))\,dx = e^x\sin(x)\mathbf{[2]}.$$

**(155)** [**0506; 0708R**] Find $\displaystyle\int x^2e^{-2x}\,dx$. **(4 marks)**

**Solution:** The answer has the form $(ax^2 + bx + c)e^{-2x}$ for some constants $a$, $b$ and $c$ **[1]**. Differentiation gives

$$\begin{aligned} x^2e^{-2x} &= \frac{d}{dx}(ax^2 + bx + c)e^{-2x}) \\ &= (2ax + b)e^{-2x} - 2(ax^2 + bx + c)e^{-2x} \\ &= (-2ax^2 + (2a - 2b)x + (b - 2c))e^{-2x}\mathbf{[1]}. \end{aligned}$$

By comparing coefficients, we see that $-2a = 1$ and $2a - 2b = 0 = b - 2c$, so $a = -1/2$ and $b = a = -1/2$ and $c = b/2 = -1/4$ [**1**]. We conclude that

$$\int x^2 e^{-2x}\, dx = -\frac{1}{4}(2x^2 + 2x + 1)e^{-2x}[\mathbf{1}].$$

**(156) [0405R]** Find $\int (x^3 + x^2 + x + 1)e^{-x}\, dx$. **(6 marks)**

**Solution:** For general reasons, we know that

$$\int (x^3 + x^2 + x + 1)e^{-x}\, dx = (ax^3 + bx^2 + cx + 1)e^{-x}$$

for some constants $a, b, c, d$ [**2**]. Differentiating this gives

$$(x^3+x^2+x+1)e^{-x} = (3ax^2+2bx+c)e^{-x}-(ax^3+bx^2+cx+d)e^{-x} = (-ax^3+(3a-b)x^2+(2b-c)x+(c-d))e^{-x}.[\mathbf{1}]$$

Comparing coefficients gives $-a = 1$ and $3a - b = 1$ and $2b - c = 1$ and $c - d = 1$ [**1**], so $a = -1$ and $b = -4$ and $c = -9$ and $d = -10$ [**1**]. Thus

$$\int (x^3 + x^2 + x + 1)e^{-x}\, dx = -(x^3 + 4x^2 + 9x + 10)e^{-x}.[\mathbf{1}]$$

**(157)** Find $\int e^x(\sin(x) + \cos(x))\, dx$. **(6 marks)**

**Solution:** For general reasons we know that

$$\int e^x(\sin(x) + \cos(x))\, dx = e^x(a\sin(x) + b\cos(x))$$

for some constants $a$ and $b$ [**2**]. Differentiating this, we get

$$\begin{aligned} e^x(\sin(x) + \cos(x)) &= e^x(a\sin(x) + b\cos(x)) + e^x(a\cos(x) - b\sin(x)) \\ &= e^x((a-b)\sin(x) + (a+b)\cos(x))[\mathbf{2}]. \end{aligned}$$

By comparing coefficients, we see that $a + b = a - b = 1$, so $a = 1$ and $b = 0$. We thus have

$$\int e^x(\sin(x) + \cos(x))\, dx = e^x \sin(x)[\mathbf{2}].$$

**(158) [0405]** Find $\int (x^2 - x + 1)e^x\, dx$. **(5 marks)**

**Solution:** For general reasons, we know that

$$\int (x^2 - x + 1)e^x\, dx = (ax^2 + bx + c)e^x$$

for some constants $a$, $b$ and $c$ [**2**]. Differentiating this gives

$$(x^2 - x + 1)e^x = (2ax + b)e^x + (ax^2 + bx + c)e^x = (ax^2 + (2a + b)x + (b + c))e^x.[\mathbf{1}]$$

Comparing coefficients gives $a = 1$ and $2a + b = -1$ and $b + c = 1$, so $b = -3$ and $c = 4$ [**1**]. Thus

$$\int (x^2 - x + 1)e^x\, dx = (x^2 - 3x + 4)e^x.[\mathbf{1}]$$

**(159) [0405]** Find $\int e^{-x}\sin(3x)\,dx$. **(7 marks)**

**Solution:** For general reasons we know that

$$\int e^{-x}\sin(3x)\,dx = e^{-x}(a\sin(3x)+b\cos(3x))\mathbf{[2]}$$

for some constants $a$ and $b$. Differentiating this gives

$$\begin{aligned} e^{-x}\sin(3x) &= -e^{-x}(a\sin(3x)+b\cos(3x)) + e^{-x}(3a\cos(3x)-3b\sin(3x)) \\ &= e^{-x}((3a-b)\cos(3x)-(a+3b)\sin(3x)).\mathbf{[2]} \end{aligned}$$

By comparing coefficients, we see that $3a-b=0$ and $-(a+3b)=1$, so $a=-1/10$ and $b=-3/10$ **[2]**. It follows that

$$\int e^{-x}\sin(3x)\,dx = -\tfrac{1}{10}e^{-x}(3\cos(3x)+\sin(3x)).\mathbf{[1]}$$

**(160) [Mock2]** Find $\int x^2e^{-x}\,dx$. **(6 marks)**

**Solution:** For general reasons we know that the answer has the form $(ax^2+bx+c)e^{-x}$ for some constants $a$, $b$ and $c$. **[2]** Differentiating this gives

$$x^2e^{-x} = (2ax+b)e^{-x} + (ax^2+bx+c).(-e^{-x}) = (-ax^2+(2a-b)x+(b-c))e^{-x}.\mathbf{[2]}$$

By comparing coefficients, we see that $-a=1$ and $2a-b=0$ and $b-c=0$, so $a=-1$ and $b=-2$ and $c=-2$. Thus

$$\int x^2e^{-x}\,dx = -(x^2+2x+2)e^{-x}.\mathbf{[2]}$$

**(161) [Mock2]** Find $\int e^{\lambda x}\sin(\omega x)\,dx$. **(8 marks)**

**Solution:** For general reasons we know that

$$\int e^{\lambda x}\sin(\omega x)\,dx = e^{\lambda x}(a\sin(\omega x)+b\cos(\omega x))$$

for some constants $a$ and $b$ **[2]**. Differentiating this gives

$$\begin{aligned} e^{\lambda x}\sin(\omega x) &= \lambda e^{\lambda x}(a\sin(\omega x)+b\cos(\omega x)) + e^{\lambda x}(a\omega\cos(\omega x)-b\omega\sin(\omega x)) \\ &= e^{\lambda x}((a\lambda-b\omega)\sin(\omega x)+(b\lambda+a\omega)\cos(\omega x))\mathbf{[2]} \end{aligned}$$

By comparing coefficients, we see that $a\lambda-b\omega=1$ and $b\lambda+a\omega=0$ **[1]**. The second equation gives $b=-a\omega/\lambda$ **[1]**, which we can substitute into the first equation to give $a\lambda+a\omega^2/\lambda=1$, and so $a(\lambda^2+\omega^2)/\lambda=1$ **[2]**, so $a=\lambda/(\lambda^2+\omega^2)$. It follows that $b=-a\omega/\lambda=-\omega/(\lambda^2+\omega^2)$ **[1]**, and thus that

$$\int e^{\lambda x}\sin(\omega x)\,dx = e^{\lambda x}(\lambda\sin(\omega x)-\omega\cos(\omega x))/(\lambda^2+\omega^2).\mathbf{[1]}$$

**(162)** You may assume that

$$\int \cos(x)^{-4}\,dx = \tan(x)\left(a+b\cos(x)^{-2}\right)$$

for some constants $a$ and $b$. Find these constants. **(8 marks)**
[Hint: you will need the identities $\tan(x) = \sin(x)/\cos(x)$ and $\sin(x)^2 = 1 - \cos(x)^2$.]

**Solution:** Differentiating both sides gives

$$\cos(x)^{-4} = \tan'(x)\left(a + b\cos(x)^{-2}\right) + \tan(x).b.(-2)\cos(x)^{-3}.\cos'(x).\mathbf{[2]}$$

We know that $\tan'(x) = \cos(x)^{-2}$ **[1]**and $\cos'(x) = -\sin(x)$, so

$$\cos(x)^{-4} = a\cos(x)^{-2} + b\cos(x)^{-4} + 2b\tan(x)\cos(x)^{-3}\sin(x).\mathbf{[2]}$$

We now use the identities $\tan(x) = \sin(x)/\cos(x)$ and $\sin(x)^2 = 1 - \cos(x)^2$ to get

$$\begin{aligned}\cos(x)^{-4} &= a\cos(x)^{-2} + b\cos(x)^{-4} + 2b\frac{\sin(x)}{\cos(x)}\cos(x)^{-3}\sin(x)\mathbf{[1]}\\ &= a\cos(x)^{-2} + b\cos(x)^{-4} + 2b\sin(x)^2\cos(x)^{-4}\\ &= a\cos(x)^{-2} + b\cos(x)^{-4} + 2b(1-\cos(x)^2)\cos(x)^{-4}\\ &= (a-2b)\cos(x)^{-2} + 3b\cos(x)^{-4}.\mathbf{[1]}\end{aligned}$$

For this to match up we must have $a - 2b = 0$ and $3b = 1$, so $b = 1/3$ and $a = 2/3$. **[1]**

## 2.8 Other integration

**(163)** [**0910**] Find $\int_3^{24}\sqrt{1+x}\,dx$ **(3 marks)**

**Solution:**

$$\begin{aligned}\int_3^{24}\sqrt{1+x}\,dx &= \left[\frac{2}{3}(1+x)^{3/2}\right]_3^2 4\mathbf{[2]}\\ &= \frac{2}{3}(25^{3/2} - 4^{3/2}) = \frac{2}{3}(125-8) = 78.\mathbf{[1]}\end{aligned}$$

**(164)** [**0910**] Write down the standard integral $\int\tan(x)\,dx$; then simplify $\exp\left(\int_{\pi/4}^{\pi/3} 2\tan(x)\,dx\right)$. **(5 marks)**

**Solution:** It is standard that $\int\tan(x)\,dx = -\log(\cos(x))$ **[2]**, so

$$\int_{\pi/4}^{\pi/3} 2\tan(x)\,dx = -2\log(\cos(\pi/3)) + 2\log(\cos(\pi/4))\mathbf{[1]} = \log\left(\left(\frac{\cos(\pi/4)}{\cos(\pi/3)}\right)^2\right).$$

Now $\cos(\pi/3) = 1/2$ and $\cos(\pi/4) = 1/\sqrt{2}$ **[1]**so $(\cos(\pi/4)/\cos(\pi/3))^2 = 2$. We therefore see that $\int_{\pi/4}^{\pi/3} 2\tan(x)\,dx = \log(2)$, and so

$$\exp\left(\int_{\pi/4}^{\pi/3} 2\tan(x)\,dx\right) = 2.\mathbf{[1]}$$

**(165)** Find $\int\sin(x)^2\cos(x)^2\,dx$ **(7 marks)**

**Solution:** Note that $\sin(x)\cos(x) = \sin(2x)/2$ [**1**], so

$$\sin(x)^2\cos(x)^2 = \sin(2x)^2/4[\mathbf{1}] = (1-\cos(4x))/8[\mathbf{2}].$$

Thus

$$\begin{aligned}\int \sin(x)^2\cos(x)^2\,dx &= \tfrac{1}{8}\int 1-\cos(4x)\,dx[\mathbf{1}] \\ &= \frac{x}{8} - \frac{\sin(4x)}{32} = \frac{4x-\sin(4x)}{32}[\mathbf{2}].\end{aligned}$$

**(166)** [**0506**]

(a) What is $\tan'(x)$? (You need not write any working if you remember the answer.) **(2 marks)**

(b) What is $\arctan'(x)$? (You need not write any working if you remember the answer.) **(2 marks)**

(c) Find $\int \frac{x}{1+x^2}\,dx$ (by substitution or otherwise). **(4 marks)**

(d) Find $\int \arctan(x)\,dx$. (**Hint:** write it as $1.\arctan(x)$ and integrate by parts using (b) and (c).) **(4 marks)**

**Solution:**

(a) $\tan'(x) = \dfrac{\sin'(x)\cos(x) - \sin(x)\cos'(x)}{\cos(x)^2} = \dfrac{\cos(x)^2+\sin(x)^2}{\cos(x)^2} = \sec(x)^2$ [**2**]

(b) If $y = \arctan(x)$ then $x = \tan(y)$, so (a) gives $dx/dy = \sec(y)^2 = 1+\tan(y)^2 = 1+x^2$, so $\arctan'(x) = dy/dx = 1/(1+x^2)$. [**2**]

(c) Put $u = x^2$, [**1**]so $x\,dx = du/2$ [**1**]. Then

$$\int \frac{x}{1+x^2}\,dx = \int \frac{du/2}{1+u} = \ln(1+u)/2 = \ln(1+x^2)/2.[\mathbf{2}]$$

(d) Take $u = \arctan(x)$ and $dv/dx = 1$, [**1**]so $du/dx = 1/(1+x^2)$ and $v = x$ [**1**]. Then

$$\begin{aligned}\int \arctan(x)\,dx &= \int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx}v\,dx \\ &= x\arctan(x) - \int \frac{x}{1+x^2}\,dx = x\arctan(x) - \ln(1+x^2)/2.[\mathbf{2}]\end{aligned}$$