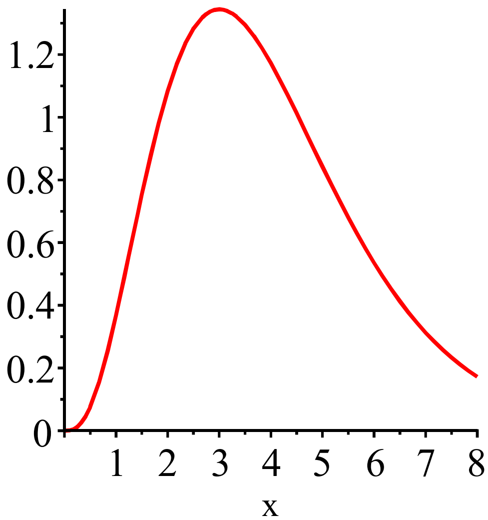
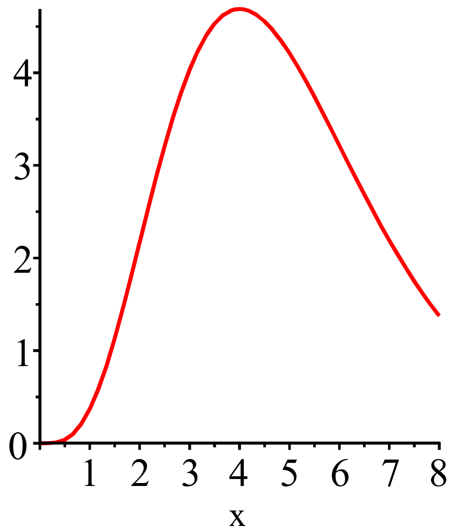
More differentiation

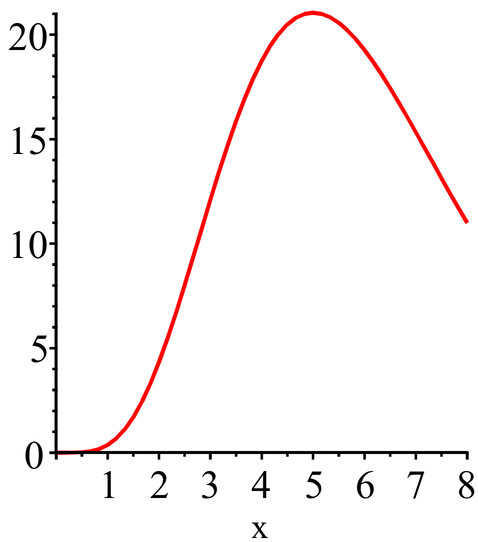
Exercise 1.1

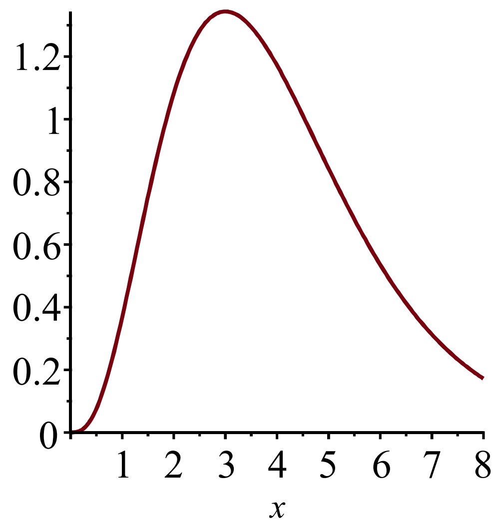
Here are the graphs of $y = x^n e^{-x}$ for $n = 3, 4$ and 5:

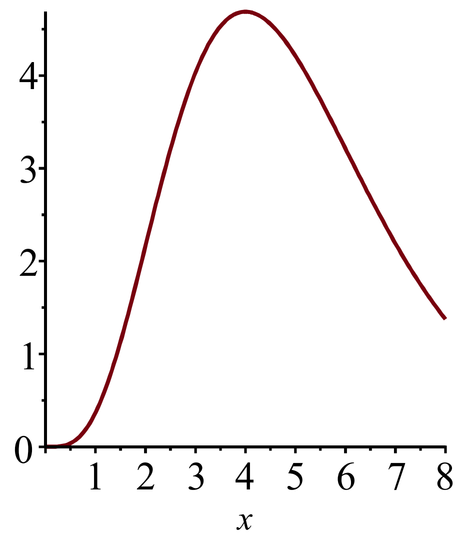
```
> plot(x^3*exp(-x),x=0..8);
  plot(x^4*exp(-x),x=0..8);
  plot(x^5*exp(-x),x=0..8);
```

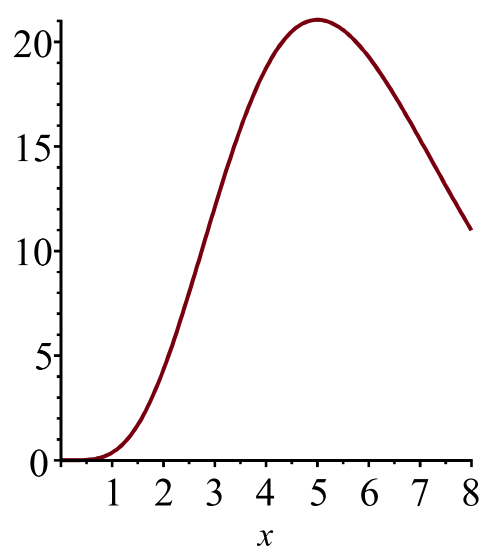












You can just about see that the peaks occur at $x=3$, $x=4$ and $x=5$, suggesting that for general n we should have a peak at $x=n$. To check this, we must solve $\frac{dy}{dx}=0$.

```
> y := x^n * exp(-x);
```

$$y := x^n e^{-x} \tag{1}$$

```
> simplify(diff(y,x));
```

$$e^{-x}\left(x^{n-1} n - x^n\right) \tag{2}$$

```
> solve(diff(y,x)=0,{x});
```

$$\{x = n\} \tag{3}$$

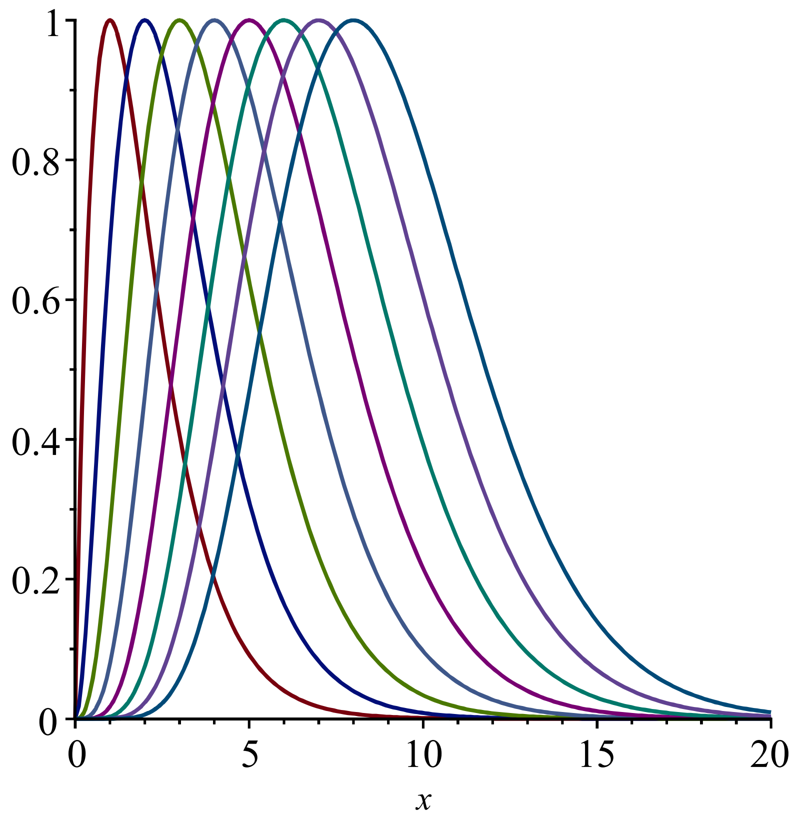
This shows that the peak does indeed occur at $x = n$. To find the height of the peak, we must put $x = n$ in y:

```
> y[max] := subs(x=n,y);
```

$$y_{max} := n^n e^{-n} \tag{4}$$

It follows that $\frac{y}{y_{max}} = \left(\frac{x}{n}\right)^n e^{n-x}$, so the maximum value of $\left(\frac{x}{n}\right)^n e^{n-x}$ is 1. We can plot these functions for different n together as follows:

```
> plot([seq((x/n)^n*exp(n-x),n=1..8)],x=0..20);
```

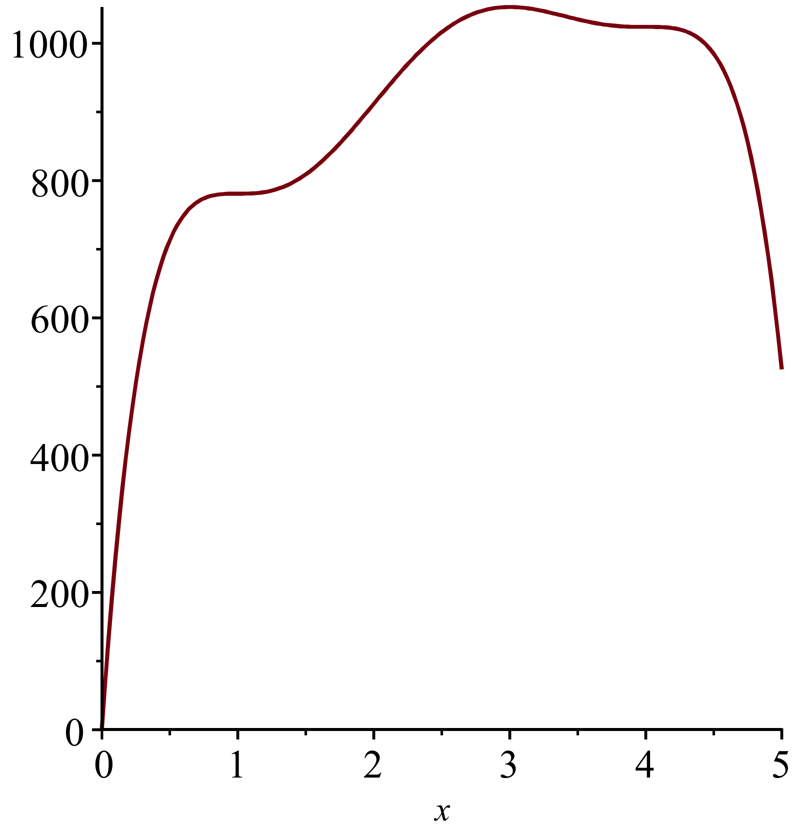


Exercise 1.2

```
> p := (x) -> -10*x^6+156*x^5-945*x^4+2780*x^3-4080*x^2+2880*x;
```

$$p := x \mapsto -10\,x^6 + 156\,x^5 - 945\,x^4 + 2780\,x^3 - 4080\,x^2 + 2880\,x \tag{5}$$

```
> plot(p(x),x=0..5);
```



We see from the picture that the maximum value is about 1000, occurring at about $x = 3$. There also seem to be two inflection points, where the curve flattens out but does not have a maximum or minimum. To check this, we solve for $p'(x) = 0$:

```
> p1 := diff(p(x),x);
```

$$p1 := -60\,x^5 + 780\,x^4 - 3780\,x^3 + 8340\,x^2 - 8160\,x + 2880 \quad (6)$$

```
> solve(p1=0,x);
```

$$3,\ 1,\ 1,\ 4,\ 4 \quad (7)$$

This shows that the critical points are at $x = 1$, $x = 3$ and $x = 4$. The numbers 1 and 4 are repeated because they are double roots of $p'(x)$, and therefore inflection points of $p(x)$. To check this, we differentiate again:

```
> p2 := diff(p(x),x,x);
```

$$p2 := -300\,x^4 + 3120\,x^3 - 11340\,x^2 + 16680\,x - 8160 \quad (8)$$

```
> subs(x=1,p2);
```

$$0 \quad (9)$$

```
> subs(x=3,p2);
```

$$-240 \quad (10)$$

```
> subs(x=4,p2);
```

$$0 \quad (11)$$

As $p'(3) = 0$ and $p''(3) < 0$ we see that $x = 3$ is a local maximum; by looking at the graph we see that it is a global maximum. The maximum value of $p(x)$ is thus given by $p(3)$:

```
> p(3);
```

$$1053 \quad (12)$$

The values of $p(x)$ at the inflection points are strictly smaller, as expected:

```
> p(1),p(4);
```

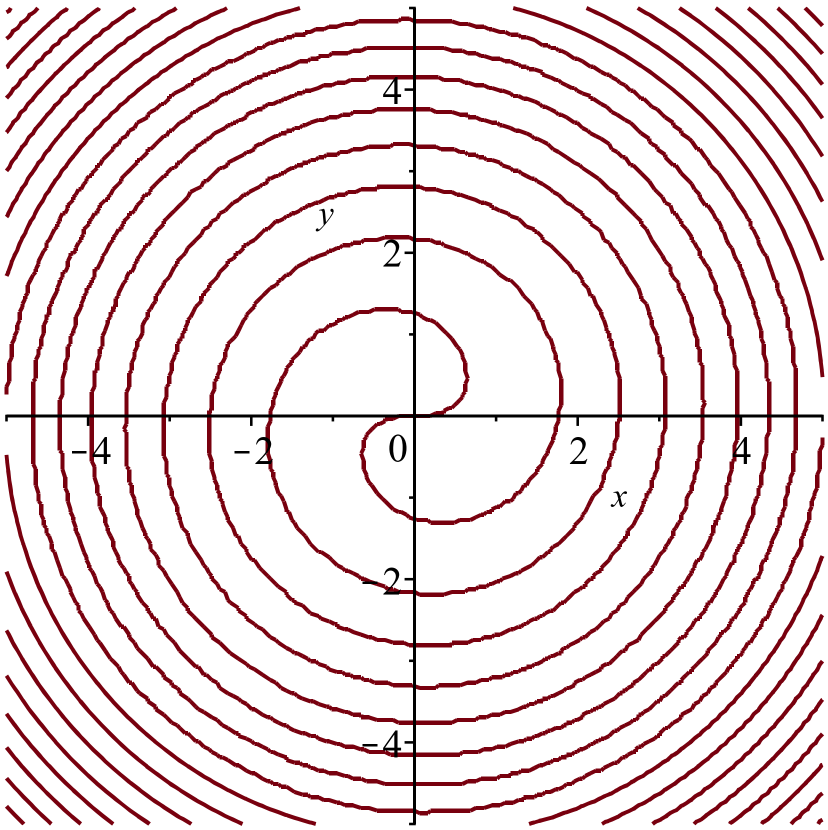
$$781, 1024 \quad (13)$$

Exercise 2.1

```
> restart;
> u := x*sin(x^2+y^2)-y*cos(x^2+y^2);
```

$$u := x\sin(x^2+y^2) - y\cos(x^2+y^2) \quad (14)$$

```
> with(plots):
> implicitplot(u=0,x=-5..5,y=-5..5,grid=[100,100]);
```




```
> slope1 := implicitdiff(u=0,y,x);
```

$$slope1 := -\frac{2\sin(x^2+y^2)\,y\,x + 2\cos(x^2+y^2)\,x^2 + \sin(x^2+y^2)}{2\,y^2\sin(x^2+y^2) + 2\,x\,y\cos(x^2+y^2) - \cos(x^2+y^2)} \quad (15)$$

The answer can be rewritten more nicely using the original relation

$$x\sin(x^2+y^2) = y\cos(x^2+y^2),$$

which allows us to rewrite $\sin(x^2+y^2)$ as $\frac{y\cos(x^2+y^2)}{x}$, after which we can cancel the cos terms.

```
> subs(sin(x^2+y^2) = (y/x)*cos(x^2+y^2),slope1);
```

$$-\frac{2y^2\cos(x^2+y^2)+2\cos(x^2+y^2)\,x^2+\frac{y\cos(x^2+y^2)}{x}}{\frac{2y^3\cos(x^2+y^2)}{x}+2\,x\,y\cos(x^2+y^2)-\cos(x^2+y^2)} \tag{16}$$

```
> simplify(%);
```

$$\frac{-2x^3-2xy^2-y}{2x^2y+2y^3-x} \tag{17}$$

```
> slope1 := %;
```

$$slope1 := \frac{-2x^3-2xy^2-y}{2x^2y+2y^3-x} \tag{18}$$

We now write the curve parametrically in terms of t:

```
> xt := t * cos(t^2); yt := t * sin(t^2);
```

$$xt := t\cos(t^2)$$
$$yt := t\sin(t^2) \tag{19}$$

To check that this really is the same curve as before, we put $x = t\cos(t^2)$ and $y = t\sin(t^2)$ in u and make sure that we get zero:

```
> subs(x=xt,y=yt,u);
```

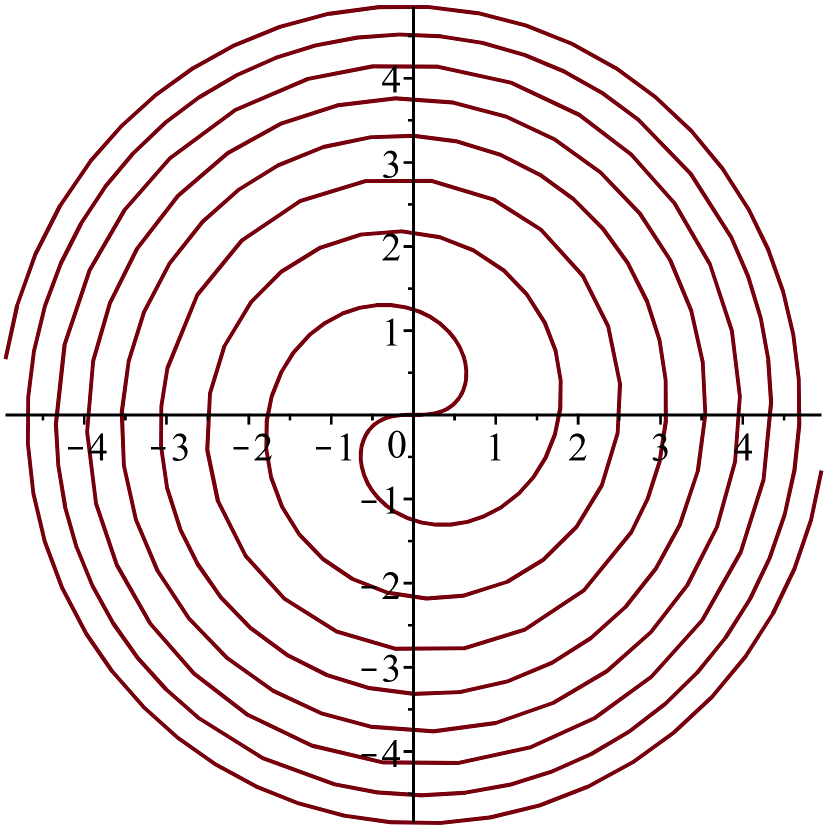
$$t\cos(t^2)\sin\left(t^2\cos(t^2)^2+t^2\sin(t^2)^2\right)-t\sin(t^2)\cos\left(t^2\cos(t^2)^2+t^2\sin(t^2)^2\right) \tag{20}$$

```
> simplify(subs(x=xt,y=yt,u));
```

$$0 \tag{21}$$

We can also check graphically:

```
> plot([xt,yt,t=-5..5]);
```



We now calculate $\frac{dy}{dx}$ again from the parametric representation:

```
> diff(xt,t);
```

$$\cos(t^2) - 2\,t^2 \sin(t^2) \quad \textbf{(22)}$$

```
> diff(yt,t);
```

$$\sin(t^2) + 2\,t^2 \cos(t^2) \quad \textbf{(23)}$$

```
> slope2 := diff(yt,t)/diff(xt,t);
```

$$slope2 := \frac{\sin(t^2) + 2\,t^2 \cos(t^2)}{\cos(t^2) - 2\,t^2 \sin(t^2)} \quad \textbf{(24)}$$

This should be the same as what we get by putting $x = t\cos(t^2)$ and $y = t\sin(t^2)$ in our earlier answer:

```
> simplify(subs(x=xt,y=yt,slope1));
```

$$\frac{\sin(t^2) + 2\,t^2 \cos(t^2)}{\cos(t^2) - 2\,t^2 \sin(t^2)} \quad \textbf{(25)}$$

Exercise 2.2

```
> restart;
```

We start by entering the definitions given in the question:

```
> u := (x^2+y^2)^2 + 85*(x^2+y^2) - 500 + 18*x*(3*y^2-x^2);
```

$$u := (x^2 + y^2)^2 + 85\,x^2 + 85\,y^2 - 500 + 18\,x\,(-x^2 + 3\,y^2) \quad \textbf{(26)}$$

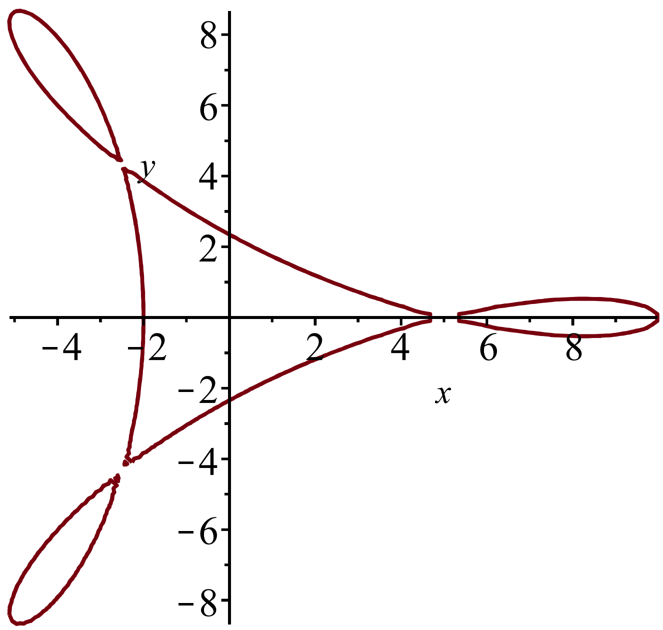
```
> xt := 6*cos(t) + 8*cos(t)^2 - 4;
  yt := 2*sin(t)*(3-4*cos(t));
```

$$xt := 6\cos(t) + 8\cos(t)^2 - 4$$

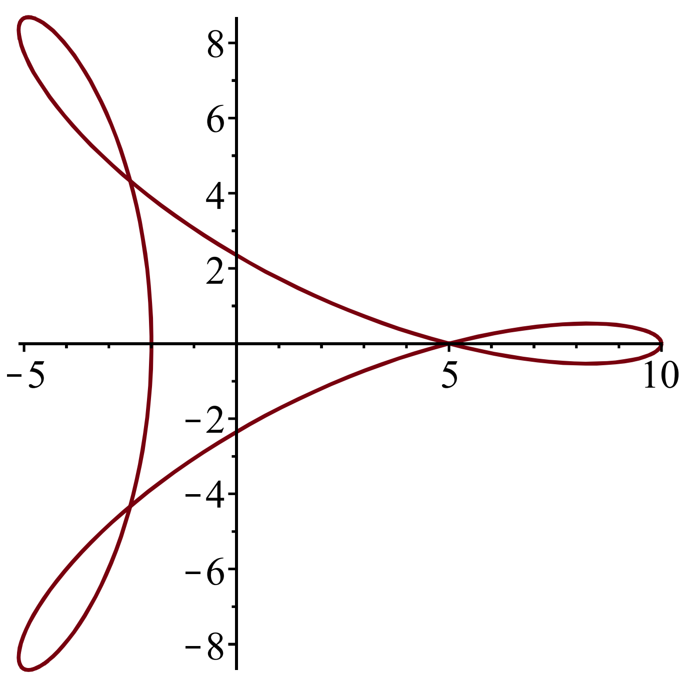
$$yt := 2\sin(t)\,(3 - 4\cos(t)) \quad \textbf{(27)}$$

We now plot the curve $u = 0$ and the curve given parametrically in terms of t:

```
> with(plots):
> implicitplot(u=0,x=-10..10,y=-10..10,grid=[100,100]);
```




```
> plot([xt,yt,t=0..2*Pi]);
```



The curves certainly look the same. This can be tested as follows:

```
> simplify(subs(x=xt,y=yt,u));
```

$$0 \quad \textbf{(28)}$$

We next find $\frac{dy}{dx}$ by implicit differentiation:

```
> slope1 := simplify(implicitdiff(u=0,y,x));
```

$$slope1 := \frac{-2x^3 - 2y^2 x + 27x^2 - 27y^2 - 85x}{y(2x^2 + 2y^2 + 54x + 85)} \quad (29)$$

Alternatively, we can find $\frac{dy}{dx}$ from the parametric representation:

```
> slope2 := simplify(diff(yt,t)/diff(xt,t));
```

$$slope2 := \frac{8\cos(t)^2 - 3\cos(t) - 4}{\sin(t)\,(8\cos(t) + 3)} \quad (30)$$

This is the same as what we get by rewriting *slope1* in terms of t:

```
> simplify(subs(x=xt,y=yt,slope1));
```

$$\frac{8\cos(t)^2 - 3\cos(t) - 4}{\sin(t)\,(8\cos(t) + 3)} \quad (31)$$

Exercise 3.1

```
> r := (n) -> diff(x^n*ln(x)/n!,x$n);
```

$$r := n \rightarrow \frac{\partial^n}{\partial x^n}\left(\frac{x^n \ln(x)}{n!}\right) \quad (32)$$

```
> seq(r(n),n=1..10);
```

$$\ln(x) + 1, \ln(x) + \frac{3}{2}, \ln(x) + \frac{11}{6}, \ln(x) + \frac{25}{12}, \ln(x) + \frac{137}{60}, \ln(x) + \frac{49}{20}, \ln(x) + \frac{363}{140}, \ln(x) + \frac{761}{280}, \ln(x) + \frac{7129}{2520}, \ln(x) + \frac{7381}{2520} \quad (33)$$

```
> seq(r(n)-r(n-1),n=2..10);
```

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10} \quad (34)$$

We see from this that $r(n) - r(n-1) = \frac{1}{n}$. We start with $r(1) = \ln(x) + 1$, and then

$$r(2) = r(1) + r(2) - r(1) = r(1) + \frac{1}{2} = \ln(x) + 1 + \frac{1}{2}$$

$$r(3) = r(2) + r(3) - r(2) = r(2) + \frac{1}{3} = \ln(x) + 1 + \frac{1}{2} + \frac{1}{3}$$

$$r(4) = r(3) + r(4) - r(3) = r(3) + \frac{1}{4} = \ln(x) + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

and so on. In general, we have

$$r(n) = \ln(x) + 1 + \frac{1}{2} + \ldots + \frac{1}{n} = \ln(x) + \sum_{k=1}^{n} \frac{1}{k}$$

Maple knows about a constant called gamma (Euler's constant, approximately 0.577) and a special function called Ψ with the property that $\sum_{k=1}^{n} \frac{1}{k} = \Psi(n+1) + \mathrm{gamma}$ for any positive integer n. (You can enter **?Psi** to find out more.) Using this, we could also write

$$r(n) = \ln(x) + \Psi(n+1) + \mathrm{gamma}$$

We can check this as follows:

```
> seq(r(n),n=1..8);
```

$$\ln(x)+1, \ln(x)+\frac{3}{2}, \ln(x)+\frac{11}{6}, \ln(x)+\frac{25}{12}, \ln(x)+\frac{137}{60}, \ln(x)+\frac{49}{20}, \ln(x)+\frac{363}{140}, \ln(x)+\frac{761}{280} \qquad (35)$$

```
> seq(ln(x)+Psi(n+1)+gamma,n=1..8);
```

$$\ln(x)+1, \ln(x)+\frac{3}{2}, \ln(x)+\frac{11}{6}, \ln(x)+\frac{25}{12}, \ln(x)+\frac{137}{60}, \ln(x)+\frac{49}{20}, \ln(x)+\frac{363}{140}, \ln(x)+\frac{761}{280} \qquad (36)$$

```
> unassign('r');
```

We can prove that the formula is valid by induction, but we will not give the details here.

Exercise 3.2

```
> y := t^2*exp(t);
```

$$y := t^2 e^t \qquad (37)$$

```
> z := simplify(
        diff(y,t,t,t) +
    a * diff(y,t,t) +
    b * diff(y,t) +
    c *y
  );
>
```

$$z := e^t \left((a+b+c+1)\, t^2 + (4\,a+2\,b+6)\, t + 2\,a+6 \right) \qquad (38)$$

We want this to be zero for all t, so the coefficients of the individual powers of t must all be zero. To find these coefficients, we use the **collect** function:

```
> collect(z,t);
```

$$e^t\,(a+b+c+1)\, t^2 + e^t\,(4\,a+2\,b+6)\, t + e^t\,(2\,a+6) \qquad (39)$$

We must therefore find a, b and c such that

$$1+a+b+c = 6+4\,a+2\,b = 6+2\,a = 0$$

```
> solve({1+a+b+c=0,6+4*a+2*b=0,6+2*a=0},{a,b,c});
```

$$\{a=-3, b=3, c=-1\} \quad \textbf{(40)}$$

The conclusion is that $y''' - 3\,y'' + 3\,y' - y = 0$.

You might think that we could do this more quickly by just solving $z=0$. This does not work, because it does not capture the fact that a, b and c are supposed to be constants. Maple gives us the following answer, in which b and c can be anything, but a depends on t:

```
> solve(z=0,{a,b,c});
```

$$\left\{a=a,\ b=b,\ c=-\frac{(t^2+4\,t+2)\,a}{t^2}-\frac{(t+2)\,b}{t}-\frac{t^2+6\,t+6}{t^2}\right\} \quad \textbf{(41)}$$

A correct approach along these lines is to replace the equation `z=0` by the expression `identity(z=0, t)` to indicate that the equation is supposed to hold for all t.
This is probably the best approach, if you can remember the syn

```
> solve(identity(z=0,t),{a,b,c});
```

$$\{a=-3, b=3, c=-1\} \quad \textbf{(42)}$$

Exercise 3.3

```
> p := (n) -> sort(expand(exp(x^2)*diff(exp(-x^2),x$n)));
```

$$p := n \rightarrow sort\left(expand\left(e^{x^2}\left(\frac{d^n}{dx^n}\,e^{-x^2}\right)\right)\right) \quad \textbf{(43)}$$

```
> p(2);
```

$$4\,x^2-2 \quad \textbf{(44)}$$

```
> p(3);
```

$$-8\,x^3+12\,x \quad \textbf{(45)}$$

```
> p(4);
```

$$16\,x^4-48\,x^2+12 \quad \textbf{(46)}$$

```
> seq(print(p(n)),n=1..10);
```

$$-2\,x$$

$$4\,x^2-2$$

$$-8\,x^3+12\,x$$

$$16\,x^4-48\,x^2+12$$

$$-32\,x^5+160\,x^3-120\,x$$

$$64\,x^6-480\,x^4+720\,x^2-120$$

$$-128\,x^7+1344\,x^5-3360\,x^3+1680\,x$$

$$256\,x^8-3584\,x^6+13440\,x^4-13440\,x^2+1680$$

$$-512\,x^9+9216\,x^7-48384\,x^5+80640\,x^3-30240\,x$$

(47)

$$1024\,x^{10} - 23040\,x^8 + 161280\,x^6 - 403200\,x^4 + 302400\,x^2 - 30240 \quad \textbf{(47)}$$

We first look at the leading term of $p(n)$. The leading term of $p(1)$ is $-2\,x$, the leading term of $p(2)$ is $4\,x^2$, the leading term of $p(3)$ is $-8\,x^3$ and so on. The signs alternate, the constant is 2^n, and the power of x is x^n. More succinctly, the leading term in $p(n)$ is $(-2\,x)^n$.

Next, note that when n is even we only get even powers of x, and when n is odd we only get odd powers of x. For example, $p(7)$ involves only x, x^3, x^5 and x^7, whereas $p(6)$ involves x^2, x^4 and x^6 (and a constant term, which we can think of as a multiple of x^0).

Now look at the last term in $p(n)$. When n is even, the last term is a constant, but when n is odd, it is a multiple of x. It is best to consider these separately, starting with the even case, where we may write $n = 2\,m$.

```
> seq(print(p(2*m)),m=1..6);
```

$$4\,x^2 - 2$$

$$16\,x^4 - 48\,x^2 + 12$$

$$64\,x^6 - 480\,x^4 + 720\,x^2 - 120$$

$$256\,x^8 - 3584\,x^6 + 13440\,x^4 - 13440\,x^2 + 1680$$

$$1024\,x^{10} - 23040\,x^8 + 161280\,x^6 - 403200\,x^4 + 302400\,x^2 - 30240$$

$$4096\,x^{12} - 135168\,x^{10} + 1520640\,x^8 - 7096320\,x^6 + 13305600\,x^4 - 7983360\,x^2 + 665280 \quad \textbf{(48)}$$

We see that the last term is $(-1)^m$ times a constant, with the sequence of constants being $2, 12, 120, 1680, 302040, 665280$ and so on. If we enter this in the Online Encyclopedia of Integer Sequences we get an answer including the line

Name: Quadruple factorial numbers: (2n)!/n!.

We need to be a little careful with this formula. The encyclopedia assumes that the numbers we entered correspond to $n = 1, 2, 3, \ldots$ and gives the formula $\frac{(2\,n)!}{n!}$. In fact our numbers correspond to $m = 1, 2, 3, \ldots$ (where $n = 2\,m$) so the right formula for the constant term in $p(2\,m)$ is $\frac{(-1)^m\,(2\,m)!}{m!}$.

Of course this kind of experimental approach does not really prove that the formula is correct, but it is very suggestive.

We now look at the case where n is odd, say $n = 2\,m - 1$:

```
> seq(print(p(2*m-1)),m=1..5);
```

$$-2\,x$$

$$-8\,x^3 + 12\,x$$

$$-32\,x^5 + 160\,x^3 - 120\,x$$

$$-128\,x^7 + 1344\,x^5 - 3360\,x^3 + 1680\,x$$

$$-512\,x^9 + 9216\,x^7 - 48384\,x^5 + 80640\,x^3 - 30240\,x \quad \textbf{(49)}$$

The final terms are the same as before, but multiplied by x. In other words, the last term in $p(2\,m-1)$

is $\frac{(-1)^m (2m)!\,x}{m!}$.

We can now predict that $p(12)$ should start with $(-2x)^{12} = 4096\,x^{12}$, and that it should contain multiples of $x^{10}, x^8, x^6, x^4, x^2$ and a constant term. This is the case where $n = 2m$ and $m = 6$, so the constant term is $\frac{(-1)^6 \cdot 12!}{6!} = 665280$. We check this as follows:

```
> p(12);
```

$$4096\,x^{12} - 135168\,x^{10} + 1520640\,x^8 - 7096320\,x^6 + 13305600\,x^4 - 7983360\,x^2 + 665280 \quad \textbf{(50)}$$

We now compare $p(n)$ with the Hermite polynomial $H_n(x)$, entered in Maple as **simplify (HermiteH(n,x))**:

```
> q := (n) -> simplify(HermiteH(n,x));
```

$$q := n \mapsto simplify(\mathrm{HermiteH}(n, x)) \quad \textbf{(51)}$$

```
> seq(print([p(n),q(n)]),n=1..6);
```

$$[-2x, 2x]$$

$$[4x^2 - 2, 4x^2 - 2]$$

$$[-8x^3 + 12x, 8x^3 - 12x]$$

$$[16x^4 - 48x^2 + 12, 16x^4 - 48x^2 + 12]$$

$$[-32x^5 + 160x^3 - 120x, 32x^5 - 160x^3 + 120x]$$

$$[64x^6 - 480x^4 + 720x^2 - 120, 64x^6 - 480x^4 + 720x^2 - 120] \quad \textbf{(52)}$$

We see that $q(n)$ is the same as $p(n)$ when n is even, and the same as $-p(n)$ when n is odd. In other words, we have $q(n) = (-1)^n p(n)$.