Mathematics with Maple (MAS100)

Introduction

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- We will learn how to use Maple, a powerful software package for solving mathematical problems.
- In the process, we will review and extend many parts of A-level mathematics, from a new perspective.

Algebraic manipulation

Skills to learn or practice:

- Expand out powers and products
- Factorize simple expressions by inspection
- Manipulate powers (using $a^n a^m = a^{n+m}$, $(a^n)^m = a^{nm}$ and so on)
- Manipulate and simplify algebraic fractions

Maple commands:

- `expand`, `factor` and `combine`
- `simplify`; the `symbolic` option
- `collect` and `coeff`

Expansion

- You should practice expanding out products and powers of algebraic expressions.
- You should check and remember the following identities:

$$\begin{aligned}(a+b)(a-b) &= a^2 - b^2\\ (a+b)^2 &= a^2 + 2ab + b^2\\ (a-b)^2 &= a^2 - 2ab + b^2.\end{aligned}$$

- Often you will need to use these when a and b are themselves complicated expressions.
- **Example:** To simplify $(w+x+y+z)^2 - (x+y+z)^2$, put $a = w+x+y+z$ and $b = x+y+z$. Then

$$\begin{aligned}(w+x+y+z)^2 - (x+y+z)^2 &= a^2 - b^2 = (a+b)(a-b)\\ &= (w+2x+2y+2z)w\\ &= w^2 + 2xw + 2yw + 2zw.\end{aligned}$$

An example: Cauchy-Schwartz

- **Problem:** Check the identity

$$\begin{aligned}(x^2+y^2+z^2)(u^2+v^2+w^2) &= (xu+yv+zw)^2 + \\ &\quad (xv-yu)^2+(yw-zv)^2+(zu-xw)^2 \\ &\geq (xu+yv+zw)^2\end{aligned}$$

- $$\begin{aligned}(x^2+y^2+z^2)(u^2+v^2+w^2) &= x^2u^2+x^2v^2+x^2w^2+ \\ &\quad y^2u^2+y^2v^2+y^2w^2+ \\ &\quad z^2u^2+z^2v^2+z^2w^2\end{aligned}$$

- $$\begin{aligned}(xu+yv+zw)^2 &= x^2u^2+y^2v^2+z^2w^2+2xyuv+2xzuw+2yzvw \\ +(xv-yu)^2 &\quad +x^2v^2-2xyuv+y^2u^2 \\ +(yw-zv)^2 &\quad +y^2w^2-2yzvw+z^2v^2 \\ +(zu-xw)^2 &\quad +z^2u^2-2xzuw+x^2w^2\end{aligned}$$

Factoring

- You should practice finding simple factorizations by inspection.
- $$\begin{aligned}a^2-b^2 &= (a+b)(a-b) \\ a^3-b^3 &= (a^2+ab+b^2)(a-b) \\ ax^2+bx^2+ay^2+by^2 &= (a+b)(x^2+y^2) \\ 1+t+t^2+t^3 &= (1+t)(1+t^2) \\ u^2-5u+6 &= (u-2)(u-3)\end{aligned}$$
- Maple's `factor` command will handle more complicated cases.

Powers

- You should practice using the basic rules for powers:

$$\begin{aligned}a^na^m &= a^{n+m} & (a^n)^m &= a^{nm} \\ a^nb^n &= (ab)^n & a^n/b^n &= (a/b)^n = a^nb^{-n} \\ (a+b)^n &\neq a^n+b^n & (a+b)^n &= \textstyle\sum_{k=0}^n \frac{n!}{k!(n-k)!}a^kb^{n-k}\end{aligned}$$

- **Warning:** the rule $(a^n)^m = a^{nm}$ has exceptions, for example:

$$((-3)^4)^{\frac{1}{4}} = (81)^{\frac{1}{4}} = +3 \qquad \text{but} \qquad (-3)^{4\times\frac{1}{4}} = (-3)^1 = -3.$$

However, the rule works whenever $a > 0$ or n and m are integers.

- **Example:**

$$\begin{aligned}(2^{1/2}3^{1/3}4^{1/4})^3 &= 2^{3/2}3^{3/3}4^{3/4} \\ &= 2^{3/2}(2^2)^{3/4}3 \\ &= 2^{3/2}2^{3/2}3 \\ &= 2^33 = 24\end{aligned}$$

Algebraic fractions

- You should practice manipulating fractions of the form a/b, where a and b are themselves complicated algebraic expressions.
- The rules are as follows:

$$\begin{aligned}\frac{a}{b}+\frac{c}{d} &= \frac{ad+bc}{bd} \\ \frac{a}{b}-\frac{c}{d} &= \frac{ad-bc}{bd} \\ \frac{a}{b}\cdot\frac{c}{d} &= \frac{ac}{bd} \\ \frac{a}{b}\Big/\frac{c}{d} &= \frac{ad}{bc} \\ \left(\frac{a}{b}\right)^n &= \frac{a^n}{b^n} \\ \left(\frac{a}{b}\right)^{-n} &= \frac{b^n}{a^n}\end{aligned}$$

An example: the cross-ratio

- Put $\chi(a,b,c,d) = \frac{(d-a)(c-b)}{(d-b)(c-a)}$.
- **Problem:** Show that $\chi(a,b,c,d) = \chi(a^{-1},b^{-1},c^{-1},d^{-1})$.
- $$\begin{aligned}\chi(\frac{1}{a},\frac{1}{b},\frac{1}{c},\frac{1}{d}) &= \frac{\left(\frac{1}{d}-\frac{1}{a}\right)\left(\frac{1}{c}-\frac{1}{b}\right)}{\left(\frac{1}{d}-\frac{1}{b}\right)\left(\frac{1}{c}-\frac{1}{a}\right)} \\ &= \frac{\frac{a-d}{ad}\,\frac{b-c}{bc}}{\frac{b-d}{bd}\,\frac{a-c}{ac}} \\ &= \frac{(a-d)(b-c)/(abcd)}{(b-d)(a-c)/(abcd)} \\ &= \frac{--(d-a)(c-b)}{--(d-b)(c-a)} \\ &= \frac{(d-a)(c-b)}{(d-b)(c-a)} \\ &= \chi(a,b,c,d).\end{aligned}$$

Special functions

The *primary special functions* are

exp, ln, sin, cos, tan, arcsin, arccos, arctan.

Things you should know:

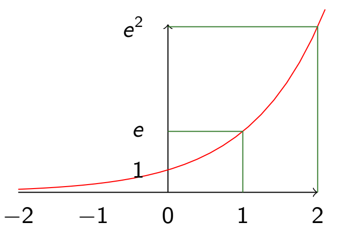
- The detailed shape of the graphs
- Domains, ranges and inverses
- Properties such as $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
- Derivatives and integrals (covered in later lectures).

The *secondary special functions* are

sec, csc, cot, sinh, cosh, tanh,
sech, csch, coth, arcsinh, arccosh, arctanh.

- You should know how these are defined in terms of the primary functions (for example, $\sinh(x) = (\exp(x) - \exp(-x))/2$, and $\sec(x) = 1/\cos(x)$)
- You should either remember the properties of the secondary functions, or be able to derive them from the properties of the primary functions

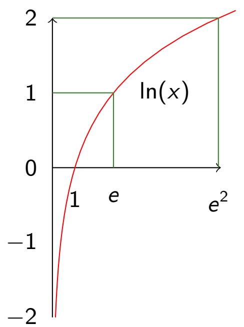
The exponential function

- $\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
  Warning: infinite sums are subtle.
- $e = \exp(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \simeq 2.71828$.
- $$\boxed{\begin{aligned}\exp(x+y) &= \exp(x)\exp(y) & \exp(x-y) &= \exp(x)/\exp(y) \\ \exp(0) &= 1 & \exp(-x) &= 1/\exp(x) \\ \exp(nx) &= \exp(x)^n & \exp(x) &= e^x\end{aligned}}$$
- 

The formula $\exp(x)\exp(y) = \exp(x+y)$

1	x	$\frac{x^2}{2!}$	$\frac{x^3}{3!}$	$\frac{x^4}{4!}$
y	$\frac{2xy}{2!}$	$\frac{3x^2y}{3!}$	$\frac{4x^3y}{4!}$	$\frac{5x^4y}{5!}$
$\frac{y^2}{2!}$	$\frac{3xy^2}{3!}$	$\frac{6x^2y^2}{4!}$	$\frac{10x^3y^2}{5!}$	$\frac{15x^4y^2}{6!}$
$\frac{y^3}{3!}$	$\frac{4xy^3}{4!}$	$\frac{10x^2y^3}{5!}$	$\frac{20x^3y^3}{6!}$	$\frac{35x^4y^3}{7!}$
$\frac{y^4}{4!}$	$\frac{5xy^4}{5!}$	$\frac{15x^2y^4}{6!}$	$\frac{35x^3y^4}{7!}$	$\frac{70x^4y^4}{8!}$

The logarithm

- The natural log function $\ln(y)$ is the inverse of the exponential.
- $\ln(y)$ is defined only when $y > 0$ (unless we use complex numbers).
- We have $\ln(\exp(x)) = \ln(e^x) = x$ for all x, and $\exp(\ln(y)) = e^{\ln(y)} = y$ when $y > 0$ (NOT $\ln(x) = 1/\exp(x)$).
- $$\boxed{\begin{aligned} \ln(xy) &= \ln(x) + \ln(y) & \ln(x/y) &= \ln(x) - \ln(y) \\ \ln(1) &= 0 & \ln(1/y) &= -\ln(y) \\ \ln(y^n) &= n\ln(y) & \ln(e) &= 1. \end{aligned}}$$
- 

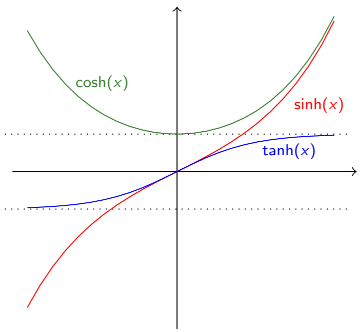
Logs to other bases

- $\log_a(y)$ is the number t such that $y = a^t$ (defined for $a, y > 0$).
- $$\begin{aligned} \log_{10}(1000) &= \log_{10}(10^3) = 3 \\ \log_2(1024) &= \log_2(2^{10}) = 10 \\ \log_{1024}(2) &= \log_{1024}(1024^{1/10}) = 1/10 \\ \log_3(1/9) &= \log_3(3^{-2}) = -2 \end{aligned}$$
- $\log_a(y) = \ln(y)/\ln(a)$
- Check: $a^{\ln(y)/\ln(a)} = (e^{\ln(a)})^{\ln(y)/\ln(a)} = e^{\ln(y)} = y$.
- $\log_{10}(y) =$ the number t such that $10^t = y$
  $\simeq$ the number of digits in y left of the decimal point.
- This is mostly of historical importance.
- $\log_2(y) =$ the number t such that $2^t = y$
  $\simeq$ the number of bits in y.
- This is of some use in computer science and information theory.
- $\log_e(y) =$ (the number t such that $e^t = y$) $= \ln(y) = \log(y)$.

Hyperbolic functions

- The hyperbolic functions are defined as follows:

$$\begin{aligned} \sinh(x) &= \tfrac{e^x - e^{-x}}{2} & \tanh(x) &= \tfrac{\sinh(x)}{\cosh(x)} & \operatorname{csch}(x) &= \tfrac{1}{\sinh(x)} \\ \cosh(x) &= \tfrac{e^x + e^{-x}}{2} & \coth(x) &= \tfrac{\cosh(x)}{\sinh(x)} & \operatorname{sech}(x) &= \tfrac{1}{\cosh(x)} \end{aligned}$$

  Use `convert(...,exp)` in Maple to rewrite in terms of exponentials.
- Properties are easily deduced from those of exp.
- These are related to trig functions using complex numbers, eg $\sin(x) = \sinh(ix)/i$, where $i = \sqrt{-1}$.
- 

Hyperbolic identities

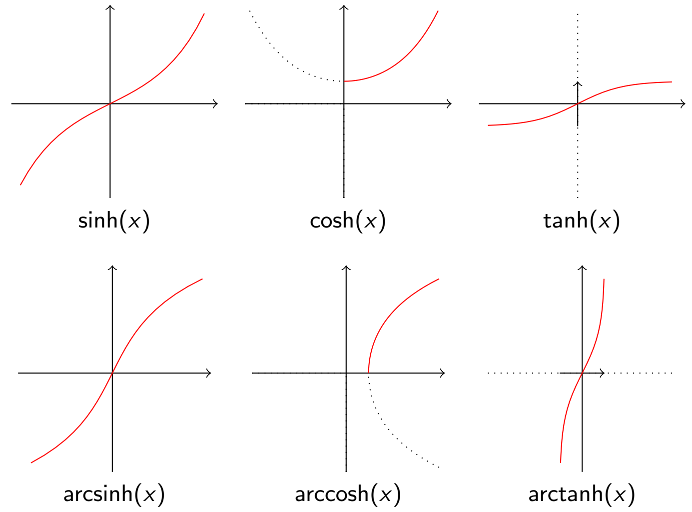
- $$\begin{aligned} \cosh(x)^2 - \sinh(x)^2 &= 1 \\ \operatorname{sech}(x)^2 + \tanh(x)^2 &= 1 \\ \sinh(x+y) &= \sinh(x)\cosh(y) + \cosh(x)\sinh(y) \\ \cosh(x+y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \end{aligned}$$
- To check these, put $u = e^x$, so $\sinh(x) = \frac{u-u^{-1}}{2}$ and $\cosh(x) = \frac{u+u^{-1}}{2}$.
- $$\begin{aligned} \cosh(x)^2 - \sinh(x)^2 &= \frac{(u+u^{-1})^2}{4} - \frac{(u-u^{-1})^2}{4} \\ &= \frac{(u^2 + 2 + u^{-2}) - (u^2 - 2 + u^{-2})}{4} \\ &= (2 - (-2))/4 = 1. \end{aligned}$$
- Now put $v = e^y$, so $uv = e^{x+y}$.
- $\sinh(x)\cosh(y) + \cosh(x)\sinh(y) = \frac{(u-u^{-1})}{2}\frac{(v+v^{-1})}{2} + \frac{(u+u^{-1})}{2}\frac{(v-v^{-1})}{2}$

$$\begin{aligned} &= \frac{(uv + uv^{-1} - u^{-1}v - u^{-1}v^{-1} + uv - uv^{-1} + u^{-1}v - u^{-1}v^{-1})}{4} \\ &= \frac{uv - (uv)^{-1}}{2} = \frac{e^{x+y} - e^{-x-y}}{2} = \sinh(x+y) \end{aligned}$$

Inverse hyperbolic functions

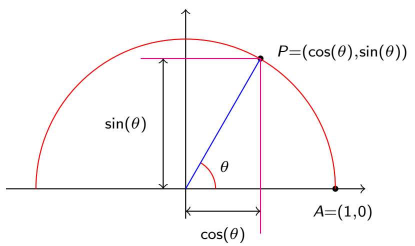
- The graph of $y = \sinh(x)$ crosses each horizontal line precisely once, which means that there is an inverse function $x = \sinh^{-1}(y) = \operatorname{arcsinh}(y)$, defined for all $y \in \mathbb{R}$.
- This can be written in terms of ln: $\operatorname{arcsinh}(y) = \ln(y + \sqrt{1+y^2})$.
- **Check:** Suppose $y = \sinh(x)$; we must show that $x = \ln(y + \sqrt{1+y^2})$.
  - We have $1 + y^2 = 1 + \sinh(x)^2 = \cosh(x)^2$ (and $\cosh(x), 1 + y^2 > 0$), so $\sqrt{1+y^2} = \cosh(x)$.
  - Thus $y + \sqrt{1+y^2} = \sinh(x) + \cosh(x) = \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} = e^x$
  - so $\ln(y + \sqrt{1+y^2}) = \ln(e^x) = x$ as required.
- Similarly, $\operatorname{arccosh}(y) = \ln(y + \sqrt{y^2 - 1})$, defined for $y \geq 1$
- and $\operatorname{arctanh}(y) = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$, defined when $-1 < y < 1$.

Graphs



Trigonometric functions

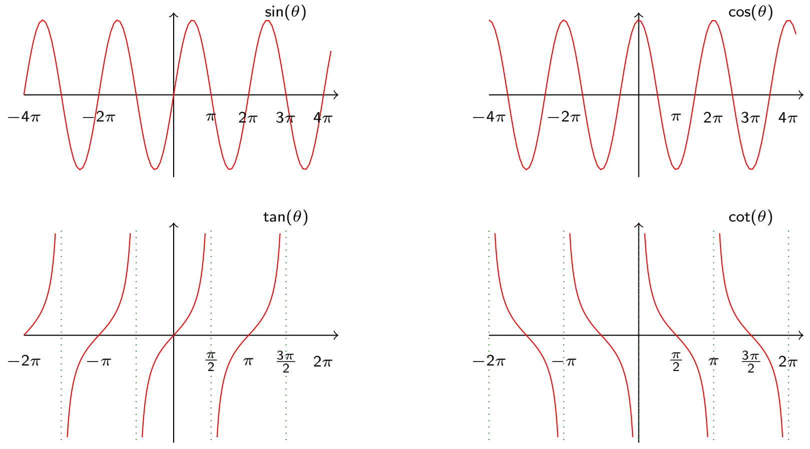
- Let P be one unit away from the origin, at an angle of θ measured anticlockwise from the point $A = (1, 0)$.



- (We measure θ in radians, so the length of the arc AP is θ.)
- The numbers $\cos(\theta)$ and $\sin(\theta)$ are *defined* to be the x and y coordinates of P.
- We also put

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \csc(x) = \frac{1}{\sin(x)}$$
$$\cot(x) = \frac{\cos(x)}{\sin(x)} \qquad \sec(x) = \frac{1}{\cos(x)}$$

Graphs



$$\begin{aligned} \sin(\pi/2 + x) &= \cos(x) & \cos(\pi/2 + x) &= -\sin(x) \\ \sin(\pi + x) &= -\sin(x) & \cos(\pi + x) &= -\cos(x) \\ \sin(2\pi + x) &= \sin(x) & \cos(2\pi + x) &= \cos(x) \\ \sin(-x) &= -\sin(x) & \cos(-x) &= \cos(x). \end{aligned}$$

Preview of complex numbers

- Complex numbers are expressions like $z = 3 + 4i$, where i satisfies $i^2 = -1$.
- You can add and subtract complex numbers in an obvious way, for example $(3 + 4i) + (7 - 3i) = 10 + i$.
- To multiply: expand out and use $i^2 = -1$. For example: $(1 + 2i)(3 + 4i) = 3 + 4i + 6i + 8i^2 = 3 + 4i + 6i - 8 = -5 + 10i$.
- Note that the powers of i repeat with period 4:

$$i^0 = 1 \quad i^1 = i \quad i^2 = -1 \quad i^3 = -i \quad i^4 = 1 \quad i^5 = i \quad i^6 = -1 \quad i^7 = -i \quad i^8 = 1.$$

- By expanding and using this we find powers of any complex number.

$$(1 + i)^2 = 1 + 2i + i^2 = 1 + 2i + (-1) = 2i$$
$$(1 + i)^8 = ((1 + i)^2)^4 = 2^4 i^4 = 2^4 = 16$$

- Note that

$$\begin{aligned}\exp(ix) &= 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{6} + \frac{(ix)^4}{24} + \frac{(ix)^5}{120} + \cdots \\ &= 1 + ix - \frac{x^2}{2} - i\frac{x^3}{6} + \frac{x^4}{24} + i\frac{x^5}{120} + \cdots \\ &= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right) + \left(x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots\right) i \\ &= \cos(x) + \sin(x)i.\end{aligned}$$

De Moivre's theorem

$$\boxed{e^{i\theta} = \exp(i\theta) = \cos(\theta) + \sin(\theta)i}$$

$$\begin{aligned}e^{-i\theta} &= \exp(-i\theta) = \cos(\theta) - \sin(\theta)i \\ \sin(\theta) &= \frac{e^{i\theta} - e^{-i\theta}}{2i} = \sinh(i\theta)/i \\ \cos(\theta) &= \frac{e^{i\theta} + e^{-i\theta}}{2} = \cosh(i\theta) \\ \tan(\theta) &= \frac{\sin(\theta)}{\cos(\theta)} = \frac{\sinh(i\theta)/i}{\cosh(i\theta)} = \tanh(i\theta)/i.\end{aligned}$$

$$\boxed{\begin{aligned}\cos(a)^2 + \sin(a)^2 &= 1 \\ \sec(a)^2 &= 1 + \tan(a)^2 \\ \sin(a + b) &= \sin(a)\cos(b) + \cos(a)\sin(b) \\ \cos(a + b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\ \sin(2a) &= 2\sin(a)\cos(a) \\ \cos(2a) &= 2\cos(a)^2 - 1 = 1 - 2\sin(a)^2.\end{aligned}}$$

Examples

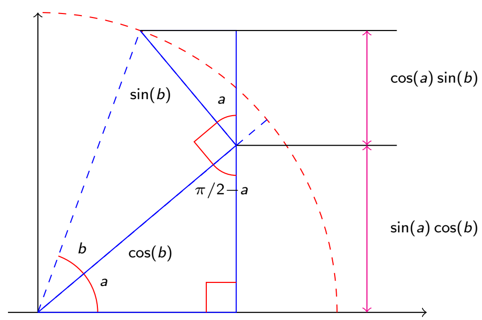
$$\begin{aligned}\cos(a)^2 + \sin(a)^2 &= \left(\frac{e^{ia} + e^{-ia}}{2}\right)^2 + \left(\frac{e^{ia} - e^{-ia}}{2i}\right)^2 \\ &= (e^{2ia} + 2 + e^{-2ia})/4 + (e^{2ia} - 2 + e^{-2ia})/(-4) \\ &= 2/4 - 2/(-4) = 1\end{aligned}$$

$$\begin{aligned}\cos(a)^2 - \sin(a)^2 &= \left(\frac{e^{ia} + e^{-ia}}{2}\right)^2 - \left(\frac{e^{ia} - e^{-ia}}{2i}\right)^2 \\ &= (e^{2ia} + 2 + e^{-2ia})/4 + (e^{2ia} - 2 + e^{-2ia})/4 \\ &= (e^{2ia} + e^{-2ia})/2 = \cos(2a)\end{aligned}$$

$$\begin{aligned}2\sin(a)\cos(a) &= 2\left(\frac{e^{ia} - e^{-ia}}{2i}\right)\left(\frac{e^{ia} + e^{-ia}}{2}\right) \\ &= \frac{2}{4i}\left(e^{2ia} + e^0 - e^0 - e^{-2ia}\right) = (e^{2ia} - e^{-2ia})/(2i) = \sin(2a)\end{aligned}$$

The addition formula

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$



$$\begin{aligned}\sin(a)\cos(b) + \cos(a)\sin(b) &= \frac{e^{ia} - e^{-ia}}{2i}\frac{e^{ib} + e^{-ib}}{2} + \frac{e^{ia} + e^{-ia}}{2}\frac{e^{ib} - e^{-ib}}{2i} \\ &= \frac{e^{i(a+b)} - e^{-i(a+b)}}{2i} = \sin(a + b)\end{aligned}$$

Finite Fourier series

- A *finite Fourier series* is a sum of constant multiples of functions of the form $\sin(nx)$ or $\cos(mx)$ (with $n, m \in \mathbb{Z}$). Note that the constant function $f(x) = a = a\cos(0x)$ is included.
- The phrase *trigonometric polynomial* means the same thing.
- Many functions can be rewritten as finite Fourier series:

$$\begin{aligned}
\sin(x)^2 &= \tfrac{1}{2} - \tfrac{1}{2}\cos(2x) \\
\sin(x)^3 &= \tfrac{3}{4}\sin(x) - \tfrac{1}{4}\sin(3x) \\
\sin(x)\sin(2x)\sin(4x) &= -\sin(x)/4 + \sin(3x)/4 + \sin(5x)/4 - \sin(7x)/4 \\
\sin(x)^4 + \cos(x)^4 &= \tfrac{3}{4} + \tfrac{1}{4}\cos(4x) \\
\sin(nx)\sin(mx) &= \tfrac{1}{2}\cos((n-m)x) - \tfrac{1}{2}\cos((n+m)x).
\end{aligned}$$

- **Method:** Rewrite using $\cos(n\theta) = (e^{in\theta} + e^{-in\theta})/2$ and $\sin(n\theta) = (e^{in\theta} - e^{-in\theta})/2i$, expand out, then rewrite using $e^{im\theta} = \cos(m\theta) + \sin(m\theta)i$.
- Once a function has been rewritten in this form, it is very easy to differentiate it or integrate it.

Examples

Problem: write $\sin(x)^4 + \cos(x)^4$ as a Fourier series.

Put $u = e^{ix}$, so $\sin(x) = (u - u^{-1})/(2i)$ and $\cos(x) = (u + u^{-1})/2$. Note that $i^2 = -1$ so $i^4 = (-1)^2 = 1$ so $(2i)^4 = 2^4 = 16$. Note also that

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

(use the binomial formula, or expand it out.) Thus

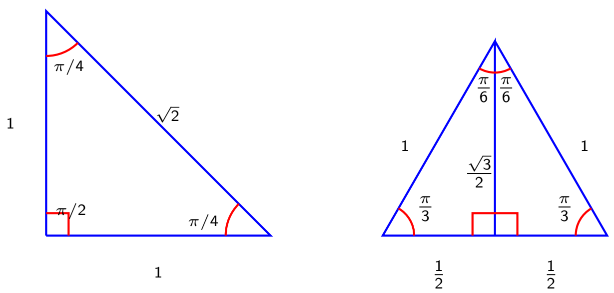
$$\begin{aligned}
\sin(x)^4 + \cos(x)^4 &= (u - u^{-1})^4/16 + (u + u^{-1})^4/16 \\
&= (u^4 - 4u^2 + 6 - 4u^{-2} + u^{-4})/16 + \\
&\quad (u^4 + 4u^2 + 6 + 4u^{-2} + u^{-4})/16 \\
&= 12/16 + 2(u^4 + u^{-4})/16 = 3/4 + ((u^4 + u^{-4})/2)/4 \\
&= (3 + \cos(4x))/4
\end{aligned}$$

Special values

You should know the following values of $\sin(\theta)$ and $\cos(\theta)$:

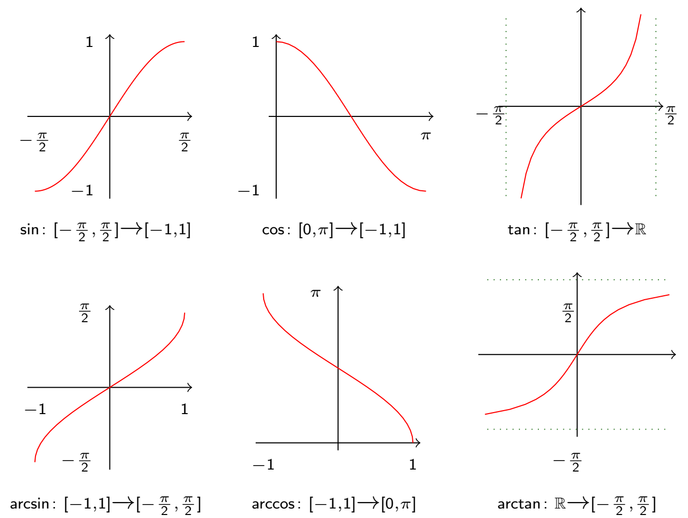
θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$\pi/2$	1	0	∞
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$

Proved by considering these triangles:



You should also be able to deduce things like $\cos(5\pi/6) = -\sqrt{3}/2$.

Inverse trigonometric functions



Differentiation

Things you should know:

- The meaning of differentiation (slopes of graphs, time-dependent and space-dependent variables, etc)
- Some derivatives from first principles: x^2, $1/x$, e^x.
- Rules for finding derivatives:
  - The product rule ($(uv)' = u'v + uv'$)
  - The quotient rule ($(u/v)' = (u'v - uv')/v^2$)
  - The chain rule ($\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$)
  - The power rule ($(u^n)' = nu^{n-1}u'$)
  - The logarithmic rule ($(\log(u))' = u'/u$)
  - The inverse function rule ($\frac{dx}{dy} = 1/\frac{dy}{dx}$)
- Derivatives of various classes of functions (eg the derivative of a rational function is another rational function.)

You must learn to find derivatives quickly and accurately.

Meaning

- Consider related variables x and y; so whenever x changes, so does y.
- Examples:
  - $p =$ price of chocolate ; $d =$ demand for chocolate .
  - $t =$ time ; $d =$ atmospheric CO_2 concentration .
  - $r =$ distance from sun ; $g =$ strength of solar gravity .
- If x changes to $x + \delta x$, then y changes to $y + \delta y$.

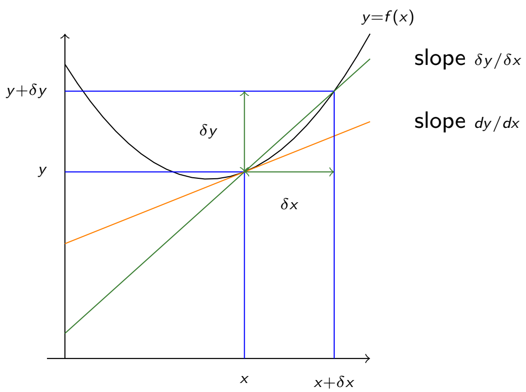
$$\frac{dy}{dx} = \lim_{\delta x \longrightarrow 0} \frac{\delta y}{\delta x} = \text{ derivative of } y \text{ with respect to } x.$$

- If $y = f(x)$, then $\delta y = f(x + \delta x) - f(x)$, so

$$f'(x) = \frac{dy}{dx} = \lim_{\delta x \longrightarrow 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

- We sometimes write y' for dy/dx (care needed).

Slopes



Consider variables x and y related by $y = f(x)$.dy/dx is the slope of the tangent line to the graph.If x changes by a small amount δx, then y will change by a small amount δy.The ratio $\delta y/\delta x$ is the slope of a chord cutting across the graph.The slope of the chord changes slightly as δx decreases.As δx approaches zero, the chord approaches the tangent, and $\delta y/\delta x$ approaches dy/dx.

The function $f(x) = x^2$

- Consider the function $f(x) = x^2$.
- Then $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$, so

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 - x^2}{h} \\ &= \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \frac{2xh + h^2}{h} \\ &= 2x + h\end{aligned}$$

- Thus

$$f'(x) = \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \longrightarrow 0} (2x + h) = 2x.$$

- Similarly:

$$\boxed{\frac{d}{dx}(x^n) = nx^{n-1} \text{ for all } n.}$$

The function $f(x) = 1/x$

- Consider the function $f(x) = 1/x$.
- $$f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x} = \frac{x - (x+h)}{x(x+h)} = \frac{-h}{x(x+h)}$$
  so
  $$\frac{f(x+h) - f(x)}{h} = \frac{-1}{x(x+h)}$$
  so
  $$f'(x) = \lim_{h\longrightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\longrightarrow 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}$$

The exponential function

- Consider the function $f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$.
- $$f(x+h) - f(x) = e^{x+h} - e^x = e^x(e^h - 1) = e^x\left(h + \tfrac{h^2}{2!} + \tfrac{h^3}{3!} + \cdots\right)$$
  so
  $$\frac{f(x+h) - f(x)}{h} = e^x\left(1 + \tfrac{h}{2!} + \tfrac{h^2}{3!} + \cdots\right)$$
  so
  $$\begin{aligned} f'(x) &= \lim_{h\longrightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\longrightarrow 0} e^x\left(1 + \tfrac{h}{2!} + \tfrac{h^2}{3!} + \cdots\right) \\ &= e^x(1 + 0 + 0 + \cdots) \\ &= e^x. \end{aligned}$$
- Conclusion: $\exp'(x) = \exp(x)$.

Special functions

$$\begin{aligned} \exp'(x) &= \exp(x) \\ \sinh'(x) &= \cosh(x) \\ \cosh'(x) &= \sinh(x) \\ \tanh'(x) &= \operatorname{sech}(x)^2 = 1 - \tanh(x)^2 \\ \sin'(x) &= \cos(x) \\ \cos'(x) &= -\sin(x) \\ \tan'(x) &= \sec(x)^2 = 1 + \tan(x)^2 \end{aligned}
\qquad
\begin{aligned} \log'(x) &= 1/x \\ \operatorname{arcsinh}'(x) &= (1 + x^2)^{-1/2} \\ \operatorname{arccosh}'(x) &= (x^2 - 1)^{-1/2} \\ \operatorname{arctanh}'(x) &= (1 - x^2)^{-1} \\ \arcsin'(x) &= (1 - x^2)^{-1/2} \\ \arccos'(x) &= -(1 - x^2)^{-1/2} \\ \arctan'(x) &= (1 + x^2)^{-1} \end{aligned}$$

- We showed earlier that $\exp'(x) = \exp(x)$
- We deduce $\sinh'(x)$ using the identity $\sinh(x) = (e^x - e^{-x})/2$. Similarly for cosh and tanh.
- Using $\cos(x) = \cosh(ix)$ etc, we find $\sin'(x)$, $\cos'(x)$ and $\tan'(x)$.
- Using $\exp'(x) = \exp(x)$ and the inverse function rule, we find that $\log'(x) = 1/x$
- The inverse function rule also gives the remaining derivatives.

The product rule

- Consider variables u and v depending on x, and put $w = uv$. Then

$$\boxed{w' = (uv)' = u'v + uv'} \qquad \boxed{\tfrac{dw}{dx} = \tfrac{d}{dx}(uv) = \tfrac{du}{dx}v + u\tfrac{dv}{dx}.}$$

- If x changes to $x + \delta x$, then u changes to $u + \delta u$ & v changes to $v + \delta v$ so w changes to

$$\begin{aligned} w + \delta w &= (u + \delta u)(v + \delta v) = uv + (\delta u)\,v + u\,(\delta v) + (\delta u)(\delta v) \\ \delta w &= (\delta u)\,v + u\,(\delta v) + (\delta u)(\delta v) \\ \frac{\delta w}{\delta x} &= \frac{\delta u}{\delta x}v + u\frac{\delta v}{\delta x} + \frac{\delta u}{\delta x}\frac{\delta v}{\delta x}\delta x \\ &\simeq \frac{du}{dx}v + u\frac{dv}{dx} + \frac{du}{dx}\frac{dv}{dx}\delta x \simeq \frac{du}{dx}v + u\frac{dv}{dx} \end{aligned}$$

(The approximations become exact in the limit as $\delta x \to 0$.)

Examples of the product rule

$$
\begin{aligned}
(uv)' &= u'v + uv' \\
\frac{d}{dx}(\sin(x)\cos(x)) &= \sin'(x)\cos(x) + \sin(x)\cos'(x) \\
&= \cos(x)\cos(x) + \sin(x)(-\sin(x)) \\
&= \cos(x)^2 - \sin(x)^2 \\
\frac{d}{dx}(x^3\log(x)) &= 3x^2\log(x) + x^3\log'(x) \\
&= 3x^2\log(x) + x^3(x^{-1}) \\
&= (3\log(x)+1)x^2 \\
\frac{d}{dx}(e^{ax}\sin(bx)) &= a\,e^{ax}\sin(bx) + e^{ax}b\cos(bx) \\
&= e^{ax}(a\sin(bx) + b\cos(bx))
\end{aligned}
$$

The quotient rule

- Consider variables u and v depending on x, and put $w = u/v$. Then

$$\boxed{w' = \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}}$$

- Indeed: $u = vw$, so $u' = v'w + vw'$ (product rule) , so

$$w' = \frac{u' - v'w}{v} = \frac{u'}{v} - \frac{v'.(u/v)}{v} = \frac{u'}{v} - \frac{uv'}{v^2} = \frac{u'v - uv'}{v^2}.$$

Examples of the quotient rule

$$\frac{d}{dx}\left(\frac{x}{\log(x)}\right) = \frac{1.\log(x) - xx^{-1}}{\log(x)^2} = \frac{\log(x)-1}{\log(x)^2} = \log(x)^{-1} - \log(x)^{-2}$$

(Aside: $x/\log(x) \simeq$ (number of primes $\leq x$))

$$\frac{d}{dx}\left(\frac{x}{1-x^2}\right) = \frac{1.(1-x^2) - x.(-2x)}{(1-x^2)^2} = \frac{1-x^2+2x^2}{(1-x^2)^2} = \frac{1+x^2}{(1-x^2)^2}$$

Now consider $\tan'(x)$, remembering that $\tan(x) = \sin(x)/\cos(x)$.

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) &= \frac{\sin'(x)\cos(x) - \sin(x)\cos'(x)}{\cos(x)^2} \\
&= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos(x)^2} \\
&= \frac{\cos(x)^2 + \sin(x)^2}{\cos(x)^2} = \frac{1}{\cos(x)^2} = \sec(x)^2
\end{aligned}
$$

The chain rule

- Suppose that y depends on u, and u depends on x. Then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

- If x changes to $x + \delta x$, then u changes to $u + \delta u$ and y changes to $y + \delta y$. Clearly

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u}\frac{\delta u}{\delta x}.$$

In the limit, δx, δu and δy all approach zero, and we get

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

- Alternative notation: suppose that $f(x) = g(h(x))$. Then

$$\boxed{f'(x) = g'(h(x))h'(x)}$$

Examples of the chain rule

- Consider $y = \cos(x^2)$. This is $y = \cos(u)$, where $u = x^2$.

$$\frac{du}{dx} = 2x \qquad \frac{dy}{du} = -\sin(u) = -\sin(x^2)$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\sin(x^2).2x = -2x\sin(x^2).$$

- Consider $f(x) = \exp(\sin(x))$.

$$f'(x) = \exp'(\sin(x)).\sin'(x) = \exp(\sin(x))\cos(x).$$

- Consider $y = a\sin(bx+c)$. Put $u = bx + c$, so $y = a\sin(u)$. Then $\frac{du}{dx} = b$ and $\frac{dy}{du} = a\cos(u)$ so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = a\cos(u).b = ab\cos(u) = ab\cos(bx+c).$$

The power rule

- If u depends on x and n does not, then

$$\boxed{\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}}$$

- Reason: If $y = u^n$ then $\frac{dy}{du} = nu^{n-1}$ so $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = nu^{n-1}\frac{du}{dx}$
- Consider $y = \sqrt{1+x^2}$. This is $y = u^{1/2}$, where $u = 1 + x^2$. Then

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{1+x^2}} \qquad \frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+x^2}}2x = \frac{x}{\sqrt{1+x^2}}.$$

- $\frac{d}{dx}(\sin(x)^5) = 5\sin(x)^4\cos(x)$
- $\frac{d}{dx}(\log(x)^3) = 3\log(x)^2x^{-1} = 3\log(x)^2/x$

The logarithmic rule

- $$\boxed{\frac{d}{dx}\log(u) = \frac{1}{u}\frac{du}{dx}} \qquad \boxed{\frac{du}{dx} = u\frac{d}{dx}\log(u)}$$

- $$\frac{d}{dx}\log(\cos(x)) = \frac{1}{\cos(x)}\cos'(x) = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$$

- $$\frac{d}{dx}\log(1+x^2) = \frac{\frac{d}{dx}(1+x^2)}{1+x^2} = \frac{2x}{1+x^2}$$

- Consider $y = x^x$, so $\log(y) = x\log(x)$. Then

$$\begin{aligned}\frac{d}{dx}\log(y) &= \frac{d}{dx}(x\log(x)) \\ &= 1.\log(x) + x.x^{-1} = \log(x) + 1 \\ \frac{dy}{dx} &= y\frac{d}{dx}\log(y) \\ &= x^x(\log(x)+1).\end{aligned}$$

The inverse function rule

- If x and y are interdependent variables, then

$$\boxed{\frac{dx}{dy} = 1/\frac{dy}{dx}}$$

- (Take limits in the obvious relation $\frac{\delta x}{\delta y} = 1/\frac{\delta y}{\delta x}$.)
- Consider $y = \log(x)$, so $x = e^y$.

$$\frac{dx}{dy} = e^y = x \qquad \frac{dy}{dx} = 1/\frac{dx}{dy} = \frac{1}{x}$$

- Alternative notation: if $y = g(x)$ then $x = f(y)$, where $f = g^{-1}$ and $g = f^{-1}$. Then

$$\boxed{g'(x) = 1/f'(g(x))}$$

- $\log'(x) = 1/\exp'(\log(x)) = 1/\exp(\log(x)) = 1/x$.

The arcsin function

- Consider $y = \arcsin(x)$, so $x = \sin(y)$.

$$\frac{dx}{dy} = \sin'(y) = \cos(y)$$
$$\frac{dy}{dx} = 1/\frac{dx}{dy} = \cos(y)^{-1}.$$

- Also $\sin(y)^2 + \cos(y)^2 = 1$, so

$$\cos(y) = \sqrt{1 - \sin(y)^2} = \sqrt{1 - x^2}$$
$$\cos(y)^{-1} = (1 - x^2)^{-1/2}$$

- So $\arcsin'(x) = \frac{dy}{dx} = (1 - x^2)^{-1/2}$.

The arctanh function

- Consider $y = \operatorname{arctanh}(x)$, so $x = \tanh(y) = \frac{\sinh(y)}{\cosh(y)}$.

$$\begin{aligned}\frac{dx}{dy} &= \tanh'(y) \\ &= \frac{\sinh'(y)\cosh(y) - \sinh(y)\cosh'(y)}{\cosh(y)^2} \\ &= \frac{\cosh(y)^2 - \sinh(y)^2}{\cosh(y)^2} \\ &= 1 - \tanh(y)^2 = 1 - x^2 \\ \frac{dy}{dx} &= 1/\frac{dx}{dy} = \frac{1}{1 - x^2}.\end{aligned}$$

- So $\operatorname{arctanh}'(x) = \frac{dy}{dx} = (1 - x^2)^{-1}$.

Classes of functions

- If $f(x)$ is a polynomial, then so is $f'(x)$.
  - Eg $f(x) = x + x^{10} + x^{100}$; $f'(x) = 1 + 10x^9 + 100x^{99}$
  - Eg $f(x) = (x-1)^4 + (x+1)^4$; $f'(x) = 4(x-1)^3 + 4(x+1)^3$
- If $f(x)$ is a rational function, then so is $f'(x)$.
  - Eg $f(x) = \frac{x^2-1}{x^2+1}$; $f'(x) = \frac{4x}{(x^2+1)^2}$
  - Eg $f(x) = \frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2}$; $f'(x) = -\frac{1}{x^2} - \frac{1}{(x+1)^2} - \frac{1}{(x+2)^2}$
- If $f(x)$ is a trigonometric polynomial, so is $f'(x)$.
  - Eg $f(x) = \sin(x) + \sin(3x)/3 + \sin(5x)/5$;
    $f'(x) = \cos(x) + \cos(3x) + \cos(5x)$.
  - Eg $f(x) = \sin(3x) + \cos(3x)$; $f'(x) = 3\cos(3x) - 3\sin(3x)$.
- If $f(x)$ is a polynomial times e^x, so is $f'(x)$.
  - Eg $f(x) = (x + x^2)e^x$; $f'(x) = (1 + 3x + x^2)e^x$.
  - Eg $f(x) = (x^4 - 4x^3 + 12x^2 - 24x + 24)e^x$; $f'(x) = x^4 e^x$.

Implicit differentiation

- Suppose that x and y are related by an equation such as $y^4 + xy = x^3$. We cannot write y as a function of x, but we can still find dy/dx.
- Differentiate both sides. Terms in the equation involving y give terms in the derivative involving dy/dx. Rearranging gives dy/dx in terms of x and y.
- Suppose that $y^4 + xy = x^3$, so

$$\tfrac{d}{dx}\left(y^4 + xy\right) = \tfrac{d}{dx}\left(x^3\right) = 3x^2.$$

Also $\frac{d}{dx}(y^4) = 4y^3\frac{dy}{dx}$ by the power rule
and $\frac{d}{dx}(xy) = \frac{dx}{dx}y + x\frac{dy}{dx} = y + x\frac{dy}{dx}$ by the product rule ; so

$$\begin{aligned}4y^3\tfrac{dy}{dx} + y + x\tfrac{dy}{dx} &= 3x^2 \\ (4y^3 + x)\tfrac{dy}{dx} &= 3x^2 - y \\ \tfrac{dy}{dx} &= \tfrac{3x^2 - y}{4y^3 + x}.\end{aligned}$$

Implicit examples

- Suppose $x + \sin(x) = y - \cos(y)$.

$$\begin{aligned} \tfrac{d}{dx}(x + \sin(x)) &= \tfrac{d}{dx}(y - \cos(y)) \\ 1 + \cos(x) &= \tfrac{dy}{dx} + \sin(y)\tfrac{dy}{dx} \\ \tfrac{dy}{dx} &= \tfrac{1+\cos(x)}{1+\sin(y)} \end{aligned}$$

- Suppose $y = \exp(x^2 + y^2)$.

$$\begin{aligned} \tfrac{dy}{dx} &= \tfrac{d}{dx}\exp(x^2+y^2) = \tfrac{d}{dx}(e^{x^2}e^{y^2}) \\ &= 2xe^{x^2}e^{y^2} + e^{x^2}.2y\tfrac{dy}{dx}e^{y^2} \\ &= 2(x + y\tfrac{dy}{dx})\exp(x^2+y^2) \\ (1 - 2y\exp(x^2+y^2))\tfrac{dy}{dx} &= 2x\exp(x^2+y^2) \\ \tfrac{dy}{dx} &= \tfrac{2x\exp(x^2+y^2)}{1-2y\exp(x^2+y^2)} \end{aligned}$$

Parametric differentiation

- Suppose that x and y are both functions of another variable t. Then

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

- Suppose that $x = 1 + t^2$ and $y = t + t^3$ (so $t = y/x$)

$$dy/dt = 1 + 3t^2 \qquad dx/dt = 2t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1+3t^2}{2t} = \frac{1+3(y/x)^2}{2(y/x)} = \frac{x^2+3y^2}{2xy}$$

- Suppose that $x = t - \sin(t)$ and $y = 1 - \cos(t)$.

$$dy/dt = \sin(t) \qquad dx/dt = 1 - \cos(t)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin(t)}{1-\cos(t)} = \frac{\sqrt{y(2-y)}}{y} = \sqrt{\frac{2-y}{y}}$$

The circle

- Consider a point (x, y) on the unit circle, so $x^2 + y^2 = 1$.
- Differentiate $x^2 + y^2 = 1$; $2x + 2y\dfrac{dy}{dx} = 0$;

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

- Parametrically: $x = \cos(t)$, $y = \sin(t)$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos(t)}{-\sin(t)} = -\frac{x}{y}$$

- Directly: $y = (1 - x^2)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(1-x^2)^{-1/2}\frac{d}{dx}(1-x^2) = \frac{1}{2}y^{-1}.(-2x) = -\frac{x}{y}.$$

Integration

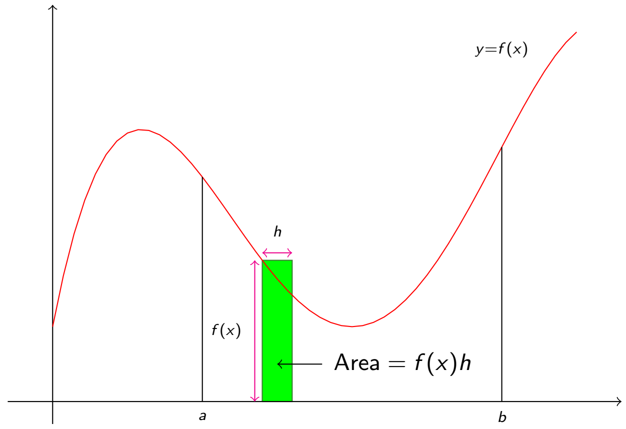
Things you should know:

- The meaning of integration (take the sum of a large number of very small contributions, and pass to the limit)
- Integration as the reverse of differentiation
- Integrals of standard functions and classes of functions
- The method of undetermined coefficients
- Integration by parts
- Integration by substitution

Meaning

- To define $\int_a^b f(x)\,dx$:
  - Divide the interval $[a,b]$ into many short intervals $[x, x+h]$.
  - For each short interval $[x, x+h]$, find $f(x)h$.
  - Add these terms together to get an approximation to $\int_a^b f(x)\,dx$.
  - For the exact value of $\int_a^b f(x)\,dx$, take the limit $h \to 0$.
- In economics, government revenue depends on time, and total revenue in the last decade is $\int_{1999}^{2009} \text{revenue(t)}\,dt$.
- If a particle moves with velocity $v(t) > 0$ at time t, then the total distance moved between times a and b is $\int_a^b v(t)\,dt$.
- A current flowing in a wire exerts a magnetic force on a moving electron. There is a formula for the force contributed by a short section of wire; to get the total force, we integrate.

Areas

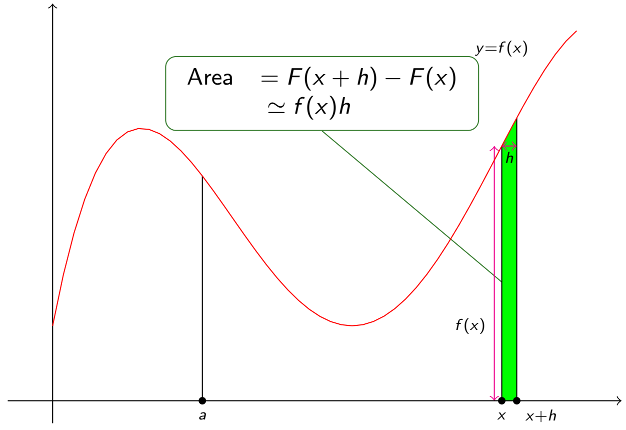


Consider the integral $\int_a^b f(x)\,dx$.For each short interval $[x, x+h] \subset [a,b]$, we have a contribution $f(x)h$. This is the area of the green rectangle.This is the contribution from one short interval, but we need to add together the contributions from many short intervals.

The Fundamental Theorem of Calculus

- An *indefinite integral* of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.
- Examples:
  - $\log(x)$ is an indefinite integral of $1/x$
  - $\sin(x)$ is an indefinite integral of $\cos(x)$
  - $F(x) = x^2 + 2x$ and $G(x) = (x+1)^2$ are indefinite integrals of $2x + 2$
- The *Fundamental Theorem of Calculus:*
  - For any number a, the function $F(x) = \int_a^x f(t)\,dt$ is an indefinite integral of $f(x)$.
  - If $F(x)$ is any indefinite integral of $f(x)$, then $\int_a^b f(x)\,dx = \big[F(x)\big]_a^b = F(b) - F(a)$.
- The functions $F(x) = \int_0^x 2t + 2\,dt = x^2 + 2x$ and $G(x) = \int_{-1}^x 2t + 2\,dt = (x+1)^2$ are both indefinite integrals of $2x + 2$.
- $\int_a^b \frac{1}{x} = \big[\log(x)\big]_a^b = \log(b) - \log(a)$

Proof of the Fundamental Theorem



$$F'(x) = \lim_{h \longrightarrow 0} (F(x+h) - F(x))/h = f(x).$$

Constants

- Is it $\int x^2\,dx = x^3/3$ or $\int x^2\,dx = x^3/3+c$?
- Either is acceptable in the exam.
  Neither one is strictly logically satisfactory.
- $x^3/3$ is *an* indefinite integral of x^2.
- *Every* indefinite integral of x^2 has the form $x^3/3+c$ for some c.
- If you just want to calculate $\int_a^b f(x)\,dx$, it does not matter which indefinite integral you use. Any two choices will give the same answer.
- In solving differential equations, it often does matter which indefinite integral you use. You must therefore include a '+c' term, and do some extra work to see what c should be.
- Maple's `int()` command will never give you a '+c' term.
  If you need one, you must insert it yourself.

Checking and Guessing

- $\boxed{\text{Integrals can easily be checked by differentiating}}$
- $\int \sin(x)^2\,dx \neq \sin(x)^3/3$, because
$$\frac{d}{dx}\left(\sin(x)^3/3\right) = 3\sin(x)^2\cos(x)/3 = \sin(x)^2\cos(x) \neq \sin(x)^2.$$
- $\int \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}\,dx = \frac{\sin(x)}{x}$, because
$$\frac{d}{dx}\left(\frac{\sin(x)}{x}\right) = \frac{\sin'(x).x - \sin(x).1}{x^2} = \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}.$$
- $\int 2x\,e^{x^2}\,dx = e^{x^2}$, because $\frac{d}{dx}e^{x^2} = 2x\,e^{x^2}$.
- $\int \frac{3x^2+2x+1}{x^3+x^2+x+1}\,dx = \log(x^3+x^2+x+1)$, because
$$\frac{d}{dx}\log(x^3+x^2+x+1) = \frac{\frac{d}{dx}(x^3+x^2+x+1)}{x^3+x^2+x+1} = \frac{3x^2+2x+1}{x^3+x^2+x+1}.$$

Undetermined coefficients

- Suppose we know that for some constants $a,\dots,d$
$$\int \log(x)^3\,dx = (a\log(x)^3 + b\log(x)^2 + c\log(x) + d)x$$
  (How could we know this? — see later)
- **Problem:** find a, b, c and d.
- $$\begin{aligned} \log(x)^3 &= \tfrac{d}{dx}\left((a\log(x)^3 + b\log(x)^2 + c\log(x) + d)x\right) \\ &= (3a\log(x)^2x^{-1} + 2b\log(x)x^{-1} + cx^{-1})x + \\ &\quad (a\log(x)^3 + b\log(x)^2 + c\log(x) + d).1 \\ &= a\log(x)^3 + (b+3a)\log(x)^2 + (c+2b)\log(x) + (d+c) \end{aligned}$$
- So $a=1$, $b+3a=0$, $c+2b=0$ and $d+c=0$ (compare coefficients)
- So $a=1$, $b=-3$, $c=6$ and $d=-6$
$$\int \log(x)^3\,dx = (\log(x)^3 - 3\log(x)^2 + 6\log(x) - 6)x.$$

Standard integrals

$$\begin{aligned} \int \exp(x)\,dx &= \exp(x) & \int 1/x\,dx &= \log(x) \\ \int \cosh(x)\,dx &= \sinh(x) & \int (1+x^2)^{-1/2}\,dx &= \operatorname{arcsinh}(x) \\ \int \sinh(x)\,dx &= \cosh(x) & \int (x^2-1)^{-1/2}\,dx &= \operatorname{arccosh}(x) \\ \int \operatorname{sech}(x)^2\,dx &= \tanh(x) & \int (1-x^2)^{-1}\,dx &= \operatorname{arctanh}(x) \\ \int \cos(x)\,dx &= \sin(x) & \int (1-x^2)^{-1/2}\,dx &= \arcsin(x) \\ \int \sin(x)\,dx &= -\cos(x) & \int (1-x^2)^{-1/2}\,dx &= -\arccos(x) \\ \int \sec(x)^2\,dx &= \tan(x) & \int (1+x^2)^{-1}\,dx &= \arctan(x) \end{aligned}$$

$$\int x^n\,dx = x^{n+1}/(n+1) \qquad (n \neq -1)$$
$$\int a^x\,dx = a^x/\log(a)$$
$$\int \log(x)\,dx = x\log(x) - x$$
$$\int \tan(x)\,dx = -\log(\cos(x))$$
$$\int \sin(x)^2\,dx = (2x - \sin(2x))/4$$
$$\int \cos(x)^2\,dx = (2x + \sin(2x))/4$$

Rational functions

- A *rational function* of x is a function defined using only constants, addition, multiplication, division and integer powers.
- No roots, fractional powers, logs, exponentials, trigonometric functions and so on can occur in a rational function.
- **Examples:** $\frac{1+x+x^2}{1-x+x^2}$ $\quad \frac{1}{x}+\frac{\pi}{x-1}+\frac{\pi^2}{x-2}$ $\quad x^2+x+1+x^{-1}+x^{-2}$
- **Non-Examples:** $e^{-x}\sin(x)$ $\quad \sqrt{1-x^2}$ $\quad \frac{\log(x)}{1+x}$ $\quad \frac{\arctan(x)}{2\pi}$.
- If $f(x)$ is a rational function, then $\int f(x)\,dx$ is a sum of terms of the following types:
  - Rational functions
  - Terms of the form $\ln(|x-u|)$
  - Terms of the form $\ln(x^2+vx+w)$
  - Terms of the form $\arctan(ux+v)$.
- $\displaystyle\int \frac{4x^3+8}{x^6-x^2}\,dx = \frac{8}{x} + 3\ln(|x-1|) - \ln(|x+1|) - \ln(x^2+1) + 4\arctan(x)$

Rational function examples

- $\displaystyle\int \frac{x^2+1}{x^2-1}\,dx = x + \ln(|x-1|) + \ln(|x+1|)$
- $\displaystyle\int \left(\frac{x+1}{x-1}\right)^3 dx = 1 + \frac{6}{x-1} + \frac{12}{(x-1)^2} + \frac{8}{(x-1)^3}$
- $\displaystyle\int \frac{2x+2}{x^2+1}\,dx = \ln(x^2+1) + 2\arctan(x)$
- $\displaystyle\int \frac{1}{x^{-1}+1+x}\,dx = \frac{1}{2}\ln(1+x+x^2) - \frac{1}{\sqrt{3}}\arctan\left(\frac{1+2x}{\sqrt{3}}\right)$
- $\displaystyle\int \frac{4}{1-x^4}\,dx = \ln(|x+1|) - \ln(|x-1|) + 2\arctan(x)$

$$\frac{d}{dx}\ln(|x-u|) = \frac{1}{x-u} \qquad \frac{d}{dx}\ln(x^2+ux+v) = \frac{2x+u}{x^2+ux+v}$$

$$\frac{d}{dx}\arctan(ux+v) = \frac{u}{1+(ux+v)^2} = \frac{u}{u^2x^2+2uvx+(v^2+1)}$$

Trigonometric polynomials

$$\boxed{\int \sin(nx)\,dx = -\cos(nx)/n \qquad \int \cos(nx)\,dx = \sin(nx)/n}$$

$$\begin{aligned}
\cos(2x) &= \cos(x)^2 - \sin(x)^2 = 2\cos(x)^2 - 1 = 1 - 2\sin(x)^2 \\
\sin(x)^2 &= 1/2 - \cos(2x)/2 \\
\int \sin(x)^2\,dx &= x/2 - \sin(2x)/4 \\
\int \cos(x)^2\,dx &= x/2 + \sin(2x)/4 \\
\sin(x)^3 &= 3\sin(x)/4 - \sin(3x)/4 \\
\int \sin(x)^3\,dx &= -3\cos(x)/4 + \cos(3x)/12 \\
\sin(x)\sin(2x)\sin(4x) &= -\sin(x)/4 + \sin(3x)/4 + \sin(5x)/4 - \sin(7x)/4 \\
\int \sin(x)\sin(2x)\sin(4x)\,dx &= \cos(x)/4 - \cos(3x)/12 - \cos(5x)/20 + \cos(7x)/28 \\
\sin(x)^4 + \cos(x)^4 &= 3/4 + \cos(4x)/4 \\
\int \sin(x)^4 + \cos(x)^4\,dx &= 3x/4 + \sin(4x)/16
\end{aligned}$$

Affine substitution

If $\displaystyle\int f(x)\,dx = g(x)$ and a, b are constant, then

$$\int f(ax+b)\,dx = g(ax+b)/a$$

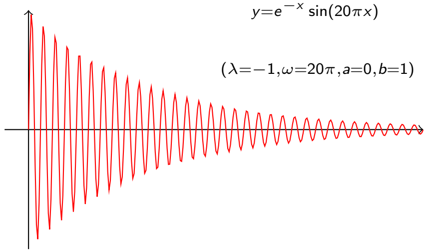
$$\int \cos(x)\,dx = \sin(x) \qquad \int \cos(2x+3)\,dx = \sin(2x+3)/2$$

$$\int e^x\,dx = e^x \qquad \int e^{-2x+7}\,dx = e^{-2x+7}/(-2)$$

$$\int \tan(x)\,dx = -\ln(\cos(x)) \qquad \int \tan(\pi x)\,dx = -\ln(\cos(\pi x))/\pi$$

Exponential oscillations

- An *exponential oscillation* is a function of the form
$$f(x) = e^{\lambda x}(a\cos(\omega x) + b\sin(\omega x)),$$
where a, b, λ and ω are constants.
- The *growth rate* is λ, and the *angular frequency* is ω.



- Special cases:

$$\begin{aligned} f(x) &= e^{\lambda x}\sin(\omega x) && (a=0, b=1)\\ f(x) &= a\cos(\omega x) + b\sin(\omega x) && (\lambda = 0)\\ f(x) &= ae^{\lambda x} && (\omega = 0). \end{aligned}$$

Integrating exponential oscillations

The integral of an EO is another EO with the same growth rate and angular frequency.

$$\boxed{\int e^{\lambda x}(a\cos(\omega x) + b\sin(\omega x))\,dx = e^{\lambda x}(A\cos(\omega x) + B\sin(\omega x))}$$

$$\boxed{A = \frac{a\lambda - b\omega}{\lambda^2+\omega^2} \qquad B = \frac{a\omega + b\lambda}{\lambda^2+\omega^2}.}$$

- Example: find
$\int e^{-2x}(5\cos(4x) - 3\sin(4x))\,dx \int e^{-2x}(5\cos(4x) - 3\sin(4x))\,dx$
  - $\lambda = -2$, $\omega = 4$, $a = 5$, $b = -3$
  - $A = \dfrac{a\lambda - b\omega}{\lambda^2+\omega^2} = \dfrac{5.(-2) - (-3).4}{(-2)^2+4^2} = 1/10$
  - $B = \dfrac{a\omega + b\lambda}{\lambda^2+\omega^2} = \dfrac{5.4 + (-3)(-2)}{(-2)^2+4^2} = 13/10$

$$\int e^{-2x}(5\cos(4x) - 3\sin(4x))\,dx = e^{-2x}(\cos(4x) + 13\sin(4x))/10$$

Integrating exponential oscillations

Alternatively:

$$\int e^{-2x}(5\cos(4x) - 3\sin(4x))\,dx = e^{-2x}(A\cos(4x) + B\sin(4x)) \text{ for some } A, B$$

$$\begin{aligned} e^{-2x}(5\cos(4x) - 3\sin(4x)) &= \frac{d}{dx}\left(e^{-2x}(A\cos(4x) + B\sin(4x))\right)\\ &= -2e^{-2x}(A\cos(4x) + B\sin(4x)) +\\ &\quad e^{-2x}(-4A\sin(4x) + 4B\cos(4x))\\ &= e^{-2x}((4B-2A)\cos(4x) - (2B+4A)\sin(4x)) \end{aligned}$$

By comparing coefficients, we must have $4B - 2A = 5$ and $2B + 4A = 3$. These equations can be solved to give $A = 1/10$ and $B = 13/10$. Thus

$$\int e^{-2x}(5\cos(4x) - 3\sin(4x))\,dx = e^{-2x}(\cos(4x) + 13\sin(4x))/10.$$

Polynomial exponential oscillations

- A *polynomial exponential oscillation* is a function of the form
$$f(x) = e^{\lambda x}(a(x)\cos(\omega x) + b(x)\sin(\omega x)),$$
where $a(x)$ and $b(x)$ are polynomials.
- λ is the *growth rate* and ω is the *angular frequency*. The *degree* is the highest power of x that occurs in $a(x)$ or in $b(x)$.
- The function $f(x) = e^{-2x}((1+x^5)\cos(4x) + x^3\sin(4x))$ is a PEO of growth rate -2, frequency 4 and degree 5.
- The function $f(x) = e^{4x}((1+x^3+x^6)\sin(3x))$ is a PEO of growth rate 4, frequency 3 and degree 6.
- **Fact:** The integral of any PEO is another PEO with the same growth rate, frequency and degree.

Integrating PEO's — I

- $\int xe^{-x}\sin(x)\,dx$ is a PEO of degree 1, growth -1, frequency 1.
- $\int xe^{-x}\sin(x)\,dx = (Ax+B)e^{-x}\cos(x) + (Cx+D)e^{-x}\sin(x)$ for some A, B, C, D.
- $$\begin{aligned} xe^{-x}\sin(x) &= \tfrac{d}{dx}\left((Ax+B)e^{-x}\cos(x) + (Cx+D)e^{-x}\sin(x)\right) \\ &= Ae^{-x}\cos(x) - (Ax+B)e^{-x}\cos(x) - (Ax+B)e^{-x}\sin(x) + \\ &\quad Ce^{-x}\sin(x) - (Cx+D)e^{-x}\sin(x) + (Cx+D)e^{-x}\cos(x) \\ &= (-A+C)xe^{-x}\cos(x) + (A-B+D)e^{-x}\cos(x) + \\ &\quad (-A-C)xe^{-x}\sin(x) + (-B+C-D)e^{-x}\sin(x). \end{aligned}$$
- $-A+C=0$, $A-B+D=0$, $-A-C=1$, $-B+C-D=0$.
- So $A=-1/2$, $B=-1/2$, $C=-1/2$, $D=0$
- $\int xe^{-x}\sin(x)\,dx = -((x+1)e^{-x}\cos(x) + xe^{-x}\sin(x))/2$.

Integrating PEO's — II

- $\int x^3e^x\,dx$ is a PEO of degree 3, growth 1 and frequency 0.
- $\int x^3e^x\,dx = (Ax^3+Bx^2+Cx+D)e^x$ for some A, B, C, D.
- $$\begin{aligned} x^3e^x &= \tfrac{d}{dx}\left((Ax^3+Bx^2+Cx+D)e^x\right) \\ &= (3Ax^2+2Bx+C)e^x + (Ax^3+Bx^2+Cx+D)e^x \\ &= (Ax^3 + (3A+B)x^2 + (2B+C)x + (C+D))e^x. \end{aligned}$$
- $A=1$, $3A+B=0$, $2B+C=0$, $C+D=0$.
- so $A=1$, $B=-3$, $C=6$, $D=-6$
- so $\int x^3e^x\,dx = (x^3-3x^2+6x-6)e^x$.

Integration by parts — I

- Consider $\displaystyle\int xe^{x/a}\,dx$.
- Consider $\displaystyle\int xe^{x/a}\,dx$.
- $u = x$ $\qquad$ $dv/dx = e^{x/a}$
- $du/dx = 1$ $\qquad$ $v = a\,e^{x/a}$
- $\displaystyle\int xe^{x/a}\,dx = uv - \int \frac{du}{dx}v\,dx = axe^{x/a} - \int a\,e^{x/a}\,dx = axe^{x/a} - a^2e^{x/a}$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula: $$\boxed{\int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx}v\,dx}$$
- This is most useful when (a) du/dx is simpler than u (eg u polynomial) and (b) v is no more complicated than dv/dx (eg $dv/dx = \cos(x)$).

Integration by parts — II

- Consider $\displaystyle\int (1-\ln(x))x^{-2}\,dx$.
- Consider $\displaystyle\int (1-\ln(x))x^{-2}\,dx$.
- $u = 1-\ln(x)$ $\qquad$ $dv/dx = x^{-2}$
- $du/dx = -x^{-1}$ $\qquad$ $v = -x^{-1}$
- $\displaystyle\int (1-\ln(x))x^{-2}\,dx = uv - \int \frac{du}{dx}v\,dx = -(1-\ln(x))x^{-1} - \int x^{-2}\,dx$
  $= (\ln(x)-1)x^{-1} + x^{-1} = \ln(x)/x$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula: $$\boxed{\int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx}v\,dx}$$

Integration by parts — III

- Consider $\int x\sin(\omega x)\,dx$.
- Consider $\int x\sin(\omega x)\,dx$.
- $u = x$ $\qquad dv/dx = \sin(\omega x)$
- $du/dx = 1$ $\qquad v = -\omega^{-1}\cos(\omega x)$
- $\int x\sin(\omega x)\,dx = uv - \int \frac{du}{dx} v\,dx = -\omega^{-1}x\cos(\omega x) + \int \omega^{-1}\cos(\omega x)\,dx$

  $= -\omega^{-1}x\cos(\omega x) + \omega^{-2}\sin(\omega x)$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula: $\boxed{\int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx} v\,dx}$

Integration by parts — IV

- Consider $\int \arcsin(x)\,dx$.
- Consider $\int \arcsin(x).1\,dx$.
- $u = \arcsin(x)$ $\qquad dv/dx = 1$
- $du/dx = (1-x^2)^{-1/2}$ $\qquad v = x$
- $\int \arcsin(x).1\,dx = uv - \int \frac{du}{dx} v\,dx = \arcsin(x).x - \int x(1-x^2)^{-1/2}\,dx$

  $= x\arcsin(x) + (1-x^2)^{1/2}$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula: $\boxed{\int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx} v\,dx}$

Integration by substitution — I

- Consider $\int \frac{\sin(x)}{\cos(x)^n}\,dx$.
- Consider $\int \frac{\sin(x)}{\cos(x)^n}\,dx$.
- Put $u = \cos(x)$, so $du/dx = -\sin(x)$, so $dx = -du/\sin(x)$
- $$\int \frac{\sin(x)}{\cos(x)^n}\,dx = \int \frac{\sin(x)}{u^n}\frac{-du}{\sin(x)} = -\int u^{-n}\,du$$
  $$= u^{1-n}/(n-1) = \frac{\cos(x)^{1-n}}{n-1}$$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx, and rearrange to express dx in terms of x and du.
- Rewrite the integral in terms of u and du.
- Evaluate the integral, then rewrite the result in terms of x.

Integration by substitution — II

- Consider $\int xe^{-4x^2}\,dx$.
- Consider $\int xe^{-4x^2}\,dx$.
- Put $u = -4x^2$, so $du/dx = -8x$, so $dx = -du/(8x)$
- $$\int xe^{-4x^2}\,dx = \int -xe^u\frac{du}{8x} = -\frac{1}{8}\int e^u\,du$$
  $$= -e^u/8 = -e^{-4x^2}/8$$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx, and rearrange to express dx in terms of x and du.
- Rewrite the integral in terms of u and du.
- Evaluate the integral, then rewrite the result in terms of x.

Integration by substitution — III

- Consider $\int \frac{dx}{4x^2+4x+2}$.
- Consider $\int \frac{dx}{4x^2+4x+2} = \int \frac{dx}{(2x+1)^2+1}$.
- Put $u = 2x+1$, so $du/dx = 2$, so $dx = du/2$
- $$\begin{aligned}\int \frac{dx}{4x^2+4x+2} &= \int \frac{du/2}{u^2+1} \\ &= \arctan(u)/2 = \arctan(2x+1)/2\end{aligned}$$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx, and rearrange to express dx in terms of x and du.
- Rewrite the integral in terms of u and du.
- Evaluate the integral, then rewrite the result in terms of x.

Integration by substitution — IV

- Consider $\int \frac{dx}{\sqrt{x-x^2}}$.
- Put $x = t^2$, so $dx/dt = 2t$, so $dx = 2t\,dt$

$$\sqrt{x-x^2} = \sqrt{t^2-t^4} = t\sqrt{1-t^2}$$
$$\begin{aligned}\int \frac{dx}{\sqrt{x-x^2}} &= \int \frac{2t\,dt}{t\sqrt{1-t^2}} = 2\int \frac{dt}{\sqrt{1-t^2}} \\ &= 2\arcsin(t) = 2\arcsin(\sqrt{x})\end{aligned}$$

- To find $\int f(x)\,dx$, put x equal to some function of t.
- Find dx/dt, and rearrange to express dx in terms of t and dt.
- Rewrite the integral in terms of t and dt.
- Evaluate the integral, then rewrite the result in terms of x.

Integration by substitution — V

- Consider $\int \log(x)^2\,dx$.
- Put $x = e^t$, so $dx/dt = e^t$, so $dx = e^t\,dt$

$$\begin{aligned}\int \log(x)^2\,dx &= \int \log(e^t)^2 e^t\,dt = \int t^2 e^t\,dt \\ &= (t^2-2t+2)e^t = (\log(x)^2 - 2\log(x)+2)x\end{aligned}$$

- To find $\int f(x)\,dx$, put x equal to some function of t.
- Find dx/dt, and rearrange to express dx in terms of t and dt.
- Rewrite the integral in terms of t and dt.
- Evaluate the integral, then rewrite the result in terms of x.

Examples I

- $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{\cos'(x)}{\cos(x)}\,dx = -\log(\cos(x))$.
- Consider $\int x^2\tan(x^3)\,dx$. Put $u = x^3$, so $du = 3x^2\,dx$, so $dx = du/(3x^2)$.

$$\begin{aligned}\int x^2\tan(x^3)\,dx &= \int x^2\tan(u)\frac{du}{3x^2} = \frac{1}{3}\int \tan(u)\,du = -\log(\cos(u))/3 \\ &= -\log(\cos(x^3))/3\end{aligned}$$

- Consider $\int xe^{\sqrt{x}}\,dx$. Put $t = \sqrt{x}$, so $x = t^2$, so $dx = 2t\,dt$.

$$\begin{aligned}\int xe^{\sqrt{x}}\,dx &= \int t^2e^t.2t\,dt = 2\int t^3e^t\,dt = 2(t^3-3t^2+6t-6)e^t \\ &= (2x^{3/2} - 6x + 12x^{1/2} - 12)e^{\sqrt{x}}\end{aligned}$$

Examples II

- $$\int (2(x^2+1)e^x)^2\,dx = \int (4x^4+8x^2+4)e^{2x}\,dx$$
$$= (Ax^4+Bx^3+Cx^2+Dx+E)e^{2x}$$
$$(4x^4+8x^2+4)e^{2x} = \frac{d}{dx}((Ax^4+Bx^3+Cx^2+Dx+E)e^{2x})$$
$$= (4Ax^3+3Bx^2+2Cx+D)e^{2x} +$$
$$(Ax^4+Bx^3+Cx^2+Dx+E).2e^{2x}$$
$$= e^{2x}(2Ax^4+(4A+2B)x^3+(3B+2C)x^2+$$
$$(2C+2D)x+(D+2E))$$

So $4=2A$, $0=4A+2B$, $8=3B+2C$, $0=2C+2D$, $4=D+2E$
So $A=2,\ B=-4,\ C=10,\ D=-10,\ E=7$

$$\int (2(x^2+1)e^x)^2\,dx = (2x^4-4x^3+10x^2-10x+7)e^{2x}.$$

Examples III

- $$\int 1+\cosh(x)+\cosh(x)^2\,dx = \int 1+\frac{e^x+e^{-x}}{2}+\left(\frac{e^x+e^{-x}}{2}\right)^2\,dx$$
$$= \frac{1}{4}\int 4+2e^x+2e^{-x}+e^{2x}+2+e^{-2x}\,dx$$
$$= \frac{1}{4}\left(6x+2e^x-2e^{-x}+\tfrac{1}{2}e^{2x}-\tfrac{1}{2}e^{-2x}\right)$$
$$= \frac{3}{2}x+\frac{e^x-e^{-x}}{2}+\frac{1}{4}\frac{e^{2x}-e^{-2x}}{2}$$
$$= \frac{3}{2}x+\sinh(x)+\frac{1}{4}\sinh(2x).$$

Examples IV

- To show that $\displaystyle\int \frac{dx}{\cos(x)} = \log\left(\frac{1+\sin(x)}{\cos(x)}\right)$:

$$\frac{d}{dx}\left(\frac{1+\sin(x)}{\cos(x)}\right) = \frac{\cos(x).\cos(x)-(1+\sin(x))(-\sin(x))}{\cos(x)^2}$$
$$= \frac{\cos(x)^2+\sin(x)^2+\sin(x)}{\cos(x)^2} = \frac{1+\sin(x)}{\cos(x)^2}$$
$$\frac{d}{dx}\log\left(\frac{1+\sin(x)}{\cos(x)}\right) = \left(\frac{1+\sin(x)}{\cos(x)}\right)^{-1}\frac{d}{dx}\left(\frac{1+\sin(x)}{\cos(x)}\right)$$
$$= \frac{\cos(x)}{1+\sin(x)}\frac{1+\sin(x)}{\cos(x)^2} = \frac{1}{\cos(x)}$$

Examples V

- $$\int 8x\sin(x)\cos(x)\,dx = \int 4x\sin(2x)\,dx$$
$$= -2x\cos(2x)+\int 2\cos(2x)\,dx$$
$$= -2x\cos(2x)+\sin(2x).$$

- Consider $\displaystyle\int 10e^{-x}\sin(x)^2\,dx = \int 5e^{-x}\,dx+\int -5e^{-x}\cos(2x)\,dx.$

$$\int -5e^{-x}\cos(2x)\,dx = e^{-x}(A\cos(2x)+B\sin(2x))$$
$$-5e^{-x}\cos(2x) = e^{-x}((2B-A)\cos(2x)-(2A+B)\sin(2x))$$
$$A=1,\qquad B=-2$$
$$\int 10e^{-x}\sin(x)^2\,dx = -5e^{-x}+e^{-x}\cos(2x)-2e^{-x}\sin(2x).$$

Taylor series

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{k=0}^{\infty} kx^k \qquad (\text{ for } |x| < 1)$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

- For any reasonable function $f(x)$, there are coefficients a_k such that

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

(when x is sufficiently small). This is the *Taylor series* for $f(x)$.

Exceptions

Not every function has a Taylor series.

- $f_0(x) = 1/x$ does not, because $f_0(0)$ is undefined.
- $f_1(x) = |x|$ and $f_2(x) = x^{1/3}$ do not, because the slopes $f_1'(0)$ and $f_2'(0)$ are not defined.
- $f_3(x) = \ln(x)$ does not, because $f_3'(x)$ is undefined for $x < 0$.
- $f_4(x) = e^{-1/x^2}$ does not, for a more subtle reason.

For a full explanation, see Level 3 complex analysis.

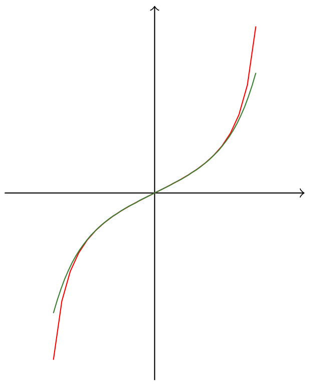
Truncated series

Often we only calculate with finitely many terms of the Taylor series.

$$\tan(x) = x + x^3/3 + 2x^5/15 + O(x^7)$$

The notation $O(x^7)$ means that there are extra terms involving powers x^k with $k \geq 7$. The above is the *7th order Taylor series* for $\tan(x)$. It is a good approximation to $\tan(x)$ if x is sufficiently small.

$\tan(x) = x + x^3/3 + 2x^5/15 + 17x^7/315 + O(x^9)$



Finding coefficients

$$\boxed{y = \sum_{k=0}^{\infty} a_k x^k, \quad \text{where} \quad a_k = \frac{1}{k!} \left.\frac{d^k y}{dx^k}\right|_{x=0}}$$

$$\boxed{f(x) = \sum_{k=0}^{\infty} a_k x^k, \quad \text{where} \quad a_k = f^{(k)}(0)/k!}$$

Example:

$$\exp^{(k)}(x) = \cdots = \exp'''(x) = \exp''(x) = \exp'(x) = \exp(x) = e^x$$

$$\exp^{(k)}(0) = \cdots = \exp'''(0) = \exp''(0) = \exp'(0) = \exp(0) = 1$$

Thus $a_k = 1/k!$, and $\exp(x) = \sum_k x^k/k!$.

Another example

Take $f(x) = \sin(x)$.

$$
\begin{array}{llll}
f(x) = \sin(x) & f'(x) = \cos(x) & f''(x) = -\sin(x) & f'''(x) = -\cos(x) \\
f^{(4)}(x) = \sin(x) & f^{(5)}(x) = \cos(x) & f^{(6)}(x) = -\sin(x) & f^{(7)}(x) = -\cos(x) \\
f^{(8)}(x) = \sin(x) & f^{(9)}(x) = \cos(x) & f^{(10)}(x) = -\sin(x) & f^{(11)}(x) = -\cos(x)
\end{array}
$$

$$
\begin{array}{llll}
f(0) = 0 & f'(0) = 1 & f''(0) = 0 & f'''(0) = -1 \\
f^{(4)}(0) = 0 & f^{(5)}(0) = 1 & f^{(6)}(0) = 0 & f^{(7)}(0) = -1 \\
f^{(8)}(0) = 0 & f^{(9)}(0) = 1 & f^{(10)}(0) = 0 & f^{(11)}(0) = -1
\end{array}
$$

$$
\begin{array}{llll}
a_0 = 0 & a_1 = 1 & a_2 = 0 & a_3 = -1/3! \\
a_4 = 0 & a_5 = 1/5! & a_6 = 0 & a_7 = -1/7! \\
a_8 = 0 & a_9 = 1/9! & a_{10} = 0 & a_{11} = -1/11!
\end{array}
$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$
\begin{aligned}
e^{-x^2} &= \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!} \\
\cosh(x) &= (e^x + e^{-x})/2 = \sum_k \frac{x^k + (-x)^k}{2\ (k!)} = \sum_{k \text{even}} \frac{x^k}{k!} = \sum_j \frac{x^{2j}}{(2j)!} \\
\sinh(x)/x &= (e^x - e^{-x})/(2x) = \sum_k \frac{x^k - (-x)^k}{2x\ (k!)} = \sum_{k \text{odd}} \frac{x^{k-1}}{k!} = \sum_j \frac{x^{2j}}{(2j+1)!} \\
1/(1-x) &= 1 + x + x^2 + x^3 + \ldots = \sum_k x^k \\
x \frac{d}{dx}\left(\frac{1}{1-x}\right) &= x \frac{d}{dx} \sum_k x^k = x \sum_k k\, x^{k-1} = \sum_k k\, x^k \\
x/(1-x)^2 &= \sum_k k\, x^k.
\end{aligned}
$$

Odd and even functions

Recall that $f(x)$ is *even* if $f(-x) = f(x)$, and *odd* if $f(-x) = -f(x)$. For example, $\cos(x)$ is even and $\sin(x)$ is odd. If

$$f(x) = \sum_k a_k x^k = \sum_{k \text{ even}} a_k x^k + \sum_{k \text{ odd}} a_k x^k$$

then

$$f(-x) = \sum_k a_k (-x)^k = \sum_{k \text{ even}} a_k x^k - \sum_{k \text{ odd}} a_k x^k.$$

Thus $f(x)$ is even iff the Taylor series involves only even powers of x, and $f(x)$ is odd iff the Taylor series involves only odd powers of x.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

Algebra of series

$$\tan(x) = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + O(x^7)$$

$$
\begin{aligned}
\tan(x)^2 &= (x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5)^2 + O(x^7) \\
&= x^2 + \tfrac{1}{3}x^4 + \tfrac{2}{15}x^6 + \\
&\quad \tfrac{1}{3}x^4 + \tfrac{1}{9}x^6 + \tfrac{2}{45}x^8 \\
&\quad \tfrac{2}{15}x^6 + \tfrac{2}{45}x^8 + \tfrac{4}{225}x^{10} + O(x^7) \\
&= x^2 + \tfrac{2}{3}x^4 + \tfrac{17}{45}x^6 + O(x^7).
\end{aligned}
$$

Expansion about other points

We can also expand $f(x)$ in terms of powers $(x-\alpha)^k$, for any α. More precisely,

$$\boxed{f(x) = \sum_{k=0}^{\infty} b_k(x-\alpha)^k, \quad \text{where} \quad b_k = f^{(k)}(\alpha)/k!}$$

	$\ln'(x) = x^{-1}$	$\ln''(x) = -x^{-2}$	$\ln'''(x) = 2x^{-3}$	$\ln^{(4)}(x) = -6x^{-4}$
$\ln(1) = 0$	$\ln'(1) = 1$	$\ln''(1) = -1$	$\ln'''(1) = 2$	$\ln^{(4)}(1) = -6$
$b_0 = 0$	$b_1 = 1$	$b_2 = -1/2$	$b_3 = 2/3! = 1/3$	$b_4 = -6/4! = -1/4$

$$\ln(x) = (x-1) - (x-1)^2/2 + (x-1)^3/3 - (x-1)^4/4 + O((x-1)^5).$$

More examples

We will find the series for $\tan(x)$ near $x = \frac{\pi}{4}$.

$$f(x) = \tan(x)$$

$$f'(x) = \frac{1}{\cos(x)^2}$$

$$f''(x) = -2\cos(x)^{-3}. - \sin(x) = \frac{2\sin(x)}{\cos(x)^3}$$

$$f(\tfrac{\pi}{4}) = 1 \qquad f'(\tfrac{\pi}{4}) = \frac{1}{(2^{-1/2})^2} = 2 \qquad f''(\tfrac{\pi}{4}) = \frac{2.2^{-1/2}}{(2^{-1/2})^3} = 4$$

$$a_0 = 1/0! = 1 \qquad a_1 = 2/1! = 2 \qquad a_2 = 4/2! = 2$$

$$\tan(x) = 1 + 2(x - \tfrac{\pi}{4}) + 2(x - \tfrac{\pi}{4})^2 + O((x - \tfrac{\pi}{4})^3).$$

More examples

Consider $y = x/(e^x - 1)$.

$$e^x = 1 + x + x^2/2 + x^3/6 + O(x^4)$$

$$e^x - 1 = x + x^2/2 + x^3/6 + O(x^4)$$

$$\frac{1}{y} = \frac{e^x - 1}{x} = 1 + x/2 + x^2/6 + O(x^3) = 1 + u + O(x^3) \qquad u = x/2 + x^2/6$$

$$y = \frac{1}{1+u} = 1 - u + u^2 + O(u^3) = 1 - u + u^2 + O(x^3)$$

$$u^2 = x^2/4 + x^3/6 + x^4/36 = x^2/4 + O(x^3)$$

$$\frac{x}{e^x - 1} = 1 - u + u^2 + O(x^3)$$

$$= 1 - x/2 - x^2/6 + x^2/4 + O(x^3) = 1 - x/2 + x^2/12 + O(x^3)$$