Mathematics with Maple
(MAS100)

Introduction

The lecturer is Professor Neil Strickland.

N.P.Strickland@sheffield.ac.uk

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- We will learn how to use Maple, a powerful software package for solving mathematical problems.
- In the process, we will review and extend many parts of A-level mathematics, from a new perspective.

Algebraic manipulation

Skills to learn or practice:

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Maple commands:

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- `expand`, `factor` and `combine`
- `simplify`; the `symbolic` option
- `collect` and `coeff`

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$$\begin{aligned}(w+x+y+z)^2 - (x+y+z)^2 &= a^2 - b^2 = (a+b)(a-b) \\ &= (w+2x+2y+2z)w \\ &= w^2 + 2xw + 2yw + 2zw.\end{aligned}$$

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- Maple's `factor` command will handle more complicated cases.

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$$((-3)^4)^{\frac{1}{4}} = (81)^{\frac{1}{4}} = +3 \qquad \text{but} \qquad (-3)^{4 \times \frac{1}{4}} = (-3)^1 = -3.$$

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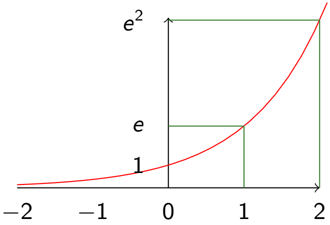

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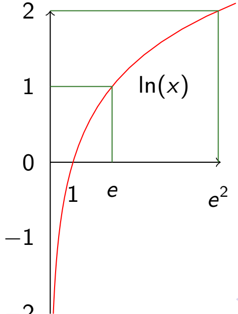
- 

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- $$\begin{aligned} \log_{10}(1000) &= \log_{10}(10^3) = 3 \\ \log_2(1024) &= \log_2(2^{10}) = 10 \\ \log_{1024}(2) &= \log_{1024}(1024^{1/10}) = 1/10 \\ \log_3(1/9) &= \log_3(3^{-2}) = -2 \end{aligned}$$

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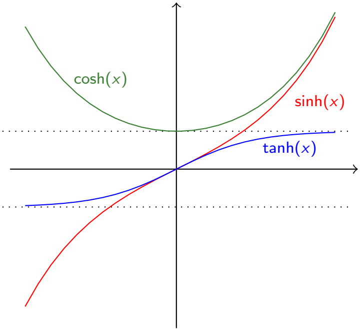
- 

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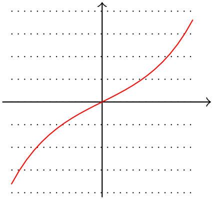

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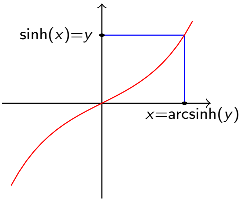

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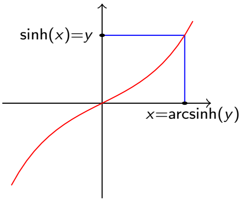

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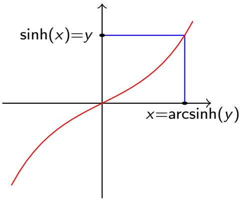

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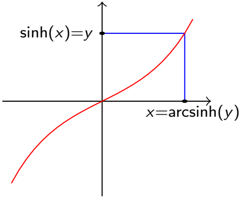

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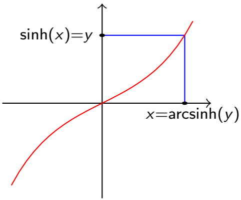

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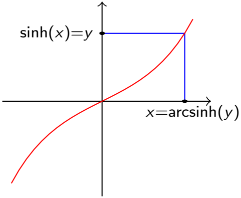

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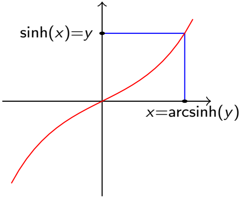

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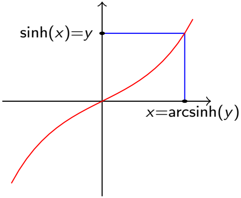

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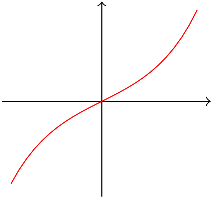
Graphs



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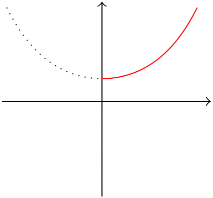

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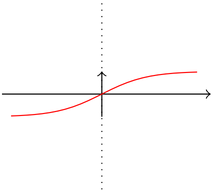

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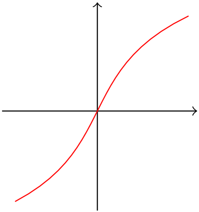

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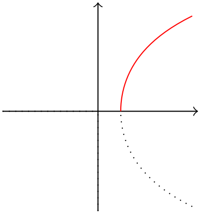

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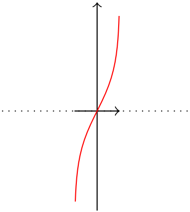

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Trigonometric functions

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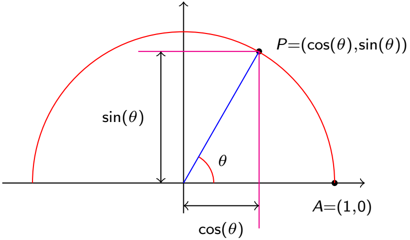

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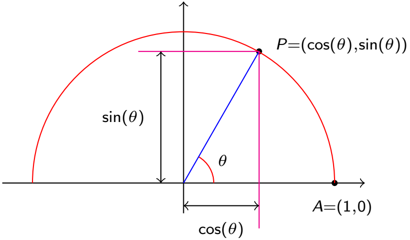

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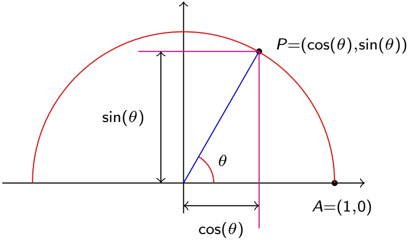

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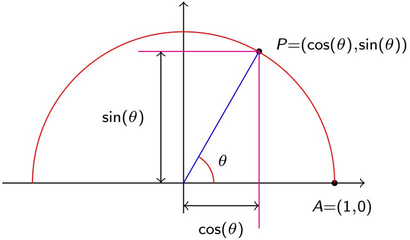

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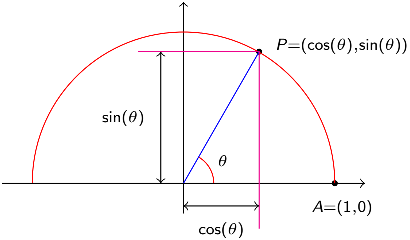

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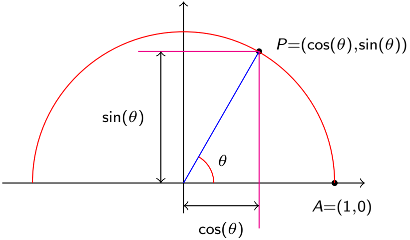

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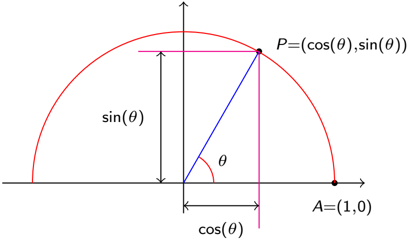

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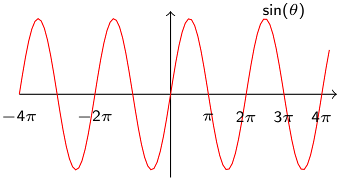
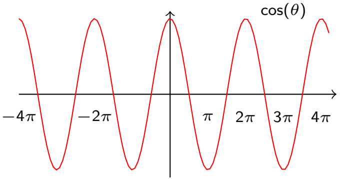


sin(θ)
−4π
−2π
π
2π
3π
4π



cos(θ)
−4π
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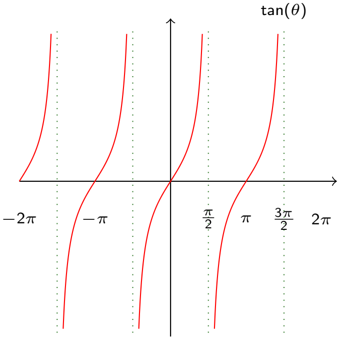
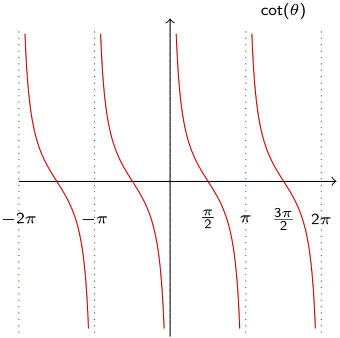
2π
3π
4π



tan(θ)
−2π
−π
π/2
π
3π/2
2π



cot(θ)
−2π
−π
π/2
π
3π/2
2π

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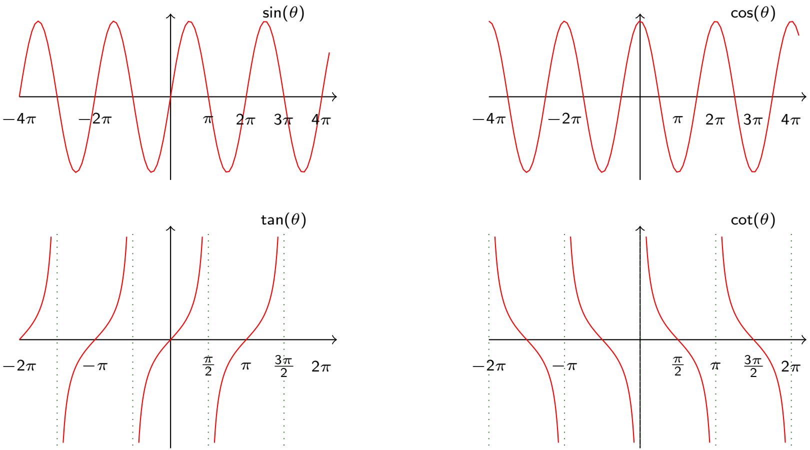

$$
\begin{aligned}
\sin(\pi/2 + x) &= \cos(x) & \cos(\pi/2 + x) &= -\sin(x)\\
\sin(\pi + x) &= -\sin(x) & \cos(\pi + x) &= -\cos(x)\\
\sin(2\pi + x) &= \sin(x) & \cos(2\pi + x) &= \cos(x)\\
\sin(-x) &= -\sin(x) & \cos(-x) &= \cos(x).
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$$\exp(ix) = 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{6} + \frac{(ix)^4}{24} + \frac{(ix)^5}{120} + \cdots$$

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$$\begin{aligned} \exp(ix) =& 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{6} + \frac{(ix)^4}{24} + \frac{(ix)^5}{120} + \cdots \\ =& 1 + ix - \frac{x^2}{2} - i\frac{x^3}{6} + \frac{x^4}{24} + i\frac{x^5}{120} + \cdots \end{aligned}$$

Preview of complex numbers

- Complex numbers are expressions like $z = 3 + 4i$, where i satisfies $i^2 = -1$.
- You can add and subtract complex numbers in an obvious way, for example $(3 + 4i) + (7 - 3i) = 10 + i$.
- To multiply: expand out and use $i^2 = -1$. For example: $(1 + 2i)(3 + 4i) = 3 + 4i + 6i + 8i^2 = 3 + 4i + 6i - 8 = -5 + 10i$.
- Note that the powers of i repeat with period 4:

$$i^0 = 1 \quad i^1 = i \quad i^2 = -1 \quad i^3 = -i \quad i^4 = 1 \quad i^5 = i \quad i^6 = -1 \quad i^7 = -i \quad i^8 = 1.$$

- By expanding and using this we find powers of any complex number.

$$(1 + i)^2 = 1 + 2i + i^2 = 1 + 2i + (-1) = 2i$$
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&= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right) + \left(x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots\right) i
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&= \cos(x) + \sin(x) i.
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De Moivre's theorem

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$$e^{i\theta} = \exp(i\theta) = \cos(\theta) + \sin(\theta)i$$

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\cos(2a) &= 2\cos(a)^2 - 1 = 1 - 2\sin(a)^2.
\end{aligned}$$

Examples

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$\cos(a)^2 + \sin(a)^2$

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$$\cos(a)^2 + \sin(a)^2 = \left(\frac{e^{ia} + e^{-ia}}{2}\right)^2 + \left(\frac{e^{ia} - e^{-ia}}{2i}\right)^2$$

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$2\sin(a)\cos(a)$

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The addition formula

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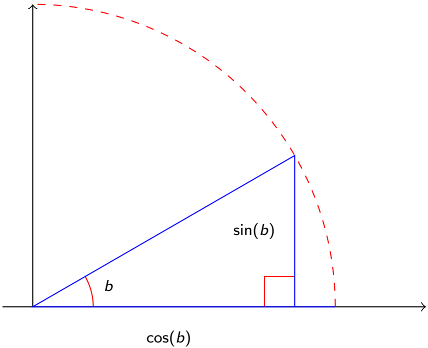

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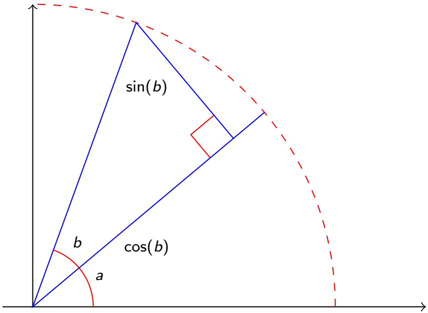

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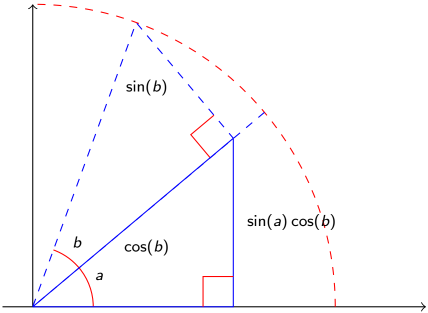

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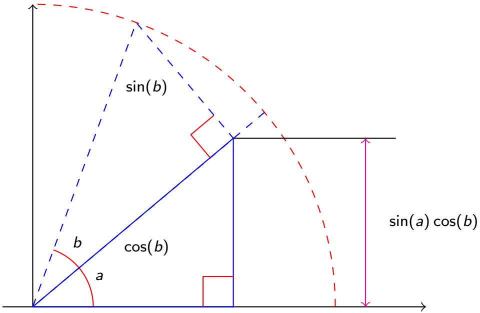

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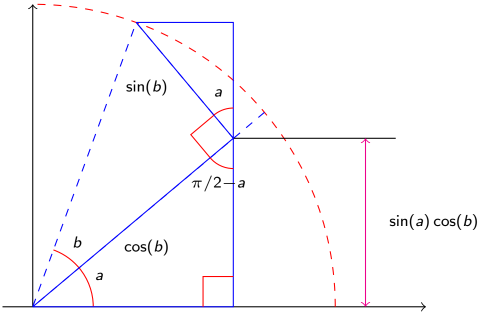

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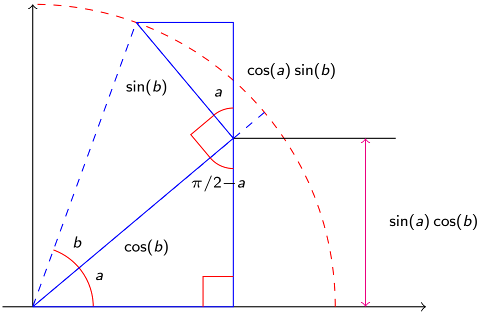

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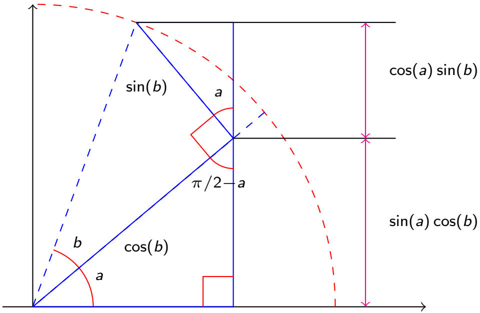

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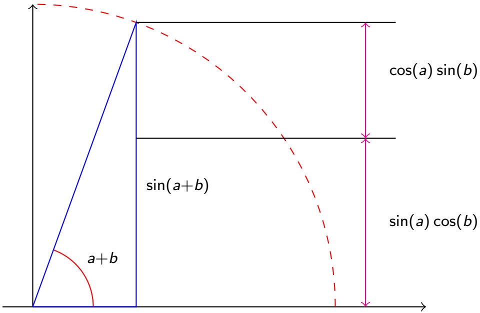

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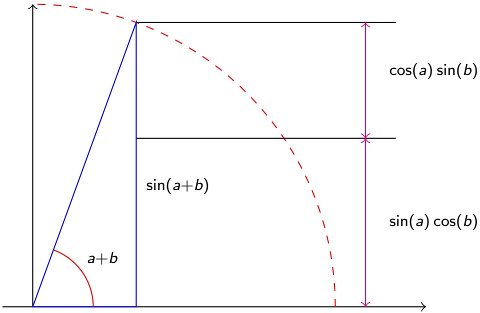

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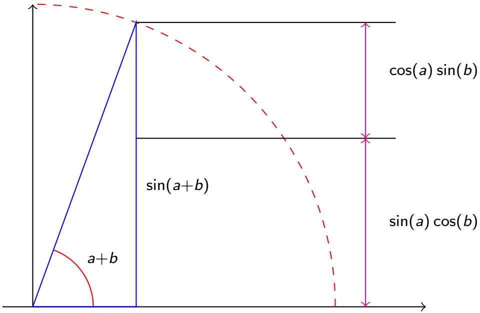

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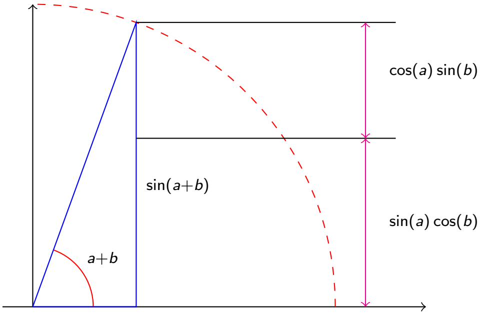

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- Once a function has been rewritten in this form, it is very easy to differentiate it or integrate it.

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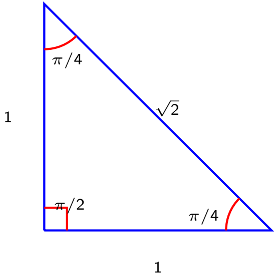
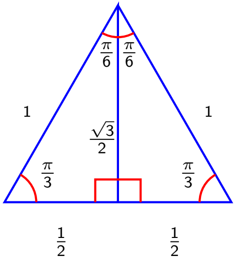
Proved by considering these triangles:





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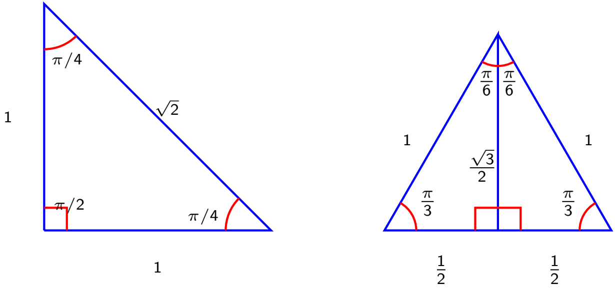
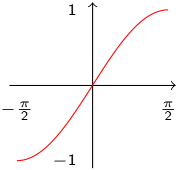
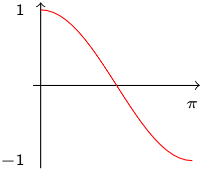


You should also be able to deduce things like $\cos(5\pi/6) = -\sqrt{3}/2$.

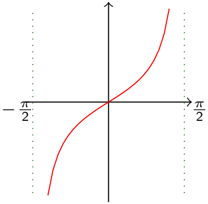
Inverse trigonometric functions



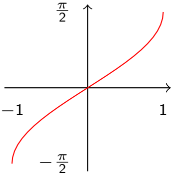
$\sin\colon [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1,1]$



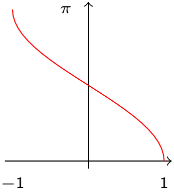
$\cos\colon [0,\pi] \longrightarrow [-1,1]$



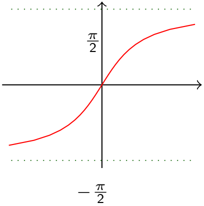
$\tan\colon [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow \mathbb{R}$



$\arcsin\colon [-1,1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$



$\arccos\colon [-1,1] \longrightarrow [0,\pi]$



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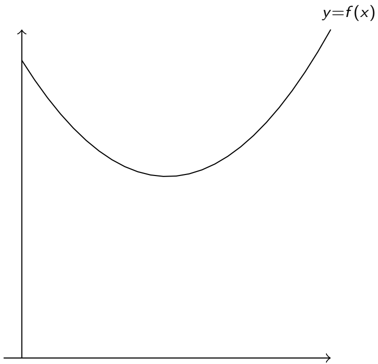
- We sometimes write y' for dy/dx (care needed).

Slopes



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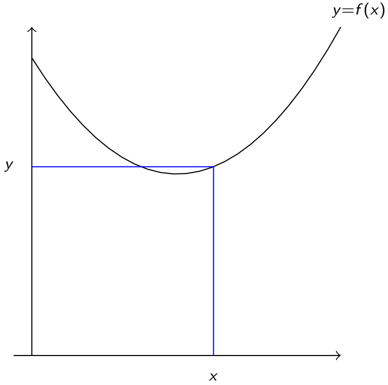
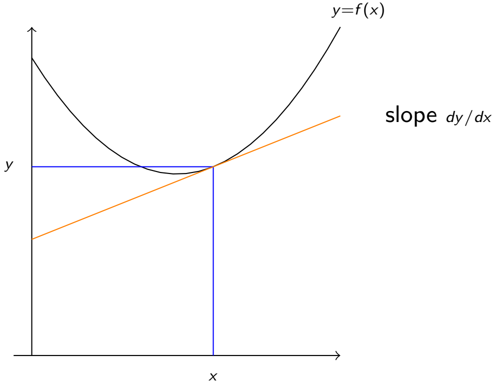


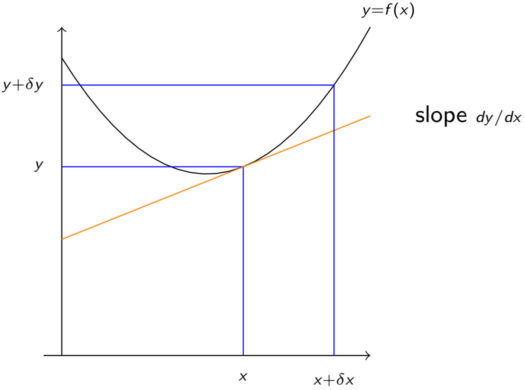
Consider variables x and y related by $y = f(x)$.

Slopes



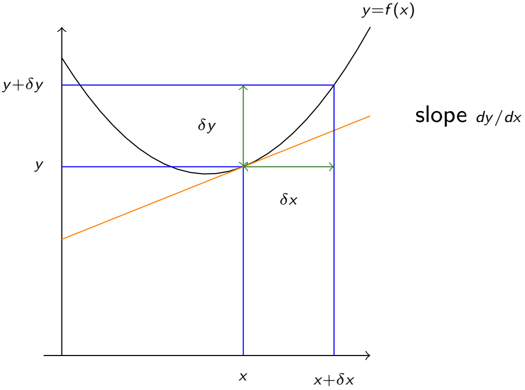
dy/dx is the slope of the tangent line to the graph.

Slopes



If x changes by a small amount δx, then y will change by a small amount δy.

Slopes



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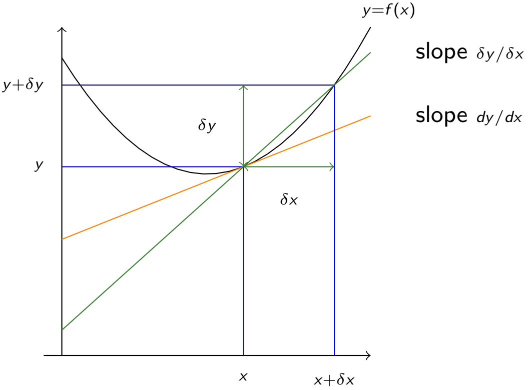
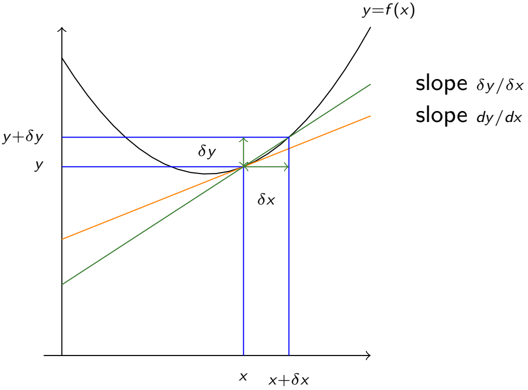


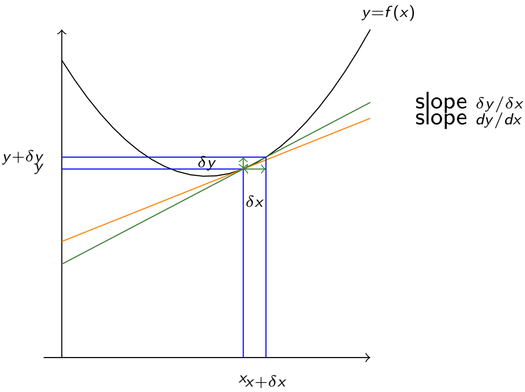
The ratio $\delta y/\delta x$ is the slope of a chord cutting across the graph.

Slopes



The slope of the chord changes slightly as δx decreases.

Slopes



As δx approaches zero, the chord approaches the tangent, and $\delta y/\delta x$ approaches dy/dx.

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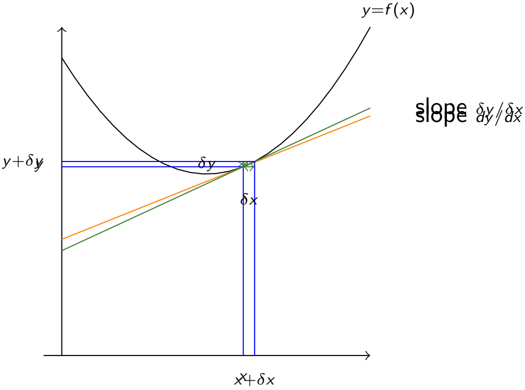

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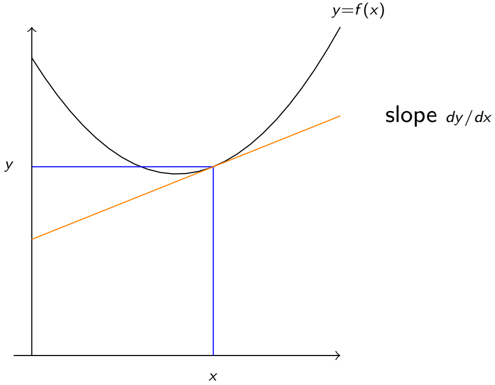

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- Similarly:

$$\boxed{\frac{d}{dx}(x^n) = nx^{n-1} \text{ for all } n.}$$

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- Conclusion: $\exp'(x) = \exp(x)$.

Special functions

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- We showed earlier that $\exp'(x) = \exp(x)$

Special functions

$$\begin{aligned}
\exp'(x) &= \exp(x) \\
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- We deduce $\sinh'(x)$ using the identity $\sinh(x) = (e^x - e^{-x})/2$. Similarly for cosh and tanh.

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$$\begin{aligned}
\log'(x) &= 1/x \\
\operatorname{arcsinh}'(x) &= (1 + x^2)^{-1/2} \\
\operatorname{arccosh}'(x) &= (x^2 - 1)^{-1/2} \\
\operatorname{arctanh}'(x) &= (1 - x^2)^{-1} \\
\arcsin'(x) &= (1 - x^2)^{-1/2} \\
\arccos'(x) &= -(1 - x^2)^{-1/2} \\
\arctan'(x) &= (1 + x^2)^{-1}
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- Using $\exp'(x) = \exp(x)$ and the inverse function rule, we find that $\log'(x) = 1/x$
- The inverse function rule also gives the remaining derivatives.

The product rule

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- Consider variables u and v depending on x, and put $w = uv$. Then

$$\boxed{w' = (uv)' = u'v + uv'}$$

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(The approximations become exact in the limit as $\delta x \to 0$.)

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The chain rule

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- Suppose that y depends on u, and u depends on x. Then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

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- Alternative notation: suppose that $f(x) = g(h(x))$. Then

$$\boxed{f'(x) = g'(h(x))h'(x)}$$

Examples of the chain rule

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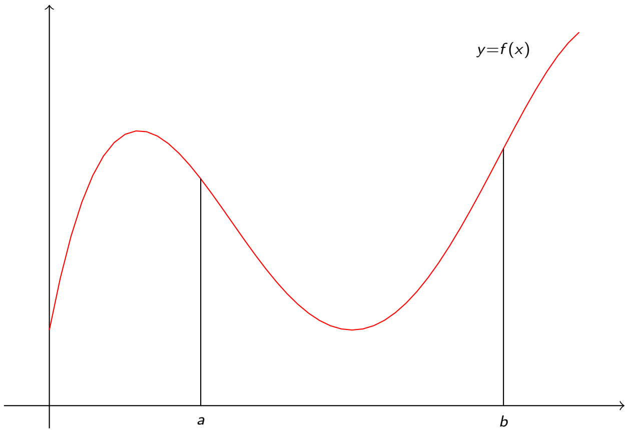
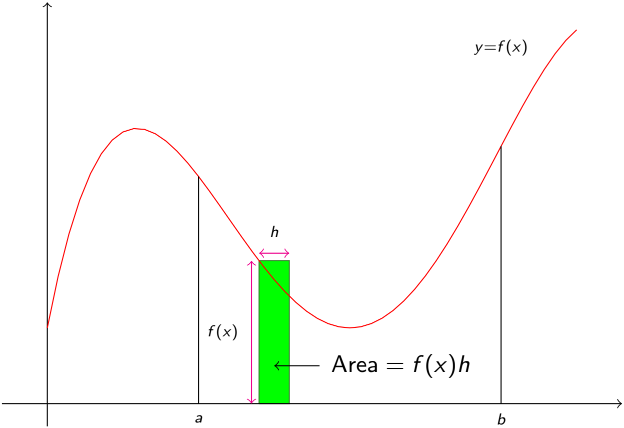
- A current flowing in a wire exerts a magnetic force on a moving electron. There is a formula for the force contributed by a short section of wire; to get the total force, we integrate.

Areas



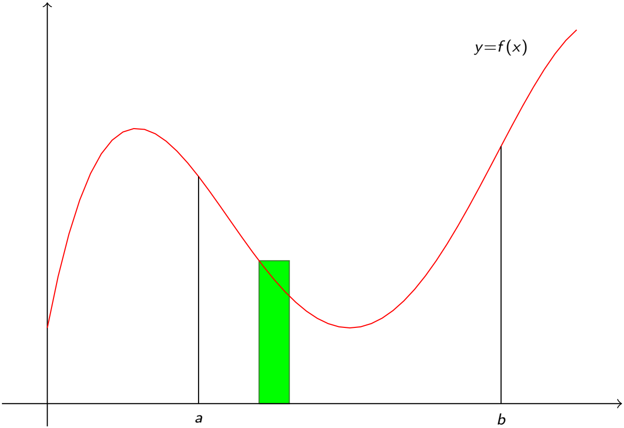
Consider the integral $\int_a^b f(x)\,dx$.

Areas



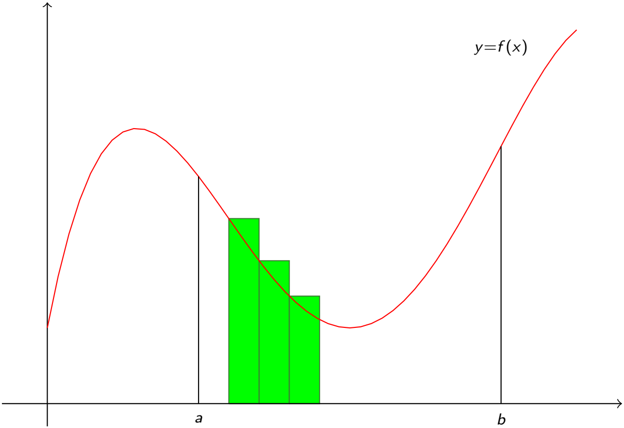
For each short interval $[x, x+h] \subset [a, b]$, we have a contribution $f(x)h$. This is the area of the green rectangle.

Areas



This is the contribution from one short interval, but we need to add together the contributions from many short intervals.

Areas



Here we have added in two more intervals

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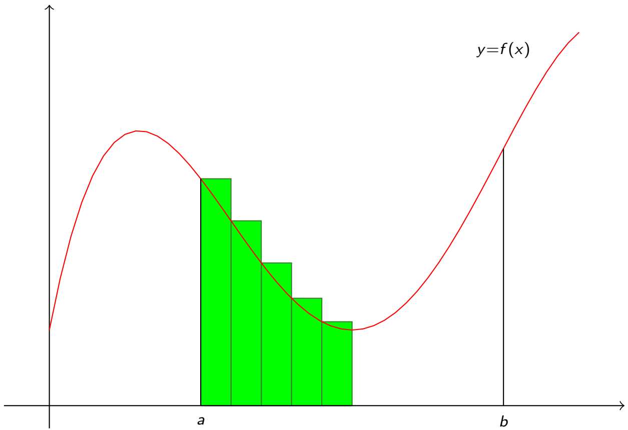

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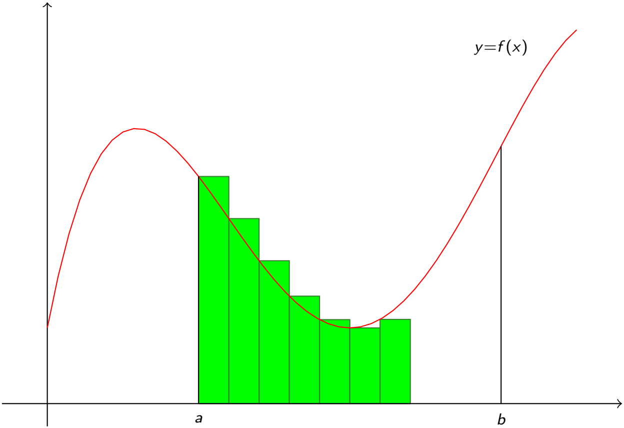

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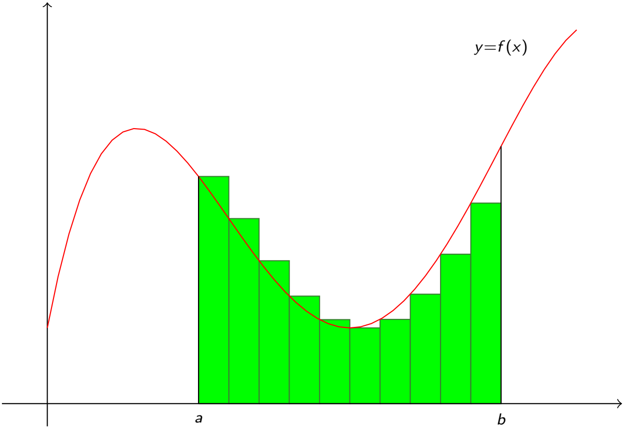
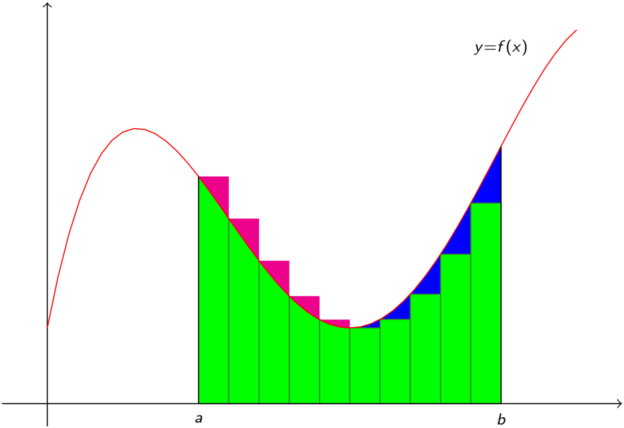


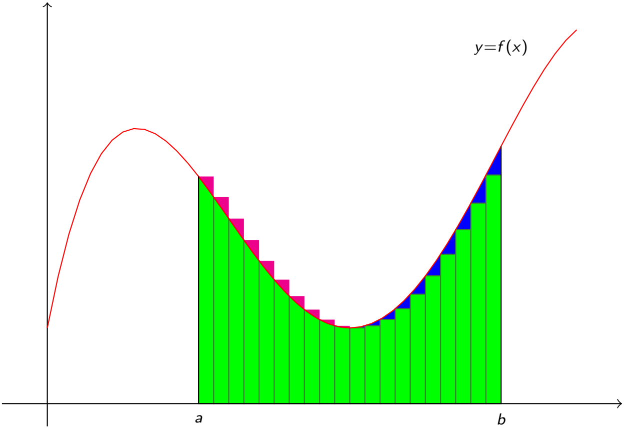
Now we have divided the whole interval $[a, b]$ into subintervals of length h. The sum of the terms $f(x)h$ is the area of the green region.

Areas



This is not exactly the same as the area under the curve, because of the regions marked in blue and pink.

Areas



However, the error decreases if we make h smaller.

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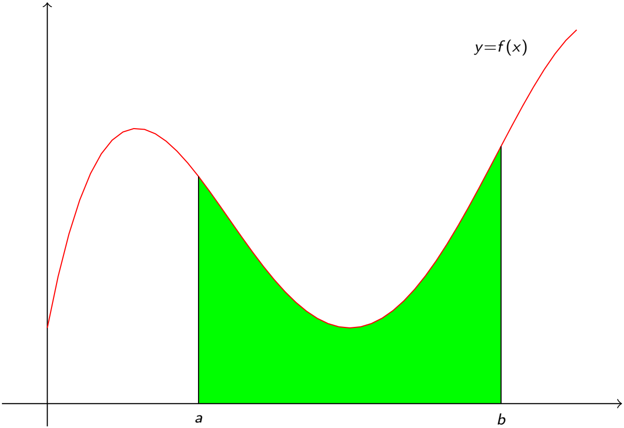

However, the error decreases if we make h smaller, and tends to zero in the limit.

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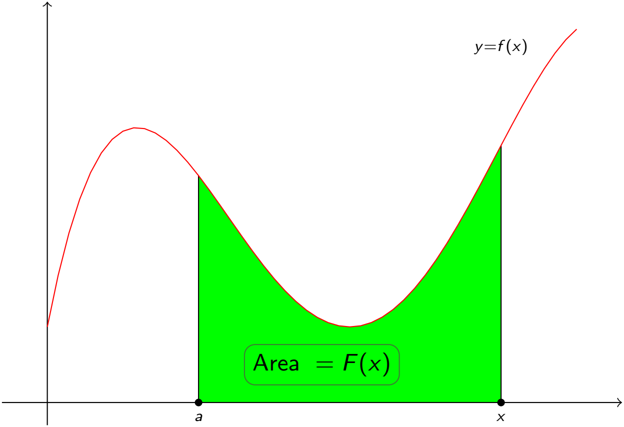
- $\int_a^b \frac{1}{x} = \left[\log(x)\right]_a^b = \log(b) - \log(a)$

Proof of the Fundamental Theorem



Choose a number a, and define $F(x) = \int_a^x f(t)\,dt$. We must show that $F'(x) = f(x)$.

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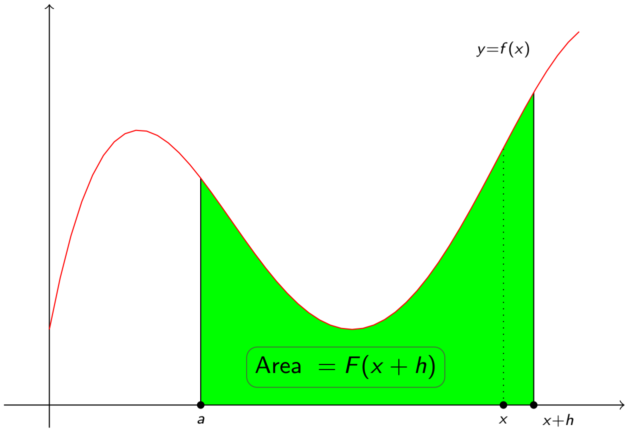

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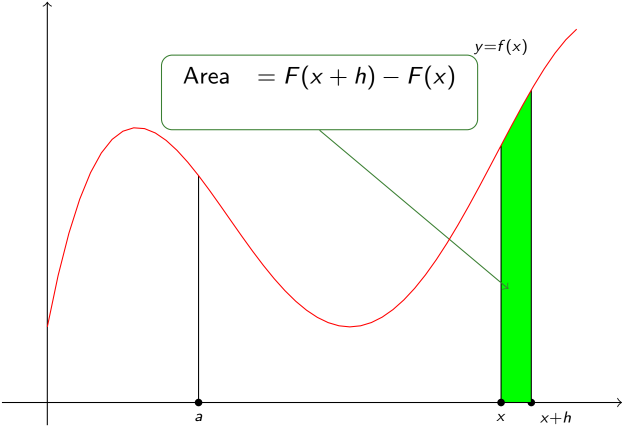

We now change x to $x + h$. The increase in $F(x)$ is $F(x + h) - F(x)$, which is the area of the thin strip as shown.

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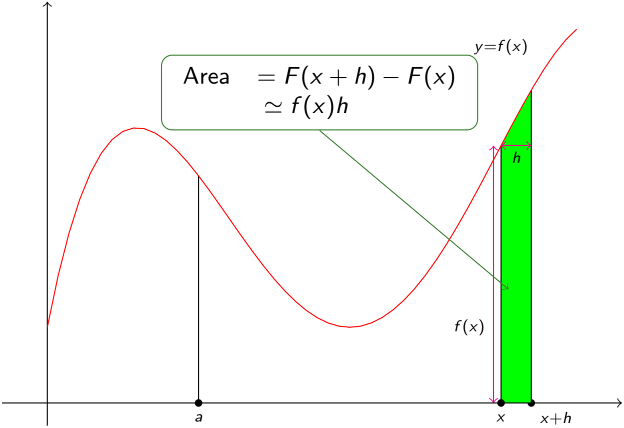

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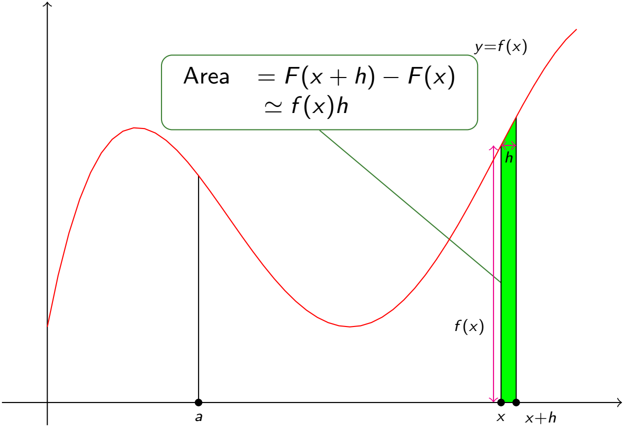

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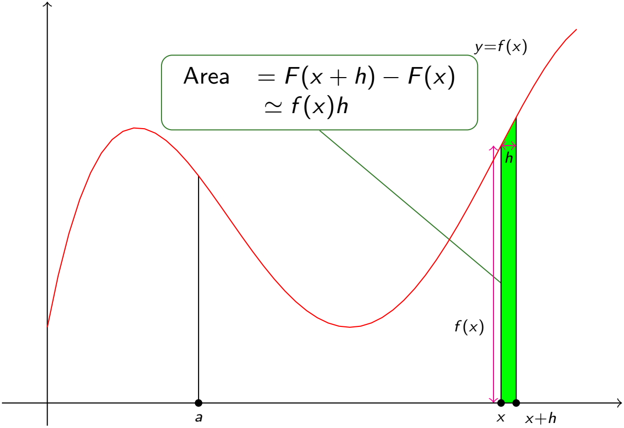

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$$F'(x) \simeq (F(x+h) - F(x))/h \simeq f(x).$$

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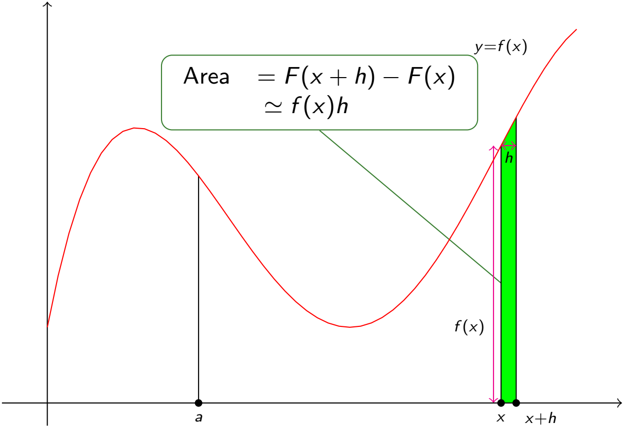

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- In solving differential equations, it often does matter which indefinite integral you use. You must therefore include a '+c' term, and do some extra work to see what c should be.
- Maple's `int()` command will never give you a '+c' term.
  If you need one, you must insert it yourself.

Checking and Guessing

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$$\frac{d}{dx}\log(x^3 + x^2 + x + 1) = \frac{\frac{d}{dx}(x^3 + x^2 + x + 1)}{x^3 + x^2 + x + 1} = \frac{3x^2 + 2x + 1}{x^3 + x^2 + x + 1}.$$

Undetermined coefficients

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- $$\begin{aligned} \log(x)^3 &= \frac{d}{dx}\Big((a \log(x)^3 + b \log(x)^2 + c \log(x) + d)x\Big) \\ &= (3a \log(x)^2 x^{-1} + 2b \log(x) x^{-1} + cx^{-1})x + \\ &\quad (a \log(x)^3 + b \log(x)^2 + c \log(x) + d).1 \end{aligned}$$

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- So $a = 1$, $b + 3a = 0$, $c + 2b = 0$ and $d + c = 0$ (compare coefficients)

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- So $a = 1$, $b + 3a = 0$, $c + 2b = 0$ and $d + c = 0$ (compare coefficients)
- So $a = 1$, $b = -3$, $c = 6$ and $d = -6$

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$$\int \log(x)^3 \, dx = (\log(x)^3 - 3\log(x)^2 + 6\log(x) - 6)x.$$

Standard integrals

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$$\begin{aligned}
\exp'(x) &= \exp(x) \\
\sinh'(x) &= \cosh(x) \\
\cosh'(x) &= \sinh(x) \\
\tanh'(x) &= \operatorname{sech}(x)^2 \\
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\cos'(x) &= -\sin(x) \\
\tan'(x) &= \sec(x)^2
\end{aligned}$$

$$\begin{aligned}
\log'(x) &= 1/x \\
\operatorname{arcsinh}'(x) &= (1+x^2)^{-1/2} \\
\operatorname{arccosh}'(x) &= (x^2-1)^{-1/2} \\
\operatorname{arctanh}'(x) &= (1-x^2)^{-1} \\
\arcsin'(x) &= (1-x^2)^{-1/2} \\
\arccos'(x) &= -(1-x^2)^{-1/2} \\
\arctan'(x) &= (1+x^2)^{-1}
\end{aligned}$$

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$$\int x^n\,dx = x^{n+1}/(n+1) \qquad (n \neq -1)$$

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\end{aligned}$$

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\int \cos(x)^2\,dx &= (2x + \sin(2x))/4
\end{aligned}
$$

Rational functions

Rational functions

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Rational function examples

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- $\displaystyle\int \frac{x^2+1}{x^2-1}\,dx = x + \ln(|x-1|) + \ln(|x+1|)$
- $\displaystyle\int \left(\frac{x+1}{x-1}\right)^3 dx = 1 + \frac{6}{x-1} + \frac{12}{(x-1)^2} + \frac{8}{(x-1)^3}$
- $\displaystyle\int \frac{2x+2}{x^2+1}\,dx = \ln(x^2+1) + 2\arctan(x)$
- $\displaystyle\int \frac{1}{x^{-1}+1+x}\,dx = \frac{1}{2}\ln(1+x+x^2) - \frac{1}{\sqrt{3}}\arctan\left(\frac{1+2x}{\sqrt{3}}\right)$
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$$\frac{d}{dx}\arctan(ux+v) = \frac{u}{1+(ux+v)^2} = \frac{u}{u^2x^2+2uvx+(v^2+1)}$$

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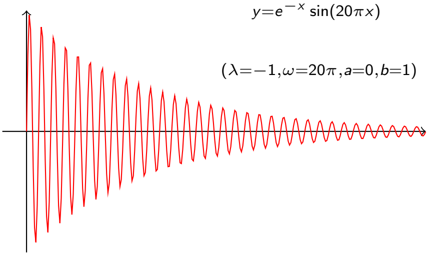

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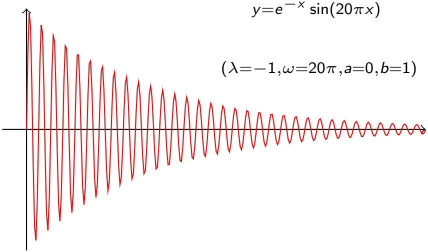

- Special cases:

$$\begin{aligned} f(x) &= e^{\lambda x}\sin(\omega x) && (a=0, b=1) \\ f(x) &= a\cos(\omega x) + b\sin(\omega x) && (\lambda = 0) \\ f(x) &= ae^{\lambda x} && (\omega = 0). \end{aligned}$$

Integrating exponential oscillations

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$$\int e^{-2x}(5\cos(4x) - 3\sin(4x))\,dx = e^{-2x}(\cos(4x) + 13\sin(4x))/10$$

Integrating exponential oscillations

Alternatively:

$$\int e^{-2x}(5\cos(4x) - 3\sin(4x))\,dx = e^{-2x}(A\cos(4x) + B\sin(4x)) \text{ for some } A, B$$

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By comparing coefficients, we must have $4B - 2A = 5$ and $2B + 4A = 3$.

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By comparing coefficients, we must have $4B - 2A = 5$ and $2B + 4A = 3$. These equations can be solved to give $A = 1/10$ and $B = 13/10$. Thus

$$\int e^{-2x}(5\cos(4x) - 3\sin(4x))\,dx = e^{-2x}(\cos(4x) + 13\sin(4x))/10.$$

Polynomial exponential oscillations

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- A *polynomial exponential oscillation* is a function of the form

$$f(x) = e^{\lambda x}(a(x)\cos(\omega x) + b(x)\sin(\omega x)),$$

where $a(x)$ and $b(x)$ are polynomials.

Polynomial exponential oscillations

- A *polynomial exponential oscillation* is a function of the form

$$f(x) = e^{\lambda x}(a(x)\cos(\omega x) + b(x)\sin(\omega x)),$$

  where $a(x)$ and $b(x)$ are polynomials.
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- **Fact:** The integral of any PEO is another PEO with the same growth rate, frequency and degree.

Integrating PEO's — I

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- $$\begin{aligned} xe^{-x}\sin(x) &= \frac{d}{dx}\left((Ax+B)e^{-x}\cos(x) + (Cx+D)e^{-x}\sin(x)\right) \\ &= Ae^{-x}\cos(x) - (Ax+B)e^{-x}\cos(x) - (Ax+B)e^{-x}\sin(x) + \\ &\quad Ce^{-x}\sin(x) - (Cx+D)e^{-x}\sin(x) + (Cx+D)e^{-x}\cos(x) \end{aligned}$$

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- $\int xe^{-x}\sin(x)\,dx = -((x+1)e^{-x}\cos(x) + xe^{-x}\sin(x))/2$.

Integrating PEO's — II

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- $A = 1$, $3A + B = 0$, $2B + C = 0$, $C + D = 0$.
- so $A = 1$, $B = -3$, $C = 6$, $D = -6$
- so $\int x^3 e^x \, dx = (x^3 - 3x^2 + 6x - 6)e^x$.

Integration by parts — I

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- Consider $\int xe^{x/a}\,dx$.

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- Consider $\int x e^{x/a}\,dx$.
- $u = x$ $\qquad dv/dx = e^{x/a}$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.

Integration by parts — I

- Consider $\int x e^{x/a}\, dx$.
- $u = x$ $\qquad$ $dv/dx = e^{x/a}$
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- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.

Integration by parts — I

- Consider $\int xe^{x/a}\,dx$.
- $u = x$ $\quad$ $dv/dx = e^{x/a}$
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- $\int xe^{x/a}\,dx = uv - \int \frac{du}{dx} v\,dx$

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- This is most useful when (a) du/dx is simpler than u (eg u polynomial) and (b) v is no more complicated than dv/dx (eg $dv/dx = \cos(x)$).

Integration by parts — II

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- Consider $\int (1 - \ln(x))x^{-2}\, dx$.

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- Consider $\int (1 - \ln(x))x^{-2}\, dx$.
- $u = 1 - \ln(x)$ $\qquad$ $dv/dx = x^{-2}$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.

Integration by parts — II

- Consider $\int (1 - \ln(x))x^{-2}\, dx$.
- $u = 1 - \ln(x)$ $\qquad dv/dx = x^{-2}$
- $du/dx = -x^{-1}$

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  $= (\ln(x) - 1)x^{-1} + x^{-1}$

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- $u = 1 - \ln(x)$ $\qquad dv/dx = x^{-2}$
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- $\int (1 - \ln(x))x^{-2}\, dx = uv - \int \frac{du}{dx} v\, dx = -(1 - \ln(x))x^{-1} - \int x^{-2}\, dx$

  $= (\ln(x) - 1)x^{-1} + x^{-1} = \ln(x)/x$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula:

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Integration by parts — III

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- Consider $\int x \sin(\omega x)\, dx$.

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- Consider $\int x \sin(\omega x)\, dx$.
- $u = x$ $\qquad$ $dv/dx = \sin(\omega x)$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.

Integration by parts — III

- Consider $\int x \sin(\omega x)\, dx$.
- $u = x$ $\qquad\qquad$ $dv/dx = \sin(\omega x)$
- $du/dx = 1$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.

Integration by parts — III

- Consider $\int x \sin(\omega x)\, dx$.
- $u = x$ $\qquad$ $dv/dx = \sin(\omega x)$
- $du/dx = 1$ $\qquad$ $v = -\omega^{-1} \cos(\omega x)$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.

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- Consider $\int x \sin(\omega x)\, dx$.
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  $= -\omega^{-1} x \cos(\omega x) + \omega^{-2} \sin(\omega x)$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
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Integration by parts — IV

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- Consider $\int \arcsin(x)\,dx$.

Integration by parts — IV

- Consider $\int \arcsin(x).1\,dx$.
- $u = \arcsin(x)$ $dv/dx = 1$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.

Integration by parts — IV

- Consider $\int \arcsin(x).1\,dx$.
- $u = \arcsin(x)$ $dv/dx = 1$
- $du/dx = (1 - x^2)^{-1/2}$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.

Integration by parts — IV

- Consider $\int \arcsin(x).1\, dx$.
- $u = \arcsin(x)$ $\quad$ $dv/dx = 1$
- $du/dx = (1 - x^2)^{-1/2}$ $\quad$ $v = x$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.

Integration by parts — IV

- Consider $\int \arcsin(x).1\,dx$.
- $u = \arcsin(x)$ $\qquad dv/dx = 1$
- $du/dx = (1-x^2)^{-1/2}$ $\qquad v = x$
- $\int \arcsin(x).1\,dx = uv - \int \frac{du}{dx} v\,dx$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
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- Consider $\int \arcsin(x).1\, dx$.
- $u = \arcsin(x)$ $\qquad dv/dx = 1$
- $du/dx = (1 - x^2)^{-1/2}$ $\qquad v = x$
- $\int \arcsin(x).1\, dx = uv - \int \frac{du}{dx} v\, dx = \arcsin(x).x - \int x(1 - x^2)^{-1/2}\, dx$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
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Integration by parts — IV

- Consider $\int \arcsin(x).1\,dx$.
- $u = \arcsin(x)$ $\qquad dv/dx = 1$
- $du/dx = (1-x^2)^{-1/2}$ $\qquad v = x$
- $\int \arcsin(x).1\,dx = uv - \int \frac{du}{dx} v\,dx = \arcsin(x).x - \int x(1-x^2)^{-1/2}\,dx$

  $= x \arcsin(x) + (1-x^2)^{1/2}$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula:

$$\int u \frac{dv}{dx}\,dx = uv - \int \frac{du}{dx} v\,dx$$

Integration by substitution — I

Integration by substitution — I

- Consider $\displaystyle\int \frac{\sin(x)}{\cos(x)^n}\,dx$.

- Consider $\int \frac{\sin(x)}{\cos(x)^n}\,dx$.
- Put $u = \cos(x)$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.

Integration by substitution — I

- Consider $\int \frac{\sin(x)}{\cos(x)^n}\,dx$.
- Put $u = \cos(x)$, so $du/dx = -\sin(x)$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx

Integration by substitution — I

- Consider $\int \frac{\sin(x)}{\cos(x)^n}\,dx$.
- Put $u = \cos(x)$, so $du/dx = -\sin(x)$, so $dx = -du/\sin(x)$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx, and rearrange to express dx in terms of x and du.

Integration by substitution — I

- Consider $\int \frac{\sin(x)}{\cos(x)^n}\,dx$.
- Put $u = \cos(x)$, so $du/dx = -\sin(x)$, so $dx = -du/\sin(x)$
- $$\int \frac{\sin(x)}{\cos(x)^n}\,dx = \int \frac{\sin(x)}{u^n}\,\frac{-du}{\sin(x)}$$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx, and rearrange to express dx in terms of x and du.
- Rewrite the integral in terms of u and du.

Integration by substitution — I

- Consider $\int \frac{\sin(x)}{\cos(x)^n}\,dx$.
- Put $u = \cos(x)$, so $du/dx = -\sin(x)$, so $dx = -du/\sin(x)$
- $$\int \frac{\sin(x)}{\cos(x)^n}\,dx = \int \frac{\sin(x)}{u^n}\frac{-du}{\sin(x)} = -\int u^{-n}\,du$$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx, and rearrange to express dx in terms of x and du.
- Rewrite the integral in terms of u and du.

Integration by substitution — I

- Consider $\int \frac{\sin(x)}{\cos(x)^n}\,dx$.
- Put $u = \cos(x)$, so $du/dx = -\sin(x)$, so $dx = -du/\sin(x)$
- $$\int \frac{\sin(x)}{\cos(x)^n}\,dx = \int \frac{\sin(x)}{u^n}\frac{-du}{\sin(x)} = -\int u^{-n}\,du$$
$$= u^{1-n}/(n-1)$$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx, and rearrange to express dx in terms of x and du.
- Rewrite the integral in terms of u and du.
- Evaluate the integral

Integration by substitution — I

- Consider $\int \frac{\sin(x)}{\cos(x)^n}\,dx$.
- Put $u = \cos(x)$, so $du/dx = -\sin(x)$, so $dx = -du/\sin(x)$
- $$\int \frac{\sin(x)}{\cos(x)^n}\,dx = \int \frac{\sin(x)}{u^n}\,\frac{-du}{\sin(x)} = -\int u^{-n}\,du$$
  $$= u^{1-n}/(n-1) = \frac{\cos(x)^{1-n}}{n-1}$$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx, and rearrange to express dx in terms of x and du.
- Rewrite the integral in terms of u and du.
- Evaluate the integral, then rewrite the result in terms of x.

Integration by substitution — II

Integration by substitution — II

- Consider $\int xe^{-4x^2}\,dx$.

Integration by substitution — II

- Consider $\displaystyle\int xe^{-4x^2}\,dx$.
- Put $u = -4x^2$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.

- Consider $\int xe^{-4x^2}\,dx$.
- Put $u = -4x^2$, so $du/dx = -8x$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx

Integration by substitution — II

- Consider $\int xe^{-4x^2}\,dx$.
- Put $u = -4x^2$, so $du/dx = -8x$, so $dx = -du/(8x)$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx, and rearrange to express dx in terms of x and du.

Integration by substitution — II

- Consider $\int xe^{-4x^2}\,dx$.
- Put $u = -4x^2$, so $du/dx = -8x$, so $dx = -du/(8x)$
- $$\int xe^{-4x^2}\,dx = \int -xe^{u}\,\frac{du}{8x}$$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx, and rearrange to express dx in terms of x and du.
- Rewrite the integral in terms of u and du.

Integration by substitution — II

- Consider $\int xe^{-4x^2}\,dx$.
- Put $u = -4x^2$, so $du/dx = -8x$, so $dx = -du/(8x)$
- $$\int xe^{-4x^2}\,dx = \int -xe^u \frac{du}{8x} = -\frac{1}{8}\int e^u\,du$$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx, and rearrange to express dx in terms of x and du.
- Rewrite the integral in terms of u and du.

Integration by substitution — II

- Consider $\int xe^{-4x^2}\, dx$.
- Put $u = -4x^2$, so $du/dx = -8x$, so $dx = -du/(8x)$
- $$\int xe^{-4x^2}\, dx = \int -xe^u \frac{du}{8x} = -\frac{1}{8}\int e^u\, du$$
$$= -e^u/8$$

- To find $\int f(x)\, dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx, and rearrange to express dx in terms of x and du.
- Rewrite the integral in terms of u and du.
- Evaluate the integral

Integration by substitution — II

- Consider $\int xe^{-4x^2}\,dx$.
- Put $u = -4x^2$, so $du/dx = -8x$, so $dx = -du/(8x)$
- $$\int xe^{-4x^2}\,dx = \int -xe^u \frac{du}{8x} = -\frac{1}{8}\int e^u\,du$$ $$= -e^u/8 = -e^{-4x^2}/8$$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx, and rearrange to express dx in terms of x and du.
- Rewrite the integral in terms of u and du.
- Evaluate the integral, then rewrite the result in terms of x.

Integration by substitution — III

Integration by substitution — III

- Consider $\displaystyle\int \frac{dx}{4x^2 + 4x + 2}$.

Integration by substitution — III

- Consider $\displaystyle\int \frac{dx}{4x^2+4x+2} = \int \frac{dx}{(2x+1)^2+1}$.
- Put $u = 2x+1$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.

Integration by substitution — III

- Consider $\int \frac{dx}{4x^2+4x+2} = \int \frac{dx}{(2x+1)^2+1}$.
- Put $u = 2x+1$, so $du/dx = 2$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx

Integration by substitution — III

- Consider $\int \frac{dx}{4x^2 + 4x + 2} = \int \frac{dx}{(2x+1)^2 + 1}$.
- Put $u = 2x + 1$, so $du/dx = 2$, so $dx = du/2$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx, and rearrange to express dx in terms of x and du.

Integration by substitution — III

- Consider $\int \frac{dx}{4x^2+4x+2} = \int \frac{dx}{(2x+1)^2+1}$.
- Put $u = 2x+1$, so $du/dx = 2$, so $dx = du/2$
- $$\int \frac{dx}{4x^2+4x+2} = \int \frac{du/2}{u^2+1}$$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx, and rearrange to express dx in terms of x and du.
- Rewrite the integral in terms of u and du.

Integration by substitution — III

- Consider $\int \frac{dx}{4x^2+4x+2} = \int \frac{dx}{(2x+1)^2+1}$.
- Put $u = 2x+1$, so $du/dx = 2$, so $dx = du/2$
- $$\begin{aligned}\int \frac{dx}{4x^2+4x+2} &= \int \frac{du/2}{u^2+1} \\ &= \arctan(u)/2\end{aligned}$$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx, and rearrange to express dx in terms of x and du.
- Rewrite the integral in terms of u and du.
- Evaluate the integral

Integration by substitution — III

- Consider $\int \frac{dx}{4x^2+4x+2} = \int \frac{dx}{(2x+1)^2+1}$.
- Put $u = 2x+1$, so $du/dx = 2$, so $dx = du/2$
- $$\int \frac{dx}{4x^2+4x+2} = \int \frac{du/2}{u^2+1}$$
  $$= \arctan(u)/2 = \arctan(2x+1)/2$$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx, and rearrange to express dx in terms of x and du.
- Rewrite the integral in terms of u and du.
- Evaluate the integral, then rewrite the result in terms of x.

Integration by substitution — IV

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- Consider $\int \frac{dx}{\sqrt{x - x^2}}$.

Integration by substitution — IV

- Consider $\int \frac{dx}{\sqrt{x - x^2}}$.
- Put $x = t^2$

- To find $\int f(x)\,dx$, put x equal to some function of t.

Integration by substitution — IV

- Consider $\int \frac{dx}{\sqrt{x - x^2}}$.
- Put $x = t^2$, so $dx/dt = 2t$

- To find $\int f(x)\,dx$, put x equal to some function of t.
- Find dx/dt

Integration by substitution — IV

- Consider $\displaystyle\int \frac{dx}{\sqrt{x - x^2}}$.
- Put $x = t^2$, so $dx/dt = 2t$, so $dx = 2t\,dt$

- To find $\int f(x)\,dx$, put x equal to some function of t.
- Find dx/dt, and rearrange to express dx in terms of t and dt.

Integration by substitution — IV

- Consider $\displaystyle\int \frac{dx}{\sqrt{x - x^2}}$.
- Put $x = t^2$, so $dx/dt = 2t$, so $dx = 2t\,dt$

$$\sqrt{x - x^2} = \sqrt{t^2 - t^4} = t\sqrt{1 - t^2}$$

- To find $\int f(x)\,dx$, put x equal to some function of t.
- Find dx/dt, and rearrange to express dx in terms of t and dt.
- Rewrite the integral in terms of t and dt.

Integration by substitution — IV

- Consider $\int \frac{dx}{\sqrt{x - x^2}}$.
- Put $x = t^2$, so $dx/dt = 2t$, so $dx = 2t\,dt$

$$\sqrt{x - x^2} = \sqrt{t^2 - t^4} = t\sqrt{1 - t^2}$$
$$\int \frac{dx}{\sqrt{x - x^2}} = \int \frac{2t\,dt}{t\sqrt{1 - t^2}}$$

- To find $\int f(x)\,dx$, put x equal to some function of t.
- Find dx/dt, and rearrange to express dx in terms of t and dt.
- Rewrite the integral in terms of t and dt.

Integration by substitution — IV

- Consider $\int \frac{dx}{\sqrt{x - x^2}}$.
- Put $x = t^2$, so $dx/dt = 2t$, so $dx = 2t\,dt$

$$\sqrt{x - x^2} = \sqrt{t^2 - t^4} = t\sqrt{1 - t^2}$$

$$\int \frac{dx}{\sqrt{x - x^2}} = \int \frac{2t\,dt}{t\sqrt{1 - t^2}} = 2\int \frac{dt}{\sqrt{1 - t^2}}$$

- To find $\int f(x)\,dx$, put x equal to some function of t.
- Find dx/dt, and rearrange to express dx in terms of t and dt.
- Rewrite the integral in terms of t and dt.

Integration by substitution — IV

- Consider $\int \frac{dx}{\sqrt{x - x^2}}$.
- Put $x = t^2$, so $dx/dt = 2t$, so $dx = 2t\,dt$

$$\sqrt{x - x^2} = \sqrt{t^2 - t^4} = t\sqrt{1 - t^2}$$

$$\int \frac{dx}{\sqrt{x - x^2}} = \int \frac{2t\,dt}{t\sqrt{1 - t^2}} = 2\int \frac{dt}{\sqrt{1 - t^2}}$$

$$= 2\arcsin(t)$$

- To find $\int f(x)\,dx$, put x equal to some function of t.
- Find dx/dt, and rearrange to express dx in terms of t and dt.
- Rewrite the integral in terms of t and dt.
- Evaluate the integral

Integration by substitution — IV

- Consider $\int \frac{dx}{\sqrt{x - x^2}}$.
- Put $x = t^2$, so $dx/dt = 2t$, so $dx = 2t\,dt$

$$\sqrt{x - x^2} = \sqrt{t^2 - t^4} = t\sqrt{1 - t^2}$$
$$\int \frac{dx}{\sqrt{x - x^2}} = \int \frac{2t\,dt}{t\sqrt{1 - t^2}} = 2\int \frac{dt}{\sqrt{1 - t^2}}$$
$$= 2\arcsin(t) = 2\arcsin(\sqrt{x})$$

- To find $\int f(x)\,dx$, put x equal to some function of t.
- Find dx/dt, and rearrange to express dx in terms of t and dt.
- Rewrite the integral in terms of t and dt.
- Evaluate the integral, then rewrite the result in terms of x.

Integration by substitution — V

Integration by substitution — V

- Consider $\int \log(x)^2 \, dx$.

Integration by substitution — V

- Consider $\int \log(x)^2 \, dx$.
- Put $x = e^t$

- To find $\int f(x) \, dx$, put x equal to some function of t.

Integration by substitution — V

- Consider $\int \log(x)^2\, dx$.
- Put $x = e^t$, so $dx/dt = e^t$

- To find $\int f(x)\, dx$, put x equal to some function of t.
- Find dx/dt

Integration by substitution — V

- Consider $\int \log(x)^2 \, dx$.
- Put $x = e^t$, so $dx/dt = e^t$, so $dx = e^t \, dt$

- To find $\int f(x)\, dx$, put x equal to some function of t.
- Find dx/dt, and rearrange to express dx in terms of t and dt.

Integration by substitution — V

- Consider $\int \log(x)^2\,dx$.
- Put $x = e^t$, so $dx/dt = e^t$, so $dx = e^t\,dt$

$$\int \log(x)^2\,dx = \int \log(e^t)^2 e^t\,dt$$

- To find $\int f(x)\,dx$, put x equal to some function of t.
- Find dx/dt, and rearrange to express dx in terms of t and dt.
- Rewrite the integral in terms of t and dt.

Integration by substitution — V

- Consider $\int \log(x)^2 \, dx$.
- Put $x = e^t$, so $dx/dt = e^t$, so $dx = e^t \, dt$

$$\int \log(x)^2 \, dx = \int \log(e^t)^2 e^t \, dt = \int t^2 e^t \, dt$$

- To find $\int f(x) \, dx$, put x equal to some function of t.
- Find dx/dt, and rearrange to express dx in terms of t and dt.
- Rewrite the integral in terms of t and dt.

Integration by substitution — V

- Consider $\int \log(x)^2\,dx$.
- Put $x = e^t$, so $dx/dt = e^t$, so $dx = e^t\,dt$

$$\int \log(x)^2\,dx = \int \log(e^t)^2 e^t\,dt = \int t^2 e^t\,dt$$
$$= (t^2 - 2t + 2)e^t$$

- To find $\int f(x)\,dx$, put x equal to some function of t.
- Find dx/dt, and rearrange to express dx in terms of t and dt.
- Rewrite the integral in terms of t and dt.
- Evaluate the integral

Integration by substitution — V

- Consider $\int \log(x)^2 \, dx$.
- Put $x = e^t$, so $dx/dt = e^t$, so $dx = e^t \, dt$

$$\int \log(x)^2 \, dx = \int \log(e^t)^2 e^t \, dt = \int t^2 e^t \, dt$$
$$= (t^2 - 2t + 2)e^t = (\log(x)^2 - 2\log(x) + 2)x$$

- To find $\int f(x) \, dx$, put x equal to some function of t.
- Find dx/dt, and rearrange to express dx in terms of t and dt.
- Rewrite the integral in terms of t and dt.
- Evaluate the integral, then rewrite the result in terms of x.

Examples I

Examples I

- $\int \tan(x)\, dx$

Examples I

- $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$

Examples I

- $\int \tan(x)\, dx = \int \frac{\sin(x)}{\cos(x)}\, dx = -\int \frac{\cos'(x)}{\cos(x)}\, dx$

Examples I

- $\int \tan(x)\, dx = \int \frac{\sin(x)}{\cos(x)}\, dx = -\int \frac{\cos'(x)}{\cos(x)}\, dx = -\log(\cos(x)).$

Examples I

- $\int \tan(x)\, dx = \int \frac{\sin(x)}{\cos(x)}\, dx = -\int \frac{\cos'(x)}{\cos(x)}\, dx = -\log(\cos(x))$.
- Consider $\int x^2 \tan(x^3)\, dx$.

Examples I

- $\displaystyle\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{\cos'(x)}{\cos(x)}\,dx = -\log(\cos(x)).$
- Consider $\int x^2 \tan(x^3)\,dx$. Put $u = x^3$, so $du = 3x^2\,dx$, so $dx = du/(3x^2)$.

Examples I

- $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{\cos'(x)}{\cos(x)}\,dx = -\log(\cos(x))$.
- Consider $\int x^2 \tan(x^3)\,dx$. Put $u = x^3$, so $du = 3x^2\,dx$, so $dx = du/(3x^2)$.

$$\int x^2 \tan(x^3)\,dx = \int x^2 \tan(u) \frac{du}{3x^2}$$

Examples I

- $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{\cos'(x)}{\cos(x)}\,dx = -\log(\cos(x)).$
- Consider $\int x^2 \tan(x^3)\,dx$. Put $u = x^3$, so $du = 3x^2\,dx$, so $dx = du/(3x^2)$.

$$\int x^2 \tan(x^3)\,dx = \int x^2 \tan(u) \frac{du}{3x^2} = \frac{1}{3}\int \tan(u)\,du$$

Examples I

- $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{\cos'(x)}{\cos(x)}\,dx = -\log(\cos(x))$.
- Consider $\int x^2 \tan(x^3)\,dx$. Put $u = x^3$, so $du = 3x^2\,dx$, so $dx = du/(3x^2)$.

$$\int x^2 \tan(x^3)\,dx = \int x^2 \tan(u)\frac{du}{3x^2} = \frac{1}{3}\int \tan(u)\,du = -\log(\cos(u))/3$$

Examples I

- $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{\cos'(x)}{\cos(x)}\,dx = -\log(\cos(x)).$
- Consider $\int x^2 \tan(x^3)\,dx$. Put $u = x^3$, so $du = 3x^2\,dx$, so $dx = du/(3x^2)$.

$$\int x^2 \tan(x^3)\,dx = \int x^2 \tan(u)\frac{du}{3x^2} = \frac{1}{3}\int \tan(u)\,du = -\log(\cos(u))/3$$
$$= -\log(\cos(x^3))/3$$

Examples I

- $\displaystyle\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{\cos'(x)}{\cos(x)}\,dx = -\log(\cos(x)).$
- Consider $\int x^2 \tan(x^3)\,dx$. Put $u = x^3$, so $du = 3x^2\,dx$, so $dx = du/(3x^2)$.

$$\begin{aligned}\int x^2 \tan(x^3)\,dx &= \int x^2 \tan(u)\frac{du}{3x^2} = \frac{1}{3}\int \tan(u)\,du = -\log(\cos(u))/3 \\ &= -\log(\cos(x^3))/3\end{aligned}$$

- Consider $\int xe^{\sqrt{x}}\,dx$.

Examples I

- $\displaystyle\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{\cos'(x)}{\cos(x)}\,dx = -\log(\cos(x)).$
- Consider $\int x^2 \tan(x^3)\,dx$. Put $u = x^3$, so $du = 3x^2\,dx$, so $dx = du/(3x^2)$.

$$\int x^2 \tan(x^3)\,dx = \int x^2 \tan(u)\frac{du}{3x^2} = \frac{1}{3}\int \tan(u)\,du = -\log(\cos(u))/3$$
$$= -\log(\cos(x^3))/3$$

- Consider $\int xe^{\sqrt{x}}\,dx$. Put $t = \sqrt{x}$, so $x = t^2$, so $dx = 2t\,dt$.

Examples I

- $\displaystyle\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{\cos'(x)}{\cos(x)}\,dx = -\log(\cos(x)).$
- Consider $\int x^2 \tan(x^3)\,dx$. Put $u = x^3$, so $du = 3x^2\,dx$, so $dx = du/(3x^2)$.

$$\begin{aligned}\int x^2 \tan(x^3)\,dx &= \int x^2 \tan(u)\frac{du}{3x^2} = \frac{1}{3}\int \tan(u)\,du = -\log(\cos(u))/3 \\ &= -\log(\cos(x^3))/3\end{aligned}$$

- Consider $\int xe^{\sqrt{x}}\,dx$. Put $t = \sqrt{x}$, so $x = t^2$, so $dx = 2t\,dt$.

$$\int xe^{\sqrt{x}}\,dx = \int t^2 e^t .2t\,dt$$

Examples I

- $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{\cos'(x)}{\cos(x)}\,dx = -\log(\cos(x)).$
- Consider $\int x^2 \tan(x^3)\,dx$. Put $u = x^3$, so $du = 3x^2\,dx$, so $dx = du/(3x^2)$.

$$\int x^2 \tan(x^3)\,dx = \int x^2 \tan(u)\frac{du}{3x^2} = \frac{1}{3}\int \tan(u)\,du = -\log(\cos(u))/3$$
$$= -\log(\cos(x^3))/3$$

- Consider $\int xe^{\sqrt{x}}\,dx$. Put $t = \sqrt{x}$, so $x = t^2$, so $dx = 2t\,dt$.

$$\int xe^{\sqrt{x}}\,dx = \int t^2 e^t.2t\,dt = 2\int t^3 e^t\,dt$$

Examples I

- $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{\cos'(x)}{\cos(x)}\,dx = -\log(\cos(x)).$
- Consider $\int x^2 \tan(x^3)\,dx$. Put $u = x^3$, so $du = 3x^2\,dx$, so $dx = du/(3x^2)$.

$$\int x^2 \tan(x^3)\,dx = \int x^2 \tan(u)\frac{du}{3x^2} = \frac{1}{3}\int \tan(u)\,du = -\log(\cos(u))/3$$
$$= -\log(\cos(x^3))/3$$

- Consider $\int xe^{\sqrt{x}}\,dx$. Put $t = \sqrt{x}$, so $x = t^2$, so $dx = 2t\,dt$.

$$\int xe^{\sqrt{x}}\,dx = \int t^2 e^t.2t\,dt = 2\int t^3 e^t\,dt = 2(t^3 - 3t^2 + 6t - 6)e^t$$

Examples I

- $\displaystyle\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{\cos'(x)}{\cos(x)}\,dx = -\log(\cos(x)).$
- Consider $\int x^2 \tan(x^3)\,dx$. Put $u = x^3$, so $du = 3x^2\,dx$, so $dx = du/(3x^2)$.

$$\begin{aligned}\int x^2 \tan(x^3)\,dx &= \int x^2 \tan(u)\frac{du}{3x^2} = \frac{1}{3}\int \tan(u)\,du = -\log(\cos(u))/3 \\ &= -\log(\cos(x^3))/3\end{aligned}$$

- Consider $\int xe^{\sqrt{x}}\,dx$. Put $t = \sqrt{x}$, so $x = t^2$, so $dx = 2t\,dt$.

$$\begin{aligned}\int xe^{\sqrt{x}}\,dx &= \int t^2 e^t.2t\,dt = 2\int t^3 e^t\,dt = 2(t^3 - 3t^2 + 6t - 6)e^t \\ &= (2x^{3/2} - 6x + 12x^{1/2} - 12)e^{\sqrt{x}}\end{aligned}$$

Examples II

Examples II

- $$\int (2(x^2+1)e^x)^2\,dx$$

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- $$\int (2(x^2+1)e^x)^2\,dx = \int (4x^4+8x^2+4)e^{2x}\,dx$$

Examples II

- $$\int (2(x^2+1)e^x)^2\,dx = \int (4x^4+8x^2+4)e^{2x}\,dx$$
  $$= (Ax^4+Bx^3+Cx^2+Dx+E)e^{2x}$$

Examples II

- $$\int (2(x^2+1)e^x)^2\,dx = \int (4x^4+8x^2+4)e^{2x}\,dx$$
  $$= (Ax^4+Bx^3+Cx^2+Dx+E)e^{2x}$$
  $$(4x^4+8x^2+4)e^{2x} = \frac{d}{dx}((Ax^4+Bx^3+Cx^2+Dx+E)e^{2x})$$

Examples II

- $$\begin{aligned}\int (2(x^2+1)e^x)^2\,dx &= \int (4x^4+8x^2+4)e^{2x}\,dx \\ &= (Ax^4+Bx^3+Cx^2+Dx+E)e^{2x} \\ (4x^4+8x^2+4)e^{2x} &= \frac{d}{dx}((Ax^4+Bx^3+Cx^2+Dx+E)e^{2x}) \\ &= (4Ax^3+3Bx^2+2Cx+D)e^{2x}+ \\ &\quad (Ax^4+Bx^3+Cx^2+Dx+E).2e^{2x}\end{aligned}$$

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  So $4=2A$, $0=4A+2B$, $8=3B+2C$, $0=2C+2D$, $4=D+2E$

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  So $4 = 2A$, $0 = 4A+2B$, $8 = 3B+2C$, $0 = 2C+2D$, $4 = D+2E$
  So $A = 2,\ B = -4,\ C = 10,\ D = -10,\ E = 7$

Examples II

- $$\int (2(x^2+1)e^x)^2\,dx = \int (4x^4+8x^2+4)e^{2x}\,dx$$
$$= (Ax^4+Bx^3+Cx^2+Dx+E)e^{2x}$$
$$(4x^4+8x^2+4)e^{2x} = \frac{d}{dx}((Ax^4+Bx^3+Cx^2+Dx+E)e^{2x})$$
$$= (4Ax^3+3Bx^2+2Cx+D)e^{2x} + (Ax^4+Bx^3+Cx^2+Dx+E).2e^{2x}$$
$$= e^{2x}(2Ax^4+(4A+2B)x^3+(3B+2C)x^2+(2C+2D)x+(D+2E))$$

So $4=2A$, $0=4A+2B$, $8=3B+2C$, $0=2C+2D$, $4=D+2E$

So $A=2,\ B=-4,\ C=10,\ D=-10,\ E=7$

$$\int (2(x^2+1)e^x)^2\,dx = (2x^4-4x^3+10x^2-10x+7)e^{2x}.$$

Examples III

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- $$\int 1 + \cosh(x) + \cosh(x)^2 \, dx = \int 1 + \frac{e^x + e^{-x}}{2} + \left(\frac{e^x + e^{-x}}{2}\right)^2 \, dx$$

Examples III

- $$\begin{aligned}\int 1+\cosh(x)+\cosh(x)^2\,dx &= \int 1+\frac{e^x+e^{-x}}{2}+\left(\frac{e^x+e^{-x}}{2}\right)^2\,dx\\ &= \frac{1}{4}\int 4+2e^x+2e^{-x}+e^{2x}+2+e^{-2x}\,dx\end{aligned}$$

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  $$= \frac{1}{4} \left(6x + 2e^x - 2e^{-x} + \tfrac{1}{2} e^{2x} - \tfrac{1}{2} e^{-2x}\right)$$

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\frac{d}{dx}\left(\frac{1+\sin(x)}{\cos(x)}\right) &= \frac{\cos(x).\cos(x) - (1+\sin(x))(-\sin(x))}{\cos(x)^2} \\
&= \frac{\cos(x)^2 + \sin(x)^2 + \sin(x)}{\cos(x)^2} = \frac{1+\sin(x)}{\cos(x)^2} \\
\frac{d}{dx}\log\left(\frac{1+\sin(x)}{\cos(x)}\right) &= \left(\frac{1+\sin(x)}{\cos(x)}\right)^{-1} \frac{d}{dx}\left(\frac{1+\sin(x)}{\cos(x)}\right) \\
&= \frac{\cos(x)}{1+\sin(x)} \frac{1+\sin(x)}{\cos(x)^2}
\end{aligned}$$

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- For any reasonable function $f(x)$, there are coefficients a_k such that

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

(when x is sufficiently small). This is the *Taylor series* for $f(x)$.

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For a full explanation, see Level 3 complex analysis.

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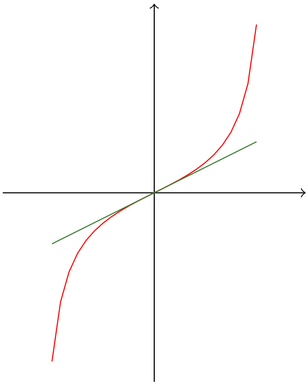
$\tan(x) = x + O(x^3)$



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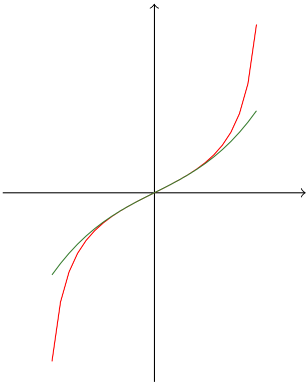

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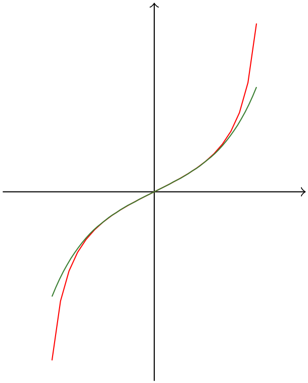

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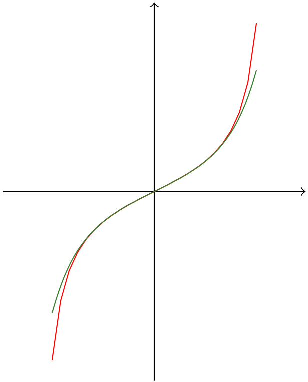
$\tan(x) = x + x^3/3 + 2x^5/15 + 17x^7/315 + O(x^9)$



Finding coefficients

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$$y = \sum_{k=0}^{\infty} a_k x^k, \qquad \text{where} \qquad a_k = \frac{1}{k!} \left. \frac{d^k y}{dx^k} \right|_{x=0}$$

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Thus $a_k = 1/k!$, and $\exp(x) = \sum_k x^k/k!$.

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$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots$$

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$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

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Odd and even functions

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Recall that $f(x)$ is *even* if $f(-x) = f(x)$, and *odd* if $f(-x) = -f(x)$.

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For example, $\cos(x)$ is even and $\sin(x)$ is odd.

Odd and even functions

Recall that $f(x)$ is *even* if $f(-x) = f(x)$, and *odd* if $f(-x) = -f(x)$. For example, $\cos(x)$ is even and $\sin(x)$ is odd. If

$$f(x) = \sum_k a_k x^k = \sum_{k \text{ even}} a_k x^k + \sum_{k \text{ odd}} a_k x^k$$

then

$$f(-x) = \sum_k a_k (-x)^k = \sum_{k \text{ even}} a_k x^k - \sum_{k \text{ odd}} a_k x^k.$$

Odd and even functions

Recall that $f(x)$ is *even* if $f(-x) = f(x)$, and *odd* if $f(-x) = -f(x)$. For example, $\cos(x)$ is even and $\sin(x)$ is odd. If

$$f(x) = \sum_k a_k x^k = \sum_{k \text{ even}} a_k x^k + \sum_{k \text{ odd}} a_k x^k$$

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$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

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\end{array}$$

$$\ln(x) = (x-1) - (x-1)^2/2 + (x-1)^3/3 - (x-1)^4/4 + O((x-1)^5).$$

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$$\begin{aligned} e^x &= 1 + x + x^2/2 + x^3/6 + O(x^4) \\ e^x - 1 &= x + x^2/2 + x^3/6 + O(x^4) \end{aligned}$$

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&= 1 - x/2 - x^2/6 + x^2/4 + O(x^3) = 1 - x/2 + x^2/12 + O(x^3)
\end{aligned}$$