PROBLEMS FOR MATHS WITH MAPLE

Week 1

Exercise 1.1. Solve the equations

$$w + x + y + z = 1234 \quad (1) \qquad w + x - y - z = 1210 \quad (2)$$
$$w - x + y - z = 1010 \quad (3) \qquad w - x - y + z = 990 \quad (4)$$

Solution: We deduce further equations as follows:

((5)=(1)-(2)) $2y + 2z = 24$

((6)=(1)-(3)) $2x + 2z = 224$

((7)=(1)-(4)) $2x + 2y = 244$

((8)=(6)-(5)) $2x - 2y = 200$

((9)=(7)+(8)) $4x = 444$

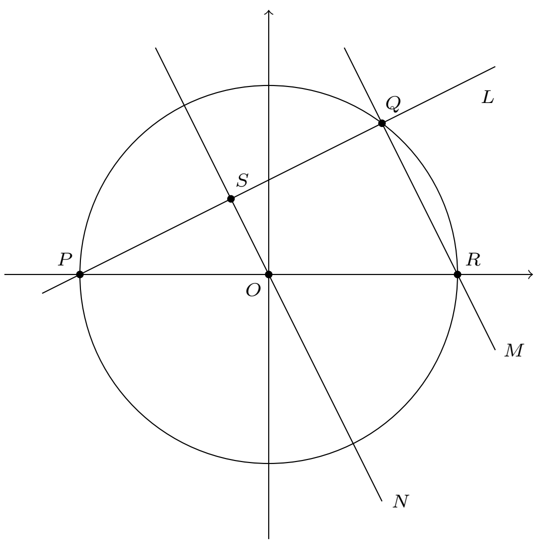
From this we get $x = 111$, which we substitute in (7) and (6) to get $y = 11$ and $z = 1$. Putting these in (1) gives $w = 1111$. In conclusion, we have

$$(w, x, y, z) = (1111, 111, 11, 1).$$

Exercise 1.2. As you work through (a) to (g) below, draw a diagram showing all the curves, points and lines involved.

(a) Let C be the curve $x^2 + y^2 = 25$. Describe this geometrically. Note that the points $P = (-5, 0)$ and $Q = (3, 4)$ lie on C.
(b) Let L be the line joining P to Q. What is the equation for L? What is its slope (or in other words, its gradient)?
(c) Let M be the line of slope -2 passing through Q. What is the equation for M?
(d) Let R be the point (other than Q) where M meets C. What are the coordinates of R?
(e) What is the angle between L and M? (Note: here and almost everywhere else in University mathematics, angles are measured in radians, not degrees.) Do you know a general geometric fact that explains this?
(f) Let S be the midpoint of the line segment PQ. What are the coordinates of S?
(g) Let N be the line joining S to the origin $O = (0, 0)$. Show that N is parallel to M.

Solution:



(a) C is the circle of radius 5 centred at the origin.

(b) The equation for L is $y = x/2 + 5/2$, with slope $1/2$. (One way to obtain this is to say that the equation must be $y = mx + b$ for some m and b. As $P = (-5, 0)$ lies on the line, we must have $0 = -5m + b$, so $b = 5m$. As $Q = (3, 4)$ lies on the line, we must $4 = 3m + b = 3m + 5m = 8m$, so $m = 1/2$, so $b = 5/2$.)

(c) The equation for M is $y - 4 = -2(x - 3)$, or equivalently $y = -2x + 10$.

(d) M meets C at the points (x, y) where $y = -2x+10$ and $x^2+y^2 = 25$, which implies $x^2+(-2x+10)^2 = 25$. This can be expanded and rearranged as $5x^2 - 40x + 75 = 0$, or $5(x - 3)(x - 5) = 0$, so $x = 3$ or $x = 5$. Using $y = -2x + 10$ we see that the intersection points are $(3, 4)$ and $(5, 0)$. The first of these is Q, so R must be $(5, 0)$.

(e) The slope of L times the slope of M is $(1/2).(-2) = -1$, which means that L and M are at right angles to each other. In fact, whenever you have a triangle with one side being the diameter of a circle and the third vertex also lying on the circle, then the angle at the third vertex is always a right angle (often stated as "the angle in a semicircle is a right angle").

(f) The point S is $(P + Q)/2 = (-5 + 3, 0 + 4)/2 = (-1, 2)$. The line N joining S to O is $y = -2x$, with slope -2. This is the same as the slope of M, so N and M are parallel.

Exercise 1.3. Differentiate the following functions, simplifying your answers as much as possible:

(a) $x+x^{10}+x^{100}$ **(b)** $(3x+2)/(4x+3)$ **(c)** $x\log(x)-x$ **(d)** $e^{-x}\sin(10x)$ **(e)** $\sin(x^2)$

Solution:

(a) Using the rule $\frac{d}{dx}(x^n) = n\,x^{n-1}$ we find that the derivative is $1 + 10x^9 + 100x^{99}$

(b) Here we use the quotient rule

$$\tfrac{d}{dx}\left(\tfrac{u}{v}\right) = \left(\tfrac{du}{dx}v - u\tfrac{dv}{dx}\right)/v^2.$$

We have $u = 3x + 2$ and $v = 4x + 3$, so $du/dx = 3$ and $dv/dx = 4$, so $\frac{du}{dx}v - u\frac{dv}{dx} = 3(4x + 3) - 4(3x + 2) = 1$. The derivative is thus $1/(4x + 3)^2$

(c) We differentiate $x\log(x)$ using the product rule $\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$, and remembering that $\frac{d}{dx}\log(x) = 1/x$. This gives $\frac{d}{dx}(x\log(x)) = 1.\log(x) + x.1/x = \log(x) + 1$. As $\frac{d}{dx}x = 1$ this gives $\frac{d}{dx}(x\log(x) - x) = \log(x) + 1 - 1 = \log(x)$.

(d) We use the product rule again, together with the facts that $\frac{d}{dx}e^{-x} = -e^{-x}$ and $\frac{d}{dx}\sin(10x) = 10\cos(10x)$. We find that the final answer is $e^{-x}(10\cos(10x) - \sin(10x))$.

(e) Write $u = x^2$ and $y = \sin(u) = \sin(x^2)$. We are asked to find dy/dx, but the chain rule tells us that this is the same as $\frac{dy}{du}\frac{du}{dx}$. Here $y = \sin(u)$ so $dy/du = \cos(u) = \cos(x^2)$, and $du/dx = 2x$, so $dy/dx = 2x\cos(x^2)$.

Exercise 1.4. Evaluate the following integrals:

$$\textbf{(a)}\ \int x^9 + x^{99} + x^{999}\,dx \quad \textbf{(b)}\ \int x\,e^{3x}\,dx \quad \textbf{(c)}\ \int x\,e^{-x^2}\,dx \quad \textbf{(d)}\ \int \frac{dx}{\sqrt{1-x^2}} \quad \textbf{(e)}\ \int_1^{e^2}\frac{dx}{x}$$

Solution:

(a) Using the rule $\int x^n\,dx = x^{n+1}/(n + 1)$ we find that the integral is $\dfrac{x^{10}}{10} + \dfrac{x^{100}}{100} + \dfrac{x^{1000}}{1000}$.

(b) Here we integrate by parts, taking $u = x$ and $dv/dx = e^{3x}$, so that $du/dx = 1$ and $v = e^{3x}/3$. This gives

$$\int x\,e^{3x}\,dx = \tfrac{1}{3}x\,e^{3x} - \int \tfrac{1}{3}e^{3x}\,dx = \tfrac{1}{3}x\,e^{3x} - \tfrac{1}{9}e^{3x} = (x/3 - 1/9)e^{3x}.$$

(c) Here we substitute $u = -x^2$, so $du = -2x\,dx$ or equivalently $dx = -du/(2x)$. This gives

$$\int x\,e^{-x^2}\,dx = \int x\,e^u\,\frac{-du}{2x} = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u = -e^{-x^2}/2.$$

(d) Here we substitute $x = \sin(t)$, so $dx = \cos(t)\,dt$. As $\sin(t)^2 + \cos(t)^2 = 1$ we have $\sqrt{1 - x^2} = \cos(t)$. It follows that

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos(t)\,dt}{\cos(t)} = \int 1\,dt = t = \arcsin(x).$$

(e) It is standard that $\int \frac{dx}{x} = \log(x)$, so $\int_1^{e^2}\frac{dx}{x} = \log(e^2) - \log(1) = 2 - 0 = 2$.

Exercise 1.5. We have $10^3 = 1000 \approx 1024 = 2^{10}$. Deduce similar approximations for 10^9 and 8×10^9 as powers of 2.

Solution:

$$10^9 = 10^{3\times 3} = (10^3)^3 \approx (2^{10})^3 = 2^{10\times 3} = 2^{30}$$
$$8 \times 10^9 = 2^3 \times 10^9 \simeq 2^3 \times 2^{30} = 2^{33}.$$

(The exact numbers are $2^{30} = 1073741824$ and $2^{33} = 8589934592$.)

According to the usual metric conventions, a "gigabyte" of memory should contain 10^9 bytes. Because digital electronics is based on binary numbers, it is easier and more natural to build memory chips with 2^{30} bytes of capacity, so that is what a gigabyte means in practice. A byte is 8 bits, so the capacity is $8 \times 2^{30} = 2^{33} \approx 8 \times 10^9$ bits.

Exercise 1.6. (a) Give a formula for the infinite sum $S = \sum_{i=0}^{\infty} x^i = 1 + x + x^2 + \cdots$. (You may assume that $|x| < 1$.)

(b) Deduce a formula for the infinite sum $T = \sum_{i=1}^{\infty} x^i = x + x^2 + x^3 + \cdots$.

(c) Consider the expression

$$y = 142857(10^{-6} + 10^{-12} + 10^{-18} + \cdots).$$

Write the terms 142857×10^{-6}, 142857×10^{-12} and 142857×10^{-18} as decimals. What is the decimal expansion of y itself?

(d) Using part (b), give an exact expression for $1/y$. (Your answer should be a whole number, not a fraction.)

Solution:

(a) The geometric progression formula says that $S = 1/(1-x)$. (The proof is as follows: note that $xS = x(1 + x + x^2 + \cdots) = x + x^2 + x^3 + \cdots = S - 1$. Rearranging this gives $(1-x)S = 1$ and so $S = 1/(1-x)$.)

(b) From the defining formulae we see that $T = xS = x/(1-x)$. Alternatively, we see from the formulae that $T = S - 1 = 1/(x-1) - 1$, but this can be simplified to $x/(1-x)$ again.

(c) We have

$$\begin{aligned} 142857 \times 10^{-6} &= 0.142857 \\ 142857 \times 10^{-12} &= 0.000000142857 \\ 142857 \times 10^{-18} &= 0.000000000000142857 \end{aligned}$$

By adding these together with all the subsequent terms in the sequence, we see that y is the recurring decimal

$$0.142857142857142857142857\ldots = 0.\dot{1}4285\dot{7}.$$

(d) Part (b) (with $x = 10^{-6}$) tells us that

$$\begin{aligned} 10^{-6} + 10^{-12} + 10^{-18} + \cdots &= 10^{-6} + (10^{-6})^2 + (10^{-6})^3 + \cdots \\ &= 10^{-6}/(1 - 10^{-6}) = 1/(10^6 - 1) = 1/999999. \end{aligned}$$

This gives $y = 142857/999999$ and so

$$y^{-1} = \left(\frac{142857}{999999}\right)^{-1} = \frac{999999}{142857} = 7.$$

Putting all this together, we conclude that $1/7 = 0.\dot{1}4285\dot{7}$.

Exercise 1.7. Consider the following functions:

$$\begin{aligned} \phi_1(x) &= x - 1 & \phi_5(x) &= x^4 + x^3 + x^2 + x + 1 \\ \phi_2(x) &= x + 1 & \phi_6(x) &= x^2 - x + 1 \\ \phi_3(x) &= x^2 + x + 1 & \phi_{12}(x) &= x^4 - x^2 + 1 \\ \phi_4(x) &= x^2 + 1 & & \end{aligned}$$

(These are called *cyclotomic polynomials*. They are very important in *Number Theory*, which means the study of prime numbers, divisibility and so on.)

Expand out the following products:

$$\phi_1(x)\phi_2(x) \qquad \phi_1(x)\phi_3(x) \qquad \phi_1(x)\phi_2(x)\phi_4(x) \qquad \phi_1(x)\phi_5(x) \qquad \phi_1(x)\phi_2(x)\phi_3(x)\phi_6(x).$$

Can you see the pattern? Can you guess what is the corresponding equation involving $\phi_{12}(x)$?

Solution:

$$\begin{aligned}
\phi_1(x)\phi_2(x) &= (x-1)(x+1) = x^2 - 1 \\
\phi_1(x)\phi_3(x) &= (x-1)(x^2+x+1) = x^3 - 1 \\
\phi_1(x)\phi_2(x)\phi_4(x) &= (x-1)(x+1)(x^2+1) \\
&= (x^2-1)(x^2+1) = x^4 - 1 \\
\phi_1(x)\phi_5(x) &= (x-1)(x^4+x^3+x^2+x+1) = x^5 - 1 \\
\phi_1(x)\phi_2(x)\phi_3(x)\phi_6(x) &= (x-1)(x+1)(x^2+x+1)(x^2-x+1) \\
&= (x^2-1)(x^4+x^2+1) = x^6 - 1.
\end{aligned}$$

The general rule so far is as follows: for any number n, we take all the numbers d that divide n, take the corresponding polynomials $\phi_d(x)$, multiply them together, and we get $x^n - 1$. (Consider the case $n = 6$, for example: the divisors of 6 are 1, 2, 3 and 6, the corresponding polynomials are $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$ and $\phi_6(x)$, and we saw above that if we multiply these together, we get $x^6 - 1$.) This suggests that we should have

$$\phi_1(x)\phi_2(x)\phi_3(x)\phi_4(x)\phi_6(x)\phi_{12}(x) = x^{12} - 1.$$

If we feed in the case $n = 6$ which we have already worked out, the left hand side becomes $(x^6 - 1)\phi_4(x)\phi_{12}(x)$. It is easy to check that $\phi_4(x)\phi_{12}(x) = x^6+1$, so the left hand side becomes $(x^6-1)(x^6+1)$ which is $x^{12} - 1$ as expected.

Exercise 1.8. Put

$$\begin{aligned}
b_3(x) &= x(x-1)(x-2)/6 \\
b_4(x) &= x(x-1)(x-2)(x-3)/24 \\
b_5(x) &= x(x-1)(x-2)(x-3)(x-4)/120.
\end{aligned}$$

- Simplify $b_4(x+1) - b_4(x)$. If you do this in the right way, it will take only a few simple steps; if you do it the wrong way, you will have to work much harder. Do not expand anything out if you do not have to.
- Simplify $b_5(x+1) - b_5(x)$.
- What is the general pattern? (Your answer should include a definition of $b_n(x)$ for all n.)

Solution: We have

$$\begin{aligned}
b_4(x+1) &= (x+1)(x+1-1)(x+1-2)(x+1-3)/24 \\
&= (x+1)x(x-1)(x-2)/24 \\
b_4(x) &= x(x-1)(x-2)(x-3)/24,
\end{aligned}$$

so

$$\begin{aligned}
b_4(x+1) - b_4(x) &= ((x+1)-(x-3))x(x-1)(x-2)/24 \\
&= 4x(x-1)(x-2)/24 \\
&= x(x-1)(x-2)/6 = b_3(x).
\end{aligned}$$

Similarly,

$$\begin{aligned}
b_5(x+1) &= (x+1)x(x-1)(x-2)(x-3)/120 \\
b_5(x) &= x(x-1)(x-2)(x-3)(x-4)/120,
\end{aligned}$$

so

$$\begin{aligned}
b_5(x+1) - b_5(x) &= ((x+1)-(x-4))x(x-1)(x-2)(x-3)/120 \\
&= x(x-1)(x-2)(x-3)/24 = b_4(x).
\end{aligned}$$

The general pattern is that $b_n(x+1) - b_n(x) = b_{n-1}(x)$, where

$$b_n(x) = x(x-1)\dots(x-n+1)/n!.$$

Exercise 2.1. Solve the equation

$$(((x^3+1)^3+1)^3+1)^3+1=9,$$

where x is a real number. (It is not helpful to expand out the left hand side.)

Solution:

$$\begin{aligned}
&(((x^3+1)^3+1)^3+1)^3+1=9\\
\Leftrightarrow\ &(((x^3+1)^3+1)^3+1)^3=8\\
\Leftrightarrow\ &((x^3+1)^3+1)^3+1=8^{1/3}=2\\
\Leftrightarrow\ &((x^3+1)^3+1)^3=1\\
\Leftrightarrow\ &(x^3+1)^3+1=1^{1/3}=1\\
\Leftrightarrow\ &(x^3+1)^3=0\\
\Leftrightarrow\ &x^3+1=0 \Leftrightarrow x^3=-1 \Leftrightarrow x=-1.
\end{aligned}$$

Thus, the only solution is $x=-1$. Another way to write this is to use the function $f(t)=t^3+1$ and the inverse function $f^{-1}(s)=(s-1)^{1/3}$. The given equation says $f(f(f(f(x))))=9$, and the solution is $x=f^{-1}(f^{-1}(f^{-1}(f^{-1}(9))))$. We have $f^{-1}(9)=2$ and $f^{-1}(2)=1$ and $f^{-1}(1)=0$ and $f^{-1}(0)=-1$, so the conclusion is that $x=-1$.

If you ask Maple to do this problem, it will think for a while and then report a long list of 81 solutions. In fact, some of these solutions are mentioned more than once, so there are only 75 different solutions. Moreover, 74 of them are complex numbers and so are not allowed in this question. The only real solution is $x=-1$.

Exercise 2.2. Solve the equation $\ln(x)^3=\ln(x)$.

Solution:

$$\begin{aligned}
&\ln(x)^3=\ln(x)\\
\Leftrightarrow\ &\ln(x)^3-\ln(x)=0\\
\Leftrightarrow\ &\ln(x)(\ln(x)-1)(\ln(x)+1)=0\\
\Leftrightarrow\ &\ln(x)\in\{0,1,-1\}\\
\Leftrightarrow\ &x\in\{1,e,e^{-1}\}.
\end{aligned}$$

Exercise 2.3. Find a solution to the equation $\cos(\theta)^2+2\cos(\theta)=3$ where θ is a real number and $\theta>10$.

Solution: We first rearrange as $\cos(\theta)^2+2\cos(\theta)-3=0$, which factors as $(\cos(\theta)-1)(\cos(\theta)+3)=0$. As $\cos(\theta)$ only ranges between -1 and $+1$, the factor $\cos(\theta)+3$ is never zero, so we must instead have $\cos(\theta)-1=0$, so $\cos(\theta)=1$. This means that $\theta=2n\pi$ for some integer n. Note that $2\pi\simeq 6.28$ and $4\pi\simeq 12.56$. Thus, for $\theta>10$, we must have $n\geq 2$. The smallest possible answer is thus $\theta=4\pi$.

Exercise 2.4. You are given that $x,y>0$ and that

$$(A)\qquad x^3y^2=e \qquad\qquad (B)\qquad x^4y^3=e^2.$$

Find x and y. (One method is to take logs.)

Solution: If we take the log of both sides of the equation $x^3y^2=e$, we get $\ln(x^3y^2)=\ln(e)$, or equivalently $3\ln(x)+2\ln(y)=1$. Treating the other equation similarly, we have

$$\begin{aligned}
3\ln(x)+2\ln(y)&=1 &&(\ln(A))\\
4\ln(x)+3\ln(y)&=2 &&(\ln(B))
\end{aligned}$$

These equations can be solved for $\ln(x)$ and $\ln(y)$:

$$\begin{aligned}
\ln(x)&=-1 &&(3\ln(A)-2\ln(B))\\
\ln(y)&=2 &&(3\ln(B)-4\ln(A))
\end{aligned}$$

This gives $x=e^{-1}$ and $y=e^2$.

For an alternative approach, we can divide the cube of equation (A) by the square of equation (B):

$$\begin{aligned} x^9y^6 &= e^3 && (A^3) \\ x^8y^6 &= e^4 && (B^2) \\ x &= e^{-1} && (A^3/B^2). \end{aligned}$$

We can substitute this back into equation (A) to get $y^2 = e^4$ and so $y = e^2$ again.

Exercise 2.5. Solve the equation $\ln(e^{3t} - 7) = 0$.

Solution: Start with the given equation $\ln(e^{3t} - 7) = 0$. We can take the exponential of both sides to get $e^{\ln(e^{3t}-7)} = e^0 = 1$. Using the rule $e^{\ln(x)} = x$ we can simplify the left hand side to $e^{3t} - 7$, so $e^{3t} - 7 = 1$, so $e^{3t} = 8$. We can now take the logarithm of both sides to give $3t = \ln(8)$, or equivalently $t = \ln(8)/3$. This can be simplified further if we note that $8 = 2^3$, so $\ln(8) = 3\ln(2)$; this gives $t = \ln(2)$.

Note that the rule $\ln(a - b) = \ln(a) - \ln(b)$ is **not** valid, so we **cannot** start by converting $\ln(e^{3t} - 7)$ to $\ln(e^{3t}) - \ln(7)$.

Exercise 2.6. Solve the equation $\ln(e^x + 1) = x + 1$.

Solution: Start with the given equation $\ln(e^x + 1) = x + 1$. We can take the exponential of both sides to get $e^{\ln(e^x+1)} = e^{x+1}$. We can use the rule $e^{\ln(a)} = a$ to simplify the left hand side to $e^x + 1$. We can use the rules $e^{a+b} = e^a e^b$ and $e^1 = e$ to simplify the right hand side to $e^x e$. We now have $e^x + 1 = e^x e$, which can be rearranged as $e^x(e - 1) = 1$, so $e^x = 1/(e - 1)$. We now take logs on both sides to get $x = \ln(1/(e - 1)) = -\ln(e - 1)$.

Note that the rule $\ln(a + b) = \ln(a) + \ln(b)$ is **not** valid, so we **cannot** start by converting $\ln(e^x + 1)$ to $\ln(e^x) + \ln(1)$.

Exercise 2.7. Solve the following equations:

$$x = \sin(t) \qquad y = \cos(t) \qquad z = 6t/\pi \qquad x^2 + y^2 + z^2 = 10.$$

(In physics, this determines the time and place where an electron moving in a magnetic field hits the wall of a spherical chamber.)

Solution: We have $x^2+y^2 = \sin(t)^2+\cos(t)^2 = 1$, so the last equation simplifies to $z^2 = 9$, giving $z = \pm 3$. The third equation rearranges to give $t = \pi z/6 = \pm\pi/2$. It follows in turn that $x = \sin(\pm\pi/2) = \pm 1$ and $y = \cos(\pm\pi/2) = 0$. Thus, the two possible solutions are $(x, y, z, t) = (1, 0, 3, \pi/2)$ and $(x, y, z, t) = (-1, 0, -3, -\pi/2)$.

Exercise 2.8. Given a constant $a > 1$, solve the following equations:

$$x^2 + y^2 = 1 \qquad (a - x)x - y^2 = 0.$$

If x and y satisfy the equations, what is $\sqrt{(a - x)^2 + y^2}$?

(Let C be the circle of radius one centred at the origin, and let L be one of the two lines through $(a, 0)$ that just touches C; then L meets C at (x, y), where x and y satisfy the equations above.)

Solution: The second equation can be rearranged to give $ax = x^2 + y^2$, so $ax = 1$, so $x = a^{-1}$. The first equation then gives $y = \sqrt{1 - x^2} = \sqrt{1 - a^{-2}}$. (We are given that $a > 1$, so $a^{-2} < 1$, so $1-a^{-2}$ is positive and we can meaningfully take its square root.) Thus, the two solutions are $(x, y) = (a^{-1}, \sqrt{1 - a^{-2}})$ and $(x, y) = (a^{-1}, -\sqrt{1 - a^{-2}})$. In either case we have

$$\begin{aligned} (a - x)^2 + y^2 &= (a - a^{-1})^2 + 1 - a^{-2} \\ &= a^2 - 2 + a^{-2} + 1 - a^{-2} = a^2 - 1 \\ \sqrt{(a - x)^2 + y^2} &= \sqrt{a^2 - 1}. \end{aligned}$$

Exercise 2.9. Consider the equations

$$(1) \quad a + 2b + 3c = 123 \qquad (2) \quad 2a + 3b + c = 231 \qquad (3) \quad 3a + b + 2c = 312.$$

Can you see a solution just by looking at the equations? Solve the equations by a more systematic method, and so verify that the visible solution is the only solution.

Solution: It is easy to see that if we put $a = 100$ and $b = 10$ and $c = 1$, then the equations are satisfied. For the more systematic approach, form the equations 2.(1) − (2) and 3.(1) − (3):

$$b + 5c = 15 \tag{4}$$

$$5b + 7c = 57 \tag{5}$$

Now take 5.(4) − (5):

$$18c = 18 \tag{6}$$

This gives $c = 1$, which we substitute into (4) to get $b = 10$, which we substitute into (1) to get $a = 100$.

Exercise 2.10. Given a constant a, consider the following equations for x and y:

$$(A) \qquad x + ay = 1 \qquad\qquad (B) \qquad ax + y = a^2$$

(i) Solve the equations. Try to write your solution in a way that makes sense for $a = 1$.
(ii) What happens when $a = -1$? (You should go back to the original equations to answer this, rather than starting from your solution for the general case.)
(iii) What happens when $a = 1$? (First go back to the original equations, then compare with what you get by putting $a = 1$ in your solution to (i).)

Solution:

(i) Form the equations $(A) - a(B)$ and $(B) - a(A)$:

$$(C) \qquad (1 - a^2)x = 1 - a^3 = (1 - a)(1 + a + a^2)$$
$$(D) \qquad (1 - a^2)y = a^2 - a = -a(1 - a).$$

Now divide by $1 - a^2$, noting that $1 - a^2 = (1 - a)(1 + a)$: this gives

$$x = (1 - a^3)/(1 - a^2) = (1 + a + a^2)/(1 + a)$$
$$y = (a^2 - 1)/(1 - a^2) = -a/(1 + a).$$

Note that our final answers make sense when $a = 1$, but the intermediate calculation does not: we divided by $1 - a^2$, which is not valid when $a = 1$. When $a = -1$, even the final answer does not make sense.

(ii) Putting $a = -1$ in the original equations gives $x - y = 1$ and $-x + y = 1$, which is equivalent to $x - y = -1$. These equations are inconsistent, so there are no solutions.

(iii) If we put $a = 1$ then both the original equations become $x + y = 1$. This means that x can take any value, and y is given by $1 - x$. If we put $a = 1$ in our solution to (i) we get $x = 3/2$ and $y = 1 - x = -1/2$; this is only one of the many solutions.

Exercise 2.11. The hydrogen atoms in a molecule of methane lie at the points $(0, 0, 1)$, $(a, 0, -b)$, $(-c, d, -b)$ and $(-c, -d, -b)$, where $a, b, c, d > 0$ and the following equations are satisfied:

$$(1) \quad c^2 + d^2 = a^2 \qquad (2) \quad c^2 - d^2 = -ac \qquad (3) \quad a^2 + b^2 = 1 \qquad (4) \quad b^2 + b = ac.$$

Solve these equations to find a, b, c and d.

(**Hint:** find an equation involving only a and c, and put it in the form (something) = 0. Factorise it, and note that one of the factors is > 0, so the other one must be zero. This will let you write c in terms of a, and thus remove c from equation (4). You can then combine (3) and (4) to get an equation involving only b.)

Solution: After adding (1) and (2) and rearranging, we get $2c^2 + ac - a^2 = 0$, which factors as $(2c - a)(c + a) = 0$. As $a, c > 0$ we have $c + a > 0$ and so $2c - a = 0$, so $c = a/2$. We can now subtract (1) and (2) to get $2d^2 = a^2 + ac = 3a^2/2$, so $d^2 = 3a^2/4$ and $d = \sqrt{3}a/2$. We can now rewrite (4) as $b^2 + b = a^2/2$, but (3) gives $a^2 = 1 - b^2$, so $b^2 + b = (1 - b^2)/2$. This can be rearranged as $3b^2 + 2b - 1 = 0$, which factors as $(3b - 1)(b + 1) = 0$. As $b + 1 > 0$ we have $3b - 1 = 0$, so $b = 1/3$. Feeding this into (3) gives $a = \sqrt{1 - b^2} = 2\sqrt{2}/3$, so $c = a/2 = \sqrt{2}/3$, and $d = \sqrt{3}a/2 = \sqrt{2/3} = \sqrt{6}/3$. In conclusion, we have

$$a = \frac{2\sqrt{2}}{3} \qquad b = \frac{1}{3} \qquad c = \frac{\sqrt{2}}{3} \qquad d = \frac{\sqrt{6}}{3}$$

Exercise 2.12. Solve the following equations for a, b, c and λ (assuming that all of these are positive):

$$bc = \lambda a \qquad ac = \lambda b \qquad ab = \lambda c \qquad a^2 + b^2 + c^2 = 1$$

Hence find the quantity $V = 8abc$. (This is the largest possible volume for a cuboid contained in a sphere of radius one. The maximising cuboid has sides of length $2a$, $2b$ and $2c$.)

Solution: If we multiply the first two equations we get $abc^2 = \lambda^2 ab$. As $a, b > 0$ we can divide by ab to get $c^2 = \lambda^2$. As $\lambda, c > 0$ it follows that $c = \lambda$. Similarly, the first and third equations give $b = \lambda$, and the first and second equations give $a = \lambda$. The last equation then gives $3\lambda^2 = 1$, so $\lambda = 1/\sqrt{3}$. It follows that $V = 8(1/\sqrt{3})^3 = 8/(3\sqrt{3}) = 8\sqrt{3}/9$.

Week 3

Exercise 3.1. Calculate the following:

$$\log_{10}(10000) \qquad \log_{100}(10000) \qquad \log_{1000}(10000) \qquad \log_{1/10}(\sqrt{1000})$$

Solution:

- $10000 = 10^4$, so $\log_{10}(10000) = 4$.
- $10000 = 100^2$, so $\log_{100}(10000) = 2$.
- $10000 = 10^4 = ((1000)^{1/3})^4 = 1000^{4/3}$, so $\log_{1000}(10000) = 4/3$.
- $10000 = 10^4 = ((1/10)^{-1})^4 = (1/10)^{-4}$, so $\log_{1/10}(10000) = -4$.

These calculations can also be written slightly differently, using the identity $\log_a(x) = \ln(x)/\ln(a)$. For example, in the last case we have

$$\log_{1/10}(10000) = \frac{\ln(10000)}{\ln(1/10)} = \frac{4\ln(10)}{-\ln(10)} = -4.$$

Exercise 3.2. Some of the following statements are true, but others are either false or not meaningful. Decide which is which, and explain why. For each statement that is false, give an explicit counterexample. If there is a straightforward way to correct the statement, then do so.

(a) $\ln(e^x + e^y) = x + y$
(b) If $\ln(x) = a$, then $\ln(-x) = -a$
(c) $\log_a(b)\log_b(a) = 1$
(d) $\exp(\ln(x) - \ln(y)) = x - y$
(e) $\ln(\ln(\exp(\exp(x)))) = x$
(f) $\exp(\sqrt{x})^2 = \exp(x)$

Solution:

(a) This is false. For example, when $x = y = 0$ the left hand side is $\ln(e^0 + e^0) = \ln(2)$ but the right hand side is 0. A correct statement along the same lines is that $\ln(e^x e^y) = x + y$.
(b) This is not really meaningful, because $\ln(x)$ is only defined when $x > 0$, so $\ln(x)$ and $\ln(-x)$ are never both defined, so we cannot compare them. (If we allow complex numbers then $\ln(x)$ is defined for $x < 0$ but the formula is still false, despite many new subtleties that we shall not discuss.)
(c) This is true, because $\log_a(b) = \ln(b)/\ln(a)$ and $\log_b(a) = \ln(a)/\ln(b)$.
(d) This is false. For example, when $x = y = 1$ the left hand side is $\exp(0-0) = \exp(0) = 1$, whereas the right hand side is $1-1 = 0$. A true statement along the same lines is that $\exp(\ln(x)-\ln(y)) = x/y$.
(e) This is true. For any number w, we have $\ln(\exp(w)) = w$. Taking $w = \exp(x)$, we see that $\ln(\exp(\exp(x))) = \exp(x)$. Applying ln to both sides of this equation gives $\ln(\ln(\exp(\exp(x)))) = \ln(\exp(x)) = x$.
(f) This is false. For example, when $x = 1$ the left hand side is $\exp(1)^2 = e^2$, whereas the right hand side is just e. A correct statement along the same lines is that $\exp(x/2)^2 = \exp(x)$.

Exercise 3.3. The following statements are false, but they can be corrected by changing some of the signs. Do so.

(a) $\sinh(x)^2 + \cosh(x)^2 = 1$
(b) $\cosh(2x) = \cosh(x)^2 - \sinh(x)^2$
(c) $\sinh(x + y) = -\sinh(x)\cosh(y) - \cosh(x)\sinh(y)$

Solution: The correct versions are as follows:

$$\begin{aligned}
-\sinh(x)^2 + \cosh(x)^2 &= 1 \\
\cosh(2x) &= \cosh(x)^2 + \sinh(x)^2 \\
\sinh(x+y) &= \sinh(x)\cosh(y) + \cosh(x)\sinh(y)
\end{aligned}$$

Indeed, these are equivalent to the following equations, which can easily be checked by expanding everything out:

$$\begin{aligned}
-(e^x - e^{-x})^2/4 + (e^x + e^{-x})^2/4 &= 1 \\
(e^{2x} + e^{-2x})/2 &= (e^x - e^{-x})^2/4 + (e^x + e^{-x})^2/4 \\
(e^{x+y} - e^{-x-y})/2 &= (e^x - e^{-x})(e^y + e^{-y})/4 + (e^x + e^{-x})(e^y - e^{-y})/4
\end{aligned}$$

Exercise 3.4. Simplify the following expressions (using the identities $\ln(xy) = \ln(x) + \ln(y)$, $\exp(x)^n = \exp(nx)$ and so on).

(a) $\ln(xy^2z^3)$
(b) $\ln(e^{(x^2)}e^{2xy}e^{(y^2)})$
(c) $\dfrac{\ln(a^n)\ln(a^m)}{\ln(a^p)\ln(a^q)}$
(d) $\dfrac{\exp(a + b\ln(t))}{\exp(a - b\ln(t))}$
(e) $\dfrac{\ln(a^n)\ln(b^m)}{\ln(a^m)\ln(b^n)}$

Solution:

(a)

$$\begin{aligned}
\ln(xy^2z^3) &= \ln(x) + \ln(y^2) + \ln(z^3) \\
&= \ln(x) + 2\ln(y) + 3\ln(z).
\end{aligned}$$

At the first stage we used the rule $\ln(uv) = \ln(u) + \ln(v)$, and at the second stage we used the rule $\ln(u^n) = n\ln(u)$.

(b)

$$\begin{aligned}
\ln(e^{(x^2)}e^{2xy}e^{(y^2)}) &= \ln(e^{(x^2+2xy+y^2)}) \\
&= x^2 + 2xy + y^2 = (x+y)^2.
\end{aligned}$$

At the first stage we used the rule $e^u e^v = e^{u+v}$, and at the second stage we used the rule $\ln(e^u) = u$.

(c)

$$\begin{aligned}
\frac{\ln(a^n)\ln(a^m)}{\ln(a^p)\ln(a^q)} &= \frac{n\ln(a).m\ln(a)}{p\ln(a).q\ln(a)} \\
&= \frac{nm\ln(a)^2}{pq\ln(a)^2} = \frac{nm}{pq}.
\end{aligned}$$

Here we just used the rule $\ln(u^n) = n\ln(u)$.

(d)

$$\begin{aligned}
\frac{\exp(a + b\ln(t))}{\exp(a - b\ln(t))} &= \exp((a + b\ln(t)) - (a - b\ln(t))) \\
&= \exp(2b\ln(t)) = \exp(\ln(t))^{2b} = t^{2b}.
\end{aligned}$$

Here we used the rule $\exp(u)/\exp(v) = \exp(u - v)$, then the rule $\exp(pq) = e^{pq} = \exp(p)^q$, then the rule $\exp(\ln(u)) = u$.

(e)

$$\begin{aligned}
\frac{\ln(a^n)\ln(b^m)}{\ln(a^m)\ln(b^n)} &= \frac{n\ln(a).m\ln(b)}{m\ln(a).n\ln(b)} \\
&= \frac{nm\ln(a)\ln(b)}{nm\ln(a)\ln(b)} = 1.
\end{aligned}$$

Exercise 3.5. Check the identities $\cosh(2x) = 2\cosh(x)^2 - 1$ and $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$.

Solution: First, we put $u = e^x$, so $u^2 = e^{2x}$. We also have $\sinh(x) = (u - u^{-1})/2$ and $\cosh(x) = (u + u^{-1})/2$, so

$$\begin{aligned}
2\cosh(x)^2 - 1 &= 2\left(\frac{u+u^{-1}}{2}\right)^2 - 1 = 2\frac{u^2+2+u^{-2}}{4} - 1 \\
&= \frac{u^2+2+u^{-2}-2}{2} = \frac{u^2+u^{-2}}{2} \\
&= \frac{e^{2x}+e^{-2x}}{2} = \cosh(2x).
\end{aligned}$$

Next, we put $v = e^y$, so $uv = e^{x+y}$. We also have $\sinh(y) = (v - v^{-1})/2$ and $\cosh(y) = (v + v^{-1})/2$, so

$$\begin{aligned}
\sinh(x)\cosh(y) + \cosh(x)\sinh(y) &= \frac{u-u^{-1}}{2}\frac{v+v^{-1}}{2} + \frac{u+u^{-1}}{2}\frac{v-v^{-1}}{2} \\
&= \frac{1}{4}\left(uv + uv^{-1} - u^{-1}v - u^{-1}v^{-1} + uv - uv^{-1} + u^{-1}v - u^{-1}v^{-1}\right) \\
&= \frac{1}{4}(2uv - 2u^{-1}v^{-1}) = \frac{e^{x+y} - e^{-x-y}}{2} \\
&= \sinh(x+y).
\end{aligned}$$

Exercise 3.6. Simplify the expressions $4\cosh(x)^3 - 3\cosh(x)$ and $\sinh(x)^2\cosh(x)$.

Solution: Put $u = e^x$. Then

$$\begin{aligned}
4\cosh(x)^3 - 3\cosh(x) &= 4\frac{u^3+3u+3u^{-1}+u^{-3}}{8} - 3\frac{u+u^{-1}}{2} \\
&= \frac{u^3+u^{-3}}{2} = \cosh(3x) \\
\sinh(x)^2\cosh(x) &= \frac{(u-u^{-1})^2}{4}\frac{u+u^{-1}}{2} \\
&= \frac{1}{8}(u^2-2+u^{-2})(u+u^{-1}) = \frac{1}{8}(u^3 - 2u + u^{-1} + u - 2u^{-1} + u^{-3}) \\
&= \frac{u^3+u^{-3}}{8} - \frac{u+u^{-1}}{8} = \cosh(3x)/4 - \cosh(x)/4.
\end{aligned}$$

Exercise 3.7. Using the fact that $\sin(x) = (e^{ix} - e^{-ix})/(2i)$ (where $i^2 = -1$), show that

$$\begin{aligned}
4\sin(x)^3 &= 3\sin(x) - \sin(3x) \\
4\sin(4x)\sin(2x)\sin(x) &= -\sin(x) + \sin(3x) + \sin(5x) - \sin(7x).
\end{aligned}$$

Solution: Put $u = e^{ix}$, so $\sin(x) = (u - u^{-1})/(2i)$ and $\sin(3x) = (u^3 - u^{-3})/(2i)$. Note that $i^3 = i^2.i = -i$. Then

$$\begin{aligned}
4\sin(x)^3 &= 4(u-u^{-1})^3/(8i^3) \\
&= (u^3 - 3u^2u^{-1} + 3uu^{-2} - u^{-3})/(-2i) \\
&= (-u^3 + 3u - 3u^{-1} + u^{-3})/(2i) \\
&= 3\frac{u-u^{-1}}{2i} - \frac{u^3-u^{-3}}{2i} \\
&= 3\sin(x) - \sin(3x).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
4\sin(4x)\sin(2x)\sin(x) &= 4\frac{u^4-u^{-4}}{2i}\,\frac{u^2-u^{-2}}{2i}\,\frac{u-u^{-1}}{2i} \\
&= \frac{4}{8i^3}(u^4-u^{-4})(u^2-u^{-2})(u-u^{-1}) \\
&= \frac{-1}{2i}(u^4-u^{-4})(u^3-u-u^{-1}+u^{-3}) \\
&= \frac{-1}{2i}(u^7-u^5-u^3+u-u^{-1}+u^{-3}+u^{-5}-u^{-7}) \\
&= -\sin(7x)+\sin(5x)+\sin(3x)-\sin(x).
\end{aligned}$$

Exercise 3.8. In this exercise, we show that $\operatorname{arcsinh}(y)=\ln(y+\sqrt{y^2+1})$.

(a) Suppose that $y=\sinh(x)$. Using the relation $\cosh(x)^2-\sinh(x)^2=1$ and the definitions of sinh and cosh, simplify $y+\sqrt{y^2+1}$. Deduce that $x=\ln(y+\sqrt{y^2+1})$.
(b) Show in a similar way that if $z=\tanh(x)$, then $x=\ln((1+z)/(1-z))/2$.
(c) Now start instead with the formula $x=\ln(\sqrt{y^2+1}+y)$. Simplify $(\sqrt{y^2+1}+y)(\sqrt{y^2+1}-y)$, and rearrange to express $-x$ as ln(something else).
(d) Deduce that $\sinh(x)=y$.

Solution:

(a) Using $\cosh(x)^2-\sinh(x)^2=1$ we get $1+y^2=\cosh(x)^2$, so $\sqrt{1+y^2}=\cosh(x)=(e^x+e^{-x})/2$. By adding this to $y=\sinh(x)=(e^x+e^{-x})/2$, we get $y+\sqrt{1+y^2}=e^x$, and so $x=\ln(y+\sqrt{1+y^2})$.

Here we have implicitly used some things about signs. Firstly, $1+y^2$ is always positive, so the expression $\sqrt{1+y^2}$ is meaningful. As always, $\sqrt{1+y^2}$ refers to the positive square root. Moreover, e^x and e^{-x} are always positive, so $\cosh(x)$ is positive. Thus, when we take square roots in the equation $1+y^2=\cosh(x)^2$, we must have $\sqrt{1+y^2}=+\cosh(x)$; there is no possibility of a minus sign creeping in.

(b) Now suppose that $z=\tanh(x)$. Then

$$\begin{aligned}
z=\tanh(x) &= \frac{e^x-e^{-x}}{e^x+e^{-x}} \\
1+z=1+\frac{e^x-e^{-x}}{e^x+e^{-x}} &= \frac{2e^x}{e^x+e^{-x}} \\
1-z=1-\frac{e^x-e^{-x}}{e^x+e^{-x}} &= \frac{2e^{-x}}{e^x+e^{-x}} \\
\frac{1+z}{1-z} &= \frac{2e^x}{e^x+e^{-x}}\Big/\frac{2e^{-x}}{e^x+e^{-x}} \\
&= (2e^x)/(2e^{-x})=e^{2x} \\
\frac{1}{2}\log\left(\frac{1+z}{1-z}\right) &= \frac{1}{2}\log(e^{2x})=\frac{1}{2}.2x=x,
\end{aligned}$$

as required.

(c) By the difference of squares formula,

$$(\sqrt{y^2+1}+y)(\sqrt{y^2+1}-y)=y^2+1-y^2=1.$$

Taking logs and rearranging, we deduce that $-x=-\ln(\sqrt{y^2+1}+y)=\ln(\sqrt{y^2+1}-y)$.

(d) We now have

$$\begin{aligned}
\sinh(x) &= \frac{1}{2}(e^x-e^{-x}) \\
&= \frac{1}{2}\Big(\exp(\ln(\sqrt{y^2+1}+y))-\exp(\ln(\sqrt{y^2+1}-y))\Big) \\
&= \frac{1}{2}\Big(\ln(\sqrt{y^2+1}+y)-\ln(\sqrt{y^2+1}-y)\Big)=y.
\end{aligned}$$

Exercise 3.9. Write $s=\sin(\theta)$ and $c=\cos(\theta)$, for brevity, so that $s^2+c^2=1$. Consider the expression

$$x=(2sc)^2+(c^2-s^2)^2.$$

Expand this out, factorise and simplify, and deduce that $x=1$. What is the simple explanation for this?

Solution: We have

$$\begin{aligned} x &= 4s^2c^2 + (c^4 - 2s^2c^2 + s^4) \\ &= c^4 + 2s^2c^2 + s^4 \\ &= (c^2 + s^2)^2 = 1^2 = 1. \end{aligned}$$

The real reason is that $2sc = \sin(2\theta)$ and $c^2 - s^2 = \cos(2\theta)$, so $x = \sin(2\theta)^2 + \cos(2\theta)^2 = 1$.

Exercise 3.10. Write $f_n(x) = (1 + 2^{-n}x)^{2^n}$. Simplify $f_{n+1}(2x)f_n(x)^{-2}$.

Solution: First, we have

$$f_{n+1}(2x) = (1 + 2^{-(n+1)}.2x)^{2^{n+1}},$$

but $2^{-(n+1)}.2 = 2^{-n-1}.2^1 = 2^{-n}$, so

$$f_{n+1}(2x) = (1 + 2^{-n}x)^{2^{n+1}}.$$

On the other hand, we have

$$\begin{aligned} f_n(x)^2 &= \left((1 + 2^{-n}x)^{2^n}\right)^2 \\ &= (1 + 2^{-n}x)^{2^n \times 2} \\ &= (1 + 2^{-n}x)^{2^{n+1}}, \end{aligned}$$

which is just the same. Thus $f_{n+1}(2x)f_n(x)^{-2} = f_{n+1}(2x)/f_n(x)^2 = 1$.

It is an important fact that $f_n(x)$ tends to e^x as n tends to infinity. The proof is not too hard, and may be explained in MAS170 (Practical Calculus). If we let n tend to infinity in our relation $f_{n+1}(2x) = f_n(x)^2$, we get $e^{2x} = (e^x)^2$. Of course we knew that anyway, but it is nice to see how it all fits together.

Exercise 3.11. Consider the number $a = (\sqrt{6} + \sqrt{2})/4$.

(a) Simplify $2a^2 - 1$.
(b) What is $\cos(\pi/6)$?
(c) Give a formula relating $\cos(2\theta)$ to $\cos(\theta)$. What does this tell us about $\cos(\pi/6)$ and $\cos(\pi/12)$?
(d) Deduce that $\cos(\pi/12) = a$.

Solution:

(a) First, we have

$$\begin{aligned} a^2 &= (\sqrt{6}^2 + 2\sqrt{6}\sqrt{2} + \sqrt{2}^2)/16 \\ &= (6 + 2\sqrt{12} + 2)/16 = (8 + 4\sqrt{3})/16 \\ &= \frac{1}{2} + \frac{\sqrt{3}}{4}. \end{aligned}$$

This gives $2a^2 - 1 = \sqrt{3}/2$.

(b) It is a standard fact that $\cos(\pi/6) = \sqrt{3}/2$, or in other words $\cos(\pi/6) = 2a^2 - 1$.
(c) Using the formula $\cos(2\theta) = 2\cos(\theta)^2 - 1$, we see that $\cos(\pi/6) = 2\cos(\pi/12)^2 - 1$.
(d) The comparison between (b) and (c) certainly suggests that $\cos(\pi/12) = a$, but we should not be too hasty. We can rearrange (b) to show that $a^2 = (1+\cos(\pi/6))/2$, so $a = \pm\sqrt{(1 + \cos(\pi/6))/2}$. Similarly, we can rearrange (c) to show that $\cos(\pi/12)^2 = \pm\sqrt{(1 + \cos(\pi/6))/2}$, and it follows that $\cos(\pi/12) = \pm a$. On the other hand, we certainly have $a > 0$, and $\pi/12$ is a rather small angle (15 degrees) so it is clear that $\cos(\pi/12) > 0$. This means we cannot have $a = -\cos(\pi/12)$, so we must have $a = \cos(\pi/12)$ after all.

Week 4

Exercise 4.1. Find $\sin(-7\pi/3)$ and $\tan(9999\pi/4)$. (You should give exact answers, not decimal approximations.)

Solution: Note that $\sin(x)$ repeats with period 2π, so

$$\sin(-7\pi/3) = \sin(-7\pi/3 + 2\pi) = \sin(-\pi/3) = -\sin(\pi/3) = -\sqrt{3}/2.$$

Similarly, $\tan(x)$ repeats with period π, so

$$\tan(9999\pi/4) = \tan(9999\pi/4 - 2500\pi) = \tan(-\pi/4) = -1.$$

Exercise 4.2. If $t = \tan(\theta/2)$, show that $1 + t^2 = \cos(\theta/2)^{-2}$, and thus that

$$\sin(\theta) = \frac{2t}{1+t^2} \qquad \cos(\theta) = \frac{1-t^2}{1+t^2} \qquad \tan(\theta) = \frac{2t}{1-t^2}.$$

(It follows that any trigonometric function of θ can be rewritten as a rational function of t.)

Solution: We have $t = \sin(\theta/2)/\cos(\theta/2)$, so

$$1 + t^2 = 1 + \frac{\sin(\theta/2)^2}{\cos(\theta/2)^2} = \frac{\cos(\theta/2)^2 + \sin(\theta/2)^2}{\cos(\theta/2)^2} = \frac{1}{\cos(\theta/2)^2}.$$

Alternatively, we can reduce everything to complex exponentials. Put $u = e^{i\theta/2}$, so

$$t = \frac{\sin(\theta/2)}{\cos(\theta/2)} = \frac{(u - u^{-1})/(2i)}{(u + u^{-1})/2} = \frac{u - u^{-1}}{i(u + u^{-1})}.$$

This gives

$$\begin{aligned}
t^2 &= \frac{(u - u^{-1})^2}{i^2(u + u^{-1})^2} = -\frac{u^2 - 2 + u^{-2}}{u^2 + 2 + u^{-2}} \\
1 + t^2 &= 1 - \frac{u^2 - 2 + u^{-2}}{u^2 + 2 + u^{-2}} \\
&= \frac{(u^2 + 2 + u^{-2}) - (u^2 - 2 + u^{-2})}{u^2 + 2 + u^{-2}} \\
&= \frac{4}{u^2 + 2 + u^{-2}} = \left(\frac{2}{u + u^{-1}}\right)^2 = \left(\frac{u + u^{-1}}{2}\right)^{-2} \\
&= \cos(\theta/2)^{-2}.
\end{aligned}$$

It now follows that $1/(1 + t^2) = \cos(\theta/2)^2$, so

$$\frac{2t}{1+t^2} = 2\frac{\sin(\theta/2)}{\cos(\theta/2)}\cos(\theta/2)^2 = 2\sin(\theta/2)\cos(\theta/2) = \sin(\theta)$$

(using the double-angle formula $\sin(2\phi) = 2\sin(\phi)\cos(\phi)$). Next, we have

$$1 - t^2 = 1 - \frac{\sin(\theta/2)^2}{\cos(\theta/2)^2} = \frac{\cos(\theta/2)^2 - \sin(\theta/2)^2}{\cos(\theta/2)^2} = \frac{\cos(\theta)}{\cos(\theta/2)^2}.$$

(using the double-angle formula $\cos(2\phi) = \cos(\phi)^2 - \sin(\phi)^2$). We can divide this by the equation $1+t^2 = \cos(\theta/2)^{-2}$ to get $(1 - t^2)/(1 + t^2) = \cos(\theta)$. Finally, we can divide the equation $\sin(\theta) = (2t)/(1 + t^2)$ by the equation $\cos(\theta) = (1 - t^2)/(1 + t^2)$ to get $\tan(\theta) = (2t)/(1 - t^2)$.

Exercise 4.3. Simplify the expression $\sin(x)(\cos(x) + \cos(3x) + \cos(5x) + \cos(7x))$. Can you find a similar equation with five terms instead of four terms? What about six terms or seven terms or n terms?

Solution: Put $u = e^{ix}$, so $\sin(x) = (u - u^{-1})/(2i)$ and $\cos(kx) = (u^k + u^{-k})/2$. It follows that

$$\begin{aligned}
&\sin(x)(\cos(x) + \cos(3x) + \cos(5x) + \cos(7x)) \\
=&\tfrac{1}{4i}(u - u^{-1})(u + u^{-1} + u^3 + u^{-3} + u^5 + u^{-5} + u^7 + u^{-7}) \\
=&\tfrac{1}{4i}(u^2 + 1 + u^4 + u^{-2} + u^6 + u^{-4} + u^8 + u^{-6} \\
&\quad - 1 - u^{-2} - u^2 - u^{-4} - u^4 - u^{-6} - u^6 - u^{-8}) \\
=&(u^8 - u^{-8})/(4i) = \sin(8x)/2.
\end{aligned}$$

In the same way, one can check that

$$\begin{aligned}
\sin(x)(\cos(x) + \cos(3x) + \cos(5x) + \cos(7x) + \cos(9x)) &= \sin(10x)/2 \\
\sin(x)(\cos(x) + \cos(3x) + \cos(5x) + \cos(7x) + \cos(9x) + \cos(11x)) &= \sin(12x)/2 \\
\sin(x)(\cos(x) + \cos(3x) + \cos(5x) + \cos(7x) + \cos(9x) + \cos(11x) + \cos(13x)) &= \sin(14x)/2 \\
\sin(x)(\cos(x) + \cos(3x) + \cdots + \cos((2n-1)x)) &= \sin(2nx)/2.
\end{aligned}$$

Exercise 4.4. Show that $\tan(x+y) = (\tan(x)+\tan(y))/(1-\tan(x)\tan(y))$. (It is easiest to do this by writing $\tan(z) = \sin(z)/\cos(z)$ and using the addition formulae for sin and cos, but you have to remember the signs correctly to make that work. You can also prove the formula be rewriting everything in terms of complex exponentials.)

Solution: For the first method, we have

$$\begin{aligned}
\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)} &= \frac{\frac{\sin(x)}{\cos(x)}+\frac{\sin(y)}{\cos(y)}}{1-\frac{\sin(x)\sin(y)}{\cos(x)\cos(y)}} \\
&= \frac{\sin(x)\cos(y)+\cos(x)\sin(y)}{\cos(x)\cos(y)-\sin(x)\sin(y)} \\
&= \frac{\sin(x+y)}{\cos(x+y)} = \tan(x+y).
\end{aligned}$$

For the second method, write $u = e^{ix}$ and $v = e^{iy}$. Then

$$\begin{aligned}
\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)} &= \frac{\frac{u-u^{-1}}{i(u+u^{-1})}+\frac{v-v^{-1}}{i(v+v^{-1})}}{1-\frac{u-u^{-1}}{i(u+u^{-1})}\frac{v-v^{-1}}{i(v+v^{-1})}} \\
&= \frac{1}{i}\frac{\frac{u-u^{-1}}{u+u^{-1}}+\frac{v-v^{-1}}{v+v^{-1}}}{1+\frac{u-u^{-1}}{(u+u^{-1})}\frac{v-v^{-1}}{(v+v^{-1})}} \qquad \text{(using } i^2=-1\text{)} \\
&= \frac{1}{i}\frac{(u-u^{-1})(v+v^{-1})+(u+u^{-1})(v-v^{-1})}{(u+u^{-1})(v+v^{-1})+(u-u^{-1})(v-v^{-1})} \\
&= \frac{1}{i}\frac{uv+uv^{-1}-u^{-1}v-u^{-1}v^{-1}+uv-uv^{-1}+u^{-1}v-u^{-1}v^{-1}}{uv+uv^{-1}+u^{-1}v+u^{-1}v^{-1}+uv-uv^{-1}-u^{-1}v+u^{-1}v^{-1}} \\
&= \frac{1}{i}\frac{2uv-2u^{-1}v^{-1}}{2uv+2u^{-1}v^{-1}} = \frac{1}{i}\frac{uv-u^{-1}v^{-1}}{uv+u^{-1}v^{-1}} = \tan(x+y).
\end{aligned}$$

Exercise 4.5. Show that $\cos(2\pi/5) = (\sqrt{5}-1)/4$, as follows. Put $u = e^{2\pi i/5}$ and $c = \cos(2\pi/5) = (u+u^{-1})/2$. What is u^5? Expand out $(u-1)u^2(4c^2+2c-1)$, and deduce that $4c^2+2c-1=0$. This gives two possibilities for c; explain why one of them can be rejected.

Solution: First, we have

$$u^5 = e^{2\pi i} = \cos(2\pi)+\sin(2\pi)i = 1+0i = 1.$$

Next, we have

$$\begin{aligned}
(u-1)u^2(4c^2+2c-1) &= (u-1)u^2((u+u^{-1})^2+u+u^{-1}-1) = (u-1)u^2(u^2+2+u^{-2}+u+u^{-1}-1) \\
&= (u-1)(u^4+u^3+u^2+u+1) = u^5-1 = 0.
\end{aligned}$$

Clearly $u \neq 0$ and $u \neq 1$, so $u-1 \neq 0$, so we can divide the relation $(u-1)u^2(4c^2+2c-1) = 0$ by $u^2(u-1)$ to get $4c^2+2c-1=0$. We can solve this quadratic to give $c = (-2\pm\sqrt{20})/8 = (-1\pm\sqrt{5})/4$. Moreover, we have $2\pi/5 < \pi/2$ so $\cos(2\pi/5) > 0$ (look at the graph, or think about triangles). We therefore cannot have $c = (-1-\sqrt{5})/4$, so we must have $c = (-1+\sqrt{5})/4$.

Exercise 4.6. Show that for $0 \leq \theta \leq \pi$ we have $\arcsin(\cos(\theta)) = \pi/2-\theta$.

Solution: Using the rule $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$ and the values $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$, we get

$$\sin(\pi/2-\theta) = 1.\cos(\theta)-0.\sin(\theta) = \cos(\theta).$$

(This can also be read off from the graphs, or deduced from the geometry of triangles.) The idea now is just to take arcsin of both sides to get

$$\pi/2-\theta = \arcsin(\sin(\pi/2-\theta)) = \arcsin(\cos(\theta)).$$

Strictly speaking, this is not quite right, because $\arcsin(\sin(\phi))$ is not always equal to ϕ. For example, $\sin(100\pi) = 0$, so $\arcsin(\sin(100\pi)) = \arcsin(0) = 0 \neq 100\pi$. The precise definition of $\arcsin(x)$ is that it is the unique angle α between $-\pi/2$ and $+\pi/2$ (inclusive) such that $\sin(\alpha) = x$. This means that if $-\pi/2 \leq \phi \leq \pi/2$, then $\arcsin(\sin(\phi)) = \phi$. In particular, if $0 \leq \theta \leq \pi$ then $-\pi/2 \leq \pi/2-\theta \leq \pi/2$, so $\arcsin(\sin(\pi/2-\theta)) = \pi/2-\theta$ and the argument above is correct in this case.

Exercise 4.7. (a) Simplify $(1+\tan(\theta)^2)^{-1/2}$ and $\tan(\theta)(1+\tan(\theta)^2)^{-1/2}$

(b) Deduce that $\cos(\arctan(t)) = (1+t^2)^{-1/2}$ and $\sin(\arctan(t)) = t(1+t^2)^{-1/2}$.

(c) Now suppose that $A, B > 0$ and $C = \sqrt{A^2+B^2}$ and $\phi = \arctan(B/A)$. Simplify $C\cos(\phi)$ and $C\sin(\phi)$.

(d) Deduce that $A\sin(\theta) + B\cos(\theta) = C\sin(\theta+\phi)$.

Solution:

(a) We have

$$1+\tan(\theta)^2 = 1+\frac{\sin(\theta)^2}{\cos(\theta)^2} = \frac{\cos(\theta)^2+\sin(\theta)^2}{\cos(\theta)^2} = \frac{1}{\cos(\theta)^2},$$

so $(1+\tan(\theta)^2)^{-1/2} = \cos(\theta)$. We can multiply this by the equation $\tan(\theta) = \sin(\theta)/\cos(\theta)$ to get $\tan(\theta)(1+\tan(\theta)^2)^{-1/2} = \sin(\theta)$.

(b) Now take $\theta = \arctan(t)$, so $\tan(\theta) = t$. The above equations then read

$$\begin{aligned}(1+t^2)^{-1/2} &= (1+\tan(\theta)^2)^{-1/2} = \cos(\theta) = \cos(\arctan(t))\\ t(1+t^2)^{-1/2} &= \tan(\theta)(1+\tan(\theta)^2)^{-1/2} = \sin(\theta) = \sin(\arctan(t)).\end{aligned}$$

(c) We have

$$\begin{aligned}C\cos(\phi) &= \sqrt{A^2+B^2}\cos(\arctan(B/A)) = \sqrt{A^2+B^2}(1+B^2/A^2)^{-1/2}\\ &= \sqrt{\frac{A^2+B^2}{1+B^2/A^2}} = \sqrt{A^2} = A\\ C\sin(\phi) &= C\cos(\phi)\tan(\phi) = A\frac{B}{A} = B.\end{aligned}$$

(Alternatively, we can simplify $C\sin(\phi)$ by the same method as we used for $C\cos(\phi)$.)

(d) This now gives

$$C\sin(\theta+\phi) = C\sin(\theta)\cos(\phi) + C\cos(\theta)\sin(\phi) = A\sin(\theta) + B\cos(\theta).$$

Exercise 4.8. Let C be the curve given by equations $x = \cos(t)/(1+e\cos(t))$ and $y = \sin(t)/(1+e\cos(t))$. Show that $((1-e^2)x+e)^2 + (1-e^2)y^2 = 1$. (This actually implies that C is an ellipse; this calculation comes up in the theory of planetary motion. The number e is called the eccentricity, which measures how far the ellipse is from being a circle.)

Solution: Write $s = \sin(t)$ and $c = \cos(t)$, for brevity, and recall that $s^2+c^2 = 1$. We have

$$\begin{aligned}((1-e^2)x+e)^2 + (1-e^2)y^2 &= \left(\frac{(1-e^2)c}{1+ec}+e\right)^2 + (1-e^2)\frac{s^2}{(1+ec)^2}\\ &= \left(\frac{c-e^2c+e(1+ec)}{1+ec}\right)^2 + \frac{(1-e^2)(1-c^2)}{(1+ec)^2}\\ &= \frac{(c+e)^2+1-e^2-c^2+e^2c^2}{(1+ec)^2}\\ &= \frac{c^2+2ec+e^2+1-e^2-c^2+e^2c^2}{(1+ec)^2} = \frac{1+2ec+e^2c^2}{(1+ec)^2} = 1.\end{aligned}$$

Exercise 4.9. Show that $\sin(x)^4 + \cos(x)^4 = 1 - \frac{1}{2}\sin(2x)^2$ for all x.

Solution: We first do this by the standard method: put $u = e^{ix}$ and just expand everything out. On one side we have

$$\begin{aligned}\sin(x)^4+\cos(x)^4 &= \left(\frac{u-u^{-1}}{2i}\right)^4 + \left(\frac{u+u^{-1}}{2}\right)^4\\ &= \frac{u^4-4u^2+6-4u^{-2}+u^{-4}}{16} + \frac{u^4+4u^2+6+4u^{-2}+u^{-4}}{16}\\ &= \frac{2u^4+12+2u^{-4}}{16}\\ &= \frac{u^4+6+u^{-4}}{8}.\end{aligned}$$

On the other side we have

$$\begin{aligned}1 - \frac{1}{2}\sin(2x)^2 &= 1 - \frac{1}{2}\left(\frac{u^2 - u^{-2}}{2i}\right)^2 \\ &= 1 - \frac{u^4 - 2 + u^{-4}}{-8} \\ &= \frac{8 + u^4 - 2 + u^{-4}}{8} \\ &= \frac{u^4 + 6 + u^{-4}}{8},\end{aligned}$$

which is the same.

For this particular identity, there is another approach that is quicker but less systematic. If we square the identity $\sin(x)^2 + \cos(x)^2 = 1$ we get $\sin(x)^4 + 2\sin(x)^2\cos(x)^2 + \cos(x)^4 = 1$. On the other hand, we know that $\sin(2x) = 2\sin(x)\cos(x)$, so $\sin(x)^2\cos(x)^2 = \sin(2x)^2/4$. If we substitute this into the previous identity we get $\sin(x)^4 + 2\sin(2x)^2/4 + \cos(x)^4 = 1$, which we can rearrange to get $\sin(x)^4 + \cos(x)^4 = 1 - \sin(2x)^2/2$ as claimed.

Exercise 4.10. Let n be an integer bigger than one, and let R and r be numbers with $R > r > 0$. Expand and simplify the following equation, and thus find all the solutions:

$$(R\cos(x) + r\cos(nx))^2 + (R\sin(x) + r\sin(nx))^2 = (R + r)^2.$$

Solution: The left hand side can be expanded as follows:

$$\begin{aligned}&(R\cos(x) + r\cos(nx))^2 + (R\sin(x) + r\sin(nx))^2 \\ =&R^2\cos(x)^2 + 2Rr\cos(x)\cos(nx) + r^2\cos(nx)^2 + R^2\sin(x)^2 + 2Rr\sin(x)\sin(nx) + r^2\sin(nx)^2 \\ =&R^2(\cos(x)^2 + \sin(x)^2) + r^2(\cos(nx)^2 + \sin(nx)^2) + 2Rr(\cos(x)\cos(nx) + \sin(x)\sin(nx)).\end{aligned}$$

We then use the standard identities

$$\begin{aligned}\cos(x)^2 + \sin(x)^2 &= 1 \\ \cos(nx)^2 + \sin(nx)^2 &= 1 \\ \cos(x)\cos(nx) + \sin(x)\sin(nx) &= \cos(nx - x) = \cos((n-1)x)\end{aligned}$$

to simplify the left hand side to $R^2 + r^2 + 2Rr\cos((n-1)x)$. On the other hand, the right hand side is just $(R + r)^2 = R^2 + r^2 + 2Rr$. The equation is thus equivalent to $2Rr\cos((n-1)x) = 2Rr$, or $\cos((n-1)x) = 1$. This means that $(n-1)x = 2k\pi$ for some integer k, or equivalently $x = 2k\pi/(n-1)$.

Exercise 4.11. Do not use Maple or a calculator for this problem, but instead analyse the situation logically. Sketch the graphs of the functions $\cos(x)^{100}$ and $\sin(x)^{100}$, thinking carefully about the maximum and minimum values and the overall shape. What exactly is the maximum value of $\cos(x)^{100}\sin(x)^{100}$? (You could use calculus for this, but it is not actually necessary.) How does this relate to your sketches?

Solution: As 100 is even we see that $\cos(x)^{100}$ is always positive or zero. As $\cos(x)$ lies between -1 and 1, we see that $\cos(x)^{100}$ is between 0 and 1. When x is a multiple of π we see that $\cos(x) = \pm 1$ and so $\cos(x)^{100} = 1$. However, we only need to move a little way away from multiples of π for $|\cos(x)|$ to be noticeably less than one, and then $\cos(x)^{100}$ will be very small. This means that the graph consists of thin spikes of height 1 near where x is a multiple of π, and is very close to zero apart from these spikes. Similarly, the graph of $\sin(x)^{100}$ consists of thin spikes of height one near points of the form $x = (k + \frac{1}{2})\pi$. This means that wherever we look, either $\cos(x)^{100}$ is very small, or $\sin(x)^{100}$ is very small, or both. This means that the product $\cos(x)^{100}\sin(x)^{100}$ is always very small. To be more precise, we have $\cos(x)\sin(x) = \sin(2x)/2$, so

$$\cos(x)^{100}\sin(x)^{100} = (\sin(2x)/2)^{100} = \sin(2x)^{100}/2^{100}.$$

The extreme values of $\sin(2x)$ are ± 1, so the maximum value of $\sin(2x)^{100}$ is one, so the maximum value of $\cos(x)^{100}\sin(x)^{100}$ is $1/2^{100}$, which is very small, as expected.

Exercise 5.1. Find $\frac{d}{dx}\left(\frac{x^2}{\ln(x)}\right)$.

Solution: The quotient rule gives

$$\frac{d}{dx}\left(\frac{x^2}{\ln(x)}\right) = \frac{2x.\ln(x) - x^2.\ln'(x)}{\ln(x)^2} = \frac{2x\ln(x) - x^2.x^{-1}}{\ln(x)^2}$$
$$= \frac{2x}{\ln(x)} - \frac{x}{\ln(x)^2}.$$

(Note here that $\ln(x)^2$ is not the same as $\ln(x^2) = 2\ln(x)$. For example, $\ln(e) = 1$, so $\ln(e)^2 = 1^2 = 1$, but $\ln(e^2) = 2$.)

Exercise 5.2. Let a, b, c and d be constants and put $y = (ax+b)/(cx+d)$. Calculate dy/dx, simplifying your answer as much as possible.

Solution: Put $u = ax + b$ and $v = cx + d$, so $u' = a$ and $v' = c$. We then have

$$\begin{aligned} y' &= (u/v)' = (u'v - uv')/v^2 \\ &= ((acx + ad) - (acx + bc))/(cx + d)^2 \\ &= \frac{ad - bc}{(cx + d)^2}. \end{aligned}$$

Exercise 5.3. Let a, b, c and d be constants. Find $\frac{d}{dx}\left(\frac{ax + bx^{-1}}{cx + dx^{-1}}\right)$.

Solution: Put $u = ax + b/x$ and $v = cx + d/x$ and $y = u/v$; we must find y'. Note that

$$\begin{aligned} u' &= a - b/x^2 \\ v' &= c - d/x^2 \\ u'v - uv' &= (a - b/x^2)(cx + d/x) - (ax + b/x)(c - d/x^2) \\ &= acx + ad/x - bc/x - bd/x^3 - acx + ad/x - bc/x + bd/x^3 \\ &= 2(ad - bc)/x, \\ y' &= \frac{u'v - uv'}{v^2} = \frac{2(ad - bc)}{x(cx + d/x)^2}. \end{aligned}$$

Exercise 5.4. Simplify $(x^2 + y^2)^{-1}\frac{dx}{dt}$, where $x = \cos(t)/(1 + a\cos(t))$ and $y = \sin(t)/(1 + a\cos(t))$.

Solution: First, recall that $\cos'(t) = -\sin(t)$, so $\frac{d}{dt}(1+a\cos(t)) = -a\sin(t)$. Using this and the quotient rule, we get

$$\begin{aligned} \frac{dx}{dt} &= \frac{(-\sin(t))(1 + a\cos(t)) - \cos(t)(-a\sin(t))}{(1 + a\cos(t))^2} = \frac{-\sin(t) - a\sin(t)\cos(t) + a\sin(t)\cos(t)}{(1 + a\cos(t))^2} \\ &= -\frac{\sin(t)}{(1 + a\cos(t))^2}. \end{aligned}$$

Next, we have

$$x^2 + y^2 = \frac{\cos(t)^2 + \sin(t)^2}{(1 + a\cos(t))^2} = \frac{1}{(1 + a\cos(t))^2},$$

so $(x^2 + y^2)^{-1} = (1 + a\cos(t))^2$. Multiplying these two results together, we get

$$(x^2 + y^2)^{-1}\frac{dx}{dt} = -\sin(t).$$

Exercise 5.5. Calculate the derivatives of the functions $(x^2 - 2x + 2)e^x$, $(x^3 - 3x^2 + 6x - 6)e^x$ and $(x^4 - 4x^3 + 12x^2 - 24x + 24)e^x$. What is the next thing in this sequence?

Solution: We use the product rule, and the fact that $\frac{d}{dx}e^x = e^x$. This gives

$$\begin{aligned} \tfrac{d}{dx}((x^2 - 2x + 2)e^x) &= (2x - 2)e^x + (x^2 - 2x + 2)e^x = x^2e^x \\ \tfrac{d}{dx}((x^3 - 3x^2 + 6x - 6)e^x) &= (3x^2 - 6x + 6)e^x + (x^3 - 3x^2 + 6x - 6)e^x = x^3e^x \\ \tfrac{d}{dx}((x^4 - 4x^3 + 12x^2 - 24x + 24)e^x) &= (4x^3 - 12x^2 + 24x - 24)e^x + (x^4 - 4x^3 + 12x^2 - 24x + 24)e^x = x^4e^x. \end{aligned}$$

The next thing in the sequence is

$$\tfrac{d}{dx}((x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120)e^x) = x^5 e^x.$$

The coefficients here are 1, -5, $(-5)\times(-4) = 20$, $(-5)\times(-4)\times(-3) = -60$, $(-5)\times(-4)\times(-3)\times(-2) = 120$ and $(-5)\times(-4)\times(-3)\times(-2)\times(-1) = -120$.

Exercise 5.6. Suppose that $y = (pq)/(rs)$, where p, q, r and s all depend on x. Simplify y'/y. (You should write your answer as a sum of four terms, not as a single fraction.) Hence find y'/y when $y = x(x+3)/((x+1)(x+2))$, simplifying your answer as much as possible.

Solution: The most efficient method is as follows:

$$\begin{aligned} y'/y &= \ln(y)' \\ &= (\ln(p) + \ln(q) - \ln(r) - \ln(s))' \\ &= p'/p + q'/q - r'/r - s'/s. \end{aligned}$$

Alternatively, we have

$$\begin{aligned} y' &= \frac{(pq)'rs - pq(rs)'}{(rs)^2} \\ &= \frac{p'qrs + pq'rs - pqr's - pqrs'}{r^2s^2}, \end{aligned}$$

so

$$\begin{aligned} \frac{y'}{y} &= \frac{p'qrs + pq'rs - pqr's - pqrs'}{r^2s^2}\,\frac{rs}{pq} \\ &= \frac{p'qrs + pq'rs - pqr's - pqrs'}{pqrs} \\ &= p'/p + q'/q - r'/r - s'/s. \end{aligned}$$

Now take $p = x$, $q = x+3$, $r = x+1$ and $s = x+2$, so $y = x(x+3)/((x+1)(x+2))$. We then have $p' = q' = r' = s' = 1$, so

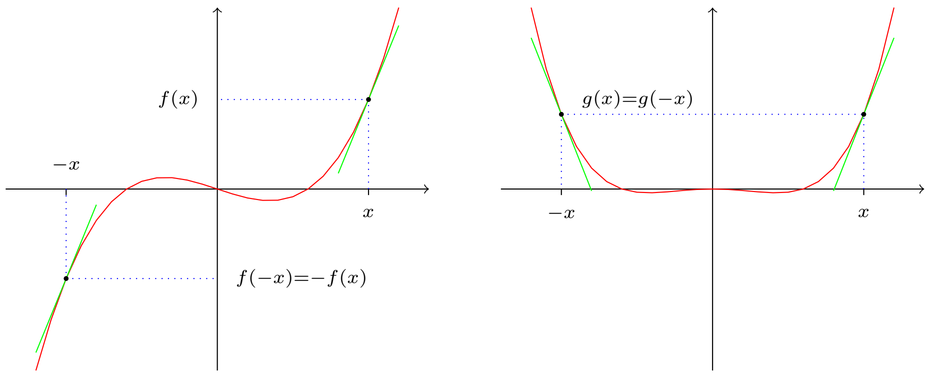
$$\begin{aligned} y &= \frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3} \\ &= \frac{1}{x(x+1)} - \frac{1}{(x+2)(x+3)} = \frac{(x+2)(x+3) - x(x+1)}{x(x+1)(x+2)(x+3)} \\ &= \frac{4x+6}{x(x+1)(x+2)(x+3)}. \end{aligned}$$

Exercise 5.7. Recall that a function $f(x)$ is *even* if $f(-x) = f(x)$ for all x, and *odd* if $f(-x) = -f(x)$. Note that most functions are neither even nor odd.

(a) By drawing pictures, convince yourself that if $f(x)$ is odd then $f'(x)$ is even.
(b) By drawing pictures, convince yourself that if $f'(x)$ is odd then $f(x)$ is even.
(c) Find a function $f(x)$ that is neither even nor odd. Try to make your answer as simple as possible.
(d) Find a function $f(x)$ such that $f'(x)$ is even but $f(x)$ is not odd. Try to make your answer as simple as possible.

Solution:

(a),(b) The left hand picture shows a typical odd function $f(x)$. The tangent lines at x and $-x$ have the same slope, so $f'(-x) = f'(x)$, so $f'(x)$ is even.



The right hand picture shows a typical even function $g(x)$. The tangent lines at x and $-x$ have opposite slopes, so $g'(-x) = -g'(x)$, so $g'(x)$ is an odd function.

(c),(d) Take $f(x) = 1 + x$. This is neither even nor odd (because $f(1) = 2$ and $f(-1) = 0$ which is neither $+2$ nor -2). However, $f'(x)$ is the constant function 1, which is even. Thus this $f(x)$ answers both (c) and (d).

Exercise 5.8. Find a function $f(x)$ such that $f'(-1) = f'(0) = f'(1) = 0$ and $f(0) < f(1)$. Try to make your answer as simple as possible.

Solution: Probably the simplest answer is $f(x) = -\cos(\pi x)$, so $f'(x) = \pi\sin(\pi x)$, so $f'(n) = 0$ for all integers n. We also have $f(0) = -1$ and $f(1) = 1$, so $f(0) < f(1)$ as required. Another approach is to take $f(x) = x^2/2 - x^4/4$, so $f'(x) = x - x^3 = x(1 + x)(1 - x)$, so again $f'(0) = f'(1) = f'(-1) = 0$. In this case we have $f(0) = 0$ and $f(1) = 1/4$, so $f(0) < f(1)$.

Exercise 5.9. Find a function $f(x)$ such that

(a) $f(x)$ is defined for all x (without any division by zero, square roots of negative numbers, or other horrors.)
(b) $f'(x) < 0$ for all x (so $f(x)$ is continuously decreasing)
(c) $f(x) > 0$ for all x (so although $f(x)$ is decreasing, it never reaches zero.)

Solution: The obvious example is $f(x) = e^{-x}$ (so $f'(x) = -e^{-x} < 0$). Other examples include $1 - \frac{x}{\sqrt{1+x^2}}$ and $1 - \tanh(x)$.

Exercise 5.10. Consider the function $f(x) = ax^2 + bx + c$, where $a, b, c > 0$. Suppose that there is a point x_0 where $f(x_0) = f'(x_0) = 0$. Give formulae for x_0 and c in terms of a and b. (You might like to draw some pictures first.)

Solution: We have $f'(x) = 2ax + b$ for all x. As $f'(x_0) = 0$, we must have $2ax_0 + b = 0$, so $x_0 = -b/(2a)$. We also have $f(x_0) = 0$, so $x_0 = -b/(2a) \pm \sqrt{b^2 - 4ac}/(2a)$. These two expressions for x_0 can only be compatible if $\sqrt{b^2 - 4ac}/(2a) = 0$, so $b^2 - 4ac = 0$, so $c = b^2/(4a)$.

Exercise 5.11. Let c be a positive constant, and put $g(v) = (1 - v^2/c^2)^{-1/2}$. Calculate $g'(v)$.

Solution:

$$\begin{aligned} g'(v) &= -\tfrac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2} \frac{d}{dv}\left(1 - \frac{v^2}{c^2}\right) && \text{(power rule)} \\ &= -\tfrac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2} (-2v/c^2) \\ &= vc^{-2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2}. \end{aligned}$$

Exercise 5.12. Calculate $\frac{d}{dx}\left(e^{-a^2x^2}\sin(\omega x)\right)$, where a and ω are constants.

Solution: Put $t = -a^2x^2$ and $u = \exp(t) = e^{-ax^2}$ and $v = \sin(\omega x)$ and $y = uv = e^{-a^2x^2}\sin(\omega x)$; we must find dy/dx. We have

$$\begin{aligned}
\tfrac{dt}{dx} &= -2a^2x \\
\tfrac{du}{dt} &= \tfrac{d}{dt}e^t = e^t = e^{-a^2x^2} \\
\tfrac{du}{dx} &= \tfrac{dt}{dx}\tfrac{du}{dt} = -2a^2xe^{-a^2x^2} \\
\tfrac{dv}{dx} &= \omega\cos(\omega x) \\
\tfrac{dy}{dx} &= u\tfrac{dv}{dx} + \tfrac{du}{dx}v \\
&= e^{-a^2x^2}\omega\cos(\omega x) - 2a^2xe^{-a^2x^2}\sin(\omega x) \\
&= e^{-a^2x^2}(\omega\cos(\omega x) - 2a^2x\sin(\omega x)).
\end{aligned}$$

(It is a common mistake for things like $2a$ to creep in. If a were a variable then we would have $\frac{d}{da}(a^2) = 2a$. However, a is in fact a constant, and we are using $\frac{d}{dx}$ rather than $\frac{d}{da}$, so the equation $\frac{d}{da}(a^2) = 2a$ is not relevant.)

Exercise 5.13. Let p and q be nonzero constants, and put $y = (x^p - x^q)^{1/pq}$. Simplify $x(x^p - x^q)\frac{dy}{dx}$.

Solution: Put $u = x^p - x^q$, so $y = u^{1/pq}$. Then

$$du/dx = px^{p-1} - qx^{q-1} = x^{-1}(px^p - qx^q)$$

and

$$\frac{dy}{du} = \frac{1}{pq}u^{1/pq-1} = \frac{1}{pq}(x^p - y^q)^{1/pq-1}.$$

We therefore have

$$\begin{aligned}
x(x^p - x^q)\frac{dy}{dx} &= xu\frac{dy}{du}\frac{du}{dx} \\
&= x(x^p - x^q)\frac{1}{pq}(x^p - x^q)^{1/pq-1}x^{-1}(px^p - qx^q) \\
&= (x^p - x^q)^{1/pq}(px^p - qx^q)/(pq) \\
&= (x^p - x^q)^{1/pq}(x^p/q - x^q/p).
\end{aligned}$$

Exercise 5.14. Let a, b and n be constants. Find $f'(x)$, where $f(x) = \left(\frac{x-a}{x-b}\right)^n$.

Solution: Put $u = (x-a)/(x-b)$ and $y = f(x) = u^n$. Then

$$\frac{du}{dx} = \frac{1.(x-b) - (x-a).1}{(x-b)^2} = \frac{a-b}{(x-b)^2},$$

so

$$f'(x) = \frac{dy}{dx} = nu^{n-1}\frac{du}{dx} = n(a-b)\left(\frac{x-a}{x-b}\right)^{n-1}(x-b)^{-2} = n(a-b)(x-a)^{n-1}(x-b)^{-n-1}.$$

Exercise 5.15. Put $y = a_1x + a_2x^2 + a_3x^3 + a_4x^4$.

(a) Find $x\frac{dy}{dx}$.
(b) Find $x\frac{d}{dx}\left(x\frac{dy}{dx}\right)$.
(c) Find $x\frac{d}{dx}\left(x\frac{d}{dx}\left(x\frac{dy}{dx}\right)\right)$.
(d) What is the general rule?

Solution: It is convenient to introduce the notation $Lz = x\frac{dz}{dx}$. The question then asks us to find Ly, LLy and $LLLy$. We have

$$\begin{aligned}
Ly &= x\frac{dy}{dx} = x(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3) \\
&= a_1x + 2a_2x^2 + 3a_3x^3 + 4a_4x^4 \\
LLy &= x(a_1 + 2\times 2a_2x + 3\times 3a_3x^{+}4\times 4a_4x^3) \\
&= a_1x + 4a_2x^2 + 9a_3x^3 + 16a_4x^4 \\
&= a_1x + 2^2a_2x^2 + 3^2a_3x^3 + 4^2a_4x^4 \\
LLLy &= x(a_1 + 2\times 4a_2x + 3\times 9a_3x^2 + 4\times 16a_4x^3) \\
&= a_1x + 8a_2x^2 + 27a_3x^3 + 64a_4x^4 \\
&= a_1x + 2^3a_2x^2 + 3^3a_3x^3 + 4^3a_4x^4.
\end{aligned}$$

The general rule is clearly that if $y = \sum_k a_kx^k$, then $L^ny = \sum_k k^na_kx^k$.

Exercise 5.16. Given that y is a function of x, simplify the following expressions:

(a) $e^{-x}\frac{d}{dx}(e^xy)$
(b) $e^{-x}\frac{d^2}{dx^2}(e^xy)$
(c) $e^{-x}\frac{d^3}{dx^3}(e^xy)$

Can you guess the general rule? Can you prove it?

Solution: We first calculate the successive derivatives:

$$\begin{aligned}
\frac{d}{dx}(e^xy) &= e^xy + e^xy' \\
\frac{d^2}{dx^2}(e^xy) &= (e^xy + e^xy') + (e^xy' + e^xy'') \\
&= e^xy + 2e^xy' + e^xy'' \\
\frac{d^3}{dx^3}(e^xy) &= (e^xy + e^xy') + 2(e^xy' + e^xy'') + (e^xy'' + e^xy''') \\
&= e^xy + 3e^xy' + 3e^xy'' + e^xy'''.
\end{aligned}$$

It follows that

$$\begin{aligned}
e^{-x}\frac{d}{dx}(e^xy) &= y + y' \\
e^{-x}\frac{d^2}{dx^2}(e^xy) &= y + 2y' + y'' \\
e^{-x}\frac{d^3}{dx^3}(e^xy) &= y + 3y' + 3y'' + y'''.
\end{aligned}$$

You should recognize the numbers here as binomial coefficients; they are the same as in the formulae

$$\begin{aligned}
(1+t)^1 &= 1 + t \\
(1+t)^2 &= 1 + 2t + t^2 \\
(1+t)^3 &= 1 + 3t + 3t^2 + t^3.
\end{aligned}$$

The pattern seems to be that

$$\begin{aligned}
e^{-x}\frac{d^n}{dx^n}(e^xy) &= y + \binom{n}{1}\frac{dy}{dx} + \binom{n}{2}\frac{d^2y}{dx^2} + \cdots + \frac{d^ny}{dx^n} \\
&= \sum_{k=0}^{n}\binom{n}{k}\frac{d^ky}{dx^k}.
\end{aligned}$$

One way to prove this is by induction; we omit the details.

Here is another, more abstract approach; you can ignore it if you are not interested. Consider the operators $Dz = z'$ and $Lz = e^{-x}\frac{d}{dx}(e^xz)$. We have seen that $Lz = z + z' = (1+D)z$, so $L = 1 + D$, so

$$L^n = (1+D)^n = \sum_{k=0}^{n}\binom{n}{k}D^k,$$

so

$$L^n y = \sum_k \binom{n}{k} D^k y = \sum_k \binom{n}{k} \frac{d^k y}{dx^k}.$$

On the other hand, we have

$$\begin{aligned}
L^2 y = L(Ly) &= e^{-x} \frac{d}{dx}(e^x Ly) \\
&= e^{-x} \frac{d}{dx} \left(e^x \, e^{-x} \frac{d}{dx} (e^x y) \right) \\
&= e^{-x} \frac{d^2}{dx^2}(e^x y) \\
L^3 y &= e^{-x} \frac{d}{dx} \left(e^x \, e^{-x} \frac{d}{dx} \left(e^x \, e^{-x} \frac{d}{dx} (e^x y) \right) \right) \\
&= e^{-x} \frac{d^3}{dx^3}(e^x y)
\end{aligned}$$

and so on. It follows that

$$e^{-x} \frac{d^n}{dx^n}(e^x y) = L^n y = \sum_k \binom{n}{k} \frac{d^k y}{dx^k},$$

as claimed.

Exercise 5.17. Put $f(x) = x/\sqrt{1+x^2}$. Simplify $\sqrt{1+x^2} f'(x)$, and hence find a constant c such that $f'(x) = (1+x^2)^c$.

Solution:

$$\begin{aligned}
\frac{d}{dx} \left(\frac{x}{\sqrt{1+x^2}} \right) &= \frac{1.(1+x^2)^{1/2} - x.\frac{1}{2}(1+x^2)^{-1/2}.2x}{1+x^2} \\
&= \frac{(1+x^2)^{1/2} - x^2(1+x^2)^{-1/2}}{1+x^2},
\end{aligned}$$

so

$$(1+x^2)^{1/2} \frac{d}{dx} \left(\frac{x}{\sqrt{1+x^2}} \right) = \frac{(1+x^2) - x^2}{1+x^2} = \frac{1}{1+x^2} = (1+x^2)^{-1}$$

so

$$\frac{d}{dx} \left(\frac{x}{\sqrt{1+x^2}} \right) = (1+x^2)^{-3/2}.$$

In other words, we have $c = -3/2$.

Exercise 5.18. Calculate the derivatives of the following functions:

$$\begin{aligned}
f(x) &= x + x^2/2 + x^3/3 + x^4/4 + x^5/5 \\
g(x) &= \frac{1+x^2+x^4}{x+x^3} \\
h(x) &= \tan(x)^7 \\
k(x) &= \arcsin(x) \\
m(x) &= \ln(\cos(x)).
\end{aligned}$$

(In the case of $k(x)$ you should give an argument starting with your knowledge of $\sin'(x)$, rather than just quoting the answer from tables or your memory.)

Solution:

$$f'(x) = 1 + x + x^2 + x^3 + x^4 = (x^5 - 1)/(x - 1)$$

$$g'(x) = \frac{(x + x^3)\frac{d}{dx}(1 + x^2 + x^4) - (1 + x^2 + x^4)\frac{d}{dx}(x + x^3)}{(x + x^3)^2} = \frac{(x + x^3)(2x + 4x^3) - (1 + x^2 + x^4)(1 + 3x^2)}{(x + x^3)^2}$$

$$= \frac{2x^2 + 4x^4 + 2x^4 + 4x^6 - 1 - 3x^2 - x^2 - 3x^4 - x^4 - 3x^6}{(x + x^3)^2} = \frac{x^6 + 2x^4 - 2x^2 - 1}{(x + x^3)^2}$$

$$h'(x) = 7\tan(x)^6 \tan'(x) = 7\tan(x)^6(1 + \tan(x)^2)$$

$$= 7\left(\frac{\sin(x)}{\cos(x)}\right)^6 \frac{1}{\cos(x)^2} = 7\sin(x)^7/\cos(x)^9.$$

$$k'(x) = 1/\sin'(\arcsin(x)) = 1/\cos(\arcsin(x)) = (1 - x^2)^{-1/2}$$

$$m'(x) = \ln'(\cos(x))\cos'(x) = \frac{1}{\cos(x)}.(-\sin(x)) = -\frac{\sin(x)}{\cos(x)} = -\tan(x).$$

Week 6

Exercise 6.1. Differentiate the following functions, simplifying your answers as much as possible:

(a) $x+x^{10}+x^{100}$ **(b)** $(3x+2)/(4x+3)$ **(c)** $x\ln(x)-x$ **(d)** $e^{-x}\sin(10x)$ **(e)** $\sin(x^2)$

Solution:

(a) $1 + 10x^9 + 100x^{99}$

(b) Use the quotient rule, with $u = 3x + 2$ and $v = 4x + 3$, so $u' = 3$ and $v' = 4$. This gives $(u/v)' = (u'v - uv')/v^2 = (12x + 9 - 12x - 8)/(4x + 3)^2 = 1/(4x + 3)^2$.

(c) For the first term use the product rule with $u = x$ and $v = \ln(x)$, so $u' = 1$ and $v' = 1/x$. This gives $\frac{d}{dx}(x\ln(x)) = u'v + uv' = 1.\ln(x) + x.(1/x) = \ln(x) + 1$, so $\frac{d}{dx}(x\ln(x) - x) = \ln(x) + 1 - 1 = \ln(x)$.

(d) Use the product rule, with $u = \exp(-x)$ and $v = \sin(10x)$. Using the affine case of the chain rule we get $u' = \exp'(-x).\frac{d}{dx}(-x) = -\exp(-x)$ and similarly $v' = \sin'(10x).\frac{d}{dx}(10x) = 10\cos(10x)$. The product rule therefore gives

$$\frac{d}{dx}(e^{-x}\sin(10x)) = -e^{-x}\sin(10x) + e^{-x}.(10\cos(10x)) = e^{-x}(10\cos(10x) - \sin(10x)).$$

(e) The chain rule gives $\frac{d}{dx}\sin(x^2) = \sin'(x^2)\frac{d}{dx}(x^2) = \cos(x^2).2x = 2x\cos(x^2)$.

Exercise 6.2. Find $\dfrac{d}{dx}\log(x + 2x^2 + 3x^3 + 4x^4)$.

Solution: Put $u = x + 2x^2 + 3x^3 + 4x^4$ and $y = \log(u)$, so

$$y' = \frac{u'}{u} = \frac{1 + 4x + 9x^2 + 16x^3}{x + 2x^2 + 3x^3 + 4x^4}.$$

Exercise 6.3. Find $\dfrac{d}{dx}\log(\cos(x))$.

Solution: By the logarithmic rule, we have

$$\frac{d}{dx}\log(\cos(x)) = \frac{\cos'(x)}{\cos(x)} = -\frac{\sin(x)}{\cos(x)} = -\tan(x).$$

Exercise 6.4. Let a, n and m be constants. Find $f'(x)$, where $f(x) = (x^n + a)^m$.

Solution: Put $u = x^n + a$ and $y = f(x) = u^m$. Then $du/dx = nx^{n-1}$ and $dy/du = mu^{m-1}$, so

$$f'(x) = \frac{dy}{dx} = mu^{m-1}\frac{du}{dx} = mn(x^n + a)^{m-1}x^{n-1}.$$

Exercise 6.5. Let a be a constant. Find $f'(x)$, where $f(x) = x^2\, e^{-1/(x+a)}$.

Solution: First put $u = -1/(x+a)$, so $du/dx = 1/(x+a)^2 = (x+a)^{-2}$. Then put $v = \exp(u) = e^{-1/(x+a)}$, so the chain rule gives

$$\frac{dv}{dx} = (x+a)^{-2}e^{-1/(x+a)}.$$

Finally, we apply the product rule:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^2\,v) = 2x.v + x^2\frac{dv}{dx} \\ &= (2x + x^2(x+a)^{-2})e^{-1/(x+a)} \\ &= (2x^2 + (4a+1)x + 2a^2)x(x+a)^{-2}e^{-1/(x+a)}. \end{aligned}$$

Exercise 6.6. Find $\dfrac{d}{dx}\cos\left(\left(\dfrac{x+1}{2}\right)^2\right)$.

Solution: By the chain rule, we have

$$\frac{d}{dx}\cos\left(\left(\frac{x+1}{2}\right)^2\right) = -\sin\left(\left(\frac{x+1}{2}\right)^2\right).\frac{d}{dx}\left(\frac{x+1}{2}\right)^2 = -\sin\left(\left(\frac{x+1}{2}\right)^2\right).\frac{x+1}{2}.$$

Exercise 6.7. Let α, ω, a and b be constants, and put

$$\begin{aligned} f(t) &= \sin((\omega + a\sin(\alpha t))t) \\ g(t) &= (1 + b\sin(\alpha t))\sin(\omega t) \end{aligned}$$

(These are FM and AM radio signals.) Find $f'(t)$ and $g'(t)$.

Solution: Put $p(t) = (\omega + a\sin(\alpha t))t$, so $f(t) = \sin(p(t))$ and

$$p'(t) = \omega + a\sin(\alpha t) + a\alpha t\cos(\alpha t),$$

so

$$\begin{aligned} f'(t) &= \sin'(p(t))p'(t) = \cos(p(t))p'(t) \\ &= \cos((\omega + a\sin(\alpha t))t)(\omega + a\sin(\alpha t) + a\alpha t\cos(\alpha t)). \end{aligned}$$

We also have

$$g'(t) = b\alpha\cos(\alpha t)\sin(\omega t) + (1 + b\sin(\alpha t))\omega\cos(\omega t).$$

Exercise 6.8. Let a, b and ω be constants. Find $f'(x)$, where $f(x) = e^{-(x-a)^2/b}\sin(\omega x)$.

Solution: First put $u = -(x-a)^2/b$, so $du/dx = -2(x-a)/b$. Then put $v = \exp(u) = e^{-(x-a)^2/b}$, so the chain rule gives

$$\frac{dv}{dx} = -2(x-a)b^{-1}e^{-(x-a)^2/b}.$$

Now put $w = \sin(\omega x)$, so $dw/dx = \omega\cos(\omega x)$. Finally, put $y = vw = e^{-(x-a)^2/b}\sin(\omega x)$ and apply the product rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dv}{dx}w + v\frac{dw}{dx} \\ &= -2(x-a)b^{-1}e^{-(x-a)^2/b}\sin(\omega x) + e^{-(x-a)^2/b}\omega\cos(\omega x) \\ &= e^{-(x-a)^2/b}(\omega\cos(\omega x) - 2(x-a)b^{-1}\sin(\omega x)). \end{aligned}$$

With practice you can leave out some of these steps, but it is always safest to write them all out carefully.

Exercise 6.9. If $y = \sqrt{2\pi}x^{x-1/2}e^{-x}$, show that $y'/y = \log(x) - 1/(2x)$.

Solution: Put $y = \sqrt{2\pi}x^{x-1/2}e^{-x}$, so

$$\log(y) = \log(\sqrt{2\pi}) + (x - 1/2)\log(x) - x,$$

so

$$\begin{aligned} \frac{y'}{y} &= \log(y)' \\ &= 0 + 1.\log(x) + (x - 1/2).\log'(x) - 1 \\ &= \log(x) + (x - 1/2)/x - 1 = \log(x) + 1 - 1/(2x) - 1 \\ &= \log(x) - 1/(2x). \end{aligned}$$

Exercise 6.10. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be functions such that $f(1) = -1$ and $g(-1) = 1$. Show that $(f \circ g)'(-1) = (g \circ f)'(1)$. (**Hint:** You will need to use the chain rule, which can be written as $(p \circ q)'(x) = p'(q(x))\, q'(x)$.)

Solution:

$$(f \circ g)'(-1) = f'(g(-1))g'(-1) = f'(1)g'(-1)$$
$$(g \circ f)'(1) = g'(f(1))f'(1) = g'(-1)f'(1) = f'(1)g'(-1).$$

Exercise 6.11. Put $y = \exp(\exp(\exp(x)))$.

(a) Find dy/dx.
(b) Express x in terms of y.
(c) Working from (b), find dx/dy.
(d) Check that $\frac{dy}{dx}\frac{dx}{dy} = 1$.

Solution:

(a) The chain rule gives

$$\begin{aligned} dy/dx &= \exp'(\exp(\exp(x)))\ \exp'(\exp(x))\ \exp'(x) \\ &= \exp(\exp(\exp(x)))\ \exp(\exp(x))\ \exp(x). \end{aligned}$$

(b) We have $\log(y) = \exp(\exp(x))$, so $\log(\log(y)) = \exp(x)$, so $x = \log(\log(\log(y)))$.
(c) Using the chain rule again, we have

$$\begin{aligned} dx/dy &= \log'(\log(\log(y)))\ \log'(\log(y))\ \log'(y) \\ &= \log(\log(y))^{-1}\ \log(y)^{-1}\ y^{-1}. \end{aligned}$$

(d) Using the equations in (b), we can rewrite (a) as

$$dy/dx = y\ \log(y)\ \log(\log(y)).$$

When combined with (c), this clearly tells us that $(dy/dx).(dx/dy) = 1$.

Exercise 6.12. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function with $f(0) = 0$. Put $g(x) = f(f(f(f(x))))$. Use the chain rule to express $g'(x)$ in terms of derivatives of f, and thus show that $g'(0) \geq 0$.

Solution: By the chain rule, we have

$$g'(x) = f'(f(f(f(x))))f'(f(f(x)))f'(f(x))f'(x).$$

Now put $x = 0$, so $f(x) = f(0) = 0$, so $f(f(x)) = 0$, so $f(f(f(x))) = 0$. This gives

$$\begin{aligned} g'(0) &= f'(f(f(f(0))))f'(f(f(0)))f'(f(0))f'(0) \\ &= f'(0)f'(0)f'(0)f'(0) = f'(0)^4. \end{aligned}$$

The fourth power of any real number is nonnegative, so $g'(0) \geq 0$.

Exercise 6.13. Let A, a, b and c be positive constants. What is the maximum value of the function $f(x) = A\exp(-ax^2 - bx - c)$?

Solution: We have

$$\begin{aligned} f'(x) &= A\exp(-ax^2 - bx - c)\frac{d}{dx}(-ax^2 - bx - c) \\ &= -A\exp(-ax^2 - bx - c)(2ax + b). \end{aligned}$$

The exponential term can never be zero, so $f'(x)$ is only zero when $2ax+b = 0$, which means $x = -b/(2a)$. In that case we have

$$\begin{aligned} ax^2 + bx + c &= a\frac{b^2}{4a^2} + b\frac{-b}{2a} + c = -\frac{b^2}{4a} + c = -\frac{b^2 - 4ac}{4a} \\ f(x) &= A\exp(-(ax^2 + bx + c)) = A\exp\left(\frac{b^2 - 4ac}{4a}\right). \end{aligned}$$

This is therefore the maximum value of $f(x)$.

Exercise 6.14. What is the maximum value of the function $f(x) = \sin(x) + \cos(x)$?

Solution: We have $f'(x) = \cos(x) - \sin(x)$, so the maximum must occur at a point where $f'(x) = 0$ and so $\sin(x) = \cos(x)$. This means that $2\sin(x)^2 = \sin(x)^2 + \cos(x)^2 = 1$, so $\cos(x) = \sin(x) = \pm 1/\sqrt{2}$, so

$$f(x) = \cos(x) + \sin(x) = \pm 2/\sqrt{2} = \pm\sqrt{2}.$$

At the maximum, the sign must clearly be plus. Thus, the maximum value of $f(x)$ is $\sqrt{2}$. (In fact, one can show that $f(x) = \sqrt{2}\sin(x + \pi/4)$, which makes the maximum value quite obvious.)

Exercise 6.15. Find dy/dx, where x and y are related as follows:

(a) $x^2 + xy + y^2 = 1$
(b) $e^x + e^y = x - y$
(c) $x\ln(x) + y\ln(y) = -1$

Solution:

(a) If we apply $\frac{d}{dx}$ to the equation $x^2+xy+y^2 = 1$ we get $2x+y+x\frac{dy}{dx}+2y\frac{dy}{dx} = 0$. This rearranges to give $(x + 2y)dy/dx = -(2x + y)$, so $dy/dx = -(2x + y)/(x + 2y)$.
(b) If we apply $\frac{d}{dx}$ to the equation $e^x + e^y = x - y$, we get $e^x + e^y\frac{dy}{dx} = 1 - \frac{dy}{dx}$, which rearranges to give $(1 + e^y)dy/dx = 1 - e^x$, so $dy/dx = (1 - e^x)/(1 + e^y)$.
(c) If we apply $\frac{d}{dx}$ to the equation $x\ln(x) + y\ln(y) = -1$, we get $\ln(x) + x\frac{1}{x} + \frac{dy}{dx}\ln(y) + y\frac{1}{y}\frac{dy}{dx} = 0$, which rearranges to give $(\ln(y) + 1)\frac{dy}{dx} = -(\ln(x) + 1)$, so $dy/dx = -(\ln(x) + 1)/(\ln(y) + 1)$.

Exercise 6.16. Suppose that $x, y > 0$ and $x^{3/2} + y^{3/2} = 1$. Find dy/dx by implicit differentiation, and show that

$$\left(1 - \left(\frac{dy}{dx}\right)^3\right)^2 = \frac{1}{y^3}.$$

Solution: If we differentiate the equation $x^{3/2} + y^{3/2} = 1$ with respect to x, we get

$$\frac{3}{2}x^{1/2} + \frac{3}{2}y^{1/2}\frac{dy}{dx} = 0,$$

which gives $dy/dx = -(x/y)^{1/2}$. This implies that

$$1 - \left(\frac{dy}{dx}\right)^3 = 1 + \left(\frac{x}{y}\right)^{3/2} = \frac{y^{3/2} + x^{3/2}}{y^{3/2}}.$$

Using the original equation $x^{3/2} + y^{3/2} = 1$, this simplifies to $y^{-3/2}$. Squaring both sides, we get

$$\left(1 - \left(\frac{dy}{dx}\right)^3\right)^2 = \frac{1}{y^3}$$

as claimed.

Exercise 6.17. In each of the following cases, find dy/dx. In two of the cases you should use $dy/dx = (dy/dt)/(dx/dt)$; in the other case, there is a much simpler way.

(a) $x = a\cos(nt)$, $y = b\sin(mt)$.
(b) $x = \tan(t)^{-1/2}$, $y = \tan(t)^{1/2}$.
(c) $x = e^t(\sin(t) + \cos(t))$, $y = e^t(\sin(t) - \cos(t))$.

Solution:

(a) Here $dy/dt = mb\cos(mt)$ and $dx/dt = -na\sin(nt)$ so $dy/dx = -(mb\cos(mt))/(na\sin(nt))$.
(b) Here $y = 1/x$, so $dy/dx = -1/x^2 = -\tan(t)$. Alternatively

$$\begin{aligned}
\frac{dy}{dt} &= \frac{1}{2}\tan(t)^{-1/2}\tan'(t) = \frac{1}{2}\tan(t)^{-1/2}(1 + \tan(t)^2) \\
\frac{dx}{dt} &= \frac{-1}{2}\tan(t)^{-3/2}\tan'(t) = \frac{-1}{2}\tan(t)^{-3/2}(1 + \tan(t)^2) \\
\frac{dy}{dx} &= \frac{\frac{1}{2}\tan(t)^{-1/2}(1 + \tan(t)^2)}{\frac{-1}{2}\tan(t)^{-3/2}(1 + \tan(t)^2)} \\
&= -\tan(t)^{(-1/2)-(-3/2)} = -\tan(t).
\end{aligned}$$

(c)

$$\begin{aligned}
dx/dt &= e^t(\sin(t)+\cos(t)) + e^t(\sin'(t)+\cos'(t)) = e^t(\sin(t)+\cos(t)) + e^t(\cos(t)-\sin(t)) \\
&= 2e^t\cos(t) \\
dy/dt &= e^t(\sin(t)-\cos(t)) + e^t(\sin'(t)-\cos'(t)) = e^t(\sin(t)-\cos(t)) + e^t(\cos(t)+\sin(t)) \\
&= 2e^t\sin(t) \\
\frac{dy}{dx} &= \frac{2e^t\sin(t)}{2e^t\cos(t)} = \frac{\sin(t)}{\cos(t)} = \tan(t).
\end{aligned}$$

Exercise 6.18. Suppose we have a thin rope wound around a pillar of radius one, and we hold the end taut and unwind it. After unwinding a length t of rope, it works out that the end of the rope is at position

$$(x,y) = (\cos(t) + t\sin(t), \sin(t) - t\cos(t)).$$

(a) Calculate dx/dt and dy/dt, and so find dy/dx in terms of t.
(b) Simplify x^2+y^2-1, and so express dy/dx in terms of x and y.

Solution: First, we have

$$\begin{aligned}
dx/dt &= -\sin(t) + (1.\sin(t) + t.\cos(t)) = t\cos(t) \\
dy/dt &= \cos(t) - (1.\cos(t) + t.(-\sin(t))) = t\sin(t) \\
\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{t\sin(t)}{t\cos(t)} = \tan(t).
\end{aligned}$$

We also have

$$\begin{aligned}
x^2+y^2-1 =&(\cos(t)+t\sin(t))^2 + (\sin(t)-t\cos(t))^2 - 1 \\
=&\cos(t)^2 + 2t\sin(t)\cos(t) + t^2\sin(t)^2 + \\
&\sin(t)^2 - 2t\sin(t)\cos(t) + t^2\cos(t)^2 - 1 \\
=&1 + t^2 - 1 = t^2.
\end{aligned}$$

It follows that $t = \sqrt{x^2+y^2-1}$, and so

$$dy/dx = \tan(t) = \tan(\sqrt{x^2+y^2-1}).$$

Exercise 6.19. Show that if $y=\tan(x)$ then $y'''/(2y') = 2y^2+y'$.

Solution: Recall that

$$y' = \tan'(x) = \sec(x)^2 = 1 + \tan(x)^2 = 1 + y^2.$$

We differentiate the relation $y' = 1+y^2$ twice to get

$$\begin{aligned}
y'' &= (1+y^2)' = 0 + 2yy' = 2(y+y^3) \\
y''' &= 2(y+y^3)' = 2y' + 6y^2y' = (2+6y^2)(1+y^2) \\
y'''/(2y') &= \frac{(2+6y^2)(1+y^2)}{2(1+y^2)} = 1+3y^2 \\
2y^2+y' &= 2y^2 + (1+y^2) = 1+3y^2.
\end{aligned}$$

This shows that $y'''/(2y') = 2y^2+y'$ as claimed.

Exercise 6.20. Put $y = 1/(1-x)$. Calculate y', y'' and y'''. Guess a general formula for $y^{(n)}$

Solution: Put $u = 1-x$, so $du/dx = -1$ and $y = u^{-1}$. This gives

$$\begin{aligned}
y' &= \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -(-u^{-2}) = u^{-2} = (1-x)^{-2} \\
y'' &= \frac{d}{dx}(u^{-2}) = -2u^{-3}\frac{du}{dx} = 2u^{-3} = 2(1-x)^{-3} \\
y''' &= \frac{d}{dx}(2u^{-3}) = 2\times(-3)\times u^{-4}\frac{du}{dx} = 6u^{-4} = 6(1-x)^{-4}.
\end{aligned}$$

In the same way, we get $y^{(4)} = 24(1-x)^{-5}$ and $y^{(5)} = 120(1-x)^{-6}$. Note that the 24 here arises as $4\times3\times2$, and the 120 arises as $5\times4\times3\times2$. By now the pattern should be clear: we have $y^{(n)} = n!(1-x)^{-n-1}$ for all n.

Exercise 6.21. Consider the function $y = Ae^x + Be^{2x} + Ce^{3x}$. Simplify the expressions

$$\tfrac{1}{2}y'' - \tfrac{5}{2}y' + 3y$$
$$-y'' + 4y' - 3y.$$

What should be the third part of the question?

Solution: We have

$$\begin{aligned} y &= Ae^x + Be^{2x} + Ce^{3x} \\ y' &= Ae^x + 2Be^{2x} + 3Ce^{3x} \\ y'' &= Ae^x + 4Be^{2x} + 9Ce^{3x}. \end{aligned}$$

This gives

$$\begin{aligned} \tfrac{1}{2}y'' - \tfrac{5}{2}y' + 3y =& \tfrac{1}{2}Ae^x + 2Be^{2x} + \tfrac{9}{2}Ce^{3x} - \\ & \tfrac{5}{2}Ae^x - 5Be^{2x} - \tfrac{15}{2}Ce^{3x} + \\ & 3Ae^x + 3Be^{2x} + 3Ce^{3x} \\ =& (\tfrac{1}{2} - \tfrac{5}{2} + 3)Ae^x + (2 - 5 + 3)Be^{2x} + (\tfrac{9}{2} - \tfrac{15}{2} + 3)Ce^{3x} \\ =& Ae^x \\ -y'' + 4y' - 3y =& (-A + 4A - 3A)e^x + (-4B + 8B - 3B)e^{2x} + (-9C + 12C - 3C)e^{3x} \\ =& Be^{2x}. \end{aligned}$$

The third part of the question should involve a combination of y, y' and y'' that simplifies to give the remaining term Ce^{3x}. The easiest way to find this equation is to start with the equation $y = Ae^x + Be^{2x} + Ce^{3x}$ and subtract off the equations $y''/2 - 5y'/2 + 3y = Ae^x$ and $-y'' + 4y' - 3y = Be^{2x}$ to get

$$\tfrac{1}{2}y'' - \tfrac{3}{2}y' + y = Ce^{3x}.$$

Week 7

Exercise 7.1. Find $\int_0^5 (x^2 + 15)^2\, dx$

Solution:

$$\int_0^5 (x^2 + 15)^2\, dx = \int_0^5 x^4 + 30x^2 + 225\, dx = \left[\tfrac{1}{5}x^5 + 10x^3 + 225x\right]_{x=0}^5 = \tfrac{1}{5}5^5 + 10 \times 5^3 + 225 \times 5 = 3000.$$

At the last step you can use a calculator or you can recognise that $225 = 9 \times 25 = 9 \times 5^2$ so $225 \times 5 = 9 \times 5^3$ to get

$$\tfrac{1}{5}5^5 + 10 \times 5^3 + 225 \times 5 = 5^3(5 + 10 + 9) = 5^3 \times 2^3 \times 3 = 10^3 \times 3 = 3000.$$

Note incidentally that $\int (x^2 + 5)^2\, dx$ is **not** equal to $(x^2 + 5)^3/3$ or to $(x^2 + 5)^2/(6x)$, as you can easily check by differentiating. Instead, you need to expand first, as done above.

Exercise 7.2. Let λ be a positive constant. Then there are constants m and b such that for all a we have

$$\log\left[\int_a^{a+1} e^{\lambda x}\, dx\right] = ma + b.$$

Find formulae for m and b in terms of λ.

Solution: We have

$$\int_a^{a+1} e^{\lambda x}\, dx = \left[e^{\lambda x}/\lambda\right]_{x=a}^{a+1} = (e^{\lambda a + \lambda} - e^{\lambda a})/\lambda = e^{\lambda a}(e^{\lambda} - 1)/\lambda.$$

We can now take logs to get

$$\log\left[\int_a^{a+1} e^{\lambda x}\, dx\right] = \log(e^{\lambda a}(e^{\lambda} - 1)/\lambda) = \lambda a + \log(e^{\lambda} - 1) - \log(\lambda).$$

This is supposed to be the same as $ma + b$ for all a, so we must have $m = \lambda$ and $b = \log(e^{\lambda} - 1) - \log(\lambda)$.

Note here that the only thing we can do to simplify a logarithm is to use the rule $\log(uv) = \log(u) + \log(v)$, or related rules like $\log(u^n) = n\log(u)$. Thus, it was essential to write the integral as a product of terms that we could handle; if we had left the integral as a difference of two terms, we would have been stuck.

Exercise 7.3. Find an integer $n > 0$ such that $\int_{-\pi}^{\pi} \cos(x/n)\,dx = n$.

Solution: We have

$$\int_{-\pi}^{\pi} \cos(x/n)\,dx = [n\sin(x/n)]_{x=-\pi}^{\pi} = n(\sin(\pi/n) - \sin(-\pi/n)) = 2n\sin(\pi/n).$$

(Here we have used the rule $\sin(-x) = -\sin(x)$, or in other words, the fact that sin is an odd function.) We want this to be equal to n, so we must have $\sin(\pi/n) = 1/2$. As $\sin(\pi/6) = 1/2$, one solution is $n = 6$. It is not too hard to check that this is the only solution. Indeed, one can check directly that none of the numbers $1, 2, 3, 4, 5$ is a solution. For $n > 6$ we note that $0 < \pi/n < \pi/6$ and the function $\sin(x)$ is strictly increasing in this range so $0 < \sin(\pi/n) < 1/2$.

Exercise 7.4. Show that $\displaystyle\int \frac{dx}{\sin(2x)} = \tfrac{1}{2}\log(\tan(x))$, and thus calculate $\displaystyle\exp\left(\int_{\pi/6}^{\pi/3} \frac{dx}{\sin(2x)}\right)$.

(Note that the question does **not** ask you to find $\int \frac{dx}{\sin(2x)}$ from scratch; it suggests an answer, and asks you to show that it is correct.)

Solution: Recall that $\tan'(x) = \sec(x)^2 = \cos(x)^{-2}$. This gives

$$\begin{aligned}\frac{d}{dx}\left(\tfrac{1}{2}\log(\tan(x))\right) &= \frac{\tan'(x)}{2\tan(x)} = \frac{\cos(x)^{-2}}{2\sin(x)/\cos(x)} \\ &= \frac{1}{2\sin(x)\cos(x)} = \frac{1}{\sin(2x)}.\end{aligned}$$

It follows that $\int \frac{dx}{\sin(2x)} = \log(\tan(x))/2$ as claimed. Recall also that $\sin(\pi/6) = \cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \cos(\pi/6) = \sqrt{3}/2$. It follows that $\tan(\pi/3) = \sqrt{3} = 3^{1/2}$ and $\tan(\pi/6) = 3^{-1/2}$, so $\log(\tan(\pi/3)) = \log(3^{1/2}) = \frac{1}{2}\log(3)$ and $\log(\tan(\pi/6)) = \log(3^{-1/2}) = -\frac{1}{2}\log(3)$. This gives

$$\begin{aligned}\int_{\pi/6}^{\pi/3} \frac{dx}{\sin(2x)} &= \left[\tfrac{1}{2}\log(\tan(x))\right]_{x=\pi/6}^{\pi/3} \\ &= \tfrac{1}{4}\log(3) - (-\tfrac{1}{4})\log(3) = \tfrac{1}{2}\log(3) = \log(\sqrt{3}).\end{aligned}$$

We can now take the exponential of both sides to get

$$\exp\left(\int_{\pi/6}^{\pi/3} \frac{dx}{\sin(2x)}\right) = \sqrt{3}.$$

Exercise 7.5. Show that $\int \tanh(x)\,dx = \log(e^{2x} + 1) - x$.

Solution: Again, we are offered an integral and asked to show that it is correct, so we need only differentiate. Note that

$$\begin{aligned}\frac{d}{dx}\left(\log(e^{2x}+1) - x\right) &= \frac{2e^{2x}}{e^{2x}+1} - 1 \\ &= \frac{e^{2x}-1}{e^{2x}+1} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{(e^x - e^{-x})/2}{(e^x + e^{-x})/2} \\ &= \frac{\sinh(x)}{\cosh(x)} = \tanh(x).\end{aligned}$$

It follows that

$$\int \tanh(x)\,dx = \log(e^{2x}+1) - x$$

as claimed.

Exercise 7.6. For any function $f(x)$, evaluate $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$.

Solution: Put $u = f(x)$, so $du/dx = f'(x)$, so $f'(x)\,dx = du$. This gives

$$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{du}{u} = \log(u) = \log(f(x)).$$

Exercise 7.7. Find $\frac{d}{dx}(e^{3x}(A\sin(4x)+B\cos(4x)))$. Find A and B such that this derivative is equal to $e^{3x}\sin(4x)$. Hence find $\int_0^{2\pi} e^{3x}\sin(4x)\,dx$.

Solution:

$$\begin{aligned}
&\frac{d}{dx}\left(e^{3x}(A\cos(4x)+B\sin(4x))\right)\\
&=3e^{3x}(A\cos(4x)+B\sin(4x))+e^{3x}(-4A\sin(4x)+4B\cos(4x))\\
&=e^{3x}((3A+4B)\cos(4x)+(3B-4A)\sin(4x)),
\end{aligned}$$

For this to equal $e^{3x}\sin(4x)$ for all x, we must have $3A+4B=0$ and $3B-4A=1$. This gives $A=-4B/3$ so $1=3B-4A=3B+16B/3=25B/3$, so $B=3/25$, so $A=-4B/3=-4/25$. The conclusion is that

$$\int e^{3x}\sin(4x)\,dx = e^{3x}(3\sin(4x)-4\cos(4x))/25.$$

In particular, when $x=0$ we have $\sin(x)=0$ and $\cos(x)=1$ and $e^{3x}=1$ so $e^{3x}(3\sin(4x)-4\cos(4x))/25=-4/25$. Similarly, when $x=2\pi$ we have $e^{3x}(3\sin(4x)-4\cos(4x))/25=-4e^{6\pi}/25$. It follows that

$$\int_0^{2\pi} e^{3x}\sin(4x)\,dx=\left[e^{3x}(3\sin(4x)-4\cos(4x))/25\right]_{x=0}^{2\pi}=(-4e^{6\pi}/25)-(-4/25)=4(1-e^{6\pi})/25.$$

Exercise 7.8. The indefinite integral $\int \ln(x)^2-\ln(x)+1\,dx$ has the form $(u\ln(x)^2+v\ln(x)+w)x$ for some constants u, v and w. Find these constants.

Solution: We have

$$\begin{aligned}
\frac{d}{dx}((u\ln(x)^2+v\ln(x)+w)x)&=(u.2\ln(x).x^{-1}+v.x^{-1})x+(u\ln(x)^2+v\ln(x)+w).1\\
&=u\ln(x)^2+(2u+v)\ln(x)+(v+w).
\end{aligned}$$

For this to equal $\ln(x)^2-\ln(x)+1$ for all x, we must have $u=1$ and $2u+v=-1$ and $v+w=1$. This gives $v=-3$ and $w=4$. We conclude that

$$\int \ln(x)^2-\ln(x)+1\,dx=(\ln(x)^2-3\ln(x)+4)x.$$

Exercise 7.9. Find $\frac{d}{dx}(x^a\ln(x)^b)$. By choosing a and b suitably, find the integral $\displaystyle\int \ln(x)^{-2}-\ln(x)^{-1}\,dx$.

Solution: Put $u=\frac{d}{dx}\left(x^a\ln(x)^b\right)$. We then have

$$u=ax^{a-1}\ln(x)^b+x^a.b\ln(x)^{b-1}.x^{-1}=ax^{a-1}\ln(x)^b+bx^{a-1}\ln(x)^{b-1}.$$

We want to choose a and b to make u match up with the expression $v=\ln(x)^{-2}-\ln(x)^{-1}$. Note that u involves a factor x^{a-1}, but there is no power of x in u; this suggests that we should have $a=1$, so that the x^{a-1} factor is just 1. Note also that u involves $\ln(x)^b$ and $\ln(x)^{b-1}$, whereas v involves $\ln(x)^{-1}$ and $\ln(x)^{-2}$; this suggests that we should try $b=-1$. Using these values, we have

$$\frac{d}{dx}\left(\frac{x}{\ln(x)}\right)=\ln(x)^{-1}-\ln(x)^{-2}=-u.$$

It follows that

$$\int \ln(x)^{-2}-\ln(x)^{-1}\,dx=-\frac{x}{\ln(x)}.$$

Exercise 7.10. Find $\int xe^x\cos(x)\,dx$.

Solution: This has growth rate 1, frequency 1 and degree 1. The general form is thus

$$\int xe^x\cos(x)\,dx=e^x((Ax+B)\sin(x)+(Cx+D)\cos(x)).$$

To find the coefficients, we differentiate:

$$\begin{aligned}
\frac{d}{dx}\left((Ax+B)\sin(x)\right) =& \frac{d}{dx}(Ax+B)\ \sin(x) + (Ax+B)\frac{d}{dx}\sin(x) = A\sin(x) + (Ax+B)\cos(x) \\
\frac{d}{dx}\left((Cx+D)\cos(x)\right) =& \frac{d}{dx}(Cx+D)\ \cos(x) + (Cx+D)\frac{d}{dx}\cos(x) = C\cos(x) - (Cx+D)\sin(x) \\
& \frac{d}{dx}\left(e^x((Ax+B)\sin(x) + (Cx+D)\cos(x))\right) \\
=& \left(\frac{d}{dx}e^x\right)((Ax+B)\sin(x) + (Cx+D)\cos(x)) + e^x\frac{d}{dx}((Ax+B)\sin(x) + (Cx+D)\cos(x)) \\
=& e^x((Ax+B)\sin(x) + (Cx+D)\cos(x)) + \\
& e^x(A\sin(x) + (Ax+B)\cos(x) + C\cos(x) - (Cx+D)\sin(x)) \\
=& e^x((A-C)x\sin(x) + (B+A-D)\sin(x) + (C+A)x\cos(x) + (D+B+C)\cos(x)).
\end{aligned}$$

This must be the same as the function that we were trying to integrate, which was

$$xe^x\cos(x) = e^x(0.x\sin(x) + 0.\sin(x) + 1.x\cos(x) + 0.\cos(x))$$

By comparing coefficients, we see that $A-C=0$ and $B+A-D=0$ and $C+A=1$ and $D+B+C=0$. The first and third of these equations give $A=C=1/2$. Putting these values in the other two equations gives $B-D=-1/2$ and $B+D=-1/2$, from which we see that $B=-1/2$ and $D=0$. It follows that

$$\int xe^x\cos(x)\,dx = \tfrac{1}{2}e^x((x-1)\sin(x) + x\cos(x)).$$

Exercise 7.11. Find $\int f(x)\,dx$, where $f(x) = x^3e^{3x}$. You may assume that the answer has the form $F(x) = (Ax^3 + Bx^2 + Cx + D)e^{3x}$ for some constants A, B, C and D.

Solution: Differentiating the relation $\int f(x)\,dx = F(x)$ gives

$$\begin{aligned}
x^3e^{3x} &= f(x) = F'(x) \\
&= (3Ax^2 + 2Bx + C)e^{3x} + (Ax^3 + Bx^2 + Cx + D).3e^{3x} \\
&= (3Ax^3 + (3B+3A)x^2 + (3C+2B)x + (3D+C))e^{3x}.
\end{aligned}$$

By comparing coefficients, we see that

$$\begin{aligned}
3A &= 1 \\
3B + 3A &= 0 \\
3C + 2B &= 0 \\
3D + C &= 0
\end{aligned}$$

so $A = 1/3$ and $B = -A = -1/3$ and $C = -2B/3 = 2/9$ and $D = -C/3 = -2/27$. It follows that

$$\int x^3e^{3x}\,dx = \left(\tfrac{1}{3}x^3 - \tfrac{1}{3}x^2 + \tfrac{2}{9}x - \tfrac{2}{27}\right)e^{3x} = (9x^3 - 9x^2 + 6x - 2)e^{3x}/27.$$

Exercise 7.12. Let $p(x)$ be a polynomial of degree 3, say

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3.$$

(a) Show that $p''''(x) = 0$.
(b) Put $q(x) = p(x) + p'(x) + p''(x) + p'''(x)$. Show that $q(x) - q'(x) = p(x)$.
(c) Deduce that $\int p(x)e^{-x}\,dx = -q(x)e^{-x}$.
(d) How should this be adjusted for polynomials of degree 4, or of degree n for arbitrary n?

Solution:

(a) We have

$$\begin{aligned}
p'(x) &= a_1 + 2a_2x + 3a_3x^2 \\
p''(x) &= 2a_2 + 6a_3x \\
p'''(x) &= 6a_3 \\
p''''(x) &= 0.
\end{aligned}$$

(b)

$$\begin{aligned} q(x) - q'(x) &= p(x) + p'(x) + p''(x) + p'''(x) - \\ &\quad (p'(x) + p''(x) + p'''(x) + p''''(x)) \\ &= p(x) - p''''(x) = p(x) - 0 = p(x). \end{aligned}$$

(c) We now note that $\frac{d}{dx}(-q(x)e^{-x}) = -q'(x)e^{-x} - q(x).(-e^{-x}) = (q(x) - q'(x))e^{-x} = p(x)e^{-x}$, so $\int p(x)e^{-x}\,dx = -q(x)e^{-x}$.

(d) If $p(x)$ is a polynomial of degree n, $p^{(m)}(x) = 0$ for all $m > n$. Moreover, we have $\int p(x)e^{-x}\,dx = -q(x)e^{-x}$, where

$$q(x) = p(x) + p'(x) + \cdots + p^{(n)}(x).$$

Exercise 7.13. You may assume that $\displaystyle\int x^2 \log(x)^2\,dx = x^3(a\log(x)^2 + b\log(x) + c)$ for some constants a, b and c. Find these constants, and thus evaluate $\int_1^e x^2\log(x)^2\,dx$.

Solution: We first note that

$$\begin{aligned} \frac{d}{dx}\left(x^3(a\log(x)^2 + b\log(x) + c)\right) &= 3x^2(a\log(x)^2 + b\log(x) + c) + x^3(2a\log(x)/x + b/x) \\ &= x^2(3a\log(x)^2 + (3b + 2a)\log(x) + (3c + b)). \end{aligned}$$

This must also be equal to $x^2\log(x)^2$ for all x, so we must have

$$\begin{aligned} 3a &= 1 \\ 3b + 2a &= 0 \\ 3c + b &= 0, \end{aligned}$$

so $a = 1/3$ and $b = -2/9$ and $c = 2/27$, giving

$$\int x^2\log(x)^2\,dx = x^3(\log(x)^2/3 - 2\log(x)/9 + 2/27).$$

It follows that

$$\begin{aligned} \int_1^e x^2\log(x)^2\,dx &= \left[x^3(\log(x)^2/3 - 2\log(x)/9 + 2/27)\right]_1^e \\ &= e^3(1/3 - 2/9 + 2/27) - 1^3(0/3 - 0/9 + 2/27) \\ &= (5e^3 - 2)/27. \end{aligned}$$

Exercise 7.14. Use the general formula

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \ln(2\sqrt{a^2x^2 + abx + ac} + 2ax + b)/\sqrt{a},$$

to find the integral

$$\int \frac{dx}{\sqrt{4x^2 + 5x + 6}}.$$

Solution: We just have to put $a = 4$ and $b = 5$ and $c = 6$ in the formula, giving

$$\ln(2\sqrt{16x^2 + 20x + 24} + 8x + 5)/2.$$

Week 8

Exercise 8.1. Expand out $\cosh(x)^3$ in terms of exponentials, and thus evaluate $\int \cosh(x)^3\,dx$. Give your answer in terms of exponentials, and then rewrite it in terms of sinh and cosh.

Solution: We have $\cosh(x) = (e^x + e^{-x})/2$, so

$$\begin{aligned} \cosh(x)^3 &= \tfrac{1}{8}(e^{3x} + 3e^{2x}e^{-x} + 3e^xe^{-2x} + e^{-3x}) \\ &= \tfrac{1}{8}e^{3x} + \tfrac{3}{8}e^x + \tfrac{3}{8}e^{-x} + \tfrac{1}{8}e^{-3x} \end{aligned}$$

so

$$
\begin{aligned}
\int \cosh(x)^3\,dx &= \tfrac{1}{8}\int e^{3x}\,dx + \tfrac{3}{8}\int e^{x}\,dx + \tfrac{3}{8}\int e^{-x}\,dx + \tfrac{1}{8}\int e^{-3x}\,dx \\
&= \tfrac{1}{24}e^{3x} + \tfrac{3}{8}e^{x} - \tfrac{3}{8}e^{-x} - \tfrac{1}{24}e^{-3x} \\
&= \frac{1}{12}\,\frac{e^{3x}-e^{-3x}}{2} + \frac{3}{4}\,\frac{e^{x}-e^{-x}}{2} = \frac{1}{12}\sinh(3x) + \frac{3}{4}\sinh(x).
\end{aligned}
$$

Exercise 8.2. Find $\int \cosh(x)^2\,dx$

Solution: We have $\cosh(x) = (e^x + e^{-x})/2$, so

$$
\begin{aligned}
\cosh(x)^2 &= \tfrac{1}{4}(e^{2x} + 2 + e^{-2x}) = \tfrac{1}{2}\cosh(2x) + \tfrac{1}{2} \\
\int \cosh(x)^2\,dx &= \tfrac{1}{2}\int \cosh(2x)\,dx + \int \tfrac{1}{2}\,dx \\
&= \tfrac{1}{4}\sinh(2x) + \tfrac{1}{2}x.
\end{aligned}
$$

Exercise 8.3. Find $\int 2\sinh(3x) - (2\sinh(x))^3\,dx$.

Solution: We have

$$
\begin{aligned}
2\sinh(3x) &= e^{3x} - e^{-3x} \\
2\sinh(x) &= e^{x} - e^{-x} \\
(2\sinh(x))^3 &= (e^{x} - e^{-x})^3 = e^{3x} - 3e^{x} + 3e^{-x} - e^{-3x} \\
2\sinh(3x) - (2\sinh(x))^3 &= 3e^{x} - 3e^{-x} = 6\sinh(x) \\
\int 2\sinh(3x) - (2\sinh(x))^3\,dx &= \int 6\sinh(x)\,dx = 6\cosh(x).
\end{aligned}
$$

Exercise 8.4. Recall that

$$
\begin{aligned}
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(2x) &= \cos(x)^2 - \sin(x)^2 = 2\cos(x)^2 - 1 = 1 - 2\sin(x)^2.
\end{aligned}
$$

Use these to convert the following integrals to a form in which they can easily be evaluated, and then evaluate them.

(a) $\int \sin(x)^2\,dx$ **(b)** $\int \sin(x)\cos(x)\,dx$ **(c)** $\int \sin(x)^2\cos(x)^2\,dx$ **(d)** $\int \sin(x)\cos(x)^3 - \sin(x)^3\cos(x)\,dx$

Now check by differentiating that $\int \sin(x)\cos(x)\,dx = \sin(x)^2/2$. Is this consistent with your answer to (b)?

Solution:

$$
\cos(2x) = 1 - 2\sin(x)^2,
$$

so

$$\begin{aligned}
\sin(x)^2 &= \tfrac{1}{2} - \tfrac{1}{2}\cos(2x) \\
\int \sin(x)^2\,dx &= \int \tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\,dx \\
&= \tfrac{1}{2}x - \tfrac{1}{4}\sin(2x). \\
\sin(x)\cos(x) &= \tfrac{1}{2}\sin(2x) \\
\int \sin(x)\cos(x)\,dx &= \tfrac{1}{2}\int \sin(2x)\,dx \\
&= -\tfrac{1}{4}\cos(2x). \\
\sin(x)^2\cos(x)^2 &= \left(\tfrac{1}{2}\sin(2x)\right)^2 = \tfrac{1}{4}\sin(2x)^2 \\
&= \tfrac{1}{4}.\tfrac{1}{2}(1-\cos(4x)) \\
&= \tfrac{1}{8} - \tfrac{1}{8}\cos(4x) \\
\int \sin(x)^2\cos(x)^2\,dx &= \tfrac{1}{8}\int 1-\cos(4x)\,dx \\
&= \frac{x}{8} - \frac{\sin(4x)}{32} = \frac{4x-\sin(4x)}{32} \\
\sin(x)\cos(x)^3 - \sin(x)^3\cos(x) &= \sin(x)\cos(x)(\cos(x)^2-\sin(x)^2) = \tfrac{1}{2}\sin(2x)\cos(2x) \\
&= \tfrac{1}{4}\sin(4x) \\
\int \sin(x)\cos(x)^3 - \sin(x)^3\cos(x)\,dx &= \frac{1}{4}\int \sin(4x)\,dx = -\cos(4x)/16.
\end{aligned}$$

Next, note that the power rule gives

$$\frac{d}{dx}(\sin(x)^2/2) = 2\sin(x)\sin'(x)/2 = \sin(x)\cos(x),$$

so $\int \sin(x)\cos(x)\,dx = \sin(x)^2/2$. More precisely, this shows that $\sin(x)^2/2$ is one of the indefinite integrals of $\sin(x)\cos(x)$, and for part (b) we showed that $-\cos(2x)/4$ is another indefinite integral of $\sin(x)\cos(x)$. We know that $\cos(2x) = 1-2\sin(x)^2$, so $-\cos(2x)/4 = \sin(x)^2/2 - 1/4$, so our two indefinite integrals just differ by a constant, as expected.

Exercise 8.5. (a) Let p be an integer. Find $\int_0^{2\pi}\cos(p\theta)\,d\theta$. (Note that the case $p=0$ must be considered separately.)

(b) Let n and m be positive integers. Using the relation $\cos(\theta) = (e^{i\theta}+e^{-i\theta})/2$, show that

$$\cos(n\theta)\cos(m\theta) = \tfrac{1}{2}\cos((n+m)\theta) + \tfrac{1}{2}\cos((n-m)\theta).$$

(c) Using (a) and (b), show that $\int_0^{2\pi}\cos(n\theta)\cos(m\theta) = 0$ whenever $n,m>0$ and $n\neq m$. What is the value of the integral when $n=m$?

Solution:

(a) For $p \neq 0$, we have

$$\int_0^{2\pi}\cos(p\theta)d\theta = [\sin(p\theta)/p]_{\theta=0}^{2\pi} = \frac{1}{p}(\sin(2p\pi)-\sin(0)).$$

As p is an integer we have $\sin(2p\pi) = 0 = \sin(0)$, so $\int_0^{2\pi}\cos(p\theta)d\theta = 0$. When $p=0$ we have $\cos(p\theta)=1$ for all θ, and so $\int_0^{2\pi}\cos(p\theta)\,d\theta = \int_0^{2\pi}1\,d\theta = 2\pi$.

(b) We have

$$\begin{aligned}
\cos(n\theta)\cos(m\theta) &= \frac{e^{in\theta}+e^{-in\theta}}{2}\,\frac{e^{im\theta}+e^{-im\theta}}{2} = \tfrac{1}{4}(e^{i(n+m)\theta}+e^{i(n-m)\theta}+e^{i(m-n)\theta}+e^{i(-n-m)\theta}) \\
&= \frac{1}{2}\frac{e^{i(n+m)\theta}+e^{-i(n+m)\theta}}{2} + \frac{1}{2}\frac{e^{i(n-m)\theta}+e^{-i(n-m)\theta}}{2} \\
&= \tfrac{1}{2}\cos((n+m)\theta) + \tfrac{1}{2}\cos((n-m)\theta).
\end{aligned}$$

(c) It follows that

$$\int_0^{2\pi} \cos(n\theta)\cos(m\theta)\,d\theta = \frac{1}{2}\int_0^{2\pi} \cos((n+m)\theta)\,d\theta + \frac{1}{2}\int_0^{2\pi} \cos((n-m)\theta)\,d\theta$$

Recall that n and m were assumed to be positive, so $n+m$ can never be zero, and $n-m$ can only be zero if $n=m$. Part (a) therefore tells us that when $n \neq m$, both integrals on the right hand side are zero. If $n=m$ then the first term is still zero, but the second one is $\frac{1}{2}(2\pi) = \pi$. In conclusion, we have

$$\int_0^{2\pi} \cos(n\theta)\cos(m\theta) = \begin{cases} \pi & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

Exercise 8.6. Find $\int e^{-3x}\cos(4x)\,dx$

Solution: We know that

$$\int e^{-3x}\cos(4x)\,dx = e^{-3x}(A\cos(4x) + B\sin(4x))$$

for some A and B . To find these, we differentiate and equate coefficients:

$$\begin{aligned} e^{-3x}\cos(4x) &= \frac{d}{dx}\left(e^{-3x}(A\cos(4x)+B\sin(4x))\right) \\ &= -3e^{-3x}(A\cos(4x)+B\sin(4x)) + e^{-3x}(-4A\sin(4x)+4B\cos(4x)) \\ &= e^{3x}((-3A+4B)\cos(4x) + (-3B-4A)\sin(4x)), \end{aligned}$$

so $-3A+4B = 1$ and $-3B-4A = 0$. These equations give $A = -3/25$ and $B = 4/25$, so

$$\int e^{-3x}\cos(4x)\,dx = e^{-3x}(-3\cos(4x) + 4\sin(4x))/25.$$

Exercise 8.7. Find $\int \sqrt{2}e^{-x/\sqrt{2}}\sin(x/\sqrt{2})\,dx$

Solution: We know that

$$\int \sqrt{2}e^{-x/\sqrt{2}}\sin(x/\sqrt{2})\,dx = e^{-x/\sqrt{2}}(A\cos(x/\sqrt{2}) + B\sin(x/\sqrt{2}))$$

for some A and B. To find these, we differentiate and equate coefficients:

$$\begin{aligned} \sqrt{2}e^{-x/\sqrt{2}}\sin(x/\sqrt{2}) &= \frac{d}{dx}\left(e^{-x/\sqrt{2}}(A\cos(x/\sqrt{2}) + B\sin(x/\sqrt{2}))\right) \\ &= -\frac{1}{\sqrt{2}}e^{-x/\sqrt{2}}(A\cos(x/\sqrt{2}) + B\sin(x/\sqrt{2})) + e^{-x/\sqrt{2}}(-\frac{1}{\sqrt{2}}A\sin(x/\sqrt{2}) + \frac{1}{\sqrt{2}}B\cos(x/\sqrt{2})) \\ &= \frac{1}{\sqrt{2}}e^{-x/\sqrt{2}}\left(-A\cos(x/\sqrt{2}) - B\sin(x/\sqrt{2}) - A\sin(x/\sqrt{2}) + B\cos(x/\sqrt{2})\right) \\ &= \frac{1}{\sqrt{2}}e^{-x/\sqrt{2}}\left((B-A)\cos(x/\sqrt{2}) - (A+B)\sin(x/\sqrt{2})\right). \end{aligned}$$

If we multiply both sides by $\sqrt{2}$ and compare coefficients, we see that $B - A = 0$ and $-(A+B) = 2$, so $A = B = -1$. We conclude that

$$\int \sqrt{2}e^{-x/\sqrt{2}}\sin(x/\sqrt{2})\,dx = -e^{x/\sqrt{2}}(\sin(x/\sqrt{2}) + \cos(x/\sqrt{2})).$$

Exercise 8.8. Find $\int e^{-x}\sin(x)^2\,dx$. (The first step is to rewrite $\sin(x)^2$, just as you would if you were doing $\int \sin(x)^2\,dx$.)

Solution: First note that $\sin(x)^2 = (1-\cos(2x))/2$, so

$$e^{-x}\sin(x)^2 = \tfrac{1}{2}e^{-x} - \tfrac{1}{2}e^{-x}\cos(2x).$$

We know that

$$\int e^{-x}\cos(2x)\,dx = e^{-x}(a\cos(2x) + b\sin(2x))$$

for some a and b. Differentiating gives

$$\begin{aligned} e^{-x}\cos(2x) &= -e^{-x}(a\cos(2x)+b\sin(2x)) + e^{-x}(-2a\sin(2x)+2b\cos(2x)) \\ &= e^{-x}((2b-a)\cos(2x)-(b+2a)\sin(2x)), \end{aligned}$$

so $2b-a=1$ and $b+2a=0$, giving $a=-1/5$ and $b=2/5$. Thus

$$\begin{aligned} \int e^{-x}\sin(x)^2\,dx &= \tfrac{1}{2}\int e^{-x}\,dx - \tfrac{1}{2}\int e^{-x}\cos(2x)\,dx \\ &= -\tfrac{1}{2}e^{-x} - \tfrac{1}{2}e^{-x}(-\cos(2x)/5+2\sin(2x)/5) \\ &= \tfrac{1}{10}e^{-x}(2\sin(2x)-\cos(2x)-5). \end{aligned}$$

Exercise 8.9. The general form of $\int xe^{-x}\cos(x)$ is $e^{-x}((Ax+B)\cos(x)+(Cx+D)\sin(x))$. In the same way, write down the general form of each of the following integrals:

(a) $\int x^2e^{3x}\sin(4x)\,dx$ **(b)** $\int x^6e^{x/6}\,dx$ **(c)** $\int(1+x+x^2)\sin(x)\,dx$ **(d)** $\int(1-x)e^x(\sin(x)-\cos(x))\,dx$

(You need not find the coefficients, just write down the general form.)

Solution:

(a) $e^{3x}((A+Bx+Cx^2)\sin(4x)+(D+Ex+Fx^2)\cos(4x))$
(b) $(A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+Gx^6)e^{x/6}$
(c) $(A+Bx+Cx^2)\sin(x)+(D+Ex+Fx^2)\cos(x)$
(d) $e^x((A+Bx)\sin(x)+(C+Dx)\cos(x))$

Exercise 8.10. Find $\int 8x\sin(x)\cos(x)\,dx$

Solution: First note that $8x\sin(x)\cos(x)=4x\sin(2x)$, so

$$\begin{aligned} \int 8x\sin(x)\cos(x)\,dx &= \int 4x\sin(2x)\,dx \\ &= -2x\cos(2x)+\int 2\cos(2x)\,dx \\ &= -2x\cos(2x)+\sin(2x). \end{aligned}$$

Exercise 8.11. Find $\int x^2e^x\,dx$

Solution: We know that

$$\int x^2e^x\,dx = (ax^2+bx+c)e^x$$

for some constants a, b and c . To find these, we differentiate to get

$$\begin{aligned} x^2e^x &= \frac{d}{dx}((ax^2+bx+c)e^x) = (2ax+b)e^x+(ax^2+bx+c)e^x \\ &= (ax^2+(2a+b)x+(b+c))e^x. \end{aligned}$$

We equate coefficients to see that $a=1$ and $2a+b=b+c=0$, which gives $b=-2$ and $c=2$. We conclude that

$$\int x^2e^x\,dx = (x^2-2x+2)e^x.$$

Alternatively, we can integrate twice by parts. For the first step, put $u=x^2$ (so $du/dx=2x$) and $dv/dx=e^x$ (so $v=e^x$ as well). We then have

$$\int x^2e^x\,dx = \int u\frac{dv}{dx}\,dx = uv-\int\frac{du}{dx}v\,dx = x^2e^x-2\int x\,e^x\,dx.$$

For the second step, put $u=x$ (so $du/dx=1$) and $dv/dx=e^x$ (so $v=e^x$ as well). We then have

$$\int x\,e^x\,dx = \int u\frac{dv}{dx}\,dx = uv-\int\frac{du}{dx}v\,dx = xe^x-\int e^x\,dx = xe^x-e^x.$$

Putting this together, we get

$$\int x^2e^x\,dx = x^2e^x-2(xe^x-e^x) = (x^2-2x+2)e^x.$$

Exercise 8.12. Find $\int(4x^2+2x+1)e^{2x}\,dx$.

Solution: The general form is

$$\int (4x^2+2x+1)e^{2x}\,dx = (Ax^2+Bx+C)e^{2x}$$

for some constants A, B and C. Differentiating, we find that

$$\begin{aligned}(4x^2+2x+1)e^{2x} &= \frac{d}{dx}((Ax^2+Bx+C)e^{2x}) \\ &= (2Ax+B)e^{2x} + (Ax^2+Bx+C).2e^{2x} \\ &= (2Ax^2+(2A+2B)x+(B+2C))e^{2x},\end{aligned}$$

so $2A=4$ and $2A+2B=2$ and $B+2C=1$. It follows that $A=2$ and $B=-1$ and $C=1$, so

$$\int (4x^2+2x+1)e^{2x}\,dx = (2x^2-x+1)e^{2x}.$$

Week 9

Exercise 9.1. Use integration by parts to find $\int x\cos(\omega x)\,dx$

Solution: Put $u=x$ and $dv/dx=\cos(\omega x)$, so $du/dx=1$ and $v=\int\cos(\omega x)\,dx=\omega^{-1}\sin(\omega x)$. This gives

$$\begin{aligned}\int x\cos(\omega x)\,dx &= uv - \int \frac{du}{dx}v\,dx = \omega^{-1}x\sin(\omega x) - \omega^{-1}\int \sin(\omega x)\,dx \\ &= \omega^{-1}x\sin(\omega x)\,dx + \omega^{-2}\cos(\omega x).\end{aligned}$$

(We used the facts that $\int\cos(t)\,dt=\sin(t)$ and $\int\sin(t)\,dt=-\cos(t)$, so $\int\cos(\omega x)\,dx=\sin(\omega x)/\omega$ and $\int\sin(\omega x)\,dx=-\cos(\omega x)/\omega$.)

Exercise 9.2. Use integration by parts twice to find $\int x^2\,e^{3x}\,dx$.

Solution: For the first stage, we put $u=x^2$ and $dv/dx=e^{3x}$, so $du/dx=2x$ and $v=\int e^{3x}\,dx=e^{3x/3}$. We then have

$$\int x^2e^{3x}\,dx = uv - \int\frac{du}{dx}v\,dx = \frac{1}{3}x^2\,e^{3x} - \frac{2}{3}\int xe^{3x}\,dx.$$

To evaluate $\int xe^{3x}\,dx$, we put $u=x$ and $v=e^{3x}$, so $du/dx=1$ and $v=\int e^{3x}\,dx=e^{3x}/3$ (just as before). This gives

$$\int xe^{3x}\,dx = uv - \int\frac{du}{dx}v\,dx = \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}\,dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}.$$

Putting this together, we get

$$\begin{aligned}\int x^2e^{3x}\,dx &= \frac{1}{3}x^2\,e^{3x} - \frac{2}{3}\int xe^{3x}\,dx \\ &= \frac{1}{3}x^2\,e^{3x} - \frac{2}{3}\left(\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right) \\ &= \left(\frac{1}{3}x^2 - \frac{2}{9}x + \frac{2}{27}\right)e^{3x}.\end{aligned}$$

Exercise 9.3. Find $\int x^n\ln(x)\,dx$.

Solution: We use integration by parts. Put $u=\ln(x)$ and $\frac{dv}{dx}=x^n$, so $\frac{du}{dx}=x^{-1}$ and $v=x^{n+1}/(n+1)$. This gives

$$\begin{aligned}\int x^n\ln(x)\,dx &= \int u\,\tfrac{dv}{dx}\,dx = uv - \int \tfrac{du}{dx}v\,dx \\ &= x^{n+1}\ln(x)/(n+1) - \int x^{-1}x^{n+1}/(n+1)\,dx = x^{n+1}\ln(x)/(n+1) - \int x^n/(n+1)\,dx \\ &= x^{n+1}\ln(x)/(n+1) - x^{n+1}/(n+1)^2 = \frac{x^{n+1}}{(n+1)^2}\left((n+1)\ln(x)-1\right).\end{aligned}$$

Exercise 9.4. Put $I=\int e^x\sin(x)\,dx$ and $J=\int e^x\cos(x)\,dx$. Try to integrate I by parts; you should obtain an equation relating I and J. Now try to integrate J by parts; you should obtain another equation relating I and J. Solve these two equations simultaneously to find I and J.

Solution: In I we put $u = e^x$ and $dv/dx = \sin(x)$, so $du/dx = e^x$ and $v = \int \sin(x)\,dx = -\cos(x)$. This gives

$$I = uv - \int \frac{du}{dx} v\,dx = -e^x\cos(x) + \int e^x\cos(x)\,dx = -e^x\cos(x) + J.$$

We now try to find $J = \int e^x \cos(x)\,dx$ instead. We put $u = e^x$ and $dv/dx = \cos(x)$, so $du/dx = e^x$ and $v = \int \cos(x)\,dx = \sin(x)$. This gives

$$J = uv - \int \frac{du}{dx} v\,dx = e^x\sin(x) - \int e^x\sin(x)\,dx = e^x\sin(x) - I.$$

Rearranging these equations gives

$$\begin{aligned} J - I &= e^x\cos(x) \\ J + I &= e^x\sin(x). \end{aligned}$$

If we add these two equations and divide by 2, we get $J = e^x(\cos(x)+\sin(x))/2$. Similarly, if we subtract the above equations and divide by 2 we get $I = e^x(\sin(x) - \cos(x))/2$.

Exercise 9.5. By making suitable substitutions, find the following integrals:

(a) $\int xe^{-x^2}\,dx$ **(b)** $\int x^4\sin(x^5)\,dx$ **(c)** $\int (3x^2-1)\cos(x^3-x)\,dx$ **(d)** $\int (1-x^{-2})\ln(x+x^{-1})\,dx$

Solution:

(a) Put $u = -x^2$, so $du/dx = -2x$, so $dx = -du/(2x)$. The integral becomes

$$\int xe^{-x^2}\,dx = \int xe^u \frac{du}{-2x} = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u = -\frac{1}{2}e^{-x^2}.$$

(b) Put $u = x^5$, so $du/dx = 5x^4$, so $dx = du/(5x^4)$. The integral becomes

$$\int x^4\sin(x^5)\,dx = \int x^4\sin(u)\frac{du}{5x^4} = \frac{1}{5}\int \sin(u)\,du = -\frac{1}{5}\cos(u) = -\frac{1}{5}\cos(x^5).$$

(c) Put $u = x^3 - x$, so $du/dx = 3x^2 - 1$, so $(3x^2-1)\,dx = du$. The integral becomes

$$\int (3x^2-1)\cos(x^3-x)\,dx = \int \cos(u)\,du = \sin(u) = \sin(x^3-x).$$

(d) Put $u = x + x^{-1}$, so $du/dx = 1 - x^{-2}$, so $(1-x^{-2})dx = du$. The integral becomes

$$\int (1-x^{-2})\ln(x+x^{-1})\,dx = \int \ln(u)\,du = u(\ln(u)-1) = (x+x^{-1})(\ln(x+x^{-1})-1).$$

Exercise 9.6. Evaluate $\int_0^{\pi/4} \tan(x)\,dx$ by the substitution $u = \cos(x)$ (remembering that $\tan(x) = \sin(x)/\cos(x)$).

Solution: If $u = \cos(x)$ then $du/dx = -\sin(x)$, so $du = -\sin(x)\,dx$. Thus

$$\begin{aligned} \int \tan(x)\,dx &= \int \frac{\sin(x)\,dx}{\cos(x)} \\ &= \int \frac{-du}{u} = -\log(u) \\ &= -\log(\cos(x)). \end{aligned}$$

We now put in the limits. We have $-\log(\cos(0)) = -\log(1) = 0$ and $-\log(\cos(\pi/4)) = -\log(2^{-1/2}) = \frac{1}{2}\log(2)$ so

$$\int_0^{\pi/4} \tan(x)\,dx = [-\log(\cos(x))]_{x=0}^{\pi/4} = \tfrac{1}{2}\log(2).$$

Exercise 9.7. By putting $u = \ln(x)$, find $\int_{1/e}^{e} \frac{(1+\ln(x))^2}{x}\,dx$.

Solution: Put $u = \ln(x)$, so $du = x^{-1}dx$. Then

$$\int \frac{(1+\ln(x))^2}{x}\,dx = \int (1+u)^2 du = (1+u)^3/3$$
$$= (1+\ln(x))^3/3.$$

We now put in the limits. When $x = e$ we have $\ln(x) = 1$ so $(1+\ln(x))^3/3 = 8/3$. When $x = 1/e$ we have $\ln(x) = -1$ so $(1+\ln(x))^3/3 = 0$. It follows that $\int_{1/e}^{e}(1+\ln(x))^{-2}\,dx = 8/3 - 0 = 8/3$.

Exercise 9.8. By substituting $u = x^n$, find $\displaystyle\int \frac{dx}{x\sqrt{x^{-2n}-1}}$.

Solution: Put $u = x^n$, so $du = nx^{n-1}\,dx$, so $dx = du/(nx^{n-1})$. The integral becomes

$$\int \frac{dx}{x\sqrt{x^{-2n}-1}} = \int \frac{du}{nx^{n-1}.x\sqrt{x^{-2n}-1}} = \frac{1}{n}\int \frac{du}{x^n\sqrt{x^{-2n}-1}}$$
$$= \frac{1}{n}\int \frac{du}{u\sqrt{u^{-2}-1}} = \frac{1}{n}\int \frac{du}{\sqrt{1-u^2}} = \arcsin(u)/n$$
$$= \arcsin(x^n)/n.$$

Exercise 9.9. Find $\displaystyle\int x^{-2}e^{-2/x}\,dx$.

Solution: Put $u = -2/x$, so $du/dx = 2/x^2$, so $dx = \frac{1}{2}x^2\,du$. This gives

$$\int x^{-2}e^{-2/x}\,dx = \int x^{-2}e^u.\tfrac{1}{2}x^2\,du = \tfrac{1}{2}\int e^u\,du = \tfrac{1}{2}e^u = \tfrac{1}{2}e^{-2/x}.$$

Exercise 9.10. Find the integral $\displaystyle\int \frac{(1+x)^5}{(1-x)^7}\,dx$, by substituting $u = (1+x)/(1-x)$.

Solution: Put $u = (1+x)/(1-x)$, so

$$\frac{du}{dx} = \frac{1.(1-x)-(1+x).(-1)}{(1-x)^2} = 2(1-x)^{-2},$$

so $dx = \frac{1}{2}(1-x)^2\,du$. This gives

$$\int \frac{(1+x)^5}{(1-x)^7}\,dx = \int \frac{(1+x)^5}{2(1-x)^5}\,du = \tfrac{1}{2}\int u^5\,du = u^6/12 = \frac{1}{12}\left(\frac{1+x}{1-x}\right)^6.$$

Exercise 9.11. Find the integral $\displaystyle\int e^{e^x}\,e^x\,dx$.

Solution: Notice that

$$\frac{d}{dx}\left(e^{e^x}\right) = e^{e^x}\,e^x,$$

so

$$\int e^{e^x}\,e^x\,dx = e^{e^x}.$$

More formally, we can substitute $u = e^x$, so $du = e^x\,dx$ and $e^{e^x} = e^u$. We then have

$$\int e^{e^x}\,e^x\,dx = \int e^u\,du = e^u = e^{e^x},$$

just as before.

Exercise 9.12. Find $\displaystyle\int x^{n-1}(1+x^n)^{m-1}\,dx$ (where n and m are constants with $n, m > 0$).

Solution: We substitute $u = 1+x^n$, so $du = nx^{n-1}\,dx$, so $x^{n-1}\,dx = n^{-1}du$. The integral becomes

$$\int x^{n-1}(1+x^n)^{m-1}\,dx = \int n^{-1}u^{m-1}\,du = \frac{u^m}{nm} = \frac{(1+x^n)^m}{nm}.$$

Exercise 9.13. Find the integral $\displaystyle\int_0^1 \frac{x\,dx}{\sqrt{1-x^4}}$, by substituting $u = x^2$.

Solution: Substitute $u = x^2$, so $du = 2x\,dx$, so $x\,dx = du/2$. This gives

$$\int \frac{x\,dx}{\sqrt{1-x^4}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \arcsin(u)/2 = \arcsin(x^2)/2.$$

We now put in the limits, noting that $\arcsin(1^2) = \pi/2$ and $\arcsin(0^2) = 0$. This gives

$$\int_0^1 \frac{x\,dx}{\sqrt{1-x^4}} = \left[\arcsin(x^2)/2\right]_{x=0}^1 = (\pi/2)/2 - 0/2 = \pi/4.$$

Exercise 9.14. By making a suitable substitution, find $\int \sin(x)\ln(\cos(x))\,dx$.

Solution: Put $u = \cos(x)$, so $du = -\sin(x)\,dx$. Then

$$\begin{aligned}\int \sin(x)\ln(\cos(x))\,dx &= -\int \ln(u)\,du = -(u\ln(u) - u) = u(1-\ln(u))\\ &= \cos(x)(1-\ln(\cos(x))).\end{aligned}$$

Exercise 9.15. By substituting $u = x^{n+1}$, find $\int x^n \ln(x)\,dx$.

Solution: We have $x = u^{1/(n+1)}$, so $\ln(x) = \ln(u)/(n+1)$. We also have $du/dx = (n+1)x^n$, so $x^n\,dx = du/(n+1)$. The integral becomes

$$\int x^n \ln(x)\,dx = \int \frac{\ln(u)}{(n+1)}\,\frac{du}{(n+1)} = \frac{1}{(n+1)^2}\int \ln(u)\,du = \frac{u(\ln(u)-1)}{(n+1)^2} = \frac{x^{n+1}((n+1)\ln(x)-1)}{(n+1)^2}.$$

Exercise 9.16. Find $\int_0^\infty x\,e^{-4x^2}\,dx$.

Solution: If we put $u = -4x^2$ then $du = -8x\,dx$, so $dx = -du/(8x)$. The integral becomes

$$\int x\,e^u\,\frac{-du}{8x} = -\frac{1}{8}\int e^u\,du = -e^u/8 = -e^{-4x^2}/8.$$

We now put in the limits. When $x = 0$ we have $-e^{-4x^2}/8 = -1/8$, but as x tends to ∞ we see that $-e^{-4x^2}/8$ tends to zero. It follows that $\int_0^\infty x\,e^{-4x^2}\,dx = 0 - (-1/8) = 1/8$.

Week 10

Exercise 10.1. Find the following Taylor or MacLaurin series, to order 4 in each case.

(a) The series for $p(x) = 3e^x - 3e^{2x} + e^{3x}$ at $x = 0$.
(b) The series for $r(x) = e^x$ about $x = 1$.
(c) The series for $q(x) = \ln(1-x)$ about $x = 0$.
(d) The series for $s(x) = 1 + x + x^2 + x^3 + x^4$ about $x = 1$.
(e) The series for $t(x) = 1/\cos(x)$ about $x = 0$. (Here you should remember that $t(x)$ is an even function, and use this to simplify your calculation.)

Solution:

(a) Here it is simplest to use the known series for e^t to get

$$\begin{aligned}p(x) =&3(1 + x + x^2/2 + x^3/6)\\ &- 3(1 + 2x + 4x^2/2 + 8x^3/6)\\ &+ (1 + 3x + 9x^2/2 + 27x^3/6) + O(x^4)\\ =&(3-3+1) + (3-6+3)x + (3-12+9)x^2/2 + (3-24+27)x^3/6 + O(x^4)\\ =&1 + x^3 + O(x^4).\end{aligned}$$

Alternatively, we have

$$\begin{aligned}p(x) &= 3e^x - 3e^{2x} + e^{3x} & p(0) &= 3-3+1 = 1\\ p'(x) &= 3e^x - 6e^{2x} + 3e^{3x} & p'(0) &= 3-6+3 = 0\\ p''(x) &= 3e^x - 12e^{2x} + 9e^{3x} & p''(0) &= 3-12+9 = 0\\ p'''(x) &= 3e^x - 24e^{2x} + 27e^{3x} & p'''(0) &= 3-24+27 = 6,\end{aligned}$$

so the Taylor coefficients are

$$\begin{aligned} a_0 &= p(0)/0! = 1 \\ a_1 &= p'(0)/1! = 0 \\ a_2 &= p''(0)/2! = 0 \\ a_3 &= p'''(0)/3! = 1, \end{aligned}$$

so $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + O(x^4) = 1 + x^3 + O(x^4)$ again.

(b) We have $r(x) = r'(x) = r''(x) = r'''(x) = e^x$, so $r(1) = r'(1) = r''(1) = r'''(1) = e$, so the series is

$$\begin{aligned} r(x) &= r(1) + r'(1)(x-1) + r''(1)(x-1)^2/2 + r'''(1)(x-1)^3/6 + O((x-1)^4) \\ &= e + e(x-1) + e(x-1)^2/2 + e(x-1)^3/6 + O((x-1)^4). \end{aligned}$$

Alternatively, we have

$$r(x) = e.e^{x-1} = e.\sum_{k=0}^{\infty}(x-1)^k/k! = e + e(x-1) + e(x-1)^2/2 + e(x-1)^3/6 + O((x-1)^4).$$

(c) We have

$$q(x) = \ln(1-x) \qquad q'(x) = \frac{-1}{1-x} \qquad q''(x) = \frac{-1}{(1-x)^2} \qquad q'''(x) = \frac{-2}{(1-x)^3},$$

so

$$q(0) = 0 \qquad q'(0) = -1 \qquad q''(0) = -1 \qquad q'''(0) = -2.$$

This gives

$$q(x) = \frac{0}{0!} + \frac{-1}{1!}x + \frac{1}{2!}x^2 + \frac{2}{3!}x^3 + O(x^4) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + O(x^4).$$

This could also be obtained by integrating both sides of the equation $1/(1-x) = 1 + x + x^2 + x^3 + O(x^4)$.

(d) We have

$$\begin{aligned} s(x) &= 1 + x + x^2 + x^3 + x^4 \\ s'(x) &= 1 + 2x + 3x^2 + 4x^3 \\ s''(x) &= 2 + 6x + 12x^2 \\ s'''(x) &= 6 + 24x, \end{aligned}$$

so $s(1) = 5$, $s'(1) = 10$, $s''(1) = 20$ and $s'''(1) = 30$. Thus

$$\begin{aligned} s(x) &= s(1) + s'(1)(x-1) + s''(1)(x-1)^2/2 + s'''(1)(x-1)^3/6 + O((x-1)^4) \\ &= 5 + 10(x-1) + 10(x-1)^2 + 5(x-1)^3 + O((x-1)^4). \end{aligned}$$

Alternatively, we have the geometric progression formula $s(x) = (x^5-1)/(x-1)$. This can be written in terms of the variable $y = x-1$ as $s(x) = ((y+1)^5 - 1)/y$. We also have the binomial expansion

$$(y+1)^5 = 1 + 5y + 10y^2 + 10y^3 + 5y^4 + y^5,$$

and it follows easily that $s(x) = 5 + 10y + 10y^2 + 5y^3 + y^4$ as before.

(e) Here the easiest method is to observe that $\cos(x) = 1 - x^2/2 + O(x^4)$ and that $(1-x^2/2)(1+x^2/2) = 1 - x^4/4$, which is $1 + O(x^4)$, so

$$1/\cos(x) = 1/(1-x^2/2) + O(x^4) = 1 + x^2/2 + O(x^4).$$

The more obvious approach is as follows. We write $c = \cos(x)$ and $s = \sin(x)$ for brevity, and recall that $s^2 = 1 - c^2$.

$$\begin{aligned} t(x) &= c^{-1} \\ t'(x) &= -c^{-2}.(-s) = s\,c^{-2} \\ t''(x) &= c.c^{-2} + s.(-2c^{-3}.(-s)) = c^{-1} + 2s^2c^{-3} \\ &= c^{-1} + 2(1-c^2)c^{-3} = -c^{-1} + 2c^{-3} \\ t'''(x) &= -sc^{-2} + 2.(-3).c^{-4}.(-s) = -sc^{-2} + 6sc^{-4}. \end{aligned}$$

When we put $x = 0$ we get $s = 0$ and $c = 1$, so $t(0) = 1$, $t'(0) = 0$, $t''(0) = 1$ and $t'''(0) = 0$. This gives

$$t(x) = t(0) + t'(0)x + t''(0)x^2/2 + t'''(0)x^3/6 + O(x^4) = 1 + x^2/2 + O(x^4)$$

just as before. Note that we could have saved some work if we used the fact that $t(x)$ is even, so only even powers of x can occur in the Taylor series. In particular, the coefficient of x^3 is automatically zero, so we did not really need to find $t'''(x)$.

Exercise 10.2. Put $y = \tan(x)$ and recall that $dy/dx = 1 + \tan(x)^2 = 1 + y^2$. Differentiating this equation gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(1 + y^2) = 2y\frac{dy}{dx} = 2y(1 + y^2) = 2y + 2y^3.$$

Continue this process to find d^ky/dx^k for $k = 3$, 4 and 5.

If we put $x = 0$ then $y = \tan(x) = \tan(0) = 0$ and so

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 2 \times 0 + 2 \times 0^3 = 0.$$

In the same way, find the value of d^ky/dx^k for all $k \leq 5$, and so write down the 6th order Taylor series for $\tan(x)$ at $x = 0$.

Solution:

$$\begin{aligned}
y &= \tan(x) \\
dy/dx &= 1 + y^2 \\
d^2y/dx^2 &= 2y(1 + y^2) = 2y + 2y^3 \\
d^3y/dx^3 &= (2 + 6y^2)(1 + y^2) = 2 + 8y^2 + 6y^4 \\
d^4y/dx^4 &= (16y + 24y^3)(1 + y^2) = 16y + 40y^3 + 24y^5 \\
d^5y/dx^5 &= (16 + 120y^2 + 120y^4)(1 + y^2) = 16 + 136y^2 + 240y^4 + 120y^6.
\end{aligned}$$

If we put $x = 0$ then also $y = 0$ so the various derivatives (starting with y itself) become $0, 1, 0, 2, 0, 16$. The Taylor series is therefore

$$\begin{aligned}
\tan(x) &= 0 + 1.x + 0.\frac{x^2}{2!} + 2.\frac{x^3}{3!} + 0.\frac{x^4}{4!} + 16.\frac{x^5}{5!} + O(x^6) \\
&= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + O(x^6).
\end{aligned}$$

Exercise 10.3. Let $f(x)$ be the 5th order Taylor series for e^x at $x = 0$. Write down $f(x)$. Expand out $f(x)f(-x)$, discarding any terms involving x^k for $k \geq 5$. What do you get, and why?

Solution: First, we have

$$f(x) = \sum_{k=0}^{4} x^k/k! = 1 + x + x^2/2 + x^3/6 + x^4/24.$$

This gives

$$\begin{aligned}
f(x)f(-x) =& (1 + x + x^2/2 + x^3/6 + x^4/24)(1 - x + x^2/2 - x^3/6 + x^4/24) + O(x^5) \\
=& 1 - x + x^2/2 - x^3/6 + x^4/24 + \\
& x - x^2 + x^3/2 - x^4/6 + \\
& x^2/2 - x^3/2 + x^4/4 + \\
& x^3/6 - x^4/6 + \\
& x^4/24 + O(x^5) \\
=& 1 + (-1 + 1)x + (\tfrac{1}{2} - 1 + \tfrac{1}{2})x^2 + (-\tfrac{1}{6} + \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{6})x^3 + (\tfrac{1}{24} - \tfrac{1}{6} + \tfrac{1}{4} - \tfrac{1}{6} + \tfrac{1}{24})x^4 + O(x^5) \\
=& 1 + (-1 + 1)x + \tfrac{1-2+1}{2}x^2 + \tfrac{-1+3-3+1}{6}x^3 + \tfrac{1-4+6-4+1}{24}x^4 + O(x^5) \\
=& 1 + O(x^5).
\end{aligned}$$

This is as expected, because up to fifth-order corrections we have $f(x)f(-x) = e^x e^{-x} = 1$.

Exercise 10.4. Put $y = 1/(1-x)$.

(a) Find $\frac{d^k y}{dx^k}$ for $k = 1, 2, 3, 4$ and guess the general formula.
(b) Hence write down the 4th order Taylor series for y at $x = 0$. We will call this u.
(c) Expand out $(1-x)u$, and explain the answer.
(d) Work out the 4th order Taylor series for $1/(1-x)^2$. Check that it is the same as what you get by squaring u and expanding it out, discarding terms of the form x^k with $k \geq 4$.

Solution:

(a)

$$\begin{aligned} y &= (1-x)^{-1} \\ \frac{dy}{dx} &= -(1-x)^{-2}.(-1) = (1-x)^{-2} \\ \frac{d^2y}{dx^2} &= (-2)(1-x)^{-3}.(-1) = 2(1-x)^{-3} \\ \frac{d^3y}{dx^3} &= 2.(-3)(1-x)^{-4}.(-1) = 6(1-x)^{-4} \\ \frac{d^4y}{dx^4} &= 6.(-4)(1-x)^{-5}.(-1) = 24(1-x)^{-5} \end{aligned}$$

It should be clear from this that $d^k y/dx^k = k!(1-x)^{-k-1}$.

(b) We see from (a) that the value of the k'th derivative at $x = 0$ is $k!.(1-0)^{-k-1} = k!$. To get the corresponding coefficient a_k in the Taylor series, we must divide by $k!$, giving $a_k = 1$. Thus

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + O(x^4).$$

Of course the full Taylor series is $(1-x)^{-1} = \sum_{k=0}^{\infty} x^k$, which is just the familiar formula for the sum of a geometric progression.

(c) Expanding out $(1-x)u$ gives

$$\begin{aligned} (1-x)u &= (1-x)(1 + x + x^2 + x^3) \\ &= 1 + x + x^2 + x^3 - x - x^2 - x^3 - x^4 = 1 - x^4. \end{aligned}$$

In other words, $(1-x)u = 1$ up to 4'th order corrections, which is reasonable because $u = (1-x)^{-1}$ up to 4'th order corrections.

(d) We have

$$\begin{aligned} \frac{d}{dx}(1-x)^{-2} &= 2(1-x)^{-3} \\ \frac{d^2}{dx^2}(1-x)^{-2} &= 6(1-x)^{-4} \\ \frac{d^3}{dx^3}(1-x)^{-2} &= 24(1-x)^{-5} \end{aligned}$$

so

$$(1-x)^{-2} = \frac{1}{0!} + \frac{2}{1!}x + \frac{6}{2!}x^2 + \frac{24}{3!}x^3 + O(x^4) = 1 + 2x + 3x^2 + 4x^3 + O(x^4).$$

On the other hand, we have

$$\begin{aligned} u^2 =& (1 + x + x^2 + x^3)(1 + x + x^2 + x^3) \\ =& 1 + x + x^2 + x^3 + \\ & x + x^2 + x^3 + \\ & x^2 + x^3 + \\ & x^3 + O(x^4) \\ =& 1 + 2x + 3x^2 + 4x^3 + O(x^4), \end{aligned}$$

which is the same.

Exercise 10.5. Using the standard series

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \qquad \sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \qquad \exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

write down the 6th order Taylor series for $\cos(x)$, $\sin(x)$ and e^{ix} at $x = 0$. Check that to this order, the series for e^{ix} agrees with the series for $\cos(x) + i\sin(x)$. (Here i is the square root of -1, so $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ and so on.)

Solution:

$$\begin{aligned}
\cos(x) &= 1 + \frac{(-1)x^2}{2!} + \frac{(-1)^2x^4}{4!} + O(x^6) \\
&= 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6) \\
\sin(x) &= x + \frac{(-1)x^3}{3!} + \frac{(-1)^2x^5}{5!} + O(x^6) \\
&= x - \frac{x^3}{6} + \frac{x^5}{120} \\
e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + O(x^6) \\
&= 1 + ix - \frac{x^2}{2} - i\frac{x^3}{6} + \frac{x^4}{24} + i\frac{x^5}{120} + O(x^6) \\
&= \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) + i\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)
\end{aligned}$$

From this it is visible that the series for $\cos(x) + i\sin(x)$ is the same as the one for e^{ix}, as we expect from De Moivre's theorem.

Exercise 10.6. The Bessel function $y = J_2(x)$ has the form $y = x^2/8 + ax^4 + O(x^6)$ for some constant a, and it satisfies the differential equation $x^2y'' + xy' + (x^2 - 4)y = 0$. Starting with the given series for y, work out the series for $x^2y'' + xy' + (x^2 - 4)y$ (to order 6, again) and thus work out what the constant a must be.

Solution:

$$\begin{aligned}
y &= x^2/8 + ax^4 + O(x^6) \\
y' &= x/4 + 4ax^3 + O(x^5) \\
y'' &= 1/4 + 12ax^2 + O(x^4) \\
xy' &= x^2/4 + 4ax^4 + O(x^6) \\
x^2y'' &= x^2/4 + 12ax^4 + O(x^6) \\
(x^2 - 4)y &= x^4/8 - x^2/2 - 4ax^4 + O(x^6) \\
&= -x^2/2 + (1/8 - 4a)x^4 + O(x^6)
\end{aligned}$$

so

$$x^2y'' + xy' + (x^2 - 4)y = (\tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{2})x^2 + (4a + 12a + \tfrac{1}{8} - 4a)x^4 + O(x^6) = (12a + 1/8)x^4 + O(x^6).$$

If this is to be zero for all x, we must have $12a = -1/8$ and so $a = -1/96$.

(Notice how the order of accuracy changes in the above calculation. The series for y has correction terms like x^6 and higher, but when we differentiate them the order drops by one, so our series for y' is only accurate to order 5. This does not matter, because our formula only involves xy', and when we multiply by x the correction terms get moved up to order 6 again. Similarly, our series for y'' is only accurate to order 4, but our series for x^2y'' is accurate to order 6.)

Exercise 10.7. The 3rd order Taylor series for $\sqrt{\cos(x)}$ at $x = 0$ has the form $\sqrt{\cos(x)} = 1 + ax + bx^2 + O(x^3)$ for some constants a and b. Square this and compare with the standard series for $\cos(x)$, and hence find a and b.

Solution: On the one hand, we have

$$\begin{aligned}\cos(x) &= (1+ax+bx^2)^2 + O(x^3)\\ &= 1 + a^2x^2 + b^2x^4 + 2ax + 2bx^2 + 2abx^3 + O(x^3)\\ &= 1 + 2ax + (a^2+2b)x^2 + O(x^3).\end{aligned}$$

On the other hand, we have the standard series $\cos(x) = 1 - x^2/2 + O(x^3)$. By comparing coefficients we see that $2a = 0$ and $a^2 + 2b = -1/2$, so $a = 0$ and $b = -1/4$. This means that

$$\sqrt{\cos(x)} = 1 - x^2/4 + O(x^3).$$

Exercise 10.8. Consider the function $f(x) = (2x+3)/(3x+4)$.

(a) Calculate $f'(x)$ and $f''(x)$.
(b) Write down the third order Taylor series for $f(x)$ at $x = 0$.
(c) Observe that

$$f(x) = \frac{3+2x}{4}\,\frac{1}{1-(-3x/4)}.$$

Using this and the standard geometric progression formula $1/(1-u) = \sum_{k=0}^{\infty} u^k$, get another third order series for $f(x)$. Check that it is the same as in (b).

Solution:

(a)

$$\begin{aligned}f'(x) &= \frac{2.(3x+4)-(2x+3).3}{(3x+4)^2} = \frac{-1}{(3x+4)^2} = -(3x+4)^{-2}\\ f''(x) &= -(-2).(3x+4)^{-3}.3 = 6(3x+4)^{-3}.\end{aligned}$$

(b) From the above, we have $f(0) = 3/4$ and $f'(0) = -4^{-2} = -1/16$ and $f''(0) = 6/4^3 = 3/32$. It follows that

$$f(x) = f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2 + O(x^3) = \tfrac{3}{4} - \tfrac{1}{16}x + \tfrac{3}{64}x^2 + O(x^3).$$

(c) The geometric progression formula gives

$$\frac{1}{1-(-3x/4)} = \sum_{k=0}^{\infty}\left(-\frac{3x}{4}\right)^k = 1 - \frac{3x}{4} + \frac{9x^2}{16} + O(x^3).$$

Multiplying this by $(2x+3)/4$ gives

$$f(x) = \frac{3+2x}{4}\,\frac{1}{1-(-3x/4)} = (\tfrac{3}{4}+\tfrac{x}{2})(1-\tfrac{3}{4}x+\tfrac{9}{16}x^2) + O(x^3) \quad = \tfrac{3}{4} - \tfrac{9}{16}x + \tfrac{27}{64}x^2 + \tfrac{1}{2}x - \tfrac{3}{8}x^2 + O(x^3)$$
$$= \tfrac{3}{4} - \tfrac{1}{16}x + \tfrac{3}{24}x^2,$$

just as before.

Exercise 10.9. Find numbers a, b and c such that the function $f(x) = a\frac{x-b}{x-c}$ has Taylor series $8 + 2x + x^2 + O(x^3)$.

Solution: We have

$$\begin{aligned}f(x) &= a(x-b)/(x-c)\\ f'(x) &= a\frac{(x-c)-(x-b)}{(x-c)^2} = a(b-c)(x-c)^{-2}\\ f''(x) &= -2a(b-c)(x-c)^{-3}\\ f(0)/0! &= ab/c\\ f'(0)/1! &= a(b-c)/c^2\\ f''(0)/2! &= a(b-c)/c^3.\end{aligned}$$

To get the right Taylor series, we must therefore have $ab/c = 8$ and $a(b-c)/c^2 = 2$ and $a(b-c)/c^3 = 1$. We can divide the last two equations to get $c = 2$, and substitute in the first two equations to get $ab = 16$ and $a(b-2)/4 = 2$, or equivalently $ab - 2a = 8$. From these equations it is not hard to see that $a = 4$ and then $b = 4$. In summary, we have $f(x) = 4(x-4)/(x-2)$.

Exercise 11.1. Show that $\dfrac{1}{\tanh(x)} - \tanh(x) = \dfrac{2}{\sinh(2x)}$.

Solution: Put $u = e^x$, so

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{u - u^{-1}}{u + u^{-1}}.$$

It follows that

$$\begin{aligned}
1/\tanh(x) - \tanh(x) &= \frac{u + u^{-1}}{u - u^{-1}} - \frac{u - u^{-1}}{u + u^{-1}} \\
&= \frac{(u + u^{-1})^2 - (u - u^{-1})^2}{(u + u^{-1})(u - u^{-1})} \\
&= \frac{u^2 + 2 + u^{-2} - u^2 + 2 - u^{-2}}{u^2 - u^{-2}} \\
&= \frac{4}{u^2 - u^{-2}} = 2/((u^2 - u^{-2})/2) \\
&= 2/\sinh(2x).
\end{aligned}$$

Exercise 11.2. Show that $\dfrac{1 + \tanh(x)^2}{1 - \tanh(x)^2} = \cosh(2x)$.

Solution: Put $u = e^x$. Then

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{(u - u^{-1})/2}{(u + u^{-1})/2} = \frac{u - u^{-1}}{u + u^{-1}},$$

so

$$\begin{aligned}
1 + \tanh(x)^2 &= 1 + \left(\frac{u - u^{-1}}{u + u^{-1}}\right)^2 = 1 + \frac{u^2 - 2 + u^{-2}}{u^2 + 2 + u^{-2}} \\
&= \frac{(u^2 + 2 + u^{-2}) + (u^2 - 2 + u^{-2})}{u^2 + 2 + u^{-2}} = \frac{2u^2 + 2u^{-2}}{u^2 + 2 + u^{-2}} \\
1 - \tanh(x)^2 &= 1 - \left(\frac{u - u^{-1}}{u + u^{-1}}\right)^2 = 1 - \frac{u^2 - 2 + u^{-2}}{u^2 + 2 + u^{-2}} \\
&= \frac{(u^2 + 2 + u^{-2}) - (u^2 - 2 + u^{-2})}{u^2 + 2 + u^{-2}} = \frac{4}{u^2 + 2 + u^{-2}}
\end{aligned}$$

so

$$\frac{1 + \tanh(x)^2}{1 - \tanh(x)^2} = \frac{2u^2 + 2u^{-2}}{4} = \frac{e^{2x} + e^{-2x}}{2} = \cosh(2x).$$

Exercise 11.3. Find $\int \sinh(x)^3 \, dx$, expressing your answer in terms of hyperbolic functions.

Solution: We have

$$\sinh(x)^3 = \left(\frac{e^x - e^{-x}}{2}\right)^3 = \frac{1}{8}(e^{3x} - 3e^x + 3e^{-x} - e^{-3x})$$

so

$$\begin{aligned}
\int \sinh(x)^3 \, dx &= \frac{1}{8}\int e^{3x} - 3e^x + 3e^{-x} - e^{-3x} \, dx = \tfrac{1}{24}e^{3x} - \tfrac{3}{8}e^x - \tfrac{3}{8}e^{-x} + \tfrac{1}{24}e^{-3x} \\
&= \tfrac{1}{12}\cosh(3x) - \tfrac{3}{4}\cosh(x).
\end{aligned}$$

Exercise 11.4. Let a, b and n be constants. Find $f'(x)$, where $f(x) = \left(\dfrac{x - a}{x - b}\right)^n$.

Solution: Put $u = (x-a)/(x-b)$ and $y = f(x) = u^n$. Then

$$\frac{du}{dx} = \frac{1.(x-b)-(x-a).1}{(x-b)^2} = \frac{a-b}{(x-b)^2},$$

so

$$f'(x) = \frac{dy}{dx} = nu^{n-1}\frac{du}{dx} = n(a-b)\left(\frac{x-a}{x-b}\right)^{n-1}(x-b)^{-2} = n(a-b)(x-a)^{n-1}(x-b)^{-n-1}.$$

Exercise 11.5. Find $\frac{d}{dx}(\log(x^2+x^{-2}))$. Simplify your answer as much as possible.

Solution:

$$\frac{d}{dx}(\log(x^2+x^{-2})) = \frac{1}{x^2+x^{-2}}\frac{d}{dx}(x^2+x^{-2}) = \frac{2x-2x^{-3}}{x^2+x^{-2}} = \frac{2(x^4-1)}{x(x^4+1)}.$$

Exercise 11.6. Find $\frac{d}{dx}\left(\frac{x^4+x^2+1}{x^3-x}\right)$.

Solution:

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^4+x^2+1}{x^3-x}\right) &= \frac{(4x^3+2x)(x^3-x)-(x^4+x^2+1)(3x^2-1)}{(x^3-x)^2} \\ &= \frac{4x^6-4x^4+2x^4-2x^2-3x^6+x^4-3x^4+x^2-3x^2+1}{(x^3-x)^2} \\ &= \frac{x^6-4x^4-4x^2+1}{(x^3-x)^2}\end{aligned}$$

Exercise 11.7. Find dy/dx in terms of x and y, where x and y are related by the equation $e^{-x^2-xy-y^2}\sin(x) = a$.

Solution: Differentiate the relation to get

$$(-2x-y)e^{-x^2-xy-y^2}\sin(x) + (-x-2y)e^{-x^2-xy-y^2}\sin(x)\frac{dy}{dx} + e^{-x^2-xy-y^2}\cos(x) = 0.$$

We can now multiply by $e^{x^2+xy+y^2}$ and rearrange to get

$$\frac{dy}{dx} = \frac{\cos(x)-(2x+y)\sin(x)}{(x+2y)\sin(x)}.$$

Exercise 11.8. Find dv/du, where u and v are related by the equation $uv^2+u^2v^3 = 1$.

Solution: Apply d/du to the equation to get

$$v^2 + 2uv\frac{dv}{du} + 2uv^3 + 3u^2v^2\frac{dv}{du} = 0$$

Rearrange this to get

$$(2uv+3u^2v^2)\frac{dv}{du} = -v^2-2uv^3$$

and so

$$\frac{dv}{du} = -\frac{v^2+2uv^3}{2uv+3u^2v^2} = -\frac{v(1+2uv)}{u(2+3uv)}$$

Exercise 11.9. Find dy/dx in terms of t, where $x = t-\sin(t)$ and $y = 1-\cos(t)$.

Solution:

$$\begin{aligned} dx/dt &= 1-\cos(t) \\ dy/dt &= \sin(t) \\ \frac{dy}{dx} &= \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{\sin(t)}{1-\cos(t)}.\end{aligned}$$

Exercise 11.10. By making a suitable substitution, find $\int \sin(x)\log(\cos(x))\,dx$.

Solution: Put $u = \cos(x)$, so $du = -\sin(x)\,dx$. Then

$$\int \sin(x)\log(\cos(x))\,dx = -\int \log(u)\,du = -(u\log(u) - u) = u(1 - \log(u))$$
$$= \cos(x)(1 - \log(\cos(x))).$$

Exercise 11.11. Find $\displaystyle\int_0^1 x\sin(\pi x)\,dx$.

Solution: We integrate by parts, using $u = x$ and $dv/dx = \sin(\pi x)$, so $du/dx = 1$ and $v = -\cos(\pi x)/\pi$. This gives

$$\begin{aligned}\int_0^1 x\sin(\pi x)\,dx &= \left[-\frac{x\cos(\pi x)}{\pi}\right]_0^1 + \int_0^1 \frac{\cos(\pi x)}{\pi}\,dx \\ &= \left[-\frac{x\cos(\pi x)}{\pi} + \frac{\sin(\pi x)}{\pi^2}\right]_0^1 = \left[\frac{\sin(\pi x) - \pi x\cos(\pi x)}{\pi^2}\right]_0^1 \\ &= ((\sin(\pi) - \pi\cos(\pi)) - (\sin(0) - 0))/\pi^2 = 1/\pi\end{aligned}$$

Exercise 11.12. Find $\int (x^3 + x^2 + x + 1)e^{-x}\,dx$.

Solution: For general reasons, we know that

$$\int (x^3 + x^2 + x + 1)e^{-x}\,dx = (ax^3 + bx^2 + cx + 1)e^{-x}$$

for some constants a, b, c, d . Differentiating this gives

$$(x^3{+}x^2{+}x{+}1)e^{-x} = (3ax^2{+}2bx{+}c)e^{-x} - (ax^3{+}bx^2{+}cx{+}d)e^{-x} = (-ax^3{+}(3a{-}b)x^2{+}(2b{-}c)x{+}(c{-}d))e^{-x}.$$

Comparing coefficients gives $-a = 1$ and $3a - b = 1$ and $2b - c = 1$ and $c - d = 1$, so $a = -1$ and $b = -4$ and $c = -9$ and $d = -10$. Thus

$$\int (x^3 + x^2 + x + 1)e^{-x}\,dx = -(x^3 + 4x^2 + 9x + 10)e^{-x}.$$

Exercise 11.13. You may assume that

$$\int \cos(x)^{-4}\,dx = \tan(x)\left(a + b\cos(x)^{-2}\right)$$

for some constants a and b. Find these constants.
[Hint: you will need the identities $\tan(x) = \sin(x)/\cos(x)$ and $\sin(x)^2 = 1 - \cos(x)^2$.]

Solution: Differentiating both sides gives

$$\cos(x)^{-4} = \tan'(x)\left(a + b\cos(x)^{-2}\right) + \tan(x).b.(-2)\cos(x)^{-3}.\cos'(x).$$

We know that $\tan'(x) = \cos(x)^{-2}$ and $\cos'(x) = -\sin(x)$, so

$$\cos(x)^{-4} = a\cos(x)^{-2} + b\cos(x)^{-4} + 2b\tan(x)\cos(x)^{-3}\sin(x).$$

We now use the identities $\tan(x) = \sin(x)/\cos(x)$ and $\sin(x)^2 = 1 - \cos(x)^2$ to get

$$\begin{aligned}\cos(x)^{-4} &= a\cos(x)^{-2} + b\cos(x)^{-4} + 2b\frac{\sin(x)}{\cos(x)}\cos(x)^{-3}\sin(x) \\ &= a\cos(x)^{-2} + b\cos(x)^{-4} + 2b\sin(x)^2\cos(x)^{-4} \\ &= a\cos(x)^{-2} + b\cos(x)^{-4} + 2b(1 - \cos(x)^2)\cos(x)^{-4} \\ &= (a - 2b)\cos(x)^{-2} + 3b\cos(x)^{-4}.\end{aligned}$$

For this to match up we must have $a - 2b = 0$ and $3b = 1$, so $b = 1/3$ and $a = 2/3$.