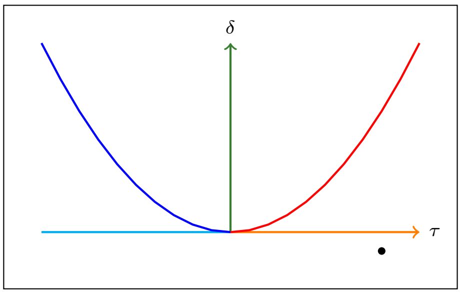
Methods for Differential Equations (NTech)
All exam questions

Question 1

(1)

(i) Suppose we have four linear systems with properties described below. In each case, find the type of equilibrium at the origin. **(8 marks)**

- The matrix for system A has characteristic polynomial $t^2 + t + 1$.
- System B corresponds to the following point in the (τ, δ) plane:



- The solution for system C involves a term $e^{4t}\cos(7t)$
- Both eigenvalues for system D are real and negative.

(ii) Consider the system $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$, where $A = \begin{bmatrix} 110 & -10 \\ 100 & 0 \end{bmatrix}$.

(a) Find the eigenvalues of A. **(4 marks)**

(b) Find a matrix P (depending on t) such that $\dot{P} = AP$, and $P = I$ when $t = 0$. **(6 marks)**

(c) Find the solution to the system for which $x = 0$ and $y = 90$ when $t = 0$. **(3 marks)**

(iii) Which of the following matrices corresponds to a system with a clockwise centre at the origin? **(4 marks)**

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \quad \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

Solution:All this is similar to questions in the lectures and on the problem sheets.

(i)
- The eigenvalues for system A are $(-1 \pm \sqrt{1-4})/2 = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}i$. These are complex with negative real part, so the system is a stable focus. **[2]**
- The (τ, δ) point for B lies below the horizontal axis, so $\delta < 0$, so the system is a saddle. **[2]**

– The solution for C involves

$$e^{4t}\cos(7t) = \tfrac{1}{2}(e^{(4+7i)t} + e^{(4-7i)t}).$$

This means that the eigenvalues must be $4 \pm 7i$, which are complex with positive real part, so the system is an unstable focus. [2]

– As the eigenvalues for D are real and negative, it is a stable node. [2]

(ii) (a) The trace and determinant are $\tau = 110$ and $\delta = 1000$ [2], so

$$\sqrt{\tau^2 - 4\delta} = \sqrt{12100 - 4000} = \sqrt{8100} = 90.$$

Thus, the eigenvalues are $(110 \pm 90)/2$, which gives $\lambda_1 = 10$ and $\lambda_2 = 100$. [2]

(b) The simplest approach is to use the standard formula

$$P = (\lambda_2 - \lambda_1)^{-1}((e^{\lambda_2 t} - e^{\lambda_1 t})A + (\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})I).\text{[3]}$$

In the present case, this becomes

$$\begin{aligned} P &= \frac{1}{90}\left((e^{100t} - e^{10t})\begin{bmatrix}110 & -10\\ 100 & 0\end{bmatrix} + (100e^{10t} - 10e^{100t})\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) \\ &= \frac{1}{90}\begin{bmatrix}110(e^{100t} - e^{10t}) + (100e^{10t} - 10e^{100t}) & -10(e^{100t} - e^{10t}) \\ 100(e^{100t} - e^{10t}) & 100e^{10t} - 10e^{100t}\end{bmatrix} \\ &= \frac{1}{90}\begin{bmatrix}100e^{100t} - 10e^{10t} & -10e^{100t} + 10e^{10t} \\ 100e^{100t} - 100e^{10t} & 100e^{100t} - 10e^{10t}\end{bmatrix}.\text{[3]} \end{aligned}$$

Alternatively, we can use the diagonalisation method. For that, we must find the eigenvectors of A. We first note that $A - 10I = \begin{bmatrix}100 & -10\\ 100 & -10\end{bmatrix}$, so the vector $v_1 = \begin{bmatrix}1\\ 10\end{bmatrix}$ is an eigenvector of eigenvalue $\lambda_1 = 10$. Similarly, we have $A - 100I = \begin{bmatrix}10 & -10\\ 100 & -100\end{bmatrix}$, so the vector $v_1 = \begin{bmatrix}1\\ 1\end{bmatrix}$ is an eigenvector of eigenvalue $\lambda_2 = 100$. It follows that $A = VDV^{-1}$, where

$$D = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} = \begin{bmatrix}10 & 0\\ 0 & 100\end{bmatrix}$$

and

$$V = \left[\, v_1 \,\middle|\, v_2 \,\right] = \begin{bmatrix}1 & 1\\ 10 & 1\end{bmatrix} \qquad V^{-1} = -\frac{1}{9}\begin{bmatrix}1 & -1\\ -10 & 1\end{bmatrix}.$$

This in turn gives $P = VEV^{-1}$, where

$$E = \begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix} = \begin{bmatrix}e^{10t} & 0\\ 0 & e^{100t}\end{bmatrix}.$$

One can check that this gives the same answer as before.

(iii) The relevant solution is

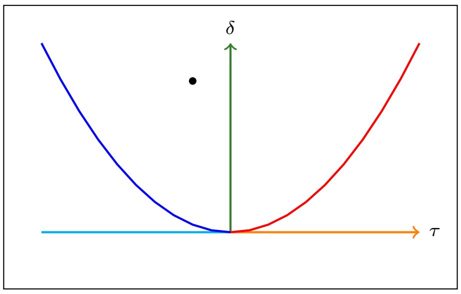
$$\begin{bmatrix}x\\ y\end{bmatrix} = P\begin{bmatrix}x_0\\ y_0\end{bmatrix} \text{[1]} = \frac{1}{90}\begin{bmatrix}100e^{100t} - 10e^{10t} & -10e^{100t} + 10e^{10t}\\ 100e^{100t} - 100e^{10t} & 100e^{100t} - 10e^{10t}\end{bmatrix}\begin{bmatrix}0\\ 90\end{bmatrix} \text{[1]} = \begin{bmatrix}-10e^{100t} + 10e^{10t}\\ 100e^{100t} - 10e^{10t}\end{bmatrix} \text{[1]}$$

(iii) Consider a matrix $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$ with $\tau = a + b$ and $\delta = ad - bc$. This corresponds to a centre if $\tau = 0$ and $\delta > 0$; if so, then the rotation is clockwise if $c < 0 < b$ and anticlockwise if $b < 0 < c$ [2]. Only the 4th and 5th matrices in the list have $c < 0$. The 4th one has $\delta = 3$, and the 5th has $\delta = -3$, and both have $\tau = 0$. It follows that the 4th matrix $\begin{bmatrix}1 & 2\\ -2 & -1\end{bmatrix}$ is the only one that gives an clockwise centre [2].

(2)

(i) Suppose we have four linear systems with properties described below. In each case, find the type of equilibrium at the origin. **(8 marks)**

- The matrix for system A has one positive eigenvalue and one negative eigenvalue.
- System B corresponds to the following point in the (τ, δ) plane:



- The solution for system C involves e^{-11t} and e^{-111t}.
- The matrix for system D has characteristic polynomial $t^2 + 10t + 100$.

(ii) Consider the system $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$, where $A = \begin{bmatrix} 16 & -25 \\ 13 & -20 \end{bmatrix}$.

(a) Find the eigenvalues of A. **(4 marks)**

(b) Find a matrix P (depending on t) such that $\dot{P} = AP$, and $P = I$ when $t = 0$. **(6 marks)**

(c) Find the solution to the system for which $x = 25$ and $y = 18$ when $t = 0$. **(3 marks)**

(iii) Which of the following matrices corresponds to a system with an anticlockwise focus at the origin? **(4 marks)**

$$\begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & -2 \\ -2 & -2 \end{bmatrix} \quad \begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix} \quad \begin{bmatrix} 2 & 2 \\ -2 & -1 \end{bmatrix}$$

Solution:All this is similar to questions in the lectures and on the problem sheets.

(i)
- Any linear system with one positive eigenvalue and one negative eigenvalue is a saddle. [2]
- The (τ, δ) point for B lies above the parabola, so $\tau^2 - 4\delta < 0$, so we have a focus. It is stable because $\tau < 0$. [2]
- The solution for C involves e^{-11t} and e^{-111t}, so the eigenvalues must be -11 and -111. These are both real and negative, so we have a stable node. [2]
- Recall that the characteristic polynomial is always $t^2 - \tau t + \delta$. Thus, for D we have $\tau = -10 < 0$ and $\delta = 100 > 0$, so $\tau^2 - 4\delta = -300 < 0$. We therefore have a stable focus, just as in system B. [2]

(ii) (a) The trace and determinant are $\tau = 16 - 20 = -4$ and $\delta = 16 \times (-20) - 13 \times (-25) = 5$ [2], so

$$\sqrt{\tau^2 - 4\delta} = \sqrt{16 - 20} = \sqrt{-4} = 2i.$$

Thus, the eigenvalues are $(-4 \pm 2i)/2$, which gives $\lambda_1 = -2 - i$ and $\lambda_2 = -2 + i$. [2]

(b) The simplest approach is to use the standard formula

$$P = e^{\lambda t}(\cos(\omega t)I + \omega^{-1}\sin(\omega t)(A - \lambda I))\,[3]$$

In the present case, we have $\lambda = -2$ and $\omega = 1$ so

$$\begin{aligned} P &= e^{-2t}\left(\cos(t)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \sin(t)\begin{bmatrix}18 & -25\\ 13 & -18\end{bmatrix}\right) \\ &= \begin{bmatrix} e^{-2t}(\cos(t)+18\sin(t)) & -25e^{-2t}\sin(t) \\ 13e^{-2t}\sin(t) & e^{-2t}(\cos(t)-18\sin(t)) \end{bmatrix}.\,[3] \end{aligned}$$

(c) The relevant solution is

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} = P\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}\,[1] &= \begin{bmatrix} e^{-2t}(\cos(t)+18\sin(t)) & -25e^{-2t}\sin(t) \\ 13e^{-2t}\sin(t) & e^{-2t}(\cos(t)-18\sin(t)) \end{bmatrix}\begin{bmatrix} 25 \\ 18 \end{bmatrix} \\ &= e^{-2t}\begin{bmatrix} 25\cos(t)+450\sin(t)-450\sin(t) \\ 325\sin(t)+18\cos(t)-324\sin(t) \end{bmatrix} = e^{-2t}\begin{bmatrix} 25\cos(t) \\ \sin(t)+18\cos(t) \end{bmatrix}.\,[2] \end{aligned}$$

(iii) The matrices in the question are as follows:

		τ	δ	$\tau^2 - 4\delta$	type
A_1	$\begin{bmatrix}-2 & 2\\ 2 & -1\end{bmatrix}$	-3	-2	17	saddle
A_2	$\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}$	3	-2	17	saddle
A_3	$\begin{bmatrix}1 & -2\\ -2 & -2\end{bmatrix}$	-1	-6	25	saddle
A_4	$\begin{bmatrix}2 & -2\\ 2 & -1\end{bmatrix}$	1	2	-7	anticlockwise unstable focus
A_5	$\begin{bmatrix}2 & -1\\ 2 & -2\end{bmatrix}$	0	-2	8	saddle
A_6	$\begin{bmatrix}2 & 2\\ -2 & -1\end{bmatrix}$	1	2	-7	clockwise unstable focus

Most of them have $\delta < 0$ so they are saddles. Matrices A_4 and A_6 have $\tau^2 - 4\delta < 0$ so they are foci (and $\tau > 0$ so they are unstable). In A_4 the bottom left entry is positive, so the rotation is anticlockwise. In A_6 the bottom left entry is negative, so the rotation is clockwise. Thus, the only anticlockwise focus is A_4.[4]

(3)

(i) Consider the equations

$$\dot{x} = (a+b)x + 2by \qquad \dot{y} = -bx + (a-b)y,$$

where a and b are nonzero real constants. Show that the system always has a focus at $(0,0)$. Give examples to show that the focus can be stable or unstable, and clockwise or anticlockwise, depending on the values of a and b. **(7 marks)**

(ii) Consider the functions

$$x = e^t + 2e^{2t} \qquad y = e^t - 2e^{2t},$$

and the vector function $u = \begin{bmatrix} x \\ y \end{bmatrix}$. Find a constant matrix A such that $\dot{u} = Au$. **(6 marks)**

(iii) For each of the following matrices A_k, find a matrix P_k (depending on t) such that $\dot{P}_k = AP_k$, and $P_k = I$ when $t = 0$. **(12 marks)**

$$A_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Solution:

(i) **Students have seen many questions like this where the matrices have numerical entries, but only a few where the matrices have symbolic entries.**

The corresponding matrix is $A = \begin{bmatrix} a+b & 2b \\ -b & a-b \end{bmatrix}$ [1], with trace $\tau = 2a$ and determinant

$$\delta = (a+b)(a-b) - 2b.(-b) = a^2 - b^2 + 2b^2 = a^2 + b^2 \text{[1]}.$$

This gives

$$\tau^2 - 4\delta = 4a^2 - 4(a^2 + b^2) = -4b^2 < 0, \text{[1]}$$

so we have a focus or centre [1]. However, we have $\tau = 2a$ and $a \neq 0$ by assumption so we cannot have a centre, and we must instead have a focus [1]. If $a < 0$ then $\tau < 0$ so the focus is stable; similarly, if $a > 0$ then the focus is unstable [1]. The direction of rotation is controlled by the bottom left entry in A, which is $-b$. If $b < 0$ then $-b > 0$ so the rotation is anticlockwise, but if $b > 0$ then the rotation is clockwise [1].

(ii) **This is unseen.**

Consider a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We then have

$$\dot{u} = \begin{bmatrix} e^t + 4e^{2t} \\ e^t - 4e^{2t} \end{bmatrix} \text{[1]}$$

$$Au = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e^t + 2e^{2t} \\ e^t - 2e^{2t} \end{bmatrix} = \begin{bmatrix} (a+b)e^t + (2a-2b)e^{2t} \\ (c+d)e^t + (2c-2d)e^{2t} \end{bmatrix} \text{[1]}$$

To get $\dot{u} = Au$, we must have $a+b = c+d = 1$ and $2a-2b = 4$ and $2c-2d = -4$ [2]. These equations can easily be solved to give $a = d = 3/2$ and $b = c = -1/2$, so

$$A = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}. \text{[2]}$$

(iii) **This is standard. The students have been told that they need to memorise the relevant formulae.**

All three matrices A_k have trace $\tau = 2$, and the determinants are $0, 1$ and 2. The corresponding values of $\tau^2 - 4\delta$ are 4, 0 and -4.

(a) For A_0, the eigenvalues are $(2 \pm \sqrt{4})/2$, which gives $\lambda_1 = 0$ and $\lambda_2 = 2$ (both real) [1]. The standard formula in this context is

$$P = \frac{1}{\lambda_2 - \lambda_1}\left((\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})I + (e^{\lambda_2 t} - e^{\lambda_1 t})A\right).[2]$$

In the present case, this becomes

$$P_0 = \frac{1}{2}\left((2e^0 - 0e^{2t})I + (e^{2t} - e^0)A_0\right) = \frac{1}{2}\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + (e^{2t} - 1)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$$
$$= \frac{1}{2}\begin{bmatrix} e^{2t}+1 & e^{2t}-1 \\ e^{2t}-1 & e^{2t}+1 \end{bmatrix}.[1]$$

(b) For A_1, the eigenvalues are $(2 \pm \sqrt{0})/2$, so $\lambda = 1$ is a repeated eigenvalue [1]. The standard formula in this context is

$$P = e^{\lambda t}(I + t(A - \lambda I)).[2]$$

In the present case, this becomes

$$P_1 = e^t\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} e^t & t\,e^t \\ 0 & e^t \end{bmatrix}.[1]$$

(c) For A_2, the eigenvalues are $(2 \pm \sqrt{-4})/2$, which gives $1 \pm i$ [1]. The standard formula in this context is

$$P = e^{\lambda t}\left(\cos(\omega t)I + \omega^{-1}\sin(\omega t)(A - \lambda I)\right).[2]$$

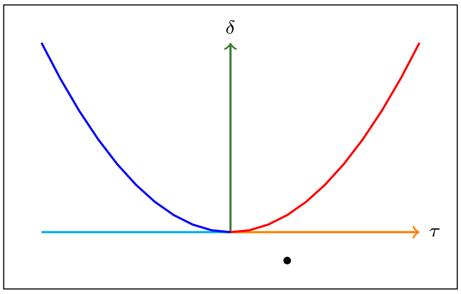
In the present case we have $\lambda = \omega = 1$, giving

$$P = e^t\left(\cos(t)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin(t)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = e^t\begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}.[1]$$

(4)

(i) Suppose we have four linear systems with properties described below. In each case, find the type of equilibrium at the origin. **(8 marks)**

- The matrix for system A has two complex eigenvalues with positive real part.
- System B corresponds to the following point in the (τ, δ) plane:



- The solution for system C involves $e^{-\pi t}\cos(3t)$.
- The matrix for system D has characteristic polynomial $t^2 + 5t - 50$.

(ii) Consider the equations

$$\dot{x} = ax + by \qquad \dot{y} = bx + ay,$$

where a and b are nonzero real constants. Give examples to show that the system can have a saddle, a stable node or an unstable node at $(0,0)$, depending on the values of a and b, but it cannot have a focus or centre. **(8 marks)**

(iii) Which of the following matrices corresponds to a system with an anticlockwise focus at the origin? **(5 marks)**

$$\begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & -2 \\ -2 & -2 \end{bmatrix} \quad \begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix} \quad \begin{bmatrix} 2 & 2 \\ -2 & -1 \end{bmatrix}$$

(iv) Let A be a 2×2 matrix with real eigenvalues λ_1 and λ_2, where $\lambda_1 \neq \lambda_2$. Give a formula for a matrix $P(t)$ satisfying $P(0) = I$ and $\dot{P} = AP$. **(4 marks)**

Solution:All this is similar to questions in the lectures, problem sheets and past papers.

(i)
- The system must be an unstable focus. [2]
- The (τ, δ) point for B lies below the τ-axis, so $\delta < 0$, so we have a saddle. [2]
- The solution for C involves $e^{-\pi t}\cos(3t)$, so the eigenvalues must be $-\pi \pm 3i$. These are complex with strictly negative real part, so we have a stable focus. [2]
- Recall that the characteristic polynomial is always $t^2 - \tau t + \delta$. Thus, for D we have $\tau = -5 < 0$ and $\delta = -50 < 0$. As $\delta < 0$, this is a saddle. [2]

(ii) The corresponding matrix is $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ [1], with trace $\tau = 2a$ [1]and determinant $\delta = a^2 - b^2$ [1]. This gives

$$\tau^2 - 4\delta = 4a^2 - 4(a^2 - b^2) = 4b^2 > 0.[1]$$

For a focus or centre we would have $\tau^2 - 4\delta < 0$, so we cannot have a focus or centre [1]. If $a^2 < b^2$ then $\delta < 0$ and we have a saddle; for example, this occurs when $a = 1$ and $b = 2$ [1]. If $a < 0$ and $a^2 > b^2$ then we have a stable node; for example, this happens when $a = -2$

and $b = -1$ [1]. If $a > 0$ and $a^2 > b^2$ then we have an unstable node; for example, this happens when $a = 2$ and $b = 1$ [1].

All this could alternatively be discussed in terms of the eigenvalues of A, which are

$$(\tau \pm \sqrt{\tau^2 - 4\delta})/2 = (2a \pm 2b)/2 = a + b \text{ and } a - b.$$

(iii) The matrices in the question are as follows:

		τ	δ	$\tau^2 - 4\delta$	type
A_1	$\begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix}$	-3	-2	17	saddle
A_2	$\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$	3	-2	17	saddle
A_3	$\begin{bmatrix} 1 & -2 \\ -2 & -2 \end{bmatrix}$	-1	-6	25	saddle
A_4	$\begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix}$	1	2	-7	anticlockwise unstable focus
A_5	$\begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix}$	0	-2	8	saddle
A_6	$\begin{bmatrix} 2 & 2 \\ -2 & -1 \end{bmatrix}$	1	2	-7	clockwise unstable focus

Most of them have $\delta < 0$ so they are saddles. Matrices A_4 and A_6 have $\tau^2 - 4\delta < 0$ so they are foci (and $\tau > 0$ so they are unstable). In A_4 the bottom left entry is positive, so the rotation is anticlockwise. In A_6 the bottom left entry is negative, so the rotation is clockwise. Thus, the only anticlockwise focus is A_4.[5]

(iv) The standard formula is

$$P = \frac{1}{\lambda_2 - \lambda_1}\left((\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})I + (e^{\lambda_2 t} - e^{\lambda_1 t})A\right) \text{ [4]}$$

(5)

(i) Consider the matrix

$$P = e^{2t}\begin{bmatrix} 7\cos(3t)+\sin(3t) & 4\cos(3t)-3\sin(3t) \\ \cos(3t)+\sin(3t) & \cos(3t) \end{bmatrix}.$$

Find a matrix A such that $\dot{P} = AP$. Is P the fundamental solution for A? **(6 marks)**

(ii) For any number $a \in \mathbb{R}$, we put

$$Q(a) = \begin{bmatrix} a/2 & 1/8 \\ a^2-1 & a/2 \end{bmatrix}.$$

Find numbers $a_1, \dots, a_5$ such that

- $Q(a_1)$ has a stable node at the origin.
- $Q(a_2)$ has a stable focus at the origin.
- $Q(a_3)$ has a centre at the origin.
- $Q(a_4)$ has an unstable focus at the origin.
- $Q(a_5)$ has an unstable node at the origin.

For $Q(a_3)$, is the rotation clockwise or anticlockwise? For $Q(a_5)$, what are the eigenvalues? **(10 marks)**

(iii) Consider the matrix

$$A = \begin{bmatrix} 0 & \ln(2)\ln(3) \\ -1 & \ln(6) \end{bmatrix}.$$

Find the eigenvalue and eigenvectors. Find a matrix P depending on t such that $\dot{P} = AP$, and $P = I$ when $t = 0$. **(9 marks)**

Solution:

(i) **This is a minor variation on a standard theme.** Write $s = \sin(3t)$ and $c = \cos(3t)$ and $A = \begin{bmatrix} m & n \\ p & q \end{bmatrix}$. We then have

$$\begin{aligned}
\dot{P} &= 2e^{2t}\begin{bmatrix} 7c+s & 4c-3s \\ c+s & c \end{bmatrix} + e^{2t}\begin{bmatrix} -21s+3c & -12s-9c \\ -3s+3c & -3s \end{bmatrix} \\
&= e^{2t}\begin{bmatrix} 17c-19s & -c-18s \\ 5c-s & 2c-3s \end{bmatrix} \text{[1]} \\
AP &= e^{2t}\begin{bmatrix} m & n \\ p & q \end{bmatrix}\begin{bmatrix} 7c+s & 4c-3s \\ c+s & c \end{bmatrix} \\
&= e^{2t}\begin{bmatrix} (7m+n)c+(m+n)s & (4m+n)c-3ms \\ (7p+q)c+(p+q)s & (4p+q)c-3ps \end{bmatrix} \text{[1]}
\end{aligned}$$

We therefore want

$$\begin{aligned}
7m+n &= 17 & m+n &= -19 \\
4m+n &= -1 & -3m &= -18 \\
7p+q &= 5 & p+q &= -1 \\
4p+q &= 2 & -3p &= -3. \text{[1]}
\end{aligned}$$

These are easily solved to give $m = 6$ and $n = -25$ and $p = 1$ and $q = -2$, so $A = \begin{bmatrix} 6 & -25 \\ 1 & -2 \end{bmatrix}$ [2]. If P was the fundamental solution for A then we would not only have $\dot{P} = AP$, but also $P = I$ when $t = 0$. In fact we have $P = \begin{bmatrix} 7 & 4 \\ 1 & 1 \end{bmatrix}$ when $t = 0$, so P is not the fundamental solution. [1]

(ii) **Students have seen similar problems.** For the matrix $Q(a)$ we have

$$\begin{aligned} \tau &= a/2 + a/2 = a\,[1] \\ \delta &= (a/2)^2 - (a^2-1)/8 = (a^2+1)/8 > 0 \\ \tau^2 - 4\delta &= a^2 - (a^2+1)/2 = (a^2-1)/2.\,[1] \end{aligned}$$

- For a stable node we need $\tau < 0$ and $\delta > 0$ and $\tau^2 - 4\delta > 0$, so $a < 0$ and $a^2 > 1$. We can therefore take $a_1 = -3$. [1]
- For a stable focus we need $\tau < 0$ and $\delta > 0$ and $\tau^2 - 4\delta < 0$, so $a < 0$ and $a^2 < 1$. We can therefore take $a_2 = -1/2$. [1]
- For a centre we need $\tau = 0$ and $\delta > 0$; the only possibility is to take $a_3 = 0$. [1]
- For an unstable focus we need $\tau > 0$ and $\delta > 0$ and $\tau^2 - 4\delta < 0$, so $a > 0$ and $a^2 < 1$. We can therefore take $a_4 = 1/2$. [1]
- For an unstable node we need $\tau > 0$ and $\delta > 0$ and $\tau^2 - 4\delta > 0$, so $a > 0$ and $a^2 > 1$. We can therefore take $a_5 = 3$. [1]

The matrix $Q(a_3)$ is $\begin{bmatrix} 0 & 1/8 \\ -1 & 0 \end{bmatrix}$. As the bottom left entry is negative and the top right entry is positive, we see that the rotation is clockwise. [1]The matrix $Q(a_5)$ has $\tau = 3$ and $\tau^2 - 4\delta = (3^2 - 1)/2 = 4$ so the eigenvalues are

$$\lambda_1, \lambda_2 = (\tau \pm \sqrt{\tau^2 - 4\delta})/2 = (3 \pm 2)/2 = 1/2,\ 5/2.\,[2]$$

(iii) **This is a standard problem.** We have $\tau = 0 + \ln(6) = \ln(2) + \ln(3)$ [1]and

$$\delta = 0 \times \ln(6) - (-1) \times \ln(2)\ln(3) = \ln(2)\ln(3).\,[1]$$

This gives

$$\chi_A(t) = t^2 - \tau t + \delta = t^2 - (\ln(2) + \ln(3))t + \ln(2)\ln(3) = (t - \ln(2))(t - \ln(3)),$$

so the eigenvalues are $\lambda_1 = \ln(2)$ and $\lambda_2 = \ln(3)$. [1]For an eigenvector $u_1 = \begin{bmatrix} p \\ q \end{bmatrix}$ of eigenvalue $\ln(2)$, we need $(A - \ln(2)I)u_2 = 0$, or equivalently

$$\begin{bmatrix} -\ln(2) & \ln(2)\ln(3) \\ -1 & \ln(3) \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This reduces to $p = \ln(3)q$, so we can take $u_1 = \begin{bmatrix} \ln(3) \\ 1 \end{bmatrix}$. [1]Similarly, we can take $u_2 =$

$\begin{bmatrix} \ln(2) \\ 1 \end{bmatrix}$. [1]This gives

$$
\begin{aligned}
U &= [u_1|u_2] = \begin{bmatrix} \ln(3) & \ln(2) \\ 1 & 1 \end{bmatrix} \text{[1]} \\
U^{-1} &= \frac{1}{\ln(3) - \ln(2)} \begin{bmatrix} 1 & -\ln(2) \\ -1 & \ln(3) \end{bmatrix} \\
D &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \ln(2) & 0 \\ 0 & \ln(3) \end{bmatrix} \text{[1]} \\
e^{tD} &= \begin{bmatrix} e^{t\ln(2)} & 0 \\ 0 & e^{t\ln(3)} \end{bmatrix} = \begin{bmatrix} 2^t & 0 \\ 0 & 3^t \end{bmatrix} \\
P &= Ue^{tD}U^{-1}\text{[1]} = \frac{1}{\ln(3) - \ln(2)} \begin{bmatrix} \ln(3) & \ln(2) \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^t & 0 \\ 0 & 3^t \end{bmatrix} \begin{bmatrix} 1 & -\ln(2) \\ -1 & \ln(3) \end{bmatrix} \\
&= \frac{1}{\ln(3) - \ln(2)} \begin{bmatrix} \ln(3)2^t - \ln(2)3^t & \ln(2)\ln(3)(3^t - 2^t) \\ 2^t - 3^t & 3^t\ln(3) - 2^t\ln(2) \end{bmatrix} .\text{[1]}
\end{aligned}
$$

(6)

(i) Draw a diagram of the (τ, δ)-plane, and show which regions correspond to saddles, stable nodes, unstable nodes, stable foci and unstable foci. **(5 marks)**

(ii) For any constant $a \in \mathbb{R}$, we can consider the system S_a given by the following equations:

$$\dot{x} = x + 2y$$
$$\dot{y} = -2x + (a-1)y.$$

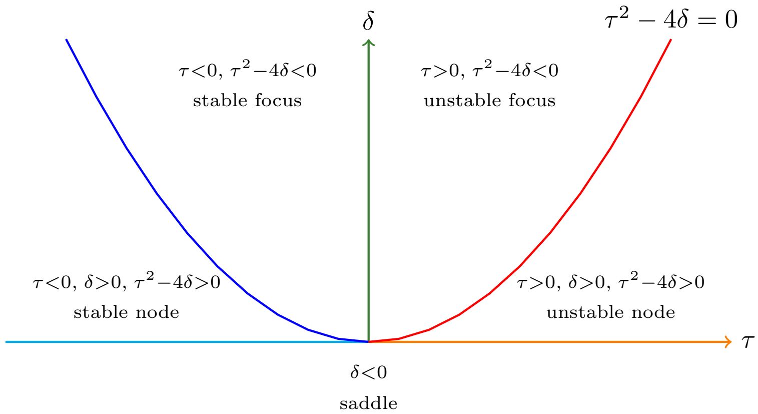
Draw a diagram to show the points corresponding to S_a in the (τ, δ)-plane. For which values of a does S_a have a saddle, stable node, unstable node, stable focus or unstable focus at the origin? **(13 marks)**

(iii) In the cases where S_a has a focus at the origin, is the rotation clockwise or anticlockwise? **(2 marks)**

(iv) Consider the matrix $A = \begin{bmatrix} a+b & -bc \\ b/c & a-b \end{bmatrix}$. Find a matrix P depending on t such that $\dot{P} = AP$, and $P = I$ when $t = 0$. **(5 marks)**

Solution:

(i) The diagram is as follows:[5]



(ii) The system S_a corresponds to the matrix $A_a = \begin{bmatrix} 1 & 2 \\ -2 & a-1 \end{bmatrix}$ with $\tau = a$ [1]and

$$\delta = 1 \times (a-1) - 2 \times (-2) = a + 3, [1]$$

which gives

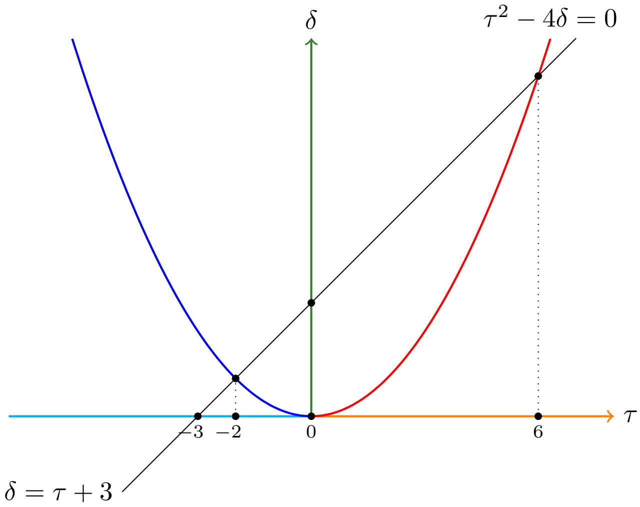
$$\tau^2 - 4\delta = a^2 - 4a - 12 = (a+2)(a-6).[1]$$

Thus:

- When $a < -3$ we have $\delta < 0$ and so the system has a saddle.
- When $-3 < a < -2$ we have $\tau < 0$ and $\delta > 0$ and $\tau^2 - 4\delta > 0$ so the system has a stable node.
- When $-2 < a < 0$ we have $\tau < 0$ and $\delta > 0$ and $\tau^2 - 4\delta < 0$ so the system has a stable focus.
- When $0 < a < 6$ we have $\tau > 0$ and $\delta > 0$ and $\tau^2 - 4\delta < 0$ so the system has an unstable focus.

- When $a > 6$ we have $\tau > 0$ and $\delta > 0$ and $\tau^2 - 4\delta > 0$ so the system has an unstable node. [5]

The corresponding points lie on the line $\delta = \tau + 3$, which can be illustrated as follows: [5]



(iii) In the matrix A_a, the bottom left entry is negative and the top right entry is positive. This means that the rotation is clockwise (in cases where there is a focus). [2]

(iv) We have $\tau = (a+b) + (a-b) = 2a$ and

$$\delta = (a+b)(a-b) - (-bc)b/c = a^2 - b^2 + b^2 = a^2,$$

so $\tau^2 - 4\delta = 0$ [2]. This means that $\lambda = a$ in the only eigenvalue. The standard formula for this case is

$$P = e^{\lambda t}(I + (A - \lambda I)t)\textbf{[2]} = e^{at}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} b & -bc \\ b/c & -b \end{bmatrix} t\right) = e^{at}\begin{bmatrix} 1+tb & -tbc \\ tb/c & 1-tb \end{bmatrix}.\textbf{[1]}$$

(7)

(i) Let θ be a constant with $-\pi/2 < \theta < \pi/2$, and consider the matrix

$$T = \begin{bmatrix} \sin(\theta) & -1 \\ 1 & -\sin(\theta) \end{bmatrix}.$$

(a) Find and simplify trace(T), $\det(T)$ and T^2. **(3 marks)**

(b) Find a matrix P depending on t such that $\dot{P} = TP$, and $P = I$ when $t = 0$. **(5 marks)**

(ii) For any number $a \in \mathbb{R}$, we put $Q(a) = \begin{bmatrix} a & 3a^2 - 2 \\ 2a^2 + 3 & 5a \end{bmatrix}$.

(a) For which values of a does $Q(a)$ have a saddle?

(b) For which values of a does $Q(a)$ have a stable node?

(c) For which values of a does $Q(a)$ have an unstable node?

(d) For which values of a does $Q(a)$ have a stable focus?

(e) For which values of a does $Q(a)$ have an unstable focus?

(f) In (d) and (e), is the rotation clockwise or anticlockwise?

(12 marks)

(iii) Consider the system

$$\dot{x} = 30x - 200y \qquad\qquad \dot{y} = 2x - 10y.$$

Find a solution of the form $x = me^{\lambda t}$ and $y = e^{\lambda t}$ for some constants m and λ. **(5 marks)**

Solution:

(i) **This is a standard problem.**

(a) We have $\tau = \text{trace}(T) = \sin(\theta) - \sin(\theta) = 0$ [1]and

$$\delta = \det(T) = -\sin^2(\theta) - (-1) = \cos^2(\theta).\text{[1]}$$

We also have

$$T^2 = \begin{bmatrix} \sin(\theta) & -1 \\ 1 & -\sin(\theta) \end{bmatrix} \begin{bmatrix} \sin(\theta) & -1 \\ 1 & -\sin(\theta) \end{bmatrix} = \begin{bmatrix} \sin^2(\theta) - 1 & \sin(\theta) - \sin(\theta) \\ \sin(\theta) - \sin(\theta) & \sin^2(\theta) - 1 \end{bmatrix} = -\cos^2(\theta)I.\text{[1]}$$

(b) The eigenvalues are $(-\tau \pm \sqrt{\tau^2 - 4\delta})/2$, which simplifies to $\pm i\cos(\theta)$ [1]. In other words, the eigenvalues are $\lambda \pm i\omega$ with $\lambda = 0$ and $\omega = \cos(\theta)$. In this context we have the standard formula

$$\begin{aligned}
P &= e^{\lambda t}(\cos(\omega t)I + \omega^{-1}\sin(\omega t)(T - \lambda I))\text{[2]} \\
&= \cos(\cos(\theta)t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\sin(\cos(\theta)t)}{\cos(\theta)} \begin{bmatrix} \sin(\theta) & -1 \\ 1 & -\sin(\theta) \end{bmatrix} \\
&= \frac{1}{\cos(\theta)} \begin{bmatrix} \cos(\cos(\theta)t)\cos(\theta) + \sin(\cos(\theta)t)\sin(\theta) & -\sin(\cos(\theta)t) \\ \sin(\cos(\theta)t) & \cos(\cos(\theta)t)\cos(\theta) - \sin(\cos(\theta)t)\sin(\theta) \end{bmatrix} \text{[2]} \\
&= \frac{1}{\cos(\theta)} \begin{bmatrix} \cos(\cos(\theta)t - \theta) & -\sin(\cos(\theta)t) \\ \sin(\cos(\theta)t) & \cos(\cos(\theta)t + \theta) \end{bmatrix}
\end{aligned}$$

(ii) **This has the same form as a question on last year's exam, but with a different matrix.** For the matrix $Q(a)$ we have

$$\begin{aligned}\tau &= a + 5a = 6a\text{[1]}\\ \delta &= 5a^2 - (2a^2+3)(3a^2-2) = 6 - 6a^4\text{[1]}\\ \Delta = \tau^2 - 4\delta &= 24a^4 + 36a^2 - 24 = 12(2a^2-1)(a^2+2).\text{[1]}\end{aligned}$$

It follows that

- $\tau > 0$ if and only if $a > 0$. [1]
- $\delta > 0$ if and only if $-1 < a < 1$. [1]
- $\Delta > 0$ if and only if $2a^2 - 1 > 0$ iff $a < -1/\sqrt{2}$ or $a > 1/\sqrt{2}$. [1]

We now see that:

(a) For a saddle we need $\delta < 0$, so $a < -1$ or $a > 1$. [1]
(b) For a stable node we need $\tau < 0$ and $\delta, \Delta > 0 > 0$ so $-1 < a < -1/\sqrt{2}$. [1]
(c) For an unstable node we need $\tau > 0$ and $\delta, \Delta > 0 > 0$ so $1/\sqrt{2} < a < 1$. [1]
(d) For a stable focus we need $\tau < 0$ and $\delta > 0$ and $\Delta < 0$, so $-1/\sqrt{2} < a < 0$. [1]
(e) For an unstable focus we need $\tau > 0$ and $\delta > 0$ and $\Delta < 0$, so $0 < a < 1/\sqrt{2}$. [1]
(f) The bottom left entry of $Q(a)$ is $2a^2 + 3 > 0$, so the rotation is anticlockwise. [1]

(iii) **This is essentially a standard problem, but phrased in a slightly unusual way.** If $x = me^{\lambda t}$ and $y = e^{\lambda t}$ then we have $\dot{x} = m\lambda e^{\lambda t}$ and $\dot{y} = \lambda e^{\lambda t}$ [1]so

$$\begin{aligned}\dot{x} - (30x - 200y) &= (m\lambda - 30m + 200)e^{\lambda t}\\ \dot{y} - (2x - 10y) &= (\lambda - 2m + 10)e^{\lambda t}.\text{[1]}\end{aligned}$$

We therefore need $m\lambda - 30m + 200 = 0$ and $\lambda - 2m + 10 = 0$ [1]. The second of these gives $\lambda = 2m - 10$, which we substitute in the first to get

$$\begin{aligned}0 &= m(2m-10) - 30m + 200 = 2m^2 - 40m + 200\\ &= 2(m^2 - 20m + 100) = 2(m-10)^2.\end{aligned}$$

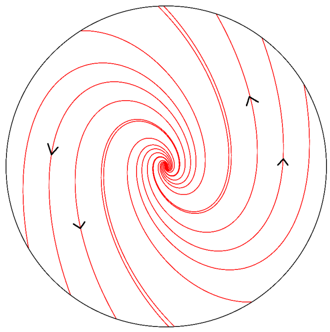
We must therefore have $m = 10$ and $\lambda = 2m - 10 = 10$, so $x = 10e^{10t}$ and $y = e^{10t}$. [2]

(An alternative approach: the corresponding matrix is $A = \begin{bmatrix} 30 & -200 \\ 2 & -10 \end{bmatrix}$, which has $\tau = 20$ and $\delta = 100$ and $\Delta = \tau^2 - 4\delta = 0$. From this we see that the only eigenvalue is $\lambda = \tau/2 = 10$. We have $A - 10I = \begin{bmatrix} 20 & -200 \\ 2 & -20 \end{bmatrix}$, and from this we see that $\begin{bmatrix} 10 \\ 1 \end{bmatrix}$ is an eigenvector. It follows that the required solution is $\begin{bmatrix} x \\ y \end{bmatrix} = e^{10t}\begin{bmatrix} 10 \\ 1 \end{bmatrix}$.)

(8)

(i) Suppose we have four planar linear systems with properties described below. In each case, find the type of equilibrium at the origin. **(12 marks)**

- System A has $\tau = \delta = 10$.
- System B has a solution with $x = e^{-3t}\cos(4t)$.
- System C has eigenvalues -10 and 11.
- System D has the following phase diagram:



- System E has a continuous conserved quantity that is not just a constant, and the equilibrium point at the origin is stable.
- System F has a strong Lyapunov function, and one of the eigenvalues is -3.

(ii) Give examples as follows. The numbers in every matrix should be real numbers. **(8 marks)**

(a) Give an example of a linear system with an anticlockwise centre at the origin.

(b) Give an example of a matrix B where $\tau = 3$ and $\delta = 0$.

(c) Give an example of a matrix C where the eigenvalues are $1 + i$ and $1 - i$.

(d) Give an example of a linear system for which the function $U = xy$ is a conserved quantity.

Solution:

(i) (A) Here $\tau, \delta > 0$ and $\tau^2 - 4\Delta = 60 > 0$ so we have an unstable node. [2]

(B) Here we must have eigenvalues $-3 \pm 4i$, giving a stable focus. [2]

(C) Here there is one negative eigenvalue and one positive eigenvalue so we have a saddle. [2]

(D) The picture shows an anticlockwise unstable focus. [2]

(E) As there is a nontrivial conserved quantity, we can only have a saddle or a centre. As saddles are unstable, we must have a centre. [2]

(F) As there is a strong Lyapunov function, the origin must be asymptotically stable, so it is a stable node or a stable focus. For a stable focus, neither eigenvalues is real. As one of the eigenvalues is -3, we must have a stable node. [2]

(ii) (a) We need $\dot{x} = ax + by$ and $\dot{y} = cx + dy$ with $\tau = a + d = 0$ and $\delta = ad - bc > 0$ and $c > 0$. The simplest way to do this is with $a = d = 0$ and $b = -1$ and $c = 1$, giving $\dot{x} = -y$ and $\dot{y} = x$. [2]

(b) We need $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a + d = 3$ and $ad - bc = 0$. The simplest way to do this is with $a = 3$ and $b = c = d = 0$ giving $B = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$. Another possibility is $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$. **[2]**

(c) We need

$$\begin{aligned} \tau &= \lambda_1 + \lambda_2 = (1 - i) + (1 + i) = 2 \\ \delta &= \lambda_1 \lambda_2 = (1 - i)(1 + i) = 1 - i^2 = 2. \end{aligned}$$

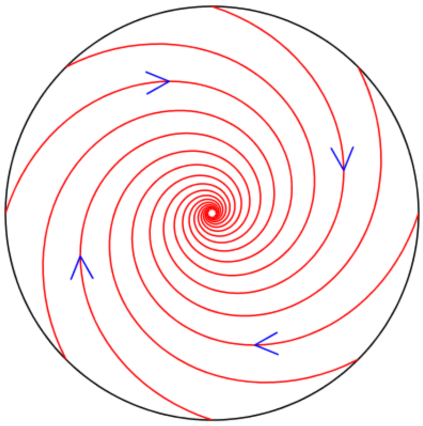
The simplest way to do this is with $D = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. **[2]**

(d) The simplest way to do this is with $\dot{x} = x$ and $\dot{y} = -y$. (This gives $\dot{U} = \dot{x}y + x\dot{y} = xy - xy = 0$. Alternatively, the solutions have the form $(x, y) = (x_0 e^t, y_0 e^{-t})$, giving $U = x_0 y_0$ for all t.) **[2]**

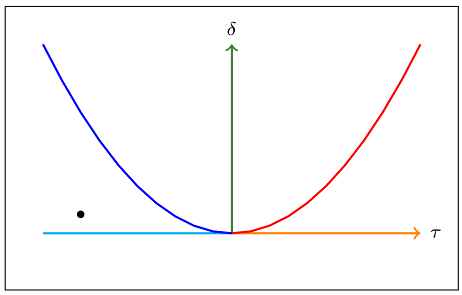
(9)

(i) Suppose we have four planar linear systems with properties described below. In each case, find the type of equilibrium at the origin. **(12 marks)**

- System A has a solution $(x, y) = (\cosh(t), \sinh(t))$.
- The matrix for system B has characteristic polynomial $t^2 - 100t + 1000$.
- System C has eigenvalues $10 + \pi i$ and $10 - \pi i$.
- System D has a continuous conserved quantity that is not just a constant, and the equilibrium point at the origin is stable.
- System E has the following phase diagram:



- System F corresponds to the following point in the (τ, δ)-plane.



(ii) Give examples as follows. The numbers in every matrix should be real numbers. **(8 marks)**

(a) Give an example of a linear system with an clockwise stable focus at the origin.

(b) Give an example of a matrix B where $\tau = 7$ and $\delta = 10$.

(c) Give an example of a matrix C where one eigenvalue is $10i$.

(d) Give an example of a linear system for which the function $U = x^2 + y^2$ is a conserved quantity.

Solution:

(i) (A) Here the exponentials involved in the colution

$$(x, y) = (\cosh(t), \sinh(t)) = (e^t + e^{-t}, e^t - e^{-t})/2$$

are e^t and e^{-t}, so the eigenvalues are $+1$ and -1. As there is one positive eigenvalue and one negative eigenvalue, we have a saddle. [2]

(B) Here $\tau = 100 > 0$ and $\delta = 1000 > 0$ and $\tau^2 - 4\delta = 6000 > 0$ so we have an unstable node. [2]

(C) Here we have two complex eigenvalues with positive real part so we have an unstable focus. [2]

(D) As there is a nontrivial conserved quantity, we can only have a saddle or a centre. As saddles are unstable, we must have a centre. [2]

(E) The picture shows an clockwise stable focus. [2]

(F) The marked point is in the region where $\tau < 0$ and $\delta > 0$ and $\tau^2 - 4\delta > 0$ so we have a stable node. [2]

(ii) (a) We need $\dot{x} = ax + by$ and $\dot{y} = cx + dy$ with $\tau = a + d < 0$ and $\delta = ad - bc > 0$ and $\tau^2 - 4\delta < 0$ and $c < 0$. The simplest way to do this is with $a = c = d = -1$ and $b = 1$, giving the matrix $A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$. [2]

(b) We need $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a + d = 7$ and $ad - bc = 10$. The simplest answer is $B = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$. There are also many other possibiities, such as $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$. [2]

(c) If one eigenvalue is $\lambda_1 = 10i$ then the other one must be $\lambda_2 = -10i$, giving $\tau = \lambda_1 + \lambda_2 = 0$ and $\delta = \lambda_1\lambda_2 = 100$. The simplest matrix like this is $C = \left[\begin{smallmatrix} 0 & 10 \\ -10 & 0 \end{smallmatrix}\right]$. [2]

(d) The simplest way to do this is with $\dot{x} = y$ and $\dot{y} = -x$. (This gives $\dot{U} = 2\dot{x}x + 2\dot{y}y = 2(yx - xy) = 0$.) [2]

Question 2

(10)

(i) Consider the system where

$$\dot{x} = x^2 + 2xy + y^2 - 1$$
$$\dot{y} = x^2 - 2xy + y^2 - 1.$$

(a) Draw a diagram showing the x-nullcline, the y-nullcline and the region where $\dot{x} < 0 < \dot{y}$. **(6 marks)**

(b) Find and classify the equilibrium points. **(6 marks)**

(ii) Consider the system where

$$\dot{x} = -2(1 + x^2 + y^2)y$$
$$\dot{y} = 3(1 + x^2 + y^2)x.$$

(a) Find p and q such that the function $U = px^2 + qy^2$ is a conserved quantity. **(3 marks)**

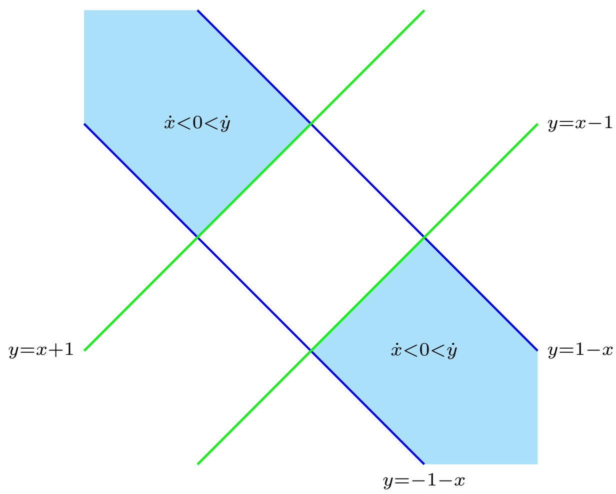
(b) Consider the flow line that has $(x, y) = (1, 6)$ at $t = 0$. Where does this cross the x-axis? **(4 marks)**

(c) Explain carefully why the origin is a stable equilibrium point, but is not asymptotically stable. **(3 marks)**

(d) Show that there are no other equilibrium points. **(3 marks)**

Solution: Most of this is similar to questions in the lectures and on the problem sheets. However, (ii)(b) which is unseen, and the students have rarely been asked for "careful explanation" as in (ii)(c).

(i) (a) The x-nullcline is given by $x^2+2xy+y^2 = 1$, or equivalently $(x+y)^2 = 1$, so $x+y = \pm 1$. Thus, it is the union of two straight lines, one with equation $y = 1 - x$ and the other with equation $y = -1 - x$ [2]. Similarly, the y-nullcline is given by $x^2 - 2xy + y^2 = 1$, or $x - y = \pm 1$. This is the union of the lines $y = x+1$ and $y = x-1$ [2]. For $\dot{x} < 0 < \dot{y}$ we must have $(x + y)^2 < 1 < (x - y)^2$, so $-1 < x + y < 1$ but $x - y$ is either less than -1 or greater than $+1$. The relevant region is shaded in the diagram below. [2]



(b) From part (i) it is clear that the equilibrium points are as follows:

$$a_1 = (0, 1) \qquad a_2 = (0, -1) \qquad a_3 = (1, 0) \qquad a_4 = (-1, 0).[2]$$

The Jacobian is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 2(x+y) & 2(x+y) \\ 2(x-y) & 2(y-x) \end{bmatrix}.[2]$$

Using this, the equilibrium points can be classified as follows:

	x	y	J	τ	δ	$\tau^2 - 4\delta$	type
a_1	0	1	$\left[\begin{smallmatrix} 2 & 2 \\ -2 & 2 \end{smallmatrix}\right]$	4	8	-16	unstable focus
a_2	0	-1	$\left[\begin{smallmatrix} -2 & -2 \\ 2 & -2 \end{smallmatrix}\right]$	-4	8	-16	stable focus
a_3	1	0	$\left[\begin{smallmatrix} 2 & 2 \\ 2 & -2 \end{smallmatrix}\right]$	0	-8	32	saddle
a_4	-1	0	$\left[\begin{smallmatrix} -2 & -2 \\ -2 & 2 \end{smallmatrix}\right]$	0	-8	32	saddle

[2]

(ii) (a) If $U = px^2 + qy^2$, then

$$\begin{aligned} \dot{U} &= U_x\dot{x} + U_y\dot{y}[1] = (2px)(-2(1+x^2+y^2)y) + (2qy)(3(1+x^2+y^2)x) \\ &= 2xy(1+x^2+y^2)(-2p+3q).[1] \end{aligned}$$

We now take $p = 3$ and $q = 2$, so $U = 3x^2+2y^2$. With these values we have $-2p+3q = 0$ so $\dot{U} = 0$, so U is a conserved quantity. [1]

(b) Suppose that when $t = 0$ we have $(x, y) = (1, 6)$, so $U = 3 \times 1^2 + 2 \times 6^2 = 75$ [1]. As U is conserved, we have $3x^2 + 2y^2 = 75$ for all t [1]. In particular, when the flow line crosses the x-axis we have $y = 0$ and so $3x^3 = 75$, which gives $x = \pm 5$. [1]Thus, the flow line crosses the x-axis at $(-5, 0)$ and $(5, 0)$ [1].

(c) It is clear that the function $U = 3x^2 + 2y^2$ is positive definite [1]. Moreover, the derivative $\dot{U} = 0$ is negative semidefinite, so U is a weak Lyapunov function, which means that the origin is a stable equilibrium [1]. However, as U is conserved, any flow line that starts away from the origin cannot converge to the origin. This means that the origin is not asymptotically stable [1].

(d) At any equilibrium point we must have $\dot{x} = -2(1+x^2+y^2)y = 0$ and $\dot{y} = 3(1+x^2+y^2)x = 0$ [1]. As $1+x^2+y^2$ is always strictly positive, we can divide by it [1], giving $-2y = 0$ and $3x = 0$, so $(x, y) = (0, 0)$. Thus, the origin is the only equilibrium point [1].

(11)

(i) Consider the system where

$$\dot{x} = x^2 + y^2 - 1$$
$$\dot{y} = x.$$

(a) Draw a diagram showing the x-nullcline, the y-nullcline and the region where $\dot{x}, \dot{y} < 0$. **(4 marks)**

(b) Find and classify the equilibrium points. **(5 marks)**

(c) Show that the function $U = e^{-2y}(x^2 + y^2 + y - \frac{1}{2})$ is a conserved quantity. **(4 marks)**

(ii) Consider the system where

$$\dot{x} = -\sin(x)\cos(y)$$
$$\dot{y} = -\cos(x)\sin(y),$$

and the functions $V = 1 - \cos(x)\cos(y)$ and $W = 2 - V = 1 + \cos(x)\cos(y)$.

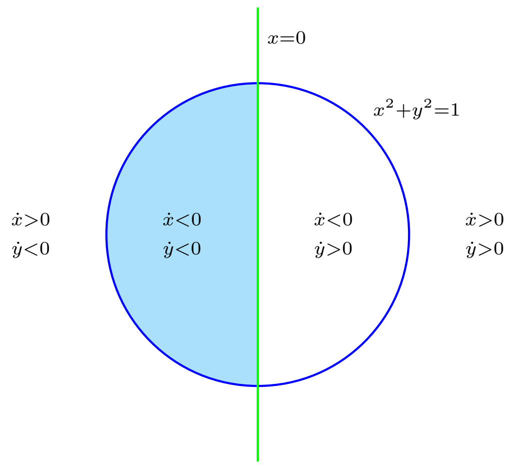
(a) Show that $V \geq 0$ and $\dot{V} \leq 0$. **(4 marks)**

(b) Find the equilibrium points, and the values of V at the equilibrium points. You should find that some equilibrium points have $V = 0$, some have $V = 1$ and some have $V = 2$. **(7 marks)**

(c) Use Lyapunov theory to show that the origin is an asymptotically stable equilibrium point. **(6 marks)**

Solution:

(i) (a) The x-nullcline is given by $x^2 + y^2 = 1$, which describes the unit circle. Inside the circle we have $\dot{x} < 0$, and outside the circle we have $\dot{x} > 0$ [1]. The y-nullcline is given by $x = 0$. In the left half plane we have $\dot{y} < 0$, and in the right half plane we have $\dot{y} > 0$ [1]. The required diagram is as follows [2]:



(b) From part (i) it is clear that the equilibrium points are as follows:

$$a_1 = (0, 1) \qquad a_2 = (0, -1).[1]$$

The Jacobian is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 1 & 0 \end{bmatrix}.[1]$$

Using this, the equilibrium points can be classified as follows:

	x	y	J	τ	δ	$\tau^2 - 4\delta$	type
a_1	0	1	$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$	0	-2	8	saddle[1]
a_2	0	-1	$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$	0	2	-8	anticlockwise centre[2]

(c) If $U = e^{-2y}(x^2 + y^2 + y - \frac{1}{2})$ then

$$\begin{aligned} U_x &= e^{-2y} \times 2x\text{[1]} \\ U_y &= -2e^{-2y}(x^2 + y^2 + y - \tfrac{1}{2}) + e^{-2y}(2y+1) \\ &= e^{-2y}(-2x^2 - 2y^2 + 2)\text{[1]} \\ \dot{U} &= (x^2 + y^2 - 1)U_x + xU_y \\ &= e^{-2y}(2x^3 + 2xy^2 - 2x - 2x^3 - 2xy^2 + 2x) = 0.\text{[2]} \end{aligned}$$

Thus, U is a conserved quantity.

(ii) (a) First, $\cos(x)$ and $\cos(y)$ both lie in the interval $[-1, 1]$, so $\cos(x)\cos(y)$ also lies in $[-1, 1]$, so the number $V = 1 - \cos(x)\cos(y)$ lies in $[0, 2]$; in particular, $V \geq 0$. [2] Next, we have

$$\begin{aligned} V_x &= \sin(x)\cos(y) \\ V_y &= \cos(x)\sin(y) \\ \dot{V} &= V_x \times (-\sin(x)\cos(y)) + V_y \times (-\cos(x)\sin(y)) \\ &= -(\sin^2(x)\cos^2(y) + \cos^2(x)\sin^2(y)) \leq 0.\text{[2]} \end{aligned}$$

(b) At an equilibrium point we must have $\dot{x} = -\sin(x)\cos(y) = 0$, so $\sin(x) = 0$ or $\cos(y) = 0$. Similarly, we must have $\dot{y} = -\cos(x)\sin(y) = 0$, so $\cos(x) = 0$ or $\sin(y) = 0$. [1]This seems to give four possibilities:

(1) $\sin(x) = 0$ and $\cos(x) = 0$
(2) $\sin(x) = 0$ and $\sin(y) = 0$
(3) $\cos(y) = 0$ and $\cos(x) = 0$
(4) $\cos(y) = 0$ and $\sin(y) = 0$.

However, $\sin^2(x) + \cos^2(x)$ is always equal to one, so $\sin(x)$ and $\cos(x)$ cannot both be zero, so case (1) is impossible. Similarly, case (4) is impossible [1]. In case (2) we have $x = n\pi$ and $y = m\pi$ for some integers n and m [1]. This gives $\cos(x) = (-1)^n$ and $\cos(y) = (-1)^m$ so $V = 1 - (-1)^{n+m}$ [1]. Thus, if $n + m$ is even then $V = 0$, and if $n + m$ is odd then $V = 2$ [1]. In case (3) we have $x = (n + \frac{1}{2})\pi$ and $y = (m + \frac{1}{2})\pi$ for some integers n and m [1], and $V = 1 - \cos(x)\cos(y) = 1$ [1].

(c) We saw in (a) that V is positive semidefinite and $\dot{V}$ is negative semidefinite everywhere in the whole plane. Now consider the region

$$R = \{(x, y) \mid -\pi/2 < x, y < \pi/2\}.$$

In this region we have $0 < \cos(x), \cos(y) \leq 1$, so the only way that the function $V = 1 - \cos(x)\cos(y)$ can be zero is if $\cos(x) = \cos(y) = 1$ which means that $x = y = 0$. Thus, V is positive definite on R [3]. We also saw that

$$\dot{V} = -(\sin^2(x)\cos^2(y) + \cos^2(x)\sin^2(y)).$$

This can only be zero if $\sin(x)\cos(y) = 0$ and also $\cos(x)\sin(y) = 0$ [1]. On R we have $\cos(x) > 0$ and $\cos(y) > 0$, so we can only have $\dot{V} = 0$ if $\sin(x) = \sin(y) = 0$, which means that $x = y = 0$ [1]. Thus, $\dot{V}$ is negative definite on R, so V is a strong Lyapunov function on R, so the origin is asymptotically stable [1].

(12)

(i) Consider the system where

$$\dot{x} = 5y + 12 - 12x^2$$
$$\dot{y} = 2y - 12 + 12x^2.$$

(a) Draw a diagram showing the x-nullcline and the y-nullcline. For each region in the diagram, say whether $\dot{x} > 0$ or $\dot{x} < 0$, and whether $\dot{y} > 0$ or $\dot{y} < 0$. **(5 marks)**

(b) Find the equilibrium points. **(2 marks)**

(c) For each equilibrium point, find the eigenvalues of the relevant matrix, and thus classify the point. **(4 marks)**

(ii) Consider the function $V = x^4 + x^2y^2 + y^4$, and the system where

$$\dot{x} = f(x, y) = 2y^3 + x^2y - x$$
$$\dot{y} = g(x, y) = -2x^3 - xy^2 - y.$$

Note that the origin is an equilibrium point.

(a) Show that $\dot{V} = -4V$. **(4 marks)**

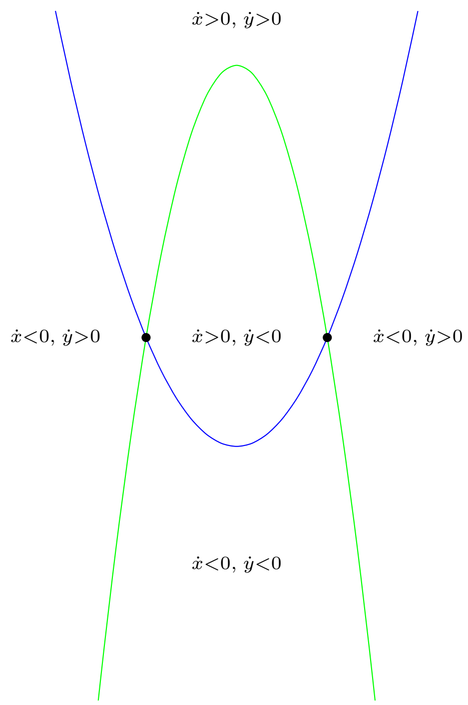
(b) What can you prove about the stability of $(0, 0)$ for the linearisation of the above system? What can you deduce about the stability of the original system? **(6 marks)**

(c) What can you prove about the stability of $(0, 0)$ using Lyapunov theory? **(4 marks)**

Solution:

(i) **This is a standard problem.**

(a) The x-nullcline is given by $y = \frac{12}{5}(x^2-1)$ [1], and the y-nullcline is given by $y = 6(1-x^2)$ [1]. Note also that $\dot{x}$ is an increasing function of y, so $\dot{x} > 0$ above the x-nullcline, and $\dot{x} < 0$ below the x-nullcline. Similarly, $\dot{y} > 0$ above the y-nullcline, and $\dot{y} < 0$ below the y-nullcline. This gives the following picture:



[3]

(b) The equilibrium points are given by $y = \frac{12}{5}(x^2 - 1) = 6(1 - x^2)$, which gives $y = x^2 - 1 = 0$, so $y = 0$ and $x = \pm 1$. Thus, the only equilibrium points are $a_1 = (1, 0)$ and $a_2 = (-1, 0)$ **[2]**.

(c) The Jacobian is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} -24x & 5 \\ 24x & 2 \end{bmatrix},$$

which has trace $\tau = 2 - 24x$ and determinant $\delta = -24 \times 7x$.

At a_1 we have $J = \begin{bmatrix} -24 & 5 \\ 24 & 2 \end{bmatrix}$, which has $\tau = -22$ and $\delta = -168$ and $\tau^2 - 4\delta = 1156 = 34^2$. The eigenvalues are $(-22 \pm 34)/2$, which gives $\lambda_1 = -28$ and $\lambda_2 = 6$ **[1]**. As one eigenvalue is negative and the other is positive, we have a saddle **[1]**.

At a_2 we have $J = \begin{bmatrix} 24 & 5 \\ -24 & 2 \end{bmatrix}$, which has $\tau = 26$ and $\delta = 168$ and $\tau^2 - 4\delta = 4 = 2^2$. The eigenvalues are $(26 \pm 2)/2$, which gives $\lambda_1 = 12$ and $\lambda_2 = 14$ **[1]**. As both eigenvalues are positive (and different) we have an unstable node **[1]**.

(ii) **This is a standard problem.**

(a) $V_x = 4x^3 + 2xy^2$ and $V_y = 2x^2y + 4y^3$ **[2]** so

$$\begin{aligned} \dot{V} &= V_x\dot{x} + V_y\dot{y}\textbf{[1]} \\ &= (4x^3 + 2xy^2)(2y^3 + x^2y - x) + (2x^2y + 4y^3)(-2x^3 - xy^2 - y) \\ &= 8x^3y^3 + 4x^5y - 4x^4 + 4xy^5 + 2x^3y^3 - 2x^2y^2 \\ &\quad - 4x^5y - 2x^3y^3 - 2x^2y^2 - 8x^3y^3 - 4xy^5 - 4y^4 \\ &= -4x^4 - 4x^2y^2 - 4y^4 = -4V.\textbf{[1]} \end{aligned}$$

(b) The Jacobian is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 2xy - 1 & x^2 + 6y^2 \\ -6x^2 - y^2 & -2xy - 1 \end{bmatrix}, \textbf{[1]}$$

and this becomes $J = -I$ at the origin **[1]**. The linearisation therefore has an asymptotically stable node at the origin **[1]**. The eigenvalues are both equal to -1, so the real part is nonzero, so the Hartman-Grobman theorem is applicable. We can thus deduce that the original system also has an asymptotically stable equilibrium at the origin **[3]**.

(c) As x^4, y^4 and x^2y^2 are always nonnegative, it is easy to see that $V \geq 0$ everywhere, and that V can only be zero at $(0, 0)$. In other words, the function V is positive definite **[1]**. It follows that the function $\dot{V} = -4V$ is negative definite **[1]**, and thus that V is a strong Lyapunov function **[1]**. This implies that the origin is an asymptotically stable equilibrium point **[1]**.

(13)

(i) Consider the system where

$$\dot{x} = e^{y+x} - e$$
$$\dot{y} = e^{y-x} - e.$$

(a) Draw a diagram showing the x-nullcline and the y-nullcline. For each region in the diagram, say whether $\dot{x} > 0$ or $\dot{x} < 0$, and whether $\dot{y} > 0$ or $\dot{y} < 0$. **(5 marks)**

(b) Find and the equilibrium point, find the eigenvalues of the relevant matrix, and thus classify the point. **(6 marks)**

(c) Explain what we can conclude from this about the stability of the original system. **(2 marks)**

(d) Sketch the flow lines. **(3 marks)**

(ii) Consider the system where

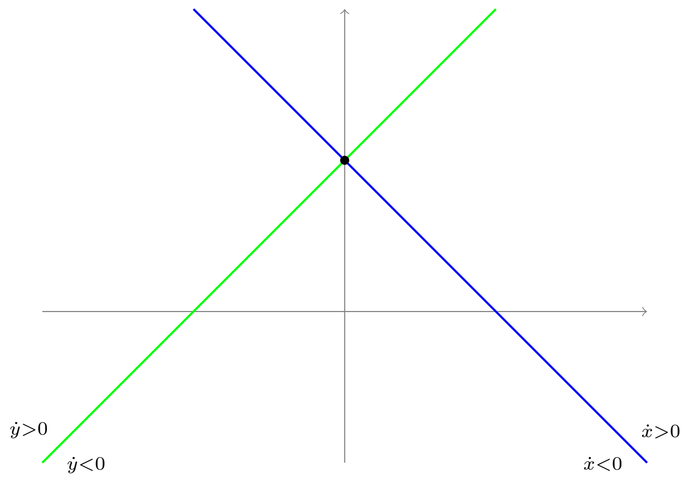
$$\dot{x} = -x + \sinh(y)$$
$$\dot{y} = -y - \sinh(x),$$

and the functions $V = \cosh(x) + \cosh(y) - 2$. Analyse the behaviour of V and $\dot{V}$, and explain what we can deduce about the stability of equilibrium points. **(9 marks)**

Solution:

(i) **This is a standard type of problem.**

(a) The x-nullcline is given by $e^{y+x} = e$, or equivalently $y + x = 1$ or $y = 1 - x$ [1]. When $y > 1 - x$ we have $\dot{x} > 0$, and when $y < 1 - x$ we have $\dot{x} < 0$ [1]. Similarly, the y-nullcline is given by $e^{y-x} = e$, or equivalently $y = 1 + x$ [1]. When $y > 1 - x$ we have $\dot{x} > 0$, and when $y < 1 - x$ we have $\dot{x} < 0$ [1]. The required diagram is as follows [1]:



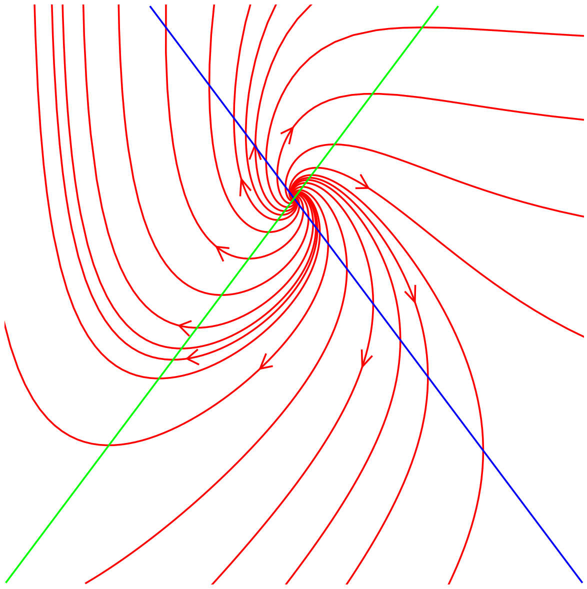
(b) For an equilibrium point we need $y = 1 + x$ and also $y = 1 - x$ which gives $x = 0$ and $y = 1$. Thus, $(0, 1)$ is the only equilibrium point [1]. The Jacobian is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} e^{y+x} & e^{y+x} \\ -e^{y-x} & e^{y-x} \end{bmatrix}. [1]$$

At $(0,1)$ this becomes $J = \begin{bmatrix} e & e \\ -e & e \end{bmatrix}$ [1]. This has $\tau = 2e$ and $\delta = 2e^2$ so $\tau^2 - 4\delta = -4e^2$. This means that the eigenvalues are $(2e \pm \sqrt{-4e^2})/2 = (1 \pm i)e$ [1]. These are complex, with positive real part, so the linearisation has an unstable focus [1]. As the bottom left entry in J is negative, the rotation is clockwise [1].

(c) As the eigenvalues have nonzero real part, the Hartman-Grobman theorem is applicable [1], and we conclude that $(0,1)$ is also an unstable equilibrium point for the original system. [1]

(d) The picture is as follows.



Note that the flow lines cross the x-nullcline vertically and the y-nullcline horizontally, in the direction indicated by the previous diagram. Note also that the flow lines spiral outwards clockwise from the equilibrium point. [3]

(ii) **This is a standard type of problem, but the students have not seen many examples with hyperbolic functions.** We have $\cosh(x) = (e^x + e^{-x})/2 \geq 1$ for all x, with equality if and only if $x = 0$. Similarly $\cosh(y) \geq 1$, with equality if and only if $y = 0$ [1]. We can thus consider V as the sum of the nonnegative functions $\cosh(x) - 1$ and $\cosh(y) - 1$, and we conclude that $V \geq 0$, we equality if and only if $x = y = 0$. In other words, V is positive definite on the whole plane [2].

Next, we have

$$\dot{V} = V_x\dot{x} + V_y\dot{y} = \sinh(x)(-x + \sinh(y)) + \sinh(y)(-y - \sinh(x)) = -x\sinh(x) - y\sinh(y).\text{[1]}$$

Now $\sinh(x) > 0$ when $x > 0$, and $\sinh(x) < 0$ when $x < 0$, so we see that $-x\sinh(x) \leq 0$, with equality if and only if $x = 0$. Similarly, $-y\sinh(y) \leq 0$, with equality if and only if $y = 0$ [2]. It follows that $\dot{V} \leq 0$, with equality if and only if $x = y = 0$. In other words, $\dot{V}$ is negative definite on the whole plane [1]. This means that V is a strong Lyapunov function on the whole plane, so the origin is the unique equilibrium point, and it is asymptotically stable [2].

(14) Consider the equations

$$\dot{x} = (x+y)^3 - (x+y) \qquad\qquad \dot{y} = (x-y)^3 - (x-y).$$

(a) Sketch the x-nullcline, the y-nullcline, and the region where $\dot{x} > 0$ and $\dot{y} > 0$. **(7 marks)**

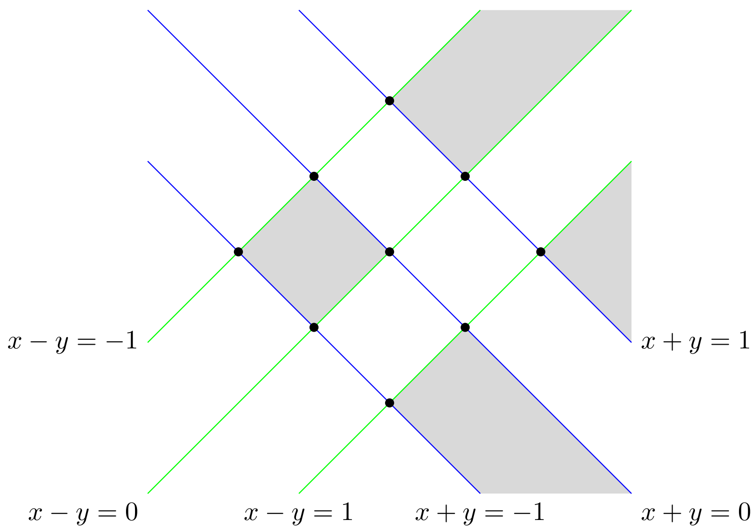
(b) Find and classify the equilibrium points. **(10 marks)**

(c) Write down the linearisation of the equations at the origin. **(2 marks)**

(d) Show that the function $V = x^2 - 2xy - y^2$ is a conserved quantity for the linearised system, but not for the original system. **(6 marks)**

Solution: This is a standard problem. Put $f = (x+y)^3 - (x+y)$ and $g = (x-y)^3 - (x-y)$, so $\dot{x} = f$ and $\dot{y} = g$.

(a) The x-nullcline is given by $(x+y)^3 = x+y$, or equivalently $x+y \in \{-1, 0, 1\}$ [1]. We have $\dot{x} > 0$ iff $(x+y)^3 > x+y$ iff $(1 < x+y < 0$ or $x+y > 1)$ [1]. Similarly, the y-nullcline is given by $x-y \in \{-1, 0, 1\}$ [1], and we have $\dot{y} > 0$ iff $x-y \in (-1, 0) \cup (1, \infty)$ [1]. This gives the following picture:



The x-nullcline is shown in blue, the y-nullcline is shown in green, and the region where $\dot{x}, \dot{y} > 0$ is shaded grey. [3]

(b) The equilibrium points are the points marked in black on the diagram, where $f = g = 0$, so $x+y \in \{-1, 0, 1\}$ and also $x-y \in \{-1, 0, 1\}$. The coordinates are $(0, 0)$ or $(\pm 1, 0)$ or $(0, \pm 1)$ or $(\pm 1/2, \pm 1/2)$ [2]. To classify these equilibrium points, we use the Jacobian matrix:

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 3(x+y)^2 - 1 & 3(x+y)^2 - 1 \\ 3(x-y)^2 - 1 & -3(x-y)^2 + 1 \end{bmatrix}. \text{[2]}$$

This gives the following table: [6]

	(x,y)	J	τ	δ	$\tau^2-4\delta$	type
a_1	$(0,0)$	$\begin{bmatrix}-1 & -1\\ -1 & 1\end{bmatrix}$	0	-2	8	saddle
a_2	$(\frac{1}{2},\frac{1}{2})$	$\begin{bmatrix}2 & 2\\ -1 & 1\end{bmatrix}$	3	4	-7	unstable focus
a_3	$(-\frac{1}{2},-\frac{1}{2})$	$\begin{bmatrix}2 & 2\\ -1 & 1\end{bmatrix}$	3	4	-7	unstable focus
a_4	$(-\frac{1}{2},\frac{1}{2})$	$\begin{bmatrix}-1 & -1\\ 2 & -2\end{bmatrix}$	-3	4	-7	stable focus
a_5	$(\frac{1}{2},-\frac{1}{2})$	$\begin{bmatrix}-1 & -1\\ 2 & -2\end{bmatrix}$	-3	4	-7	stable focus
a_6	$(1,0)$	$\begin{bmatrix}2 & 2\\ 2 & -2\end{bmatrix}$	0	-8	32	saddle
a_7	$(0,1)$	$\begin{bmatrix}2 & 2\\ 2 & -2\end{bmatrix}$	0	-8	32	saddle
a_8	$(-1,0)$	$\begin{bmatrix}2 & 2\\ 2 & -2\end{bmatrix}$	0	-8	32	saddle
a_9	$(0,-1)$	$\begin{bmatrix}2 & 2\\ 2 & -2\end{bmatrix}$	0	-8	32	saddle

(c) The linearised equations are $\dot{x} = -x - y$ and $\dot{y} = -x + y$. [2]

(d) Consider the function $V = x^2 - 2xy - y^2$, so $V_x = 2(x-y)$ and $V_y = -2(x+y)$ [1]. For the linearised system we have

$$\dot{V} = V_x\,\dot{x} + V_y\,\dot{y}[1] = 2(x-y)(-x-y) - 2(x+y)(-x+y) = 2(y^2-x^2) - 2(y^2-x^2) = 0, [1]$$

so V is conserved. For the original system we instead have

$$\begin{aligned}\dot{V} &= 2(x-y)((x+y)^3-(x+y)) - 2(x+y)((x-y)^3-(x-y))[1]\\ &= 2(x-y)(x+y)((x+y)^2-(x-y)^2) = 2(x^2-y^2).4xy\\ &= 8(x^2-y^2)xy[1] \neq 0,\end{aligned}$$

so V is not conserved. [1]

(15)

(i) Consider the system where

$$\dot{x} = -(x+1)^3(y^2-1)^3$$
$$\dot{y} = (x^2-1)^3(y+1)^3.$$

(a) Draw a diagram showing the x-nullcline and the y-nullcline. Show the region where $\dot{x} < 0 < \dot{y}$. **(5 marks)**

(b) Find all the equilibrium points, and show that there are infinitely many of them. Show that the Jacobian matrix is zero at every equilibrium point. **(4 marks)**

(c) Find a number p such that the function $V = (x-1)^p + (y-1)^p$ is a conserved quantity. **(3 marks)**

(d) Using (c), find the points where the flow line through $(1+\sqrt{3/5}, 1+\sqrt{4/5})$ passes through the line $x = 1$. **(3 marks)**

(ii) Consider the system where

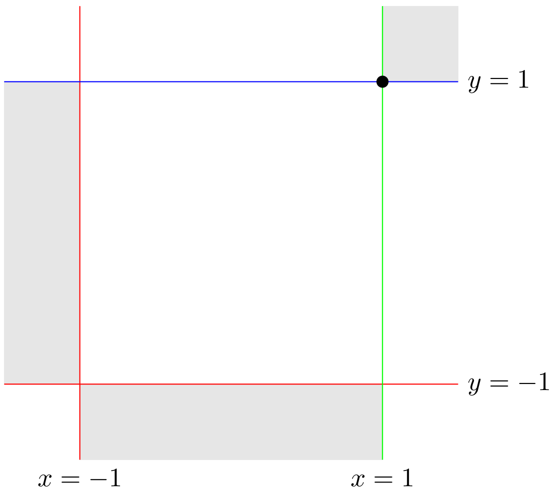
$$\dot{x} = y^2 - 1$$
$$\dot{y} = \sin(x).$$

(a) Find all the saddle points. **(5 marks)**

(b) Find functions F and G such that F only depends on x, and G depends only on y, and $F + G$ is a conserved quantity. **(5 marks)**

Solution:

(i) (a) The x-nullcline is given by $(x+1)^3(y^2-1)^3 = 0$, or equivalently ($x = -1$ or $y = -1$ or $y = 1$) [1]. The y-nullcline is given by $(x^2-1)^3(y+1)^3 = 0$, or equivalently ($x = -1$ or $y = -1$ or $x = 1$) [1].



The x-nullcline consists of the two red lines ($x = -1$ and $y = -1$) together with the blue line ($y = 1$). The y-nullcline consists of the two red lines together with the green line ($x = 1$) [1]. The shaded grey regions show where $\dot{x} < 0 < \dot{y}$ [2]. (One way to see this is to note that $\dot{x} > 0 > \dot{y}$ at the origin, and that $\dot{x}$ changes sign when we cross a red line or a blue line, whereas $\dot{y}$ changes sign when we cross a red line or a green line.)

(b) The equilibrium points are the intersection of the x-nullcline and the y-nullcline, which means the lines $x = -1$ and $y = -1$, together with one extra point at $(1, 1)$. [2] Note that

$$\partial f/\partial x = -3(x+1)^2(y^2-1)^3 \qquad \partial f/\partial y = -6y(x+1)^3(y^2-1)^2$$
$$\partial g/\partial x = 6x(x^2-1)^2(y+1)^3 \qquad \partial g/\partial y = 3(x^2-1)^3(y+1)^2.$$

If $f = 0$ then $x + 1 = 0$ or $y^2 - 1 = 0$ which means that $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$. Similarly, if $g = 0$ then $x^2 - 1 = 0$ or $y + 1 = 0$ which means that $\partial g/\partial x = 0$ and $\partial g/\partial y = 0$ [1]. At an equilibrium point we have $f = g = 0$ so all the above partial derivatives are zero, so $J = 0$ [1].

(c) Take $V = (x-1)^p + (y-1)^p$. We then have

$$\begin{aligned}\dot{V} &= V_x f + V_y g\text{[1]} = -p(x-1)^{p-1}(x+1)^3(y^2-1)^3 + p(y-1)^{p-1}(x^2-1)^3(y+1)^3 \\ &= -p(x-1)^{p-1}(x+1)^3(y-1)^3(y+1)^3 + p(x-1)^3(x+1)^3(y-1)^{p-1}(y+1)^3.\text{[1]}\end{aligned}$$

We can make this zero by taking $p = 4$. Thus, the function $V = (x-1)^4 + (y-1)^4$ is a conserved quantity. [1]

(d) Consider the flow line starting at the point $a_0 = (1 + \sqrt{3/5}, 1 + \sqrt{4/5})$. Let a_1 be a place where the flow line crosses the line $x = 1$, so $a_1 = (1, y)$ for some y. As V is a conserved quantity, it must have the same value at a_0 and a_1 [1], so

$$\sqrt{3/5}^4 + \sqrt{4/5}^4 = 0^4 + (y-1)^4.\text{[1]}$$

Expanding this out gives $(y-1)^4 = 9/25 + 16/25 = 1$, so $y - 1 = \pm 1$, so $y = 0$ or $y = 2$. It follows that $a_1 = (1, 0)$ or $a_1 = (1, 2)$ [1].

(ii) Put $f = y^2 - 1$ and $g = \sin(x)$.

(a) For an equilibrium point we need $f = g = 0$, so $x = n\pi$ for some integer n, and $y = \pm 1$ [1]. The Jacobian is then

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 0 & 2y \\ \cos(x) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2y \\ (-1)^n & 0 \end{bmatrix}.\text{[1]}$$

The determinant is $\delta = -2(-1)^n y$. For a saddle we need $\delta < 0$ [1], which means that either $y = 1$ and n is even, or $y = -1$ and n is odd. Thus the saddles are $(1, 2k\pi)$ and $(-1, (2k+1)\pi)$ for all $k \in \mathbb{Z}$ [2].

(b) Suppose that $V = F + G$, where F depends only on x, and G depends only on y. We then have $V_x = F_x$ and $V_y = G_y$ [2] so

$$\dot{V} = V_x f + G_y g = F_x.(y^2-1) + G_y.\sin(x).\text{[1]}$$

To make this zero, we can take $F_x = \sin(x)$ and $G_y = 1 - y^2$ [1], which gives $F = \int \sin(x)\,dx = -\cos(x)$ and $G = \int 1 - y^2\,dy = y - y^3/3$ [1]. We conclude that the function $V = y - y^3/3 - \cos(x)$ is a conserved quantity.

(16) Consider the equations

$$\dot{x} = 2xy + 2 \qquad\qquad \dot{y} = 3x^2 - y^2 - 2$$

(a) Find and classify the equilibrium points. **(6 marks)**

(b) Sketch the x-nullcline and the y-nullcline. In each region of the diagram, say whether $\dot{x}$ is positive or negative, and whether $\dot{y}$ is positive or negative. **(7 marks)**

(c) Find constants p, q and r such that the quantity

$$V = x^3 + px^2y + qxy^2 + r(x + y)$$

is conserved. **(9 marks)**

(d) Show that the functions $x = \tanh(2t)$ and $y = -\tanh(2t)$ give a solution. **(3 marks)**

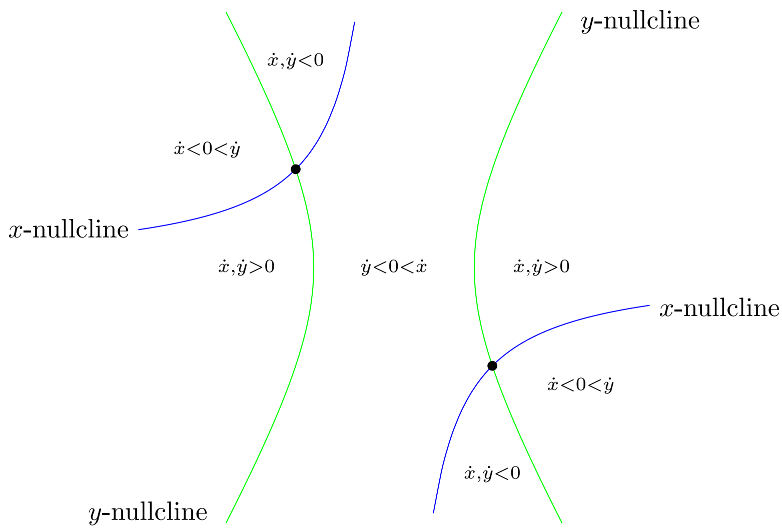
Solution: This is a standard problem. Put $f = 2(xy + 1)$ and $g = 3x^2 - y^2 - 2$, so $\dot{x} = f$ and $\dot{y} = g$.

(a) At an equilibrium point we have $f = 0$ so $y = -1/x$ [1], and also $g = 0$. Putting $y = -1/x$ in the formula for g gives $3x^2 - 1/x^2 - 2 = 0$ [1]. Multiplying by x^2 gives $3x^4 - 2x^2 - 1 = 0$, which factors as $(3x^2 + 1)(x^2 - 1) = 0$. As $3x^2 + 1 > 0$, we must have $x^2 = 1$, so $x = \pm 1$. As $y = -1/x$, we see that there are two equilibrium points, namely $a_1 = (-1, 1)$ and $a_2 = (1, -1)$ [1]. The Jacobian is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 2y & 2x \\ 6x & -2y \end{bmatrix}, \text{[2]}$$

which has $\tau = 0$ and $\delta = -4y^2 - 12x^2$. At the equilibrium points we have $x^2 = y^2 = 1$, so $\delta = -16 < 0$. This means that both equilibrium points are saddles. [1]

(b) The x-nullcline is given by $f = 0$, or equivalently $y = -1/x$ [1]. When $x > 0$ we have $f > 0$ iff $y > -1/x$, and when $x < 0$ we have $f > 0$ iff $y < -1/x$ [1]. The x-nullcline is given by $g = 0$, or equivalently $x = \pm\sqrt{(y^2 + 2)/3}$ [1]. We have $g > 0$ iff $|x| > \sqrt{(y^2 + 2)/3}$ [1]. The two nullclines meet at the equilibrium points $a_1 = (-1, 1)$ and $a_2 = (1, -1)$. The picture is as follows:



[3]

(c) Consider the function

$$V = x^3 + px^2y + qxy^2 + r(x+y),$$

so

$$\begin{aligned}
V_x &= 3x^2 + 2pxy + qy^2 + r\textbf{[1]} \\
V_y &= px^2 + 2qxy + r\textbf{[1]} \\
\dot{V} &= V_x f + V_y g\textbf{[1]} \\
&= (3x^2 + 2pxy + qy^2 + r)(2xy + 2) + (px^2 + 2qxy + r)(3x^2 - y^2 - 2) \\
&= 6x^3y + 4px^2y^2 + 2qxy^3 + 2rxy + 6x^2 + 4pxy + 2qy^2 + 2r + \\
&\qquad 3px^4 + 6qx^3y + 3rx^2 - px^2y^2 - 2qxy^3 - ry^2 - 2px^2 - 4qxy - 2r \\
&= 3px^4 + 6(q+1)x^3y + 3px^2y^2 + (6 - 2p + 3r)x^2 + (4p - 4q + 2r)xy + (2q - r)y^2.\textbf{[2]}
\end{aligned}$$

For a conserved quantity, all the coefficients in this expression must be zero. The coefficient of x^4 gives $p = 0$ **[1]**, and the coefficient of x^3y gives $q = -1$ **[1]**. After putting $p = 0$ and $q = -1$ we get

$$\dot{V} = (6 + 3r)x^2 + (4 + 2r)xy + (-2 - r)y^2.$$

Thus, if we put $r = -2$ we get $\dot{V} = 0$ **[1]**. This proves that the function

$$V = x^3 - xy^2 - 2(x+y)$$

is a conserved quantity **[1]**.

(d) Put $x = \tanh(2t)$ and $y = -\tanh(2t) = -x$. Recall that $\tanh'(s) = \operatorname{sech}^2(s) = 1 - \tanh^2(s)$ **[1]**. This gives

$$\dot{x} = 2\tanh'(2t) = 2(1 - \tanh^2(2t)) = 2(1 - x^2),$$

and similarly $\dot{y} = -\dot{x} = 2(x^2 - 1)$ **[1]**. On the other hand, we have $f = 2(xy + 1) = 2(1 - x^2)$ and $g = 3x^2 - y^2 - 2 = 2(x^2 - 1)$, so $\dot{x} = f$ and $\dot{y} = g$ as required **[1]**.

(17)

(i) Consider the system where

$$\dot{x} = 1 + xy \qquad \dot{y} = 1 - y^2.$$

(a) Draw a diagram showing the x-nullcline and the y-nullcline. Show the region where $\dot{x} < 0 < \dot{y}$. **(6 marks)**

(b) Find and classify the equilibrium points. **(6 marks)**

(c) Show that the functions $x = \arctan(\sinh(t))\cosh(t)$ and $y = \tanh(t)$ give a solution. **(6 marks)**

(d) Put $U = \sqrt{1 - y^2}$ and $V = Ux$ and

$$W = U\,\sin(V) - y\cos(V).$$

Show that $\dot{U} = -yU$ and $\dot{V} = U$, and then show that W is a conserved quantity. **(6 marks)**

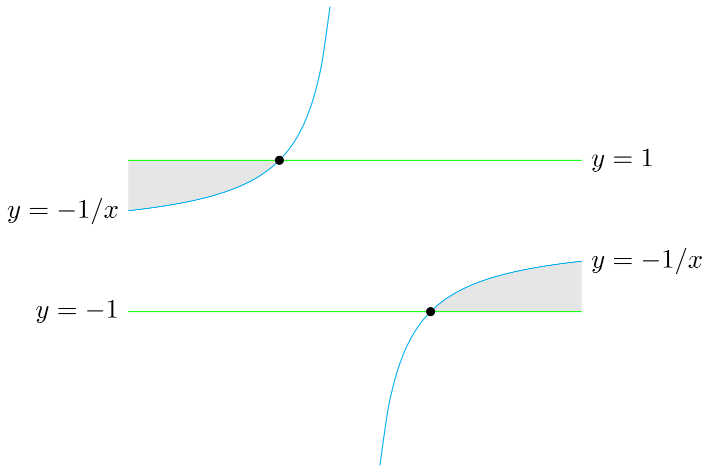
(ii) Consider the system where

$$\dot{x} = y + y^3 \qquad \dot{y} = (x^2 - 1)(x^2 - 9).$$

Show that all the centres have clockwise rotation. **(6 marks)**

Solution:

(i) (a) The x-nullcline is given by $1 + xy = 0$, or equivalently $y = -1/x$ [1]. We have $\dot{x} < 0$ when ($x > 0$ and $y < -1/x$) or ($x < 0$ and $y > -1/x$). The y-nullcline is given by $1 - y^2 = 0$, or equivalently ($y = \pm 1$) [1]. We have $\dot{y} > 0$ when $-1 < y < 1$.



Thus, we have $\dot{x} < 0 < \dot{y}$ in the shaded region. [4]

(b) The equilibrium points are the intersection of the x-nullcline and the y-nullcline, which means the points $a_1 = (-1, 1)$ and $a_2 = (1, -1)$ [2]. The Jacobian is

$$J = \begin{bmatrix} y & x \\ 0 & -2y \end{bmatrix}, \text{[1]}$$

which has determinant $\delta = -2y^2$ [1]. At a_1 and a_2 we have $\delta = -2$, so these points are saddles [2].

(c) Put $x = \arctan(\sinh(t))\cosh(t)$ and $y = \tanh(t)$. Recall that $1 + \sinh^2(t) = \cosh^2(t)$. Using this, we have

$$\begin{aligned}
\dot{x} &= \arctan'(\sinh(t))\sinh'(t)\cosh(t) + \arctan(\sinh(t))\cosh'(t) \\
&= \frac{1}{1+\sinh^2(t)}\cosh^2(t) + \arctan(\sinh(t))\sinh(t) \\
&= 1 + \arctan(\sinh(t))\sinh(t)[2]
\end{aligned}$$

$$\begin{aligned}
1 + xy &= 1 + \arctan(\sinh(t))\cosh(t)\tanh(t) = 1 + \arctan(\sinh(t))\cosh(t)\frac{\sinh(t)}{\cosh(t)} \\
&= 1 + \arctan(\sinh(t))\sinh(t)[2] = \dot{x}
\end{aligned}$$

$$\begin{aligned}
\dot{y} &= \tanh'(t) = \frac{d}{dt}\left(\frac{\sinh(t)}{\cosh(t)}\right) \\
&= \frac{\sinh'(t)\cosh(t) - \sinh(t)\cosh'(t)}{\cosh^2(t)} = \frac{\cosh^2(t)}{\cosh^2(t)} - \frac{\sinh^2(t)}{\cosh^2(t)} \\
&= 1 - \tanh^2(t)[2] = 1 - y^2.
\end{aligned}$$

Thus, we have a solution to the equations $\dot{x} = 1 + xy$ and $\dot{y} = 1 - y^2$.

(d) First, we have $\dot{y} = y^2 - 1 = -U^2$ [1]. This gives

$$\begin{aligned}
\dot{U} &= \frac{d}{dt}(1-y^2)^{1/2} = -\frac{1}{2}(1-y^2)^{-1/2} \times -2y\dot{y} = (1-y^2)^{-1/2}y\dot{y} \\
&= U^{-1}y(-U^2) = -yU[2] \\
\dot{V} &= \dot{U}x + U\dot{x} = -yUx + U(1+xy) = U[1] \\
\dot{W} &= \dot{U}\sin(V) + U\sin'(V)\dot{V} - \dot{y}\cos(V) - y\cos'(V)\dot{V}[1] \\
&= (-yU)\sin(V) + U\cos(V)U - U^2\cos(V) + y\sin(V)U = 0[1].
\end{aligned}$$

This proves that W is a conserved quantity.

(ii) Put $f = y + y^3 = y(1+y^2)$ and $g = (x^2-1)(x^2-9) = x^4 - 10x^2 + 9$. For an equilibrium point we need $f = g = 0$. As $1 + y^2 > 0$ for all y, we only have $f = 0$ when $y = 0$. Also, we have $g = 0$ for $x \in \{-3, -1, 1, 3\}$. Thus, the equilibrium points are

$$a_1 = (-3, 0) \qquad a_2 = (-1, 0) \qquad a_3 = (1, 0) \qquad a_4 = (3, 0).[2]$$

The Jacobian is

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 0 & 1+3y^2 \\ 4x^3 - 20x & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1+3y^2 \\ 4x(x^2-5) & 0 \end{bmatrix},$$

which has $\tau = 0$ [1]and $\delta = 4x(5-x^2)(1+3y^2)$ [1]. The values of J and δ at the equilibrium points a_i are

	a_1	a_2	a_3	a_4	
J	$\begin{bmatrix} 0 & 1 \\ -48 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 16 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -16 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 48 & 0 \end{bmatrix}$	[1]
δ	48	-16	16	-48	

As $\tau = 0$, we have centres where $\delta > 0$ and saddles where $\delta < 0$. Thus, we have centres at a_1 and a_3. In both cases, the bottom left entry of J is negative, so the rotation is clockwise. [1]

(18)

(i) Consider the system where

$$\dot{x} = \sin(\pi x) + y \qquad\qquad \dot{y} = \cos(\pi x) - y.$$

Find and classify the equilibrium points. (Hint: consider $\sin^2(\pi x) + \cos^2(\pi x)$.) **(11 marks)**

(ii) Consider the system where

$$\dot{x} = f = 1 - x^2 + y^2 \qquad\qquad \dot{y} = g = -2xy,$$

and the functions

$$U = \frac{y}{x^2 + y^2 - 1} \qquad\qquad V = \frac{(x-1)^2 + y^2}{(x+1)^2 + y^2} = \frac{x^2 + y^2 + 1 - 2x}{x^2 + y^2 + 1 + 2x}.$$

(a) Show that there is a stable node at $(1, 0)$. **(3 marks)**

(b) Show that U is a conserved quantity on the region where $x^2 + y^2 \neq 1$. **(4 marks)**

(c) Show that $\dot{V} = -4V$ **(6 marks)**

(d) Deduce that V is a strong Lyapunov function for the point $(1, 0)$. **(4 marks)**

Solution:

(i) At an equilibrium point we must have $\sin(\pi x) = -y$ and $\cos(\pi x) = y$. [1]This gives

$$2y^2 = \sin^2(\pi x) + \cos^2(\pi x) = 1,$$

so $y = \pm 1/\sqrt{2}$ [1]. There are two possible cases:

(a) $y = 1/\sqrt{2}$ so $(\cos(\pi x), \sin(\pi x)) = (1, -1)/\sqrt{2}$ so $x = 2n - 1/4$ for some $n \in \mathbb{Z}$.

(b) $y = -1/\sqrt{2}$ so $(\cos(\pi x), \sin(\pi x)) = (-1, 1)/\sqrt{2}$ so $x = 2n + 3/4$ for some $n \in \mathbb{Z}$. [3]

The Jacobian is

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} \pi\cos(\pi x) & 1 \\ -\pi\sin(\pi x) & -1 \end{bmatrix} \text{[1]} = \begin{bmatrix} \pi y & 1 \\ \pi y & -1 \end{bmatrix},$$

with $\tau = \pi y - 1$ and $\delta = -2\pi y$ [1]. In case (a) we see that $\delta < 0$ so we have a saddle [1]. In case (b) we have $\tau = -\pi/\sqrt{2} - 1 < 0$ and $\delta = \pi\sqrt{2} > 0$ so

$$\tau^2 - 4\delta = \pi^2/2 + \pi\sqrt{2} + 1 - 4\pi\sqrt{2} = \pi^2/2 - 3\pi\sqrt{2} + 1 \simeq -7.39 < 0.$$

This shows that we have a stable focus [2]. The bottom left entry in J is $\pi y < 0$ so the rotation is clockwise. [1]

(ii) (a) At the point $a = (1, 0)$ we have $f = 1 - 1^2 + 0^2 = 0$ and $g = -2.1.0 = 0$ so we have an equilibrium point. [1]The Jacobian at a is

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -2x & 2y \\ -2y & -2x \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \text{[1]}$$

which gives a stable node. [1]

(b) We find that

$$U_x = \frac{-2xy}{(x^2+y^2-1)^2} = \frac{g}{(x^2+y^2-1)^2}\text{[1]}$$
$$U_y = \frac{(x^2+y^2-1)-y.(2y)}{(x^2+y^2-1)^2} = \frac{x^2-y^2-1}{(x^2+y^2-1)^2} = \frac{-f}{(x^2+y^2-1)^2}.\text{[1]}$$

From this it is clear that

$$\dot{U} = U_x f + U_y g = \frac{gf-fg}{(x^2+y^2-1)^2} = 0.$$

This means that U is a conserved quantity (except on the unit circle where $x^2+y^2-1=0$ and so U is undefined). [2]

(c) We also find that

$$\begin{aligned}
V_x &= \frac{(2x-2)(x^2+y^2+1+2x)-(x^2+y^2+1-2x)(2x+2)}{(x^2+y^2+1+2x)^2} \\
&= \frac{4x^2-4y^2-4}{(x^2+y^2+1+2x)^2} = \frac{-4f}{(x^2+y^2+1+2x)^2}\text{[1]} \\
V_y &= \frac{2y(x^2+y^2+1+2x)-2y(x^2+y^2+1-2x)}{(x^2+y^2+1+2x)^2} \\
&= \frac{8xy}{(x^2+y^2+1+2x)^2} = \frac{-4g}{(x^2+y^2+1+2x)^2}\text{[1]} \\
\dot{V} &= V_x f + V_y g = \frac{-4(f^2+g^2)}{(x^2+y^2+1+2x)^2}.\text{[1]}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
f^2+g^2 &= 1+x^4+y^4-2x^2+2y^2-2x^2y^2+4x^2y^2 \\
&= 1+x^4+y^4-2x^2+2y^2+2x^2y^2 = (x^2+y^2+1+2x)^2-4x^2 \\
&= (x^2+y^2+1+2x)(x^2+y^2+1-2x).
\end{aligned}$$

This gives

$$\dot{V} = -4\frac{(x^2+y^2+1+2x)(x^2+y^2+1-2x)}{(x^2+y^2+1+2x)^2} = -4\frac{x^2+y^2+1-2x}{x^2+y^2+1+2x} = -4V\text{[3]}$$

as required.

(d) From the expression

$$V = \frac{(x-1)^2+y^2}{(x+1)^2+y^2}$$

it is clear that $V>0$ everywhere except at $(1,0)$ (where $V=0$) and $(-1,0)$ (where V is undefined). In other words, V is positive definite around $(1,0)$ [1]. As $\dot{V}=-4V$ we see that $\dot{V}$ is negative definite around $(1,0)$ [1]. As V is positive definite and $\dot{V}$ is negative definite we see that V is a strong Lyapunov function. [2]

Question 3

(19)

(i) There is a unique function $y = \sum_{k=0}^{\infty} a_k x^k$ such that $xy' - (1+x^2)y = 0$, with $y = 0$ and $y' = 1$ when $x = 0$.

(a) Find formulae for a_{2j} and a_{2j+1}. **(7 marks)**

(b) Explain why the series has infinite radius of convergence. **(3 marks)**

(c) Give a simple formula for y in terms of the exponential function. **(3 marks)**

(ii) Consider the equation

$$x^2(1-x)y'' + x(1-3x)y' - y = 0.$$

(a) Show that there is a regular singular point at $x = 0$, and find the corresponding indicial polynomial. **(4 marks)**

(b) Show that the function $z = x^{-1} + 1$ is a solution. **(2 marks)**

(c) Find the general solution. **(6 marks)**

Solution: This is fairly similar to questions in the lectures and on the problem sheets.

(i) (a) First, we have

$$\begin{aligned} xy' &= \sum_k k a_k x^k \\ -y &= \sum_k -a_k x^k \\ -x^2 y &= \sum_k -a_{k-2} x^k, \end{aligned}$$

so the differential equation $xy' - (1+x^2)y = 0$ gives $a_{k-2} = (k-1)a_k$. When $k \neq 1$ we can rewrite this as $a_k = a_{k-2}/(k-1)$ [3]. We are given that $y = 0$ and $y' = 1$ when $x = 0$, which means that $a_0 = 0$ and $a_1 = 1$. Using $a_0 = 0$ and $a_k = a_{k-2}/(k-1)$ we see that $a_k = 0$ whenever k is even, or in other words $a_{2j} = 0$ [1]. On the other hand, we have $a_{2j+1} = a_{2j-1}/(2j)$. For $j = 3$, this gives

$$a_7 = \frac{1}{2\times 3} a_5 = \frac{1}{2\times 3}\frac{1}{2\times 2} a_3 = \frac{1}{2\times 3}\frac{1}{2\times 2}\frac{1}{2\times 1} a_1 = \frac{1}{2^3 \times 3 \times 2 \times 1} = \frac{1}{2^3 3!}.$$

The pattern should be clear from this: we have $a_{2j+1} = \frac{1}{2^j j!}$ for all j. [3]

(b) By a standard variant of the ratio test, if the coefficients a_{2j} are zero and the ratios a_{2j-1}/a_{2j+1} tend to a limit L, then the radius of convergence is $\sqrt{L}$ [2]. In this case we have $a_{2j-1}/a_{2j+1} = 2j$, so $L = \infty$ and the series has infinite radius of convergence. [1]

(c) We have

$$y = \sum_{j=0}^{\infty} a_{2j+1} x^{2j+1} = \sum_{j=0}^{\infty} \frac{x^{2j+1}}{2^j j!} [1] = x \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{x^2}{2}\right)^j [1] = x\, e^{x^2/2} [1].$$

(ii) (a) First, the equation is equivalent to $y'' + Py' + Qy = 0$, where

$$\begin{aligned} P &= \frac{x(1-3x)}{x^2(1-x)} = x^{-1}\frac{1-3x}{1-x} = x^{-1} + O(1) \\ Q &= \frac{-1}{x^2(1-x)} = -x^{-2}(1-x)^{-1} = -x^{-2} + O(x^{-1}), \end{aligned}$$

so the origin is a regular singular point, with $p_0 = 1$ and $q_0 = -1$ **[3]**. The indicial polynomial is

$$\alpha^2 - \alpha + p_0\alpha + q_0 = \alpha^2 - 1 = (\alpha - 1)(\alpha + 1).\textbf{[1]}$$

(b) Consider the function $z = x^{-1} + 1$. We have $z' = -x^{-2}$ and $z'' = 2x^{-3}$ **[1]**so

$$\begin{aligned} x^2(1-x)z'' + x(1-3x)z' - z &= (x^2 - x^3) \times 2x^{-3} + (x - 3x^2) \times (-x^{-2}) - 1 - x^{-1} \\ &= 2x^{-1} - 2 - x^{-1} + 3 - 1 - x^{-1} = 0 \end{aligned}$$

as required **[1]**.

(c) As $\alpha = 1$ is the largest root of the indicial polynomial, there is a unique solution of the form $y = \sum_k a_k x^{k+1}$ with $a_0 = 1$ and $a_k = 0$ for $k < 0$ **[1]**. Now

$$\begin{aligned} x^2y'' &= \sum_k a_k(k+1)kx^{k+1} & &= \sum_j a_j(j+1)jx^{j+1} \\ -x^3y'' &= \sum_k -a_k(k+1)kx^{k+2} & &= \sum_j -a_{j-1}j(j-1)x^{j+1} \\ xy' &= \sum_k a_k(k+1)x^{k+1} & &= \sum_j a_j(j+1)x^{j+1} \\ -3x^2y' &= \sum_k -3a_k(k+1)x^{k+2} & &= \sum_j -3a_{j-1}jx^{j+1} \\ -y &= \sum_k -a_kx^{k+1} & &= \sum_j -a_jx^{j+1}\textbf{[2]}. \end{aligned}$$

Thus, the differential equation is equivalent to

$$((j+1)j + j + 1 - 1)a_j + (-j(j-1) - 3j)a_{j-1} = 0,$$

which simplifies to $(j^2 + 2j)a_j = (j^2 + 2j)a_{j-1}$. For $j > 0$ we have $j^2 + 2j > 0$ so we can divide by $j^2 + 2j$ to get $a_j = a_{j-1}$ **[2]**. As $a_0 = 1$, we see that $a_k = 1$ for all k, which gives

$$y = \sum_{k=0}^{\infty} x^{k+1} = x/(1-x).$$

This gives a second solution linearly independent from z, so the general solution is $Ay + Bz = Ax/(1-x) + B(x^{-1} + 1)$ (with A and B constant) **[1]**.

(20)

(i) There is a unique function $y = \sum_{k=0}^{\infty} a_k x^k$ such that $(1-x)y + (x-x^2)y' = 1$, with $y = 1$ when $x = 0$.

(a) Find a formula for a_k. **(5 marks)**

(b) Using (a), find the power series for $(xy)'$. **(2 marks)**

(c) Using (b), give a simple formula for y. **(2 marks)**

(ii) Consider the equation

$$xy'' + 2y' + 4xy = 0.$$

(a) Show that there is a regular singular point at $x = 0$, and find the corresponding indicial polynomial. **(3 marks)**

(b) Find two linearly independent solutions. **(13 marks)**
Hint: no logarithmic terms are needed.

Solution: This is fairly similar to questions in the lectures and on the problem sheets.

(i) (a) First, we have

$$\begin{aligned}
y &= \sum_k a_k x^k & &= \sum_j a_j x^j \\
-xy &= \sum_k -a_k x^{k+1} & &= \sum_j -a_{j-1} x^j \\
xy' &= \sum_k k a_k x^k & &= \sum_j j a_j x^j \\
-x^2 y' &= \sum_k -k a_k x^{k+1} & &= \sum_j -(j-1) a_{j-1} x^j \, [2],
\end{aligned}$$

so the differential equation $(1-x)y + (x-x^2)y' = 1$ gives $(1+j)a_j - j a_{j-1} = 0$ for $j > 0$ [1]. In the exceptional case $j = 0$, the right hand side is 1 instead of 0, so the equation becomes $a_0 = 1$. For $j > 0$ we have $a_j = \frac{j}{j+1} a_{j-1}$ [1], which gives $a_1 = \frac{1}{2}$ and $a_2 = \frac{2}{3}\frac{1}{2} = \frac{1}{3}$ and $a_3 = \frac{3}{4}\frac{1}{3} = \frac{1}{4}$ and so on, so in general we have $a_k = 1/(k+1)$ [1].

(b) We have

$$\begin{aligned}
y &= \sum_{k=0}^{\infty} \frac{x^k}{k+1} \\
xy &= \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} \; [1] \\
(xy)' &= \sum_{k=0}^{\infty} x^k \; [1]
\end{aligned}$$

(c) Part (b) gives $(xy)' = 1/(1-x)$ [1]. Integrating this gives $xy = -\ln(1-x)$, so $y = -\ln(1-x)/x$ [1].

(ii) (a) First, the equation is equivalent to $y'' + Py' + Qy = 0$, where $P = 2x^{-1}$ and $Q = 4$. Thus, the origin is a regular singular point, with $p_0 = 2$ and $q_0 = 0$ [2]. The indicial polynomial is

$$\alpha^2 - \alpha + p_0\alpha + q_0 = \alpha^2 + \alpha = \alpha(\alpha+1). [1]$$

(b) As 0 is the largest root of the indicial polynomial, the general theory says that there is a solution of the form $y = \sum_{k=0}^{\infty} a_k x^k$ with $a_0 = 1$ [1]. We then have

$$\begin{aligned} xy'' &= \sum_k k(k-1)a_k x^{k-1} \\ 2y' &= \sum_k 2k a_k x^{k-1} \\ 4xy &= \sum_j 4a_j x^{j+1} = \sum_k 4a_{k-2} x^{k-1} \text{[2]}, \end{aligned}$$

so the differential equation $xy'' + 2y' + 4xy = 0$ gives $(k(k-1) + 2k)a_k + 4a_{k-2} = 0$ [1]. For $k > 0$ we can rearrange this as $a_k = -\frac{4}{(k+1)k} a_{k-2}$ [1]. As $a_{-1} = 0$ we see inductively that $a_k = 0$ whenever k is odd [1]. For the even case, we have

$$\begin{aligned} a_0 &= 1 \\ a_2 &= -\frac{4}{3 \times 2} \\ a_4 &= +\frac{4}{5 \times 4} \frac{4}{3 \times 2} = \frac{4^2}{5!} \\ a_6 &= -\frac{4}{7 \times 6} \frac{4^2}{5!} = -\frac{4^3}{7!} \end{aligned}$$

and so on. In general, we have $a_{2j} = (-4)^j/(2j+1)!$, so

$$y = \sum_j \frac{(-4)^j x^{2j}}{(2j+1)!} = \sum_j (-1)^j \frac{(2x)^{2j}}{(2j+1)!} \text{[1]} = \frac{\sin(2x)}{2x}.$$

Next, the smaller root of the indicial polynomia is -1, and the difference between the two roots is an integer. In this situation the general theory tells us that there is another solution of the form $z = cy\ln(x) + \sum_{k=0}^{\infty} b_k x^{k-1}$ with $b_0 = 1$. The question tells us that there are no logarithmic terms, so $c = 0$ and $z = \sum_k b_k x^{k-1}$. This gives

$$\begin{aligned} xz'' &= \sum_k (k-1)(k-2) b_k x^{k-2} \\ 2z' &= \sum_k 2(k-1) b_k x^{k-2} \\ 4xz &= \sum_j 4b_j x^j = \sum_k 4b_{k-2} x^{k-2} \text{[2]}, \end{aligned}$$

so the differential equation $xy'' + 2y' + 4xy = 0$ gives $k(k-1)b_k + 4b_{k-2} = 0$ [1]. For $k = 0$ or $k = 1$ this just gives $0 = 0$, which we can ignore. For $k > 1$ it gives $b_k = -\frac{4}{k(k-1)} b_{k-2}$ [1]. We again see that $b_k = 0$ when k is odd [1], and

$$\begin{aligned} b_0 &= 1 \\ b_2 &= -\frac{4}{2 \times 1} \\ b_4 &= +\frac{4}{4 \times 3} \frac{4}{2 \times 1} = \frac{4^2}{4!} \\ b_6 &= -\frac{4}{6 \times 5} \frac{4^2}{4!} = -\frac{4^3}{6!} \end{aligned}$$

and so on. In general we have $b_{2j} = (-4)^j/(2j)!$, so

$$z = \sum_j \frac{(-4)^j x^{2j-1}}{(2j)!} = x^{-1} \sum_j (-1)^j (2x)^{2j} (2j)! \text{[1]} = \frac{\cos(2x)}{x}.$$

(21)

(i) Consider the operator

$$Lu = x^2(1+x)u'' - x(1+2x)u' + (1+2x)u.$$

(a) Find a function of the form $y = x^n$ such that $Ly = 0$. **(5 marks)**

(b) Find a function of the form $z = x^m \ln(x) + x^k$ such that $Lz = 0$. **(7 marks)**

(ii) Consider the equation

$$x^3y''' + 3x^2y'' + xy' - 27x^3y = 0.$$

There is a unique solution of the form $y = \sum_{k=0}^{\infty} a_k x^k$ such that $y = 1$ at $x = 0$. Find the coefficients a_k. **(9 marks)**

(iii) Solve the equation

$$y'' - ln(6)y' + \ln(2)\ln(3)y = 0,$$

with $y = 1$ when $x = 1$ and $y = 5$ when $x = 2$. **(4 marks)**

Solution:

(i) **Students have ssen many problems like this where one has to find a power series solution. This question is slightly unfamiliar but easier, because the relevant series has only one term.**

(a) First, if $y = x^n$ then we have

$$\begin{aligned} x^2y'' &= n(n-1)x^n \\ x^3y'' &= n(n-1)x^{n+1} \\ -xy' &= -nx^n \\ -2x^2y' &= -2nx^{n+1} \\ y &= x^n \\ 2xy &= 2x^{n+1}, \text{[2]} \end{aligned}$$

so

$$\begin{aligned} Ly &= (n^2 - n - n + 1)x^n + (n^2 - n - 2n + 2)x^{n+1} \\ &= (n-1)^2x^n + (n-1)(n-2)x^{n+1}. \text{[2]} \end{aligned}$$

Thus, to get $Ly = 0$ we must take $n = 1$ and so $y = x$. [1]

(b) Now take $u = x^m \ln(x)$. We get

$$\begin{aligned} u' &= m\,x^{m-1} \ln(x) + x^{m-1} \text{[1]} \\ u'' &= m(m-1)x^{m-2}\ln(x) + m\,x^{m-2} + (m-1)x^{m-2} = m(m-1)x^{m-2}\ln(x) + (2m-1)\,x^{m-2} \text{[1]} \\ Lu &= (x^2+x^3)(m(m-1)x^{m-2}\ln(x) + (2m-1)\,x^{m-2}) - (x+2x^2)(m\,x^{m-1}\ln(x) + x^{m-1}) + \\ &\quad (1+2x)x^m\ln(x) \\ &= (m^2-2m+1)x^m\ln(x) + (m^2-3m+2)x^{m+1}\ln(x) + 2(m-1)x^m + (2m-3)x^{m+1} \\ &= (m-1)^2x^m\ln(x) + (m-1)(m-2)x^{m+1}\ln(x) + 2(m-1)x^m + (2m-3)x^{m+1}. \text{[3]} \end{aligned}$$

Taking $m = 1$, we get $L(x\ln(x)) = -x^2$. The calculation in part (a) shows that $L(x^2) = x^2$, so $L(x\ln(x) + x^2) = 0$. Thus, we can take $z = x\ln(x) + x^2$ [2].

(ii) **Students have seen many examples where this method is used for second order equations. This problem is slightly unfamiliar because it is of third order, but the details work out in an unusually simple way.**

Take $y = \sum_k a_k x^k$. We then have

$$\begin{aligned} x^3 y''' &= \sum_k k(k-1)(k-2)a_k x^k \\ 3x^2 y'' &= \sum_k 3k(k-1)a_k x^k \\ x\,y' &= \sum_k k a_k x^k \\ -27x^3 y &= \sum_k -27a_{k-3}x^k.[3] \end{aligned}$$

Thus, we need

$$(k(k-1)(k-2) + 3k(k-1) + k)a_k = 27a_{k-3}[1]$$

for all k. The coefficient on the left hand side is

$$k^3 - 3k^2 + 2k + 3k^2 - 3k + k = k^3,$$

so the equation simplifies to $k^3 a_k = 27a_{k-3}$ [1]. For $k > 0$ this can be rearranged as $a_k = 27k^{-3}a_{k-3}$. Next, when $x = 0$ we are given that $y = 1$; this means that $a_0 = 1$ [1]. As usual, we have $a_k = 0$ for $k < 0$. It is now easy to see that $a_{3j+1} = 0$ and $a_{3j+2} = 0$ for all $j \geq 0$ [1], but for $j > 0$ we have

$$a_{3j} = 27.(3j)^{-3}a_{3j-3} = j^{-3}a_{3j-3}.$$

From this it follows inductively that $a_{3j} = j!^{-3}$ [2].

(iii) **This is fairly standard.**

This is an equation with constant coefficients, and the auxiliary polynomial is

$$t^2 - \ln(6)t + \ln(2)\ln(3) = t^2 - (\ln(2) + \ln(3))t + \ln(2)\ln(3) = (t - \ln(2))(t - \ln(3)).[1]$$

This means that $y = Ae^{\ln(2)x} + Be^{\ln(3)x}$ for some constants A and B. In other words, we have $y = A.2^x + B.3^x$. [1]Taking $x = 1$ gives $1 = 2A + 3B$, and taking $x = 2$ gives $5 = 4A + 9B$. [1]These equations can be solved to give $B = 1$ and $A = -1$, so $y = 3^x - 2^x$. [1]

(22)

(a) Suppose that y satisfies $y'' + Py' + Qy = 0$. Put $v = \int P\,dx$ and $u = \int y^{-2}e^{-v}\,dx$ and $z = uy$. Prove that we also have $z'' + Pz' + Qz = 0$. **(8 marks)**

(b) Consider the operator $Lf = x^2 f'' + xf' + (x^2 - \frac{1}{4})f$.

(i) Show that the function $y = x^{-1/2}\sin(x)$ satisfies $Ly = 0$. **(5 marks)**

(ii) Use (a) to find another function z with $Lz = 0$. $\int \frac{dx}{\sin^2(x)} = -\cot(x)$. **(7 marks)**

(iii) Now find a third solution w with $Lw = 0$ and $w(\pi) = \pi^{1/2}$ and $w'(\pi) = -\pi^{-1/2}$. **(5 marks)**

Solution:

(a) **Bookwork.** . First, we have

$$\begin{aligned} z &= uy \\ z' &= u'y + uy' [\mathbf{1}] \\ z'' &= u''y + 2u'y' + uy'', [\mathbf{1}] \end{aligned}$$

so

$$\begin{aligned} z'' + Pz' + Qz &= u''y + 2u'y' + uy'' + Pu'y + Puy' + Quy \\ &= u(y'' + Py' + Qy) + u'(2y' + Py) + u''y \\ &= u'(2y' + Py) + u''y. [\mathbf{2}] \end{aligned}$$

Next, we have $v' = P$ [**1**]and $u' = y^{-2}e^{-v}$ [**1**], and so

$$u'' = -2y^{-3}y'e^{-v} - y^{-2}e^{-v}v' = -y^{-3}e^{-v}(2y' + Py). [\mathbf{1}]$$

This gives

$$u'(2y' + Py) + u''y = y^{-2}e^{-v}(2y' + Py) - y^{-3}e^{-v}(2y' + Py)y = 0,$$

so $z'' + Pz' + Qz = 0$ as claimed. [**1**]

(b) **A standard problem.**

(i)

$$\begin{aligned} y &= x^{-1/2}\sin(x) \\ y' &= -\tfrac{1}{2}x^{-3/2}\sin(x) + x^{-1/2}\cos(x) [\mathbf{1}] \\ y'' &= \tfrac{3}{4}x^{-5/2}\sin(x) + 2 \times \tfrac{-1}{2}x^{-3/2}\cos(x) - x^{-1/2}\sin(x), [\mathbf{2}] \end{aligned}$$

so

$$\begin{aligned} x^2y'' + xy' + (x^2 - \tfrac{1}{4})y &= \tfrac{3}{4}x^{-1/2}\sin(x) - x^{1/2}\cos(x) - x^{3/2}\sin(x) - \tfrac{1}{2}x^{-1/2}\sin(x) \\ &\quad + x^{1/2}\cos(x) + x^{3/2}\sin(x) - \tfrac{1}{4}x^{-1/2}\sin(x) \\ &= 0. [\mathbf{2}] \end{aligned}$$

(ii) **A standard problem.** We can divide the equation $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$ by x^2 to get $y'' + x^{-1}y' + (1 - \frac{1}{4}x^{-2})y = 0$ [**1**]. This is of the form discussed in (a), with $P = x^{-1}$ [**1**]and $Q = 1 - 1/(4x^2)$ [**1**]. This gives $v = \int x^{-1}\,dx = \ln(x)$, so $e^{-v} = x^{-1}$ [**1**]. It follows that

$$u = \int y^{-2}e^{-v}\,dx = \int \frac{x}{\sin^2(x)}x^{-1}\,dx = \int \frac{dx}{\sin^2(x)} = -\cot(x). [\mathbf{1}]$$

Our second solution is

$$z = -\cot(x) \times x^{-1/2}\sin(x) = -x^{-1/2}\cos(x). [\mathbf{2}]$$

(iii) We must have

$$w = Ay + Bz = x^{-1/2}(A\sin(x) - B\cos(x))$$

for some constants A and B **[1]**. This gives

$$\begin{aligned} w' &= -\tfrac{1}{2}x^{-3/2}(A\sin(x) - B\cos(x)) + x^{-1/2}(A\cos(x) + B\sin(x))\textbf{[1]} \\ w(\pi) &= \pi^{-1/2}B \\ w'(\pi) &= -\tfrac{1}{2}\pi^{-3/2}B - \pi^{-1/2}A.\textbf{[1]} \end{aligned}$$

As $w(\pi) = \pi^{1/2}$ we must have $B = \pi$ **[1]**, which gives $w'(\pi) = -\frac{1}{2}\pi^{-1/2} - A\pi^{-1/2}$. As $w'(\pi) = -\pi^{-1/2}$ we must have $A = 1/2$ **[1]**. This gives

$$w = \tfrac{1}{2}y + \pi B = x^{-1/2}(\sin(x)/2 - \pi\cos(x)).$$

(23)

(i) Consider the operator

$$Lu = (1 - e^{-x})u'' + 2u' + u.$$

(a) Find $a_1, \dots, a_3$ such that the function $y = 1+a_1x+a_2x^2+a_3x^3$ satisfies $Ly = 0+O(x^3)$. **(6 marks)**

(b) Show that $Lu = ((e^x - 1)u)''.e^{-x}$, and use this to find the general solution for $Lu = 0$. **(5 marks)**

(c) Find the indicial polynomial for L, and explain the relationship with your answer for (b). **(5 marks)**

(ii) Consider the equation $xy'' + (2 - 5x)y' + (6x - 5)y = 0$. What equation is satisfied by the function $z = xy$? Use this to find the general solution for y, and then the unique solution of the form $y = 1 + O(x)$. **(9 marks)**

Solution:

(i) **This is fairly standard.**

(a) We have

$$\begin{aligned} y &= 1 + a_1x + a_2x^2 + a_3x^3 + O(x^4) \\ y' &= a_1 + 2a_2x + 3a_3x^2 + O(x^3)[1] \\ y'' &= 2a_2 + 6a_3x + O(x^2)[1] \\ e^{-x} &= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + O(x^4)[1] \\ 1 - e^{-x} &= x - \frac{x^2}{2} + \frac{x^3}{6} + O(x^4) \end{aligned}$$

so

$$\begin{aligned} (1 - e^{-x})y'' &= (2a_2 + 6a_3x)(x - \frac{x^2}{2} + \frac{x^3}{6}) + O(x^3)[1] \\ &= 2a_2x + (6a_3 - a_2)x^2 + O(x^3) \\ Ly &= 2a_2x + (6a_3 - a_2)x^2 + 2(a_1 + 2a_2x + 3a_3x^2) + (1 + a_1x + a_2x^2) + O(x^3) \\ &= (1 + 2a_1) + (a_1 + 6a_2)x + 12a_3x^2 + O(x^3).[1] \end{aligned}$$

Thus, to get $Ly = 0 + O(x^3)$ we must take $a_1 = -1/2$ and $a_2 = -a_1/6 = 1/12$ and $a_3 = 0$, which gives $y = 1 - x/2 + x^2/12$. [1]

(b) **This is a little less standard.** Next, we have

$$\begin{aligned} ((e^x - 1)u)' &= (e^x - 1)'u + (e^x - 1)u' = e^xu + (e^x - 1)u' \\ ((e^x - 1)u)'' &= (e^x - 1)''u + 2(e^x - 1)'u' + (e^x - 1)u''[1] \\ &= e^xu + 2e^xu' + (e^x - 1)u'' = e^xLu \\ ((e^x - 1)u)''.e^{-x} &= Lu.[1] \end{aligned}$$

It follows that $Lu = 0$ iff the function $v = (e^x - 1)u$ satisfies $v'' = 0$ [1], which means that v' is a constant (say $v' = A$), which means that $v = Ax + B$ for some constants A and B [1]. This gives $u = (Ax + B)/(e^x - 1)$ as the general solution for $Lu = 0$. [1]

(c) **This is standard, apart from the last mark, which asks the students to think intelligently about a standard computational method.** The equation $Lu = 0$ is equivalent to $u'' + Pu + Q = 0$, where $P = 2/(1 - e^{-x})$ and $Q = 1/(1 - e^{-x})$ [1]. Here $1 - e^{-x} = x + O(x^2)$ so $1/(1 - e^{-x}) = x^{-1} + O(1)$ [1]. Thus, we can write $P = \sum p_i x^{i-1}$ and $Q = \sum q_i x^{i-2}$ with $p_0 = 2$ and $q_0 = 0$ [1]. The indicial polynomial is thus

$$\chi(\alpha) = \alpha(\alpha - 1) + p_0\alpha + q_0 = \alpha(\alpha + 1),$$

with roots -1 and 0 [1]. Part (b) gives basic solutions $y = 1/(e^x - 1) = x^{-1} + O(1)$ and $z = x/(e^x - 1) = 1 + O(x)$, and the lowest powers of x in these solutions are the roots of the indicial polynomial. [1]

(ii) **The students have seen similar problems.** If $z = xy$ then $y = x^{-1}z$ so

$$y' = x^{-1}z' - x^{-2}z\text{[1]}$$
$$y'' = x^{-1}z'' - 2x^{-2}z' + 2x^{-3}z.\text{[1]}$$

We can substitute this in the relation $xy'' + (2 - 5x)y' + (6x - 5)y = 0$ to get

$$(z'' - 2x^{-1}z' + 2x^{-2}z) + (2 - 5x)(x^{-1}z' - x^{-2}z) + (6x - 5)x^{-1}z = 0.\text{[1]}$$

After expanding out and collecting terms we get

$$z'' + (-2x^{-1} + 2x^{-1} - 5)z' + (2x^{-2} - 2x^{-2} + 5x^{-1} + 6 - 5x^{-1})z = 0,$$

or in other words $z'' - 5z' + 6 = 0$ [1]. The auxiliary polynomial is $t^2 - 5t + 6$, which has distinct real roots $\lambda_1 = 2$ and $\lambda_2 = 3$ [1]. The general solution is $z = Ae^{2x} + Be^{3x}$ [1], giving

$$y = (Ae^{2x} + Be^{3x})/x\text{[1]} = (A + B)x^{-1} + (2A + 3B) + O(x)\text{[1]}.$$

If we want $y = 1 + O(x)$, we need $A + B = 0$ and $2A + 3B = 1$. This gives $B = 1$ and $A = -1$, so $y = (e^{3x} - e^{2x})/x$ [1].

(24)

(i) Let p and q be constants, and consider the operator

$$Lu = (1 - (p+q)x + pqx^2)u'' + (4pqx - 2p - 2q)u' + 2pqu.$$

Use power series to find the general solution for $Lu = 0$. **(13 marks)**

(ii) Consider the operator $Lf = (x - x^2)f'' + (1 - 3x)f' - f$.

(a) Calculate $L(a/(b+cx))$, and hence find one solution to the equation $Ly = 0$. **(5 marks)**

(b) Use reduction of order to find a second solution. **(7 marks)**

Solution:

(i) There is no singularity at $x = 0$, so we can just use ordinary power series [1]. Consider a function $y = \sum_{k=0}^{\infty} a_k x^k$ (and take $a_k = 0$ for $k < 0$) [1]. We then have

$$\begin{aligned}
u'' &= \sum_j (j+2)(j+1)a_{j+2}x^j \\
-(p+q)xu'' &= \sum_j -(p+q)(j+1)ja_{j+1}x^j \\
pqx^2u'' &= \sum_j pqj(j-1)a_jx^j \\
4pqxu' &= \sum_j 4pqja_jx^j \\
(-2p-2q)u' &= \sum_j -2(p+q)(j+1)a_{j+1}x^j \\
2pqu &= \sum_j 2pqa_jx^j.\text{[4]}
\end{aligned}$$

It follows that $Ly = \sum_j b_j x^j$, where

$$b_j = (j+2)(j+1)a_{j+2} - ((j+1)j + 2(j+1))(p+q)a_{j+1} + (j(j-1) + 4j + 2)pqa_j.\text{[1]}$$

This simplifies to

$$b_j = (j+1)(j+2)(a_{j+2} - (p+q)a_{j+1} + pqa_j).\text{[1]}$$

Thus, if $Ly = 0$ then for $j \geq 0$ we must have

$$a_{j+2} - (p+q)a_{j+1} + pqa_j.\text{[1]}$$

The auxiliary polynomial for this recurrence relation is $t^2 - (p+q)t + pq = (t-p)(t-q)$, so the solutions are $a_k = Ap^k + Bq^k$, where A and B are arbitrary constants [2]. This gives

$$y = A\sum_{k=0}^{\infty} p^k x^k + B\sum_{k=0}^{\infty} q^k x^k\text{[1]} = \frac{A}{1-px} + \frac{B}{1-qx}.\text{[1]}$$

(ii) (a) Put $y = a/(b+cx)$. We then have

$$\begin{aligned}
y &= a/(b+cx) \\
y' &= -ac/(b+cx)^2\text{[1]} \\
y'' &= 2ac^2/(b+cx)^3\text{[1]}
\end{aligned}$$

so

$$
\begin{aligned}
Ly &= \frac{2ac^2(x-x^2)}{(b+cx)^3} - \frac{ac(1-3x)}{(b+cx)^2} - \frac{a}{b+cx} \\
&= (b+cx)^{-3}\left(2ac^2(x-x^2) - ac(1-3x)(b+cx) - a(b+cx)^2\right) \text{ [1]} \\
&= (b+cx)^{-3}\left((-2ac^2+3ac^2-ac^2)x^2 + (2ac^2+3abc-ac^2-2abc)x + (-abc-ab^2)\right) \\
&= (b+cx)^{-3}\left((ac^2+abc)x - abc - ab^2\right) = a(b+c)(cx-b)/(b+cx)^3.\text{[1]}
\end{aligned}
$$

We can make $Ly = 0$ by choosing $a = b = 1$ and $c = -1$, which gives $y = 1/(1-x)$ [1].

(b) For the reduction of order method, we first note that $Lz = 0$ is equivalent to $z'' + Pz' + Qz = 0$, where

$$
\begin{aligned}
P &= \frac{3x-1}{x^2-x} \\
Q &= \frac{1}{x^2-x}.\text{[1]}
\end{aligned}
$$

We next need to find $v = \int P\,dx$ [1]. For this we write P in partial fraction form:

$$
P = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1)+Bx}{x^2-x} = \frac{(A+B)x - A}{x^2-x}.
$$

For the coefficients to match we must have $A + B = 3$ and $A = 1$, so $B = 2$, so $P = 1/x + 2/(x-1)$ [1]. This gives

$$
v = \ln(x) + 2\ln(x-1) = \ln(x(x-1)^2), \text{[1]}
$$

so $e^v = x(x-1)^2$. For the next step, we must find

$$
u = \int y^{-2}e^{-v}\,dx = \int (1-x)^2 . \frac{1}{x(x-1)^2}\,dx = \int x^{-1}\,dx = \ln(x).\text{[2]}
$$

It follows that the function $z = uy = \ln(x)/(1-x)$ is a second solution for $Lz = 0$. [1]

(25)

(i) Consider the operators

$$Lu = x^2u'' - 2x(x+2)u' + (x+2)^2u$$
$$L_1v = x^2v'' - 4xv' + 4v.$$

(a) Show that $L_1v = 0$ if and only if $L(e^xv) = 0$. **(4 marks)**

(b) Find the indicial polynomial for L_1, and its roots. **(4 marks)**

(c) Find the general solution for $L_1v = 0$ (the answer is very simple). **(5 marks)**

(d) Using parts (a) and (c), find a function u with $Lu = 0$ where the coefficient of x is 1 and the coefficient of x^4 is 0. **(4 marks)**

(ii) Consider the equation $(1-x^2)y'' - 4xy' - 2y = 0$. Find the solution with $y = 1 + O(x^2)$. **(8 marks)**

Solution:

(i) **This is fairly standard, except that part (c) works out much more easily than in typical cases.**

(a) If $u = e^xv$ then

$$\begin{aligned}
u' &= e^xv' + e^xv = e^x(v'+v)[1] \\
u'' &= e^xv'' + 2e^xv' + e^xv = e^x(v''+2v'+v)[1] \\
Lu &= e^x\left(x^2(v''+2v'+v) - (2x^2+4x)(v'+v) + (x^2+4x+4)v\right)\ [1] \\
&= e^x\left(x^2v'' - 4xv' + 4v\right) = e^xL_1(v).[1]
\end{aligned}$$

This shows that $L(u) = 0$ if and only if $L_1(v) = 0$, as required.

(b) The equation $L_1v = 0$ is equivalent to $v'' + Pv' + Qv = 0$, where $P = -4x^{-1}$ and $Q = 4x^{-2}$ [1]. The initial coefficients are $p_0 = -4$ and $q_0 = 4$ [1], so the indicial polynomial is

$$\chi(t) = t(t-1) + p_0t + q_0 = t^2 - 5t + 4[1] = (t-1)(t-4),$$

so the roots are $\alpha = 1$ and $\beta = 4$ [1].

(c) We have

$$L_1(x^k) = x^2.k(k-1)x^{k-2} - 4x.kx^{k-1} + 4x^k = (k(k-1) - 4k + 4)x^k = (k-1)(k-4)x^k,$$

so the basic solutions are just $y = x$ and $z = x^4$. The general solution is thus $v = Ax + Bx^4$. [5]

(d) We now see that the general solution for $L(u) = 0$ is

$$u = (Ax + Bx^4)e^x = A(x + x^2 + x^3/2 + x^4/6 + O(x^5)) + B(x^4 + O(x^5)).[1]$$

Thus, the coefficient of x is A, and the coefficient of x^4 is $A/6 + B$ [1]. Thus, to make these coefficients 1 and 0, we must have $A = 1$ and $B = -1/6$ [1]. The required solution is

$$u = (x - x^4/6)e^x.[1]$$

(ii) **The students have seen many questions of this type.** Here there is no singularity at $x = 0$ so we just have $y = \sum_{k=0}^{\infty} a_k x^k$ [1]. We want $y = 1 + O(x^2)$, which means that $a_0 = 1$ and $a_1 = 0$ [1], and as usual we take $a_k = 0$ for $k < 0$. We also have

$$\begin{aligned} y'' &= \sum_k k(k-1)a_k x^{k-2} = \sum_j (j+2)(j+1)a_{j+2}x^j \\ -x^2 y'' &= \sum_j (-j(j-1))a_j x^j \\ -4xy' &= \sum_j (-4j)a_j x^j \\ -2y &= \sum_j (-2a_j)x^j. \textbf{[3]} \end{aligned}$$

The equation $(1-x^2)y'' - 4xy' - 2y = 0$ is thus equivalent to

$$(j+2)(j+1)a_{j+2} = j(j-1)a_j + 4ja_j + 2a_j = (j^2+3j+2)a_j = (j+2)(j+1)a_j. \textbf{[1]}$$

When $j \geq 0$ we have $(j+2)(j+1) > 0$ so we can divide by this to get $a_{j+2} = a_j$ [1]. As $a_0 = 1$ and $a_1 = 0$ we see that $a_k = 1$ whenever k is even, and $a_k = 0$ when k is odd. This gives $y = \sum_i x^{2i} = 1/(1-x^2)$ [1].

(26)

(i) Consider the operator

$$Lu = (x + x^3)^2 u'' + 4(x + x^3)u' + (2 - 6x^2)u.$$

Convert this to normal form, and thus solve the equation $Ly = 0$. **(11 marks)**

(ii) Suppose that $y = \sum_k a_k x^k$ (with $a_k = 0$ for $k < 0$) and that

$$(1 - 3x + 2x^2)y'' + 2xy' - 2y = 0.$$

(a) Find an equation satisfied by the coefficients a_j, a_{j+1} and a_{j+2}. **(5 marks)**

(b) Show that if $a_{j+1} = a_j$ for some $j \geq 0$ then we also have $a_{j+2} = a_j$. **(2 marks)**

(c) Hence find a solution with $y = 1 + x + O(x^2)$. **(3 marks)**

(d) Find another solution of the form $z = x^\alpha$. **(4 marks)**

Solution:

(i) First, we have

$$\begin{aligned}
A &= (x + x^3)^2 \\
B &= 4(x + x^3) \\
C &= 2 - 6x^2 \\
P &= B/A = 4(x + x^3)^{-1}[1] \\
Q &= C/A = (2 - 6x^2)/(x + x^3)^2 \\
R &= Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2[1] = \frac{2 - 6x^2}{(x + x^3)^2} - \frac{4}{2}\frac{-1}{(x + x^3)^2}(1 + 3x^2) - \frac{1}{4}\frac{16}{(x + x^3)^2}[1] \\
&= \frac{2 - 6x^2 + 2 + 6x^2 - 4}{(x + x^3)^2} = 0.[1]
\end{aligned}$$

This means that the solutions to $Ly = 0$ have the form $y = zm$ where $z'' = 0$ and $m = e^{-v/2}$, where $v = \int P\,dx$ [1]. The equation $z'' = 0$ just gives $z = a + bx$ where A and B are constant [1]. The real problem is to find v. For this, we use partial fractions. We have $P = \alpha/x + \beta/(x^2 + 1) + \gamma x/(x^2 + 1)$ for some constants al, β, γ. This gives

$$\frac{4}{x + x^3} = P = \frac{\alpha + \alpha x^2 + \beta x + \gamma x^2}{x + x^3} = \frac{\alpha + \beta x + (\alpha + \gamma)x^2}{x + x^3}.$$

From this we see that $\alpha = 4$ and $\beta = 0$ and $\gamma = -4$, so $P = 4/x - 4x/(x^2 + 1)$ [2]. This gives

$$v = \int P\,dx = 4\ln(x) - 2\ln(x^2 + 1), [1]$$

so $m = e^{-v/2} = x^{-2}(x^2 + 1) = 1 + x^{-2}$ [1]. We now have

$$y = zm = (a + bx)(1 + x^{-2}) = ax^{-2} + bx^{-1} + a + bx.[1]$$

(ii) (a) We have

$$
\begin{aligned}
y'' &= \sum_k k(k-1)a_k x^{k-2} & &= \sum_j (j^2+3j+2)a_{j+2}x^j \\
-3xy'' &= \sum_k -3k(k-1)a_k x^{k-1} & &= \sum_j (-3j^2-3j)a_{j+1}x^j \\
2x^2y'' &= \sum_k 2k(k-1)a_k x^k & &= \sum_j (2j^2-2j)a_j x^j \\
2xy' &= \sum_k 2ka_k x^k & &= \sum_j 2ja_j x^j \\
-2y &= \sum_k -2a_k x^k & &= \sum_j -2a_j x^j.
\end{aligned}
$$

[4] Putting this together gives $Ly = \sum_j b_j x^j$, where

$$b_j = (j^2+3j+2)a_{j+2} + (-3j^2-3j)a_{j+1} + (2j^2-2)a_j.\text{[1]}$$

Thus, if $Ly = 0$ we must have $b_j = 0$ for all j.

(b) Suppose that $a_{j+1} = a_j$ for some $j \geq 0$. We then get

$$b_j = (j^2+3j+2)a_{j+2} + (-3j^2-3j+2j^2-2)a_j = (j^2+3j+2)(a_{j+2}-a_j).\text{[1]}$$

As $j \geq 0$ we have $j^2+3j+2 > 0$ and so $a_{j+2} = a_j$. [1]

(c) Now suppose that $y = 1 + x + O(x^2)$, so $a_0 = a_1 = 1$. We can then use (b) inductively to show that $a_j = 1$ for all $j \geq 0$. [1]This gives $y = \sum_k x^k = 1/(1-x)$. [2] (The power series only converges for $|x| < 1$, but in fact the function $y = 1/(1-x)$ gives a solution for all $x \neq 1$, as one can easily check directly.)

(d) For $z = x^\alpha$ to be a solution, we need

$$
\begin{aligned}
0 &= (2x^2-3x+1)\alpha(\alpha-1)x^{\alpha-2} + 2x\alpha x^{\alpha-1} - 2x^\alpha\text{[1]} \\
&= 2(\alpha^2-\alpha)x^\alpha - 3(\alpha^2-\alpha)x^{\alpha-1} + (\alpha^2-\alpha)x^{\alpha-2} + 2\alpha x^\alpha - 2x^\alpha \\
&= 2(\alpha^2-1)x^\alpha - 3(\alpha^2-\alpha)x^{\alpha-1} + (\alpha^2-\alpha)x^{\alpha-2}.\text{[2]}
\end{aligned}
$$

This means we must have $\alpha^2 - 1 = 0$ and $\alpha^2 - \alpha = 0$, which gives $\alpha = 1$. Thus, the function $z = x$ is a second solution. [1](Full marks will also be given to students who find this solution by inspection, provided that they check that it works.)

(27)

(i) Consider the operator

$$Lu = \sin^2(x)u'' + \sin(x)\cos(x)u' - u.$$

(a) Show that $y = \tan(x/2)$ gives one solution. **(6 marks)**

(b) Use reduction of order to find another solution. **(8 marks)**

You may find it helpful to write all formulae in terms of the quantities $s = \sin(x/2)$ and $c = \cos(x/2)$.

(ii) Suppose that $y = \sum_k a_k x^k$ (with $a_k = 0$ for $k < 0$) and that

$$(x^2 + x - 1)y'' + (4x + 2)y' + 2y = 0.$$

(a) Find an equation satisfied by the coefficients a_j, a_{j+1} and a_{j+2}. **(7 marks)**

(b) Suppose that $a_j = \alpha^j$ for all j. Find an equation satisfied by α. **(2 marks)**

(c) Use (b) to find and simplify two solutions to the original differential equation. **(3 marks)**

Solution:

(i) (a) We write $s = \sin(x/2)$ and $c = \cos(x/2)$ so that $y = s/c$. Note that $s' = c/2$ and $c' = -s/2$ and $\sin(x) = 2sc$ and $\cos(x) = 1 - 2s^2$ [2]. This gives

$$\begin{aligned}
y' &= (s'c - sc')/c^2 = (c^2 + s^2)/(2c^2) = \tfrac{1}{2}c^{-2}\text{[1]} \\
y'' &= -c^{-3}c' = \tfrac{1}{2}sc^{-3}\text{[1]} \\
Ly &= 4s^2c^2y'' + 2sc(1 - 2s^2)y' - y \\
&= 2s^2c^2sc^{-3} + sc(1 - 2s^2)c^{-2} - sc^{-1} \\
&= 2s^3c^{-1} + sc^{-1} - 2s^3c^{-1} - sc^{-1} = 0.\text{[2]}
\end{aligned}$$

(b) We have $Lu = Au'' + Bu' + Cu$ with $A = \sin^2(x)$ and $B = \sin(x)\cos(x)$ and $C = -1$. In the reduction of order method this gives

$$\begin{aligned}
P &= B/A = \cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{\sin'(x)}{\sin(x)}\text{[1]} \\
v &= \int P\,dx = \ln(\sin(x))\text{[2]} \\
w &= y^{-2}e^{-v}\text{[1]} = \tan(x/2)^{-2}\sin(x)^{-1} \\
&= (s/c)^{-2}(2sc)^{-1} = c/(2s^3) = s'\,s^{-3}\text{[1]} \\
u &= \int w\,dx = -\tfrac{1}{2}s^{-2}\text{[2]} \\
z &= uy = -\tfrac{1}{2}s^{-2}(s/c) = \frac{-1}{2sc} = -\csc(x).\text{[1]}
\end{aligned}$$

Thus, our second solution is $-\csc(x)$.

(ii) (a) We have

$$
\begin{aligned}
x^2y'' &= \sum_i i(i-1)a_i x^i & &= \sum_i (i^2-i)a_i x^i \\
xy'' &= \sum_i (i+1)ia_{i+1}x^i & &= \sum_i (i^2+i)a_{i+1}x^i \\
-y'' &= -\sum_i (i+2)(i+1)a_{i+2}x^i & &= \sum_i (-i^2-3i-2)a_{i+2}x^i \\
4xy' &= 4\sum_i ia_i x^i & &= \sum_i 4ia_i x^i \\
2y' &= 2\sum_i (i+1)a_{i+1}x^i & &= \sum_i (2i+2)a_{i+1}x^i \\
2y &= 2\sum_i a_i x^i & &= \sum_i 2a_i x^i.\text{[3]}
\end{aligned}
$$

Thus, the coefficient of x^i in $(x^2+x-1)y''+(4x+2)y'+2y$ is

$$
\begin{aligned}
&(i^2-i)a_i + (i^2+i)a_{i+1} + (-i^2-3i-2)a_{i+2} + 4ia_i + (2i+2)a_{i+1} + 2a_i \\
&= (i^2-i+4i+2)a_i + (i^2+i+2i+2)a_{i+1} + (-i^2-3i-2)a_{i+2} \\
&= (i^2+3i+2)(a_i+a_{i+1}-a_{i+2}) = (i+1)(i+2)(a_i+a_{i+1}-a_{i+2}).\text{[3]}
\end{aligned}
$$

This shows that the differential equation is satisfied iff the coefficients a_i satisfy the Fibonacci recurrence $a_{i+2} = a_i + a_{i+1}$ for all i. [1]

(b) Now suppose that $a_i = \alpha^i$, so $a_{i+2} - a_{i+1} - a_i = \alpha^i(\alpha^2 - \alpha - 1)$. We see that the recurrence equation is satisfied iff $\alpha^2 - \alpha - 1 = 0$. [2]

(c) The roots of the quadratic are $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$ [1]. We therefore have solutions $y = \sum_i \alpha^i x^i = 1/(1-\alpha x)$ and $z = \sum_i \beta^i x^i = 1/(1-\beta x)$. [2]

Question 4

(28)

(i) Suppose that a function y satisfies $(x^7y')'/x^7 + y = 0$. Show that the function $z = x^3y$ satisfies Bessel's equation $x^2z'' + xz' + (x^2 - n^2)z = 0$ for some value of n. **(8 marks)**

(ii) Consider a Sturm-Liouville operator $Lf = ((pf')' + qf)/r$ on the interval $[0,1]$. Show that for any two smooth functions f and g on $[0,1]$, the generalised Wronskian $W = pf'g - pfg'$ satisfies

$$W' = r((Lf)g - f(Lg)).\textbf{(6 marks)}$$

(iii) Find p, q and r such that the operator $Lf = x^2f'' - 2xf' + (x^2+2)f$ is the same as $((pf')' + qf)/r$. **(6 marks)**

(iv) Find the general solution of the equation $y'' + 6y' + 25y = 0$. **(5 marks)**

Solution: Part (i) requires a little creativity, but it is fairly similar to problems that have been seen. Part (ii) is pure bookwork. Parts (iii) and (iv) are standard problems.

(i) First, we have

$$\begin{aligned}
y &= x^{-3}z \\
y' &= x^{-3}z' - 3x^{-4}z \\
x^7y' &= x^4z' - 3x^3z \\
(x^7y')' &= x^4z'' + 4x^3z' - 3x^3z' - 9x^2z = x^4z'' + x^3z' - 9x^2z.[3]
\end{aligned}$$

As $(x^7y')'/x^7 + y = 0$ we also have $(x^7y')' = -x^7y = -x^4z$ [3]. This gives

$$x^4z'' + x^3z' - 9x^2z = -x^4z,$$

which is equivalent to

$$x^2z'' + xz' + (x^2 - 9)z = 0,$$

which is Bessel's equation with $n = 3$ [2].

(ii) First, we have

$$\begin{aligned}
Lf &= r^{-1}pf'' + r^{-1}p'f' + r^{-1}qf \\
Lg &= r^{-1}pg'' + r^{-1}p'g' + r^{-1}qg[2] \\
r.Lf.g &= pf''g + p'f'g + qfg \\
r.f.Lg &= pfg'' + p'fg' + qfg[1] \\
r.(Lf.g - f.Lg) &= p(f''g - fg'') + p'(f'g - fg')[1].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
W' &= (p'f'g + pf''g + pf'g') - (p'fg' + pf'g' + pfg'') \\
&= pf''g + p'f'g - pfg'' - p'fg',
\end{aligned}$$

which is the same [2].

(iii) The coefficients appearing in L are $A = x^2$ and $B = -2x$ and $C = x^2 + 2$. The standard formulae for conversion to Sturm-Liouville form are

$$\begin{aligned}
p &= \exp\left(\int B/A\,dx\right) = \exp(\int -2x^{-1}\,dx) = \exp(-2\log(x)) = x^{-2}\textbf{[2]} \\
r &= p/A = x^{-2}/x^2 = x^{-4}\textbf{[2]} \\
q &= rC = x^{-2} + 2x^{-4}.\textbf{[2]}
\end{aligned}$$

We therefore have

$$Lf = x^4((x^{-2}f')' + (x^{-2} + 2x^{-4})f).$$

(iv) The auxiliary polynomial is $\lambda^2 + 6\lambda + 25$ **[2]**, which has roots

$$(-6 \pm \sqrt{36 - 4 \times 25})/2 = (-6 \pm 8i)/2 = -3 \pm 4i.\textbf{[1]}$$

From this it follows that the general solution is

$$y = e^{-3x}(A\cos(4x) + B\sin(4x))$$

(where A and B are constant) **[2]**.

(29)

(i) Suppose that a function y satisfies $x^2y'' - 9xy' + x^2y = 0$. Show that the function $z = y/x^5$ satisfies Bessel's equation $x^2z'' + xz' + (x^2 - n^2)z = 0$ for some value of n. **(6 marks)**

(ii) Consider a Sturm-Liouville operator $Lf = ((pf')' + qf)/r$ on the interval $[0, 1]$. Suppose that f and g are smooth functions on $[0, 1]$ with $f(0) = g(0) = f(1) = g(1) = 0$ and $Lf = f$ and $Lg = -g$. Show that

$$\int_{x=0}^{1} r(x)\, f(x)\, g(x)\, dx = 0.$$

(5 marks) Hint: you may assume that $W' = r.((Lf)g - f(Lg))$, where $W = pf'g - pfg'$.

(iii) Find p, q and r such that the operator

$$Lf = \frac{x^2 + x + 1}{2x + 1} f'' + f' + f$$

is the same as $((pf')' + qf)/r$. **(5 marks)**

(iv) Find the general solution of the equation $y'' - 100y' - 1221y = 0$. **(4 marks)**

Solution: Part (ii) is a special case of a piece of bookwork. Parts (i), (iii) and (iv) are standard problems.

(i) First, we have

$$\begin{aligned} z &= x^{-5}y \\ z' &= x^{-5}y' - 5x^{-6}y \\ z'' &= x^{-5}y'' - 10x^{-6}y' + 30x^{-7}y, [2] \end{aligned}$$

so

$$\begin{aligned} x^2z'' + xz' + x^2z &= x^{-3}y'' - 10x^{-4}y' + 30x^{-5}y + x^{-4}y' - 5x^{-5}y + x^{-3}y[1] \\ &= x^{-5}(x^2y'' - 9xy' + x^2y + 25y) \\ &= 25x^{-5}y = 25z[1]. \end{aligned}$$

This can be rearranged to give

$$x^2z'' + xz' + (x^2 - 25)z = 0,$$

which is Bessel's equation with $n = 5$ [2].

(ii) First, as $f(0) = g(0) = f(1) = g(1) = 0$, we see that the function $W(x) = p(x)f'(x)g(x) - p(x)f(x)g'(x)$ satisfies $W(0) = 0$ and $W(1) = 0$. This means that $\int_0^1 W'(x)\, dx = W(1) - W(0) = 0$. [2] However, we also know that $W' = r\,(Lf)\,g - r\,f\,L(g)$, where $Lf = f$ and $Lg = -g$ [1]. This means that $W' = rfg - (-rfg) = 2rfg$ [1], so

$$\int_0^1 rfg\, dx = \frac{1}{2}\int_0^1 W\, dx = 0[1].$$

(iii) The coefficients appearing in L are

$$A = \frac{x^2 + x + 1}{2x + 1}$$

and $B = C = 1$ [1]. The standard formulae for conversion to Sturm-Liouville form are

$$p = \exp\left(\int B/A\,dx\right) = \exp(\int \frac{2x+1}{x^2+x+1}\,dx) = \exp(\log(x^2+x+1)) = x^2+x+1\text{[2]}$$

$$r = p/A = (x^2+x+1)\frac{2x+1}{x^2+x+1} = 2x+1\text{[1]}$$

$$q = rC = 2x+1.\text{[1]}$$

We therefore have

$$Lf = (((x^2+x+1)f')' + (2x+1)f)/(2x+1).$$

(iv) The auxiliary polynomial is $\lambda^2 - 100\lambda - 1221$ [2], which has roots

$$\frac{100 \pm \sqrt{10000 + 4 \times 1221}}{2} = \frac{100 \pm \sqrt{14884}}{2} = (100 \pm 122)/2 = -11 \text{ or } 111.$$

From this it follows that the general solution is

$$y = Ae^{-11x} + Be^{111x}$$

(where A and B are constant) [2].

(30)

(i) Let x and t be variables related by $t = x^6$, and write $u' = \frac{du}{dx}$ and $\dot{u} = \frac{du}{dt}$. Consider the differential operator

$$Ly = x^2y'' - 11x\,y' + 36(1 + x^{12})y.$$

Show that $Ly = 0$ if and only if the function $z = y/t$ satisfies $t\ddot{z} + \dot{z} + tz = 0$. Using this, find a solution for y in terms of Bessel functions. **(13 marks)**

(ii) Consider a Sturm-Liouville operator $Lf = ((pf')' + qf)/r$ on the interval $[0, 1]$. Show that for any two smooth functions f and g on $[0, 1]$, the generalised Wronskian $W = pf'g - pfg'$ satisfies $W' = r((Lf)g - f(Lg))$. **(6 marks)**

(iii) Now suppose that $Lf = \lambda f$ and $Lg = \mu g$, where λ and μ are constants with $\lambda \neq \mu$. Suppose also that f and g satisfy the boundary conditions $f'(0) = g'(0) = f'(1) = g'(1) = 0$. Prove that $\int_0^1 rfg\,dx = 0$. **(6 marks)**

Solution:

(i) **This is similar to several examples which the students have seen.**
First, for any function u we have

$$u' = \frac{du}{dx} = \frac{dt}{dx}\frac{du}{dt} = 6x^5\dot{u}, \textbf{[1]}$$

and so

$$\begin{aligned} u'' &= \frac{d}{dx}(6x^5\dot{u}) = 30x^4\dot{u} + 6x^5\frac{d}{dx}\dot{u} \\ &= 30x^4\dot{u} + 6x^5 \times 6x^5\ddot{u} = 36x^{10}\ddot{u} + 30x^4\dot{u}.\textbf{[2]} \end{aligned}$$

Now $y = tz = x^6z$, so

$$\begin{aligned} y' &= 6x^5z + x^6z' = 6x^5z + 6x^{11}\dot{z}\textbf{[1]} \\ y'' &= 30x^4z + 12x^5z' + x^6z'' \\ &= 30x^4z + 72x^{10}\dot{z} + x^6(36x^{10}\ddot{z} + 30x^4\dot{z}) \\ &= 36x^{16}\ddot{z} + 102x^{10}\dot{z} + 30x^4z.\textbf{[2]} \end{aligned}$$

This in turn gives

$$\begin{aligned} Ly &= x^2y'' - 11xy' + 36(1 + x^{12})y \\ &= 36x^{18}\ddot{z} + 102x^{12}\dot{z} + 30x^6z - 66x^6z - 66x^{12}\dot{z} + 36x^6z + 36x^{18}z \\ &= 36x^{18}\ddot{z} + 36x^{12}\dot{z} + 36x^{18}z \\ &= 36t^2(t\ddot{z} + \dot{z} + tz).\textbf{[3]} \end{aligned}$$

Thus, we have $Ly = 0$ if and only if $t\ddot{z} + \dot{z} + tz$ **[1]**. This is equivalent to $t^2\ddot{z} + t\dot{z} + t^2z = 0$, which is Bessel's equation of order 0 **[2]**, so one solution is $z = J_0(t) = J_0(x^6)$, which gives $y = x^6\,J_0(x^6)$ **[1]**.

(ii) **This is bookwork.**
First, we have

$$\begin{aligned} Lf &= r^{-1}pf'' + r^{-1}p'f' + r^{-1}qf \\ Lg &= r^{-1}pg'' + r^{-1}p'g' + r^{-1}qg\textbf{[2]} \\ r.Lf.g &= pf''g + p'f'g + qfg \\ r.f.Lg &= pfg'' + p'fg' + qfg\textbf{[1]} \\ r.(Lf.g - f.Lg) &= p(f''g - fg'') + p'(f'g - fg')\textbf{[1]}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} W' &= (p'f'g + pf''g + pf'g') - (p'fg' + pf'g' + pfg'') \\ &= pf''g + p'f'g - pfg'' - p'fg', \end{aligned}$$

which is the same **[2]**.

(iii) **This is bookwork.**
Part (ii) gives $(\lambda - \mu)rfg = W'$ **[1]**, so

$$(\lambda - \mu) \int_0^1 rfg\, dx = \int_0^1 W'\, dx = W(1) - W(0).\textbf{[2]}$$

On the other hand, we have $f'(0) = g'(0) = 0$, so

$$W(0) = p(0)f'(0)g(0) - p(0)f(0)g'(0) = 0 - 0 = 0.\textbf{[1]}$$

Similarly $W(1) = 0$, so $(\lambda - \mu) \int_0^1 rfg\, dx = 0$ **[1]**. We are also given that $\lambda \neq \mu$, so we can divide by $\lambda - \mu$ to get $\int_0^1 rfg\, dx = 0$ as claimed **[1]**.

(31)

(i) There is a unique function $y = \sum_{k=0}^{\infty} a_k x^k$ such that $xy' - (1+x^3)y = 0$, with $y = 0$ and $y' = 1$ when $x = 0$.

(a) Find a formula the coefficients a_k. **(10 marks)**

(b) Give a simple formula for y in terms of the exponential function. **(3 marks)**

(ii) Suppose that z satisfies

$$z'' - 2(1+x)z' + (11+2x)z = 0.$$

(a) Show that the function $y = e^{-x}z$ satisfies the Hermite equation

$$y'' - 2xy' + 2ny = 0$$

for a certain value of n. **(6 marks)**

(b) Hence find a solution to the original equation, of the form

$$z = (ax + bx^3 + cx^5)e^x$$

for some nonzero constants a, b and c. **(6 marks)**

Solution:

(i) **A standard problem.**

(a) First, we have

$$\begin{aligned} xy' &= \sum_k k a_k x^k \\ -y &= \sum_k -a_k x^k \\ -x^3 y &= \sum_k -a_{k-3} x^k, \text{[2]} \end{aligned}$$

so the differential equation $xy' - (1+x^3)y = 0$ gives $a_{k-3} = (k-1)a_k$. When $k \neq 1$ we can rewrite this as $a_k = a_{k-3}/(k-1)$ [2]. We are given that $y = 0$ and $y' = 1$ when $x = 0$, which means that $a_0 = 0$ and $a_1 = 1$ [1]. Using $a_0 = 0$ and $a_k = a_{k-3}/(k-1)$ we see that $a_k = 0$ whenever k has the form $k = 3j$ or $k = 3j-1$ [2]. In the remaining case, we have $a_{3j+1} = a_{3(j-1)-1}/(3j)$ [1]. The first few steps are

$$\begin{aligned} a_4 &= \frac{a_1}{3} = \frac{1}{3} \\ a_7 &= \frac{a_4}{6} = \frac{1}{3\times 6} = \frac{1}{3^2\times 1\times 2} \\ a_{10} &= \frac{a_7}{9} = \frac{1}{3\times 6\times 9} = \frac{1}{3^3\times 1\times 2\times 3} \\ a_{13} &= \frac{a_{10}}{12} = \frac{1}{3\times 6\times 9\times 12} = \frac{1}{3^4\times 1\times 2\times 3\times 4}, \end{aligned}$$

and in general we have $a_{3j+1} = 1/(3^j j!)$ [2].

(b) We have

$$y = \sum_{j=0}^{\infty} a_{3j+1}x^{3j+1} = \sum_{j=0}^{\infty} \frac{x^{3j+1}}{3^j j!}\text{[1]} = x\sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{x^3}{3}\right)^j \text{[1]} = x\, e^{x^3/3}\text{[1]}.$$

(ii) **The students have seen many examples of transforming the Bessel equation, and this is similar.**

(a) If $y = e^{-x}z$ then

$$\begin{aligned} z &= e^x y \\ z' &= e^x(y' + y) \\ z'' &= e^x(y'' + 2y' + y), \textbf{[2]} \end{aligned}$$

so

$$\begin{aligned} &z'' - 2(1+x)z' + (11+2x)z \\ =&e^x\left((y'' + 2y' + y) - 2(1+x)(y'+y) + (11+2x)y\right) \\ =&e^x\left(y'' - 2xy' + 10y\right). \textbf{[2]} \end{aligned}$$

Thus, the original equation for z is equivalent to the Hermite equation for y with $n = 5$. **[2]**

(b) If $z = (ax + bx^3 + cx^5)e^x$ then

$$\begin{aligned} y &= ax + bx^3 + cx^5 \textbf{[1]} \\ y' &= a + 3bx^2 + 5cx^4 \\ y'' &= 6bx + 20cx^3 \textbf{[1]} \\ y'' - 2xy' + 10y &= 6bx + 20cx^3 - 2ax - 6bx^3 - 10cx^5 + 10ax + 10bx^3 + 10cx^5 \\ &= (6b + 8a)x + (20c + 4b)x^3. \textbf{[1]} \end{aligned}$$

For this to be zero we must have $b = -\frac{4}{3}a$ and $c = -\frac{1}{5}b = \frac{4}{15}a$ **[1]**. Here a is arbitrary, but it will be convenient to choose $a = 15$. This gives $b = -20$ and $c = 4$ **[1]**, so $y = 15x - 20x^3 + 4x^5$ and $z = (15x - 20x^3 + 4x^5)e^x$ **[1]**.

(32)

(i) Let x and t be variables related by $x = t^2 + 1$, and write $u' = \frac{du}{dx}$ and $\dot{u} = \frac{du}{dt}$. Consider the differential operator

$$Ly = t(t^2+1)^2\ddot{y} + (t^4-1)\dot{y} + 4t^5(t^2+2)y$$

Convert the equation $Ly = 0$ to an equation involving x, y, y' and y'' instead of $t, y, \dot{y}$ and $\ddot{y}$. Using this, find a solution for y in terms of Bessel functions. **(8 marks)**

(ii) Consider a Sturm-Liouville operator $Lf = ((pf')' + qf)/r$ on the interval $[0,1]$ (with $r(x) > 0$ for all x), and define

$$\langle f, g\rangle = \int_0^1 rfg.$$

Remember that the generalised Wronskian $W = pf'g - pfg'$ satisfies $W' = r((Lf)g - f(Lg))$ (you do not need to prove this).

(a) Prove that if $f(0) = f(1) = 0$ and $g(0) = g(1) = 0$ then $\langle Lf, g\rangle = \langle f, Lg\rangle$. **(4 marks)**

(b) Now suppose that f is a nonzero complex function on $[0,1]$ with $f(0) = f(1) = 0$, and that λ is a complex number such that $Lf = \lambda f$. Prove that λ is actually real. **(8 marks)**

(c) In the case where $p = r = 1$ and $q = 0$, show that $\lambda \leq 0$. **(5 marks)**

Solution:

(i) **The students have seen many examples where they are given the equation in terms of t and asked to verify it. However, they have not seen many examples where they have to derive the transformed equation for themselves.**
First, for any function u we have

$$\dot{u} = \frac{du}{dt} = \frac{dx}{dt}\frac{du}{dx} = 2tu', \text{[1]}$$

and so

$$\begin{aligned}\ddot{u} &= \frac{d}{dt}(2tu') = 2u' + 2t\frac{d}{dt}u' \\ &= 2u' + 2t \times 2t\frac{d}{dx}u' = 2u' + 4t^2u''.\text{[2]}\end{aligned}$$

Substituting this into the relation $Ly = 0$ gives

$$t(t^2+1)^2(4t^2y'' + 2y') + (t^4-1)(2ty') + 4t^5(t^2+2)y = 0.$$

After expanding out and collecting terms we get

$$4t^3(t^2+1)^2y'' + (2t(t^2+1)^2 + 2t(t^4-1))y' + 4t^5(t^2+2)y = 0,$$

or

$$4t^3(t^2+1)^2y'' + 4t^3(t^2+1)y' + 4t^5(t^2+2)y = 0.\text{[2]}$$

If we divide by t^3 and rewrite t^2 as $x-1$, we get

$$x^2y'' + xy' + (x-1)(x+1)y = 0.\text{[1]}$$

Because $(x-1)(x+1) = x^2 - 1$, this is the same as Bessel's equation with $n = 1$ [1]. It follows that one solution is $y = J_1(x) = J_1(t^2+1)$ [1].

(ii) (a) **This is bookwork.** We have

$$\langle Lf, g\rangle - \langle f, Lg\rangle = \int_0^1 (r.Lf.g - r.f.Lg) = \int_0^1 W' = W(1) - W(0).[2]$$

However, we have $f(0) = g(0) = 0$ so

$$W(0) = p(0)f'(0)g(0) - p(0)f(0)g'(0) = 0 - 0 = 0, [1]$$

and similarly $W(1) = 0$, so $W(1) - W(0) = 0$, so $\langle Lf, g\rangle = \langle f, Lg\rangle$. [1]

(b) **This is bookwork.** We can write $f = g + ih$ and $\lambda = \mu + i\nu$, where g and h are real functions, and μ and ν are real constants [2]. The equation $Lf = \lambda f$ then becomes

$$Lg + iLh = (\mu + i\nu)(g + ih) = (\mu g - \nu h) + i(\mu h + \nu g).[1]$$

By comparing real and imaginary parts, we obtain $Lg = \mu g - \nu h$ and $Lh = \mu h + \nu g$ [1]. From part (a) we have $\langle Lg, h\rangle = \langle g, Lh\rangle$, which gives $\langle \mu g - \nu h, h\rangle = \langle g, \mu h + \nu g\rangle$ [1]. This can be rearranged to give $\nu(\langle g, g\rangle + \langle h, h\rangle) = 0$ [1]. Here

$$\langle g, g\rangle + \langle h, h\rangle = \int_0^1 r.(g^2 + h^2) = \int_0^1 r.|f|^2.$$

We are given that $f \neq 0$ and $r(x) > 0$ for all x, so $\langle g, g\rangle + \langle h, h\rangle > 0$, so we can conclude that $\nu = 0$ [1]. This means that λ is real [1].

(c) **This example has been covered in lectures, but mixed with other material that is not needed here.** Now consider the case where $p = r = 1$ and $q = 0$ so $Lu = u''$ [1]. If $\lambda > 0$ then the solutions to $Lf = \lambda f$ are $f(t) = Ae^{\sqrt{\lambda}t} + Be^{-\sqrt{\lambda}t}$ [1]. For $f(0) = 0$ we must have $B = -A$ [1], which gives $f(1) = A(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}})$, and $e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}} > 0$. Thus, for $f(0) = f(1) = 0$ we must have $A = B = 0$ and so $f = 0$ [1]. However, we are assuming that $f \neq 0$, so this is impossible. We must therefore have $\lambda \leq 0$ instead [1].

(33)

(i) Let x and t be variables related by $x = t^2$, and write $u' = \frac{du}{dx}$ and $\dot{u} = \frac{du}{dt}$. Consider the differential operator

$$Ly = 4t^2(1-t^4)\ddot{y} - 16t\dot{y} + (21-t^4)y.$$

Show that if $Ly = 0$, then the function $z = t^{-3/2}y$ satisfies the Legendre equation

$$(1-x^2)z'' - 2xz' + n(n+1)z = 0$$

for an appropriate value of n. **(13 marks)**

(ii) Consider a Sturm-Liouville operator $Lf = ((pf')' + qf)/r$ on the interval $[0,1]$. Show that for any two smooth functions f and g on $[0,1]$, the generalised Wronskian $W = pf'g - pfg'$ satisfies $W' = r((Lf)g - f(Lg))$. **(6 marks)**

(iii) Now suppose that $Lf = \lambda f$ and $Lg = \mu g$, where λ and μ are constants with $\lambda \neq \mu$. Suppose also that f and g satisfy the boundary conditions $f'(0) = g'(0) = f'(1) = g'(1) = 0$. Prove that $\int_0^1 rfg\,dx = 0$. **(6 marks)**

Solution:

(i) First, for any function u we have

$$\dot{u} = \frac{du}{dt} = \frac{dx}{dt}\frac{du}{dx} = 2tu', [1]$$

and so

$$\begin{aligned}\ddot{u} &= \frac{d}{dt}(2tu') = 2u' + 2t\frac{d}{dt}u' \\ &= 2u' + 2t \times 2t\frac{d}{dx}u' = 2u' + 4t^2u''. [2]\end{aligned}$$

Substituting this into the relation $Ly = 0$ gives

$$4t^2(1-t^4)(4t^2y'' + 2y') - 16t(2ty') + (21-t^4)y = 0. [1]$$

After expanding out and collecting terms we get

$$(16t^4 - 16t^8)y'' - (24t^2 + 8t^6)y' + (21-t^4)y = 0. [1]$$

Putting $x = t^2$, we get

$$(16x^2 - 16x^4)y'' - (24x + 8x^3)y' + (21-x^2)y = 0. [1]$$

Now put $z = x^{-3/4}y$, so $y = x^{3/4}z$ [1]. This gives

$$\begin{aligned}y' &= x^{3/4}z' + \tfrac{3}{4}x^{-1/4}z \\ &= x^{3/4}(z' + \tfrac{3}{4}x^{-1}z) [1] \\ y'' &= x^{3/4}z'' + \tfrac{3}{2}x^{-1/4}z' - \tfrac{3}{16}x^{-5/4}z \\ &= x^{3/4}(z'' + \tfrac{3}{2}x^{-1}z' - \tfrac{3}{16}x^{-2}z). [1]\end{aligned}$$

If we substitute this in our previous relation and divide by $x^{3/4}$, we get

$$(16x^2 - 16x^4)(z'' + \tfrac{3}{2}x^{-1}z' - \tfrac{3}{16}x^{-2}z) - (24x + 8x^3)(z' + \tfrac{3}{4}x^{-1}z) + (21-x^2)z = 0. [1]$$

If we expand this out and collect terms, we get

$$(16x^2 - 16x^4)z'' - 32x^3z' - 4x^2z = 0. [1]$$

Dividing by $16x^2$ gives $(1-x^2)z'' - 2xz' - \frac{1}{4}z = 0$ [1]. This is the same as the Legendre equation with parameter n provided that $n(n+1) = -1/4$, or equivalently $n^2 + n + 1/4 = 0$, and this can be solved to give $n = -1/2$. [1]

(ii) **This is bookwork.**
First, we have

$$\begin{aligned} Lf &= r^{-1}pf'' + r^{-1}p'f' + r^{-1}qf \\ Lg &= r^{-1}pg'' + r^{-1}p'g' + r^{-1}qg\textbf{[2]} \\ r.Lf.g &= pf''g + p'f'g + qfg \\ r.f.Lg &= pfg'' + p'fg' + qfg\textbf{[1]} \\ r.(Lf.g - f.Lg) &= p(f''g - fg'') + p'(f'g - fg')\textbf{[1]}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} W' &= (p'f'g + pf''g + pf'g') - (p'fg' + pf'g' + pfg'') \\ &= pf''g + p'f'g - pfg'' - p'fg', \end{aligned}$$

which is the same **[2]**.

(iii) **This is bookwork.**
Part (ii) gives $(\lambda - \mu)rfg = W'$ **[1]**, so

$$(\lambda - \mu)\int_0^1 rfg\,dx = \int_0^1 W'\,dx = W(1) - W(0).\textbf{[2]}$$

On the other hand, we have $f'(0) = g'(0) = 0$, so

$$W(0) = p(0)f'(0)g(0) - p(0)f(0)g'(0) = 0 - 0 = 0.\textbf{[1]}$$

Similarly $W(1) = 0$, so $(\lambda - \mu)\int_0^1 rfg\,dx = 0$ **[1]**. We are also given that $\lambda \neq \mu$, so we can divide by $\lambda - \mu$ to get $\int_0^1 rfg\,dx = 0$ as claimed **[1]**.

(34)

(i) Let x and t be variables related by $x = e^t$, and write $u' = \frac{du}{dx}$ and $\dot{u} = \frac{du}{dt}$. Consider the differential operator

$$Ly = (1 - e^{-t})\ddot{y} + 2\dot{y} + y.$$

(a) Convert the equation $Ly = 0$ to an equation involving x, y, y' and y'' instead of $t, y, \dot{y}$ and $\ddot{y}$. **(6 marks)**

(b) Show that $y = 1/(1 - x)$ is a solution. **(3 marks)**

(c) Use reduction of order to find another solution. **(7 marks)**

(ii) Consider the equation $x^2y'' - 20xy' + (x^2 + 110)y = 0$. Convert this to normal form, and thus find a solution with $y = 11x^{11} + O(x^{12})$. **(9 marks)**

Solution:

(i) **The students have seen many examples where they are given the equation in terms of t and asked to verify it. However, they have not seen many examples where they have to derive the transformed equation for themselves.**

(a) First, note that $dx/dt = x$. Thus, for any function u we have

$$\dot{u} = \frac{du}{dt} = \frac{dx}{dt}\frac{du}{dx} = xu', [2]$$

and so

$$\begin{aligned}\ddot{u} &= \frac{d}{dt}(xu') = \frac{dx}{dt}\frac{d}{dx}(xu') \\ &= x\frac{d}{dx}(xu') = x^2u'' + xu'.[3]\end{aligned}$$

We now substitute this (together with the relation $e^{-t} = x^{-1}$) in the equation $Ly = 0$ to get

$$(1 - x^{-1})(x^2y'' + xy') + 2xy' + y = 0.[1]$$

This simplifies to $x(x - 1)y'' + (3x - 1)y' + y = 0$.

(b) Now take $y = 1/(1 - x)$. We then have

$$\begin{aligned} y' &= 1/(1-x)^2 \\ y'' &= 2/(1-x)^3 \\ x(x-1)y'' &= (1-x)^{-2}(-2x) \\ (3x-1)y' &= (1-x)^{-2}(3x-1) \\ y &= (1-x)^{-2}(1-x).[2] \end{aligned}$$

From this it is clear that $x(x - 1)y'' + (3x - 1)y' + y = 0$, as required. [1]

(c) Our equation is $Ay'' + By' + Cy = 0$, where $A = x(x - 1)$ and $B = 3x - 1$ and $C = 1$. We need to evaluate $v = \int B/A\, dx$ [1]. Here $B/A = \frac{3x-1}{x(x-1)}$, which can be written in partial fraction form as

$$\frac{3x - 1}{x(x - 1)} = \frac{\alpha}{x} + \frac{\beta}{x - 1} = \frac{(\alpha + \beta)x - \alpha}{x(x - 1)}, [2]$$

so we must have $\alpha = 1$ and $\beta = 2$ [1]. This gives $v = \ln(x) + 2\ln(x - 1)$, and so

$$y^{-2}e^{-v} = (1 - x)^2x^{-1}(x - 1)^{-2} = x^{-1}.[1]$$

We now put $u = \int y^{-2}e^{-v}\, dx = \ln(x)$ [1]and $z = uy = \ln(x)/(1 - x)$ [1]. This is our second solution.

(ii) **This method is covered in the notes and problem sheets and in the list of formulae that the students are told to learn, but it has not previously appeared in an exam.** We first divide by x^2 to get an equation $y'' + Py' + Qy = 0$ with $P = -20x^{-1}$ and $Q = 1 + 110x^{-2}$ [1]. We then put

$$m = \exp(-\tfrac{1}{2}\int P\,dx)[\mathbf{1}] = \exp(10\,\ln(x)) = x^{10}[\mathbf{1}]$$
$$R = Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2[\mathbf{1}] = 1 + 110x^{-2} - 10x^{-2} - 100x^{-2} = 1.[\mathbf{1}]$$

The general method tells us that $y = mz$ with $z'' + Rz = 0$ [1]. As $R = 1$ we have $z'' + z = 0$ so $z = \alpha\cos(x) + \beta\sin(x)$ [1], and so $y = \alpha x^{10}\cos(x) + \beta x^{10}\sin(x)$ [1]. To get $y = 11x^{11} + O(x^{12})$ we must take $\alpha = 0$ and $\beta = 11$ giving $y = 11x^{10}\sin(x)$ [1].

(35)

(i) Find the general solution of the equation $y'' + 6y' + 25y = 0$. **(5 marks)**

(ii) Suppose that a function y satisfies $(x^7 y')'/x^7 + y = 0$. Show that the function $z = x^3 y$ satisfies Bessel's equation $x^2 z'' + xz' + (x^2 - n^2)z = 0$ for some value of n. **(8 marks)**

(iii) Consider a Sturm-Liouville operator $Lf = ((pf')' + qf)/r$ on the interval $[0, 1]$. Show that for any two smooth functions f and g on $[0, 1]$, the generalised Wronskian $W = pf'g - pfg'$ satisfies

$$W' = r((Lf)g - f(Lg)).\textbf{(6 marks)}$$

(iv) Find p, q and r such that the operator $Lf = x^2 f'' - 2xf' + (x^2 + 2)f$ is the same as $((pf')' + qf)/r$. **(6 marks)**

Solution: This is taken from an earlier paper which the students have seen. Parts (i), (ii) and (iv) are standard problems. Part (iii) is pure bookwork.

(i) The auxiliary polynomial is $\lambda^2 + 6\lambda + 25$ [2], which has roots

$$(-6 \pm \sqrt{36 - 4 \times 25})/2 = (-6 \pm 8i)/2 = -3 \pm 4i.[1]$$

From this it follows that the general solution is

$$y = e^{-3x}(A\cos(4x) + B\sin(4x))$$

(where A and B are constant) [2].

(ii) First, we have

$$\begin{aligned} y &= x^{-3}z \\ y' &= x^{-3}z' - 3x^{-4}z \\ x^7 y' &= x^4 z' - 3x^3 z \\ (x^7 y')' &= x^4 z'' + 4x^3 z' - 3x^3 z' - 9x^2 z = x^4 z'' + x^3 z' - 9x^2 z.[3] \end{aligned}$$

As $(x^7 y')'/x^7 + y = 0$ we also have $(x^7 y')' = -x^7 y = -x^4 z$ [3]. This gives

$$x^4 z'' + x^3 z' - 9x^2 z = -x^4 z,$$

which is equivalent to

$$x^2 z'' + xz' + (x^2 - 9)z = 0,$$

which is Bessel's equation with $n = 3$ [2].

(iii) First, we have

$$\begin{aligned} Lf &= r^{-1}pf'' + r^{-1}p'f' + r^{-1}qf \\ Lg &= r^{-1}pg'' + r^{-1}p'g' + r^{-1}qg[2] \\ r.Lf.g &= pf''g + p'f'g + qfg \\ r.f.Lg &= pfg'' + p'fg' + qfg[1] \\ r.(Lf.g - f.Lg) &= p(f''g - fg'') + p'(f'g - fg')[1]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} W' &= (p'f'g + pf''g + pf'g') - (p'fg' + pf'g' + pfg'') \\ &= pf''g + p'f'g - pfg'' - p'fg', \end{aligned}$$

which is the same [2].

(iv) The coefficients appearing in L are $A = x^2$ and $B = -2x$ and $C = x^2 + 2$. The standard formulae for conversion to Sturm-Liouville form are

$$p = \exp\left(\int B/A\,dx\right) = \exp(\int -2x^{-1}\,dx) = \exp(-2\log(x)) = x^{-2}\textbf{[2]}$$
$$r = p/A = x^{-2}/x^2 = x^{-4}\textbf{[2]}$$
$$q = rC = x^{-2} + 2x^{-4}.\textbf{[2]}$$

We therefore have

$$Lf = x^4((x^{-2}f')' + (x^{-2} + 2x^{-4})f).$$

(36)

(i) Find the general solution of the equation $y'' + 6y' + 25y = 0$. **(4 marks)**

(ii) Suppose that a function y satisfies the Legendre equation $(1-x^2)y'' - 2xy' + 6y = 0$. Suppose that $x = t^{-1/5}$ and write $\dot{u}$ for du/dt. Show that the function $z = t^{2/5}y$ satisfies

$$25t^2(1 - t^{-2/5})\ddot{z} + 10t\dot{z} + cz = 0$$

for some constant c. **(13 marks)**

(iii) Consider a Sturm-Liouville operator $Lf = ((pf')' + qf)/r$ on the interval $[0, 1]$. Suppose that f and g are smooth functions on $[0, 1]$ with $f(0) = g(0) = f(1) = g(1) = 0$ and $Lf = f$ and $Lg = -g$.

(a) Define the Wronskian function W. **(2 marks)**

(b) Give a formula relating W' and L. (Do not give the proof.) **(2 marks)**

(c) Show that

$$\int_{x=0}^{1} r(x)\, f(x)\, g(x)\, dx = 0.$$

(5 marks)

Solution:

(i) The auxiliary polynomial is $\lambda^2 + 6\lambda + 25$ [2], which has roots

$$(-6 \pm \sqrt{36 - 4 \times 25})/2 = (-6 \pm 8i)/2 = -3 \pm 4i.\text{[1]}$$

From this it follows that the general solution is

$$y = e^{-3x}(A\cos(4x) + B\sin(4x))$$

(where A and B are constant) [1].

(ii) First, we have $dx/dt = -\frac{1}{5}t^{-6/5}$, so $\dot{u} = -\frac{1}{5}t^{-6/5}u'$ for any u [2]. This gives

$$\begin{aligned}
\dot{z} &= \tfrac{2}{5}t^{-3/5}y + t^{2/5}(-\tfrac{1}{5}t^{-6/5}y') = \tfrac{2}{5}t^{-3/5}y - \tfrac{1}{5}t^{-4/5}y' \\
&= -\tfrac{1}{5}x^4y' + \tfrac{2}{5}x^3y\text{[2]} \\
\ddot{z} &= -\tfrac{6}{25}t^{-8/5}y + \tfrac{2}{5}t^{-3/5}(-\tfrac{1}{5}t^{-6/5}y') + \tfrac{4}{25}t^{-9/5}y' - \tfrac{1}{5}t^{-4/5}(-\tfrac{1}{5}t^{-6/5}y'') \\
&= -\tfrac{6}{25}t^{-8/5}y + \tfrac{2}{25}t^{-9/5}y' + \tfrac{1}{25}t^{-10/5}y'' \\
&= \tfrac{1}{25}x^{10}y'' + \tfrac{2}{25}x^9y' - \tfrac{6}{25}x^8y\text{[4]}
\end{aligned}$$

so

$$\begin{aligned}
25t^2(1 - t^{-2/5})\ddot{z} + 10t\dot{z} &= 25x^{-10}(1 - x^2)(\tfrac{1}{25}x^{10}y'' + \tfrac{2}{25}x^9y' - \tfrac{6}{25}x^8y) + 10x^{-5}(-\tfrac{1}{5}x^4y' + \tfrac{2}{5}x^3y) \\
&= (1 - x^2)y'' + 2(1 - x^2)x^{-1}y' - 6(1 - x^2)x^{-2}y - 2x^{-1}y + 4x^{-2}y \\
&= (1 - x^2)y'' + 2x^{-1}y' - 2xy' - 6x^{-2}y + 6y - 2x^{-1}y + 4x^{-2}y \\
&= (1 - x^2)y'' - 2xy' + 6y - 2x^{-2}y.\text{[3]}
\end{aligned}$$

As $(1-x^2)y'' - 2xy' + 6y = 0$ this simplifies to $-2x^{-2}y$, which is the same as $-2t^{2/5}y = -2z$ [1]. We thus have $25t^2(1 - t^{-2/5})\ddot{z} + 10t\dot{z} + 2z = 0$, so $c = 2$ [1].

(ii) (a) The Wronskian is $W = pf'g - pfg'$. [2]

(b) The standard formula is $W' = r.((Lf)g - f(Lg))$. [2]

(c) First, as $f(0) = g(0) = f(1) = g(1) = 0$, we see that the function $W(x) = p(x)f'(x)g(x) - p(x)f(x)g'(x)$ satisfies $W(0) = 0$ and $W(1) = 0$. This means that $\int_0^1 W'(x)\,dx = W(1) - W(0) = 0$. [2] However, we also know that $W' = r\,(Lf)\,g - r\,f\,L(g)$, where $Lf = f$ and $Lg = -g$ [1]. This means that $W' = rfg - (-rfg) = 2rfg$ [1], so

$$\int_0^1 rfg\,dx = \frac{1}{2}\int_0^1 W\,dx = 0[1].$$