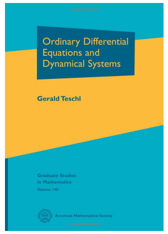
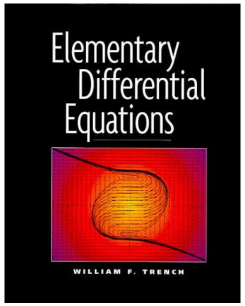
Ordinary Differential Equations

A recommended book



- This is an excellent book which is available in English and in Chinese.
- You can buy it, or you can download a copy from the author's website:
  `http://www.mat.univie.ac.at/~gerald/ftp/book-ode/`
  Both the English version and the Chinese version are there.
- Most of the book is too advanced for this course, but still it should be useful.

Another recommended book



- This is another nice book, which is only available in English.
- You can download a copy from the author's website:
  `http://ramanujan.math.trinity.edu/wtrench/texts/index.shtml`

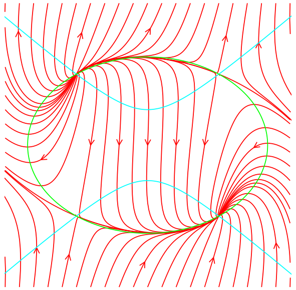
Planar Ordinary Differential Equations

Introduction

The first half of this course is about planar differential equations, like this example:

$$\dot{x} = \frac{dx}{dt} = -\frac{9}{40}x^2 + \frac{3}{10}y^2 - \frac{3}{40} \qquad \dot{y} = \frac{dy}{dt} = \frac{5}{8}x^2 + \frac{23}{20}y^2 - \frac{71}{40}$$

There is no formula for the solution. The aim is to learn how to understand the equations even without a formula for the solution. We can draw a picture:



We will start by looking quickly at some examples. Later we will develop some mathematical theory, then look at the examples again.

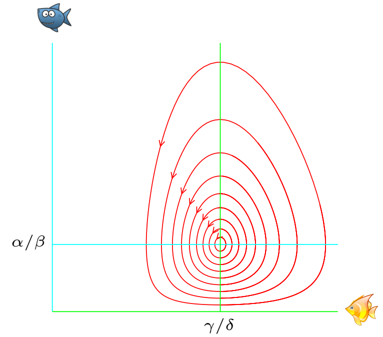
The Lotka-Volterra model

A lagoon contains F fish and S sharks. These change according to the equations

$$\dot{F} = \alpha F - \beta FS \qquad \dot{S} = -\gamma S + \delta FS,$$

where α, β, γ and δ are positive constants.
αF: fish breeding; $-\beta FS$: fish being eaten;
δFS: well-fed sharks breeding; $-\gamma S$: sharks starving.

The *phase portrait* shows how the point (F, S) moves over time:



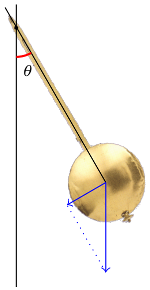
$$\dot{F} = -\beta F(S - \alpha/\beta)$$
$$\dot{S} = \delta S(F - \gamma/\delta)$$

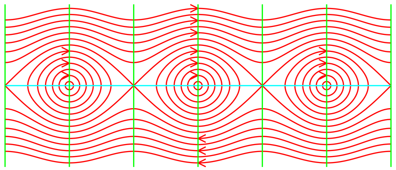
Motion of a pendulum

Consider a swinging pendulum, hanging at angle θ.
The angular velocity is $\omega = \dot{\theta}$.
The angular acceleration $\dot{\omega}$ is proportional to the component of the gravitational force perpendicular to the pendulum, which is proportional to $-\sin(\theta)$.

With suitable units, we can assume that

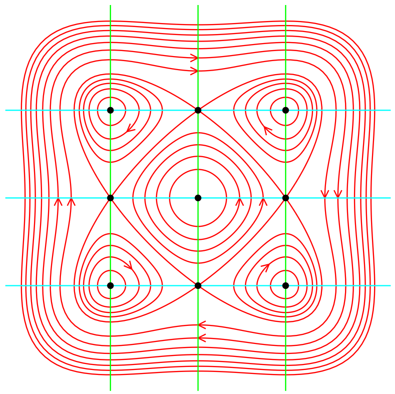
$$\dot{\theta} = \omega \qquad \dot{\omega} = -\sin(\theta).$$





A contour flow

This system has equations $\dot{x} = y^3 - y$ and $\dot{y} = x - x^3$.

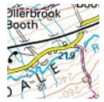


- The blue lines (x-*nullclines*) show where $y - y^3 = 0$ and so $\dot{x} = 0$.
- The green lines (y-*nullclines*) show where $x - x^3 = 0$ and so $\dot{y} = 0$.
- The black dots (*equilibrium points*) show where $\dot{x} = \dot{y} = 0$.

A contour map

This is a contour map. The height is $h(x, y)$. If you stay on one of the brown lines (contours), then you stay at the same height. That is a contour flow.

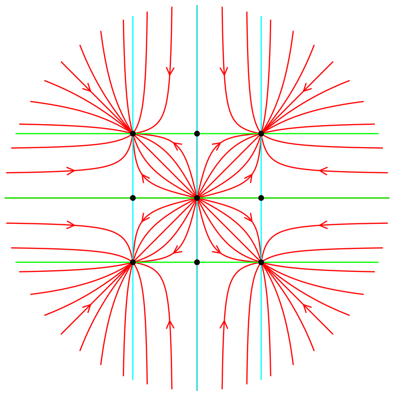
A contour map



When the contours are close together, the ground is steep.
When the contours are far apart, the ground is not steep.

A gradient flow

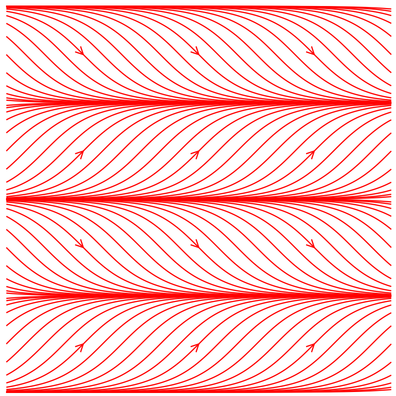
This system has equations $\dot{x} = x - x^3$ and $\dot{y} = y - y^3$.



- The blue lines (x-*nullclines*) show where $x - x^3 = 0$ and so $\dot{x} = 0$.
- The green lines (y-*nullclines*) show where $y - y^3 = 0$ and so $\dot{y} = 0$.
- The black dots (*equilibrium points*) show where $\dot{x} = \dot{y} = 0$.

Bands

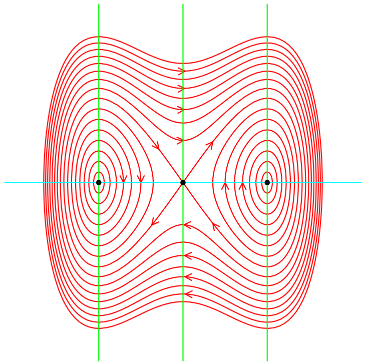
This system has equations $\dot{x} = 1$ and $\dot{y} = \sin(\pi y)$.



The solutions move steadily to the right, and converge to one of the lines where y is an odd integer.

Duffing oscillator

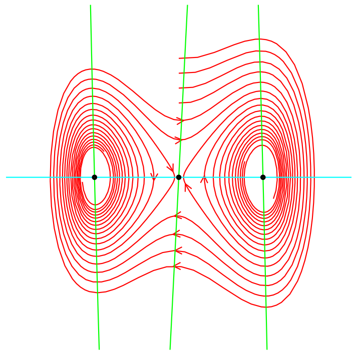
The Duffing oscillator has $\dot{x} = y$ and $\dot{y} = 2x - x^3$.



- The blue line (x-*nullcline*) shows where $y = 0$ and so $\dot{x} = 0$.
- The green lines (y-*nullclines*) show where $2x - x^3 = 0$ and so $\dot{y} = 0$.
- The black dots (*equilibrium points*) show where $\dot{x} = \dot{y} = 0$.

Damped Duffing oscillator

This system has $\dot{x} = y$ and $\dot{y} = 2x - x^3 - 0.1y$.

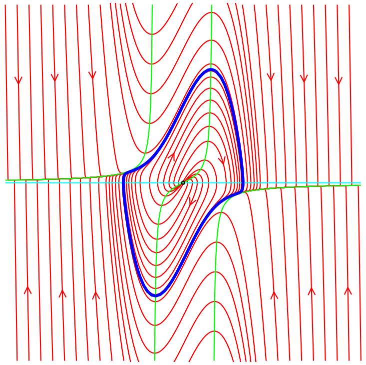


It is similar to the Duffing oscillator, but with friction or damping .

- The x-nullcline is the same as before
- But the y-nullclines have moved slightly
- The equilibrium points are unchanged.

van der Pol oscillator

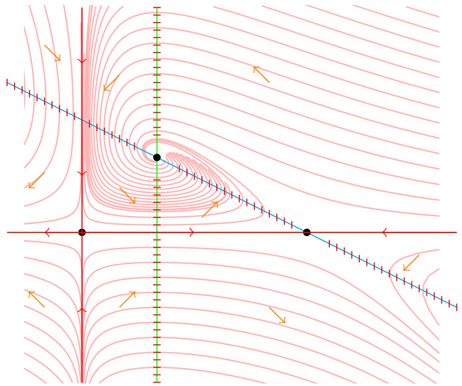
This system has $\dot{x} = y$ and $\dot{y} = 2(1 - x^2)y - x$.



- The blue line (x-*nullcline*) shows where $y = 0$ and so $\dot{x} = 0$.
- The green lines (y-*nullclines*) show where $2(1 - x^2)y - x = 0$ and so $\dot{y} = 0$.
- There is only one equilibrium point, but there is also a *limit cycle*, shown in blue. All non-constant solutions converge to the limit cycle.

Sketching a phase portrait

We will sketch the phase portrait for the system
$\dot{x} = x(3 - x - 2y), \ \dot{y} = y(x - 1)$.



A problem which we will mostly ignore

Consider the equation $\dot{x} = x^2$. This gives

$$\frac{d}{dt}x^{-1} = -x^{-2}\dot{x} = -x^{-2}.x^2 = -1$$
$$x^{-1} = x_0^{-1} - t$$
$$x = 1/(x_0^{-1} - t) = x_0/(1 - x_0 t).$$

This is not defined for all t; the solution goes to infinity as $t \to x_0^{-1}$.

A similar example in two variables: $\dot{x} = \dot{y} = xy$.
(There is a solution on the problem sheet.)

We mostly ignore this problem and consider only equations where $x(t)$ and $y(t)$ are defined for all $t \in \mathbb{R}$.

Linear systems

A *(first order, autonomous) linear system* has the form

$$\dot{x} = \frac{dx}{dt} = ax+by \qquad \dot{y} = \frac{dy}{dt} = cx+dy \qquad \begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \frac{d}{dt}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

Linear systems are the easiest kind of planar differential equations.
They will also help us to understand nonlinear systems.

Example:

Suppose $b = c = 0$, so $\begin{aligned}\dot{x} &= ax\\ \dot{y} &= dy\end{aligned}$ or $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}a & 0\\ 0 & d\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$

The solution is $\begin{aligned}x &= e^{at}x_0\\ y &= e^{dt}y_0\end{aligned}$ or $\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}e^{at} & 0\\ 0 & e^{dt}\end{bmatrix}\begin{bmatrix}x_0\\ y_0\end{bmatrix}$.

Example:

Suppose $\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x\end{aligned}$ or $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$

Solution: $\begin{aligned}x &= x_0\cos(t) + y_0\sin(t)\\ y &= y_0\cos(t) - x_0\sin(t)\end{aligned}$ or $\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}\cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{bmatrix}\begin{bmatrix}x_0\\ y_0\end{bmatrix}$.

Reminder of simple harmonic motion

Proposition: Suppose that x is a function of t such that $\ddot{x} = -\omega^2 x$; then $x(t) = A\cos(\omega t) + B\sin(\omega t)$ for some constants A and B.

Proof.
Put $A = x(0)$ and $B = \dot{x}(0)/\omega$ and $u(t) = A\cos(\omega t) + B\sin(\omega t)$ and $v(t) = x(t) - u(t)$. We want to show that $x(t) = u(t)$, so we must show that $v(t) = 0$. Note that

$$\dot{u}(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$
$$\ddot{u}(t) = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t) = -\omega^2 u(t)$$
$$\ddot{v}(t) = \ddot{x}(t) - \ddot{u}(t) = -\omega^2 x(t) + \omega^2 u(t) = -\omega^2 v(t)$$
$$v(0) = x(0) - A = 0$$
$$\dot{v}(0) = \dot{x}(0) - B\omega = 0.$$

Now put $E(t) = \omega^2 v(t)^2 + \dot{v}(t)^2$, so $E(0) = \omega^2 v(0)^2 + \dot{v}(0)^2 = 0$. Also:

$$\dot{E}(t) = 2\omega^2 v(t)\dot{v}(t) + 2\dot{v}(t)\ddot{v}(t) = 2\dot{v}(t)(\omega^2 v(t) + \ddot{v}(t)) = 0.$$

This means that E is constant, and $E(0) = 0$, so $E(t) = 0$ for all t. As squares are always nonnegative, the only way that $E(t)$ can be zero is if $v(t) = 0$ and $\dot{v}(t) = 0$. We thus have $v = 0$ as required. □

Linear systems

A *(first order, autonomous) linear system* has the form

$$\dot{x} = \frac{dx}{dt} = ax+by \qquad \dot{y} = \frac{dy}{dt} = cx+dy \qquad \begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \frac{d}{dt}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

We put $u = \begin{bmatrix}x\\ y\end{bmatrix}$ and $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$ so $\dot{u} = Au$.To solve the system, we first need to find eigenvalues and eigenvectors of A. Put

$$\tau = \mathrm{trace}(A) = a + d \qquad \delta = \det(A) = ad - bc$$

$$\chi_A(t) = \text{ characteristic polynomial } = \det(A - tI) = \det\begin{bmatrix}a-t & b\\ c & d-t\end{bmatrix}$$
$$= (a-t)(d-t) - bc = t^2 - (a+d)t + (ad-bc) = t^2 - \tau t + \delta.$$

The eigenvalues are the roots of $\chi_A(t)$, which are

$$\lambda_1 = \tfrac{1}{2}(\tau - \sqrt{\tau^2 - 4\delta}) \qquad \lambda_2 = \tfrac{1}{2}(\tau + \sqrt{\tau^2 - 4\delta}).$$

These might be real numbers or complex numbers .

$$\lambda_1 + \lambda_2 = \tau \qquad \lambda_1\lambda_2 = \delta$$

Linear systems with real eigenvalues

$$\begin{bmatrix}\dot x\\ \dot y\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} \qquad \begin{aligned}\tau &= a+b\\ \delta &= ad-bc\end{aligned} \qquad \begin{aligned}\lambda_1 &= \tfrac{1}{2}(\tau - \sqrt{\tau^2-4\delta})\\ \lambda_2 &= \tfrac{1}{2}(\tau + \sqrt{\tau^2-4\delta}).\end{aligned}$$

Suppose for the moment that $\tau^2 > 4\delta$, so λ_1 and λ_2 are real, and $\lambda_1 < \lambda_2$. We can find eigenvectors v_1 and v_2 such that $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$.

Now suppose that $u = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$ for some constants c_1 and c_2. Then

$$\dot u = c_1\lambda_1 e^{\lambda_1 t} v_1 + c_2\lambda_2 e^{\lambda_2 t} v_2 = c_1 e^{\lambda_1 t} A v_1 + c_2 e^{\lambda_2 t} A v_2 = Au,$$

so we have a solution to our system of equations.

If $\lambda_1, \lambda_2 < 0$ then $u \to 0$ as $t \to \infty$.

If $\lambda_1 < 0 < \lambda_2$ then when t is large we can ignore $c_1 e^{\lambda_1 t} v_1$ and $u \simeq c_2 e^{\lambda_2 t} v_2$.

If $0 < \lambda_1 < \lambda_2$ then both terms will be very large when t is large, but the term $c_2 e^{\lambda_2 t} v_2$ will still grow much more quickly than $c_1 e^{\lambda_1 t} v_1$.

Linear systems with real eigenvalues — example

Consider the system

$$\begin{aligned}\dot x &= 2y\\ \dot y &= x+y\end{aligned} \qquad \begin{bmatrix}\dot x\\ \dot y\end{bmatrix} = A\begin{bmatrix}x\\ y\end{bmatrix}, \text{ where } A = \begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix}$$

$$\tau = \operatorname{trace}(A) = 0+1 = 1 \qquad \delta = \det(A) = 0\times 1 - 2\times 1 = -2$$

Characteristic polynomial $\chi_A(t) = \det\begin{bmatrix}-t & 2\\ 1 & 1-t\end{bmatrix} = t^2 - t - 2 = t^2 - \tau t + \delta$.

Roots λ_1, λ_2 are $\frac{1}{2}(\tau \pm \sqrt{\tau^2 - 4\delta}) = \frac{1}{2}(1 \pm \sqrt{9}) = -1, 2$ (both real).

Eigenvector $v_1 = \begin{bmatrix}p\\ q\end{bmatrix}$ should satisfy $(A - \lambda_1 I)v_1 = 0$, or $(A+I)v_1 = 0$, or $\begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix}\begin{bmatrix}p\\ q\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$, or $p + 2q = 0$. Obvious choice is $v_1 = \begin{bmatrix}-2\\ 1\end{bmatrix}$.

Eigenvector $v_2 = \begin{bmatrix}p\\ q\end{bmatrix}$ should satisfy $(A - \lambda_2 I)v_2 = 0$, or $(A - 2I)v_2 = 0$, or $\begin{bmatrix}-2 & 2\\ 1 & -1\end{bmatrix}\begin{bmatrix}p\\ q\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$, or $p - q = 0$. Obvious choice is $v_2 = \begin{bmatrix}1\\ 1\end{bmatrix}$.

Linear systems with real eigenvalues — example

$$\begin{bmatrix}\dot x\\ \dot y\end{bmatrix} = \begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} \qquad \text{eigenvectors } \begin{bmatrix}-2\\ 1\end{bmatrix}, \begin{bmatrix}1\\ 1\end{bmatrix} \text{ with eigenvalues } -1, 2.$$

Solutions have the form

$$\begin{bmatrix}x\\ y\end{bmatrix} = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 = c_1 e^{-t}\begin{bmatrix}-2\\ 1\end{bmatrix} + c_2 e^{2t}\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{bmatrix}-2c_1 e^{-t} + c_2 e^{2t}\\ c_1 e^{-t} + c_2 e^{2t}\end{bmatrix}.$$

The values at time $t = 0$ are

$$\begin{bmatrix}x_0\\ y_0\end{bmatrix} = \begin{bmatrix}-2c_1 e^0 + c_2 e^0\\ c_1 e^0 + c_2 e^0\end{bmatrix} = \begin{bmatrix}-2c_1 + c_2\\ c_1 + c_2\end{bmatrix} = \begin{bmatrix}-2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}.$$

Often we do not know c_1 and c_2, but we do know x_0 and y_0. Then we can invert the above to find c_1 and c_2:

$$\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}-2 & 1\\ 1 & 1\end{bmatrix}^{-1}\begin{bmatrix}x_0\\ y_0\end{bmatrix} = \frac{1}{-3}\begin{bmatrix}1 & -1\\ -1 & -2\end{bmatrix}\begin{bmatrix}x_0\\ y_0\end{bmatrix} = \begin{bmatrix}y_0/3 - x_0/3\\ 2y_0/3 + x_0/3\end{bmatrix}.$$

Linear systems with real eigenvalues — example

$$\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}-2c_1 e^{-t} + c_2 e^{2t}\\ c_1 e^{-t} + c_2 e^{2t}\end{bmatrix} \qquad \begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}y_0/3 - x_0/3\\ 2y_0/3 + x_0/3\end{bmatrix}.$$

For example, suppose we know that when $t = 0$ we have $x = -1$ and $y = 1$. Put $x_0 = -1$ and $y_0 = 1$ in the right hand equation to get $c_1 = 2/3$ and $c_2 = 1/3$. Put these values in the left hand equation to get

$$u = \begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}-\frac{4}{3}e^{-t} + \frac{1}{3}e^{2t}\\ \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\end{bmatrix}.$$

To check this, note that when $t = 0$ it gives $u = \begin{bmatrix}-4/3 + 1/3\\ 2/3 + 1/3\end{bmatrix} = \begin{bmatrix}-1\\ 1\end{bmatrix} = \begin{bmatrix}x_0\\ y_0\end{bmatrix}$ as expected. Moreover:

$$\dot u = \begin{bmatrix}\frac{4}{3}e^{-t} + \frac{2}{3}e^{2t}\\ -\frac{2}{3}e^{-t} + \frac{2}{3}e^{2t}\end{bmatrix}$$

$$Au = \begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}-\frac{4}{3}e^{-t} + \frac{1}{3}e^{2t}\\ \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\end{bmatrix} = \begin{bmatrix}\frac{4}{3}e^{-t} + \frac{2}{3}e^{2t}\\ -\frac{2}{3}e^{-t} + \frac{2}{3}e^{2t}\end{bmatrix},$$

so $\dot u = Au$ as expected.

Linear systems with real eigenvalues — reformulation

Consider again a system $\dot{u} = Au$, where A has real eigenvalues $\lambda_1 < \lambda_2$ and corresponding eigenvectors v_1, v_2.

We can put v_1 and v_2 together to form a 2×2 matrix $V = \left[\begin{array}{c|c} v_1 & v_2 \end{array}\right]$.

We also put $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and $E = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$ and $P = VEV^{-1}$.

Proposition: We have $P = I$ when $t = 0$, and $\dot{P} = AP$.
Also, the solution to $\dot{u} = Au$ with $u = u_0$ at $t = 0$ is $u = Pu_0$.

Linear systems with real eigenvalues — reformulation

$$V = \left[\begin{array}{c|c} v_1 & v_2 \end{array}\right] \qquad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad E = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \qquad P = VEV^{-1}$$

Proposition: the solution to $\dot{u} = Au$ with $u = u_0$ at $t = 0$ is $u = Pu_0$.

First note that

$$AV = A\left[\begin{array}{c|c} v_1 & v_2 \end{array}\right] = \left[\begin{array}{c|c} Av_1 & Av_2 \end{array}\right] = \left[\begin{array}{c|c} \lambda_1 v_1 & \lambda_2 v_2 \end{array}\right] = \left[\begin{array}{c|c} v_1 & v_2 \end{array}\right]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = VD.$$

We can rearrange to get $A = VDV^{-1}$. This is called a *diagonalization* of A.
Now $AP = VDV^{-1}VEV^{-1} = VDEV^{-1}$. Also

$$\dot{E} = \begin{bmatrix} \lambda_1 e^{\lambda_1 t} & 0 \\ 0 & \lambda_2 e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = DE, \text{ so}$$

$$\dot{P} = V\dot{E}V^{-1} = VDEV^{-1} = VDV^{-1}VEV^{-1} = AP.$$

as claimed. Also, when $t = 0$ we have $E = I$ so $P = VV^{-1} = I$.
Now suppose we have a vector u_0, and we put $u = Pu_0$. When $t = 0$ we have $P = I$ so $u = u_0$. We also have $\dot{u} = \dot{P}u_0 = APu_0 = Au$ as required.

Linear systems with real eigenvalues — reformulated example

As before, take $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ $\quad \begin{aligned} \lambda_1 &= -1 \\ \lambda_2 &= 2 \end{aligned} \quad v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

Then

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \qquad E = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

$$V = \left[\begin{array}{c|c} v_1 & v_2 \end{array}\right] = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \qquad V^{-1} = \frac{1}{-3}\begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}.$$

$$\begin{aligned} P = VEV^{-1} &= \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \\ &= \begin{bmatrix} -2e^{-t} & e^{2t} \\ e^{-t} & e^{2t} \end{bmatrix}\begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} e^{2t} + 2e^{-t} & 2e^{2t} - 2e^{-t} \\ e^{2t} - e^{-t} & 2e^{2t} + e^{-t} \end{bmatrix} \end{aligned}$$

If $u_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ then

$$u = Pu_0 = \frac{1}{3}\begin{bmatrix} e^{2t} + 2e^{-t} & 2e^{2t} - 2e^{-t} \\ e^{2t} - e^{-t} & 2e^{2t} + e^{-t} \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} e^{2t} - 4e^{-t} \\ e^{2t} + 2e^{-t} \end{bmatrix}.$$

This is the same answer as before.

Linear systems with real eigenvalues — another example

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{aligned} \tau &= 1 + 1 = 2 \\ \delta &= 0 \end{aligned} \qquad \begin{aligned} \lambda_1 &= \tfrac{1}{2}(\tau - \sqrt{\tau^2 - 4\delta}) = 0 \\ \lambda_2 &= \tfrac{1}{2}(\tau + \sqrt{\tau^2 - 4\delta}) = 2. \end{aligned}$$

$v_1 = \begin{bmatrix} p \\ q \end{bmatrix}$ with $(A - \lambda_1)v_1 = 0$, so $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$v_2 = \begin{bmatrix} p \\ q \end{bmatrix}$ with $(A - \lambda_2)v_2 = 0$, so $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \qquad E = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{2t} \end{bmatrix}.$$

$$V = \left[\begin{array}{c|c} v_1 & v_2 \end{array}\right] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad V^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} = VEV^{-1}u_0 &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & e^{2t} \\ -1 & e^{2t} \end{bmatrix}\begin{bmatrix} x_0 - y_0 \\ x_0 + y_0 \end{bmatrix} = \begin{bmatrix} (x_0 - y_0 + (x_0 + y_0)e^{2t})/2 \\ (y_0 - x_0 + (x_0 + y_0)e^{2t})/2 \end{bmatrix} \end{aligned}$$

Determinant of the fundamental solution

Proposition: $\det(P) = e^{\mathrm{trace}(A)t}$.

Proof.
Recall that $P = VEV^{-1}$ so

$$\det(P) = \det(V)\det(E)\det(V)^{-1} = \det(E).$$

Also

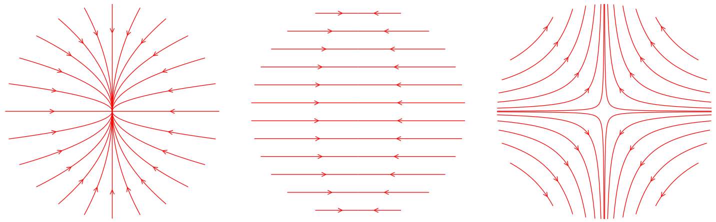
$$E = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix},$$

so

$$\det(P) = \det(E) = e^{\lambda_1 t}e^{\lambda_2 t} = e^{(\lambda_1+\lambda_2)t}.$$

We have also seen that $\lambda_1 + \lambda_2 = \tau = \mathrm{trace}(A)$, so $\det(P) = e^{\mathrm{trace}(A)t}$.

Linear systems with real eigenvalues — phase portraits



$\lambda_1 < \lambda_2 < 0$ stable node | $\lambda_1 < \lambda_2 = 0$ semistable node | $\lambda_1 < 0 < \lambda_2$ saddle

A different formula for P

The solution is $u = Pu_0$, where $P = VEV^{-1}$. To find P we need V, and to find V we need the eigenvectors. However, there is another formula which is easier.

Proposition: $P = (\lambda_2 - \lambda_1)^{-1}((\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})I + (e^{\lambda_2 t} - e^{\lambda_1 t})A)$.

Proof: First put

$$\begin{aligned}
F &= (\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})I + (e^{\lambda_2 t} - e^{\lambda_1 t})D \\
&= \begin{bmatrix} \lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} & 0 \\ 0 & \lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} \end{bmatrix} + \begin{bmatrix} \lambda_1(e^{\lambda_2 t} - e^{\lambda_1 t}) & 0 \\ 0 & \lambda_2(e^{\lambda_2 t} - e^{\lambda_1 t}) \end{bmatrix} \\
&= \begin{bmatrix} (\lambda_2 - \lambda_1)e^{\lambda_1 t} & 0 \\ 0 & (\lambda_2 - \lambda_1)e^{\lambda_2 t} \end{bmatrix} = (\lambda_2 - \lambda_1)E.
\end{aligned}$$

It follows that $P = VEV^{-1} = (\lambda_2 - \lambda_1)^{-1}VFV^{-1}$. However,

$$\begin{aligned}
VFV^{-1} &= (\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})VIV^{-1} + (e^{\lambda_2 t} - e^{\lambda_1 t})VDV^{-1} \\
&= (\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})I + (e^{\lambda_2 t} - e^{\lambda_1 t})A.
\end{aligned}$$

After multiplying by $(\lambda_2 - \lambda_1)^{-1}$ we get the claimed formula for P.

A different formula for P — example

$$P = (\lambda_2 - \lambda_1)^{-1}((\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})I + (e^{\lambda_2 t} - e^{\lambda_1 t})A).$$

Consider again $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$, so $\lambda_1 = -1$ and $\lambda_2 = 2$. Then

$$\begin{aligned}
P &= \frac{1}{3}((2e^{-t} + e^{2t})I + (e^{2t} - e^{-t})A) \\
&= \frac{1}{3}\left(\begin{bmatrix} 2e^{-t} + e^{2t} & 0 \\ 0 & 2e^{-t} + e^{2t} \end{bmatrix} + \begin{bmatrix} 0 & 2e^{2t} - 2e^{-t} \\ e^{2t} - e^{-t} & e^{2t} - e^{-t} \end{bmatrix}\right) \\
&= \frac{1}{3}\begin{bmatrix} 2e^{-t} + e^{2t} & 2e^{2t} - 2e^{-t} \\ e^{2t} - e^{-t} & e^{-t} + 2e^{2t} \end{bmatrix}
\end{aligned}$$

This is the same answer as we found previously.

A different formula for P — another example

$$P = (\lambda_2 - \lambda_1)^{-1}((\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})I + (e^{\lambda_2 t} - e^{\lambda_1 t})A).$$

Consider again $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, so $\lambda_1 = 0$ and $\lambda_2 = 2$. Then

$$\begin{aligned} P &= \frac{1}{2}((2e^{0t} - 0e^{2t})I + (e^{2t} - e^{0t})A) = \frac{1}{2}(2I + (e^{2t} - 1)A) \\ &= \frac{1}{2}\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} e^{2t}-1 & e^{2t}-1 \\ e^{2t}-1 & e^{2t}-1 \end{bmatrix}\right) \\ &= \frac{1}{2}\begin{bmatrix} e^{2t}+1 & e^{2t}-1 \\ e^{2t}-1 & e^{2t}+1 \end{bmatrix}. \end{aligned}$$

Complex eigenvalues

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\lambda_1, \lambda_2 = \frac{1}{2}(\tau \mp \sqrt{\tau^2 - 4\delta})$, where $\begin{matrix} \tau & = a + d \\ \delta & = ad - bc \end{matrix}$.

Now suppose that $\tau^2 - 4\delta < 0$, so λ_1 and λ_2 are complex numbers. Put

$$\lambda = \tau/2 \qquad \omega = \sqrt{4\delta - \tau^2}/2 \qquad \text{so } \lambda_1, \lambda_2 = \lambda \mp i\omega.$$

We can use the same method as before, remembering that

$$e^{(\lambda \mp i\omega)t} = e^{\lambda t} e^{\mp i\omega t} = e^{\lambda t}(\cos(\omega t) \mp i \sin(\omega t))$$

and

$$\cos(\omega t) = (e^{i\omega t} + e^{-i\omega t})/2 \qquad \sin(\omega t) = (e^{i\omega t} - e^{-i\omega t})/(2i).$$

Some complex numbers appear, but in the end the imaginary parts cancel.

Proposition: The solution to $\dot{u} = Au$ with $u = u_0$ at $t = 0$ is $u = Pu_0$, where

$$P = e^{\lambda t}(\cos(\omega t)I + \omega^{-1}\sin(\omega t)(A - \lambda I))$$

Complex eigenvalues — formula for P

$$\lambda_1 = \lambda - i\omega \qquad \lambda_2 = \lambda + i\omega \qquad \text{Solution: } u = Pu_0$$

Proposition: $P = e^{\lambda t}(\cos(\omega t)I + \omega^{-1}\sin(\omega t)(A - \lambda I))$

Proof: We saw before that

$$P = (\lambda_2 - \lambda_1)^{-1}((\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})I + (e^{\lambda_2 t} - e^{\lambda_1 t})A)$$

Now

$$\begin{aligned} \lambda_2 - \lambda_1 &= 2i\omega \\ e^{\lambda_2 t} - e^{\lambda_1 t} &= e^{\lambda t}e^{i\omega t} - e^{\lambda t}e^{-i\omega t} = 2ie^{\lambda t}\sin(\omega t) \\ \lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} &= (\lambda + i\omega)e^{\lambda t}e^{-i\omega t} - (\lambda - i\omega)e^{\lambda t}e^{i\omega t} \\ &= e^{\lambda t}\left(i\omega(e^{i\omega t} + e^{-i\omega t}) - \lambda(e^{i\omega t} - e^{-i\omega t})\right) \\ &= e^{\lambda t}(2i\omega\cos(\omega t) - 2i\lambda\sin(\omega t)) \\ P &= (2i\omega)^{-1}e^{\lambda t}(2i\omega\cos(\omega t)I - 2i\lambda\sin(\omega t)I + 2i\sin(\omega t)A) \\ &= e^{\lambda t}(\cos(\omega t)I + \omega^{-1}\sin(\omega t)(A - \lambda I)) \qquad \square \end{aligned}$$

Complex eigenvalues — example

Suppose that $\begin{matrix} \dot{x} & = \alpha x + \beta y \\ \dot{y} & = -\beta x + \alpha y \end{matrix}$ or $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

Then

$$\tau = 2\alpha \qquad \delta = \alpha^2 + \beta^2 \qquad \lambda = \frac{\tau}{2} = \alpha \qquad \omega = \frac{\sqrt{4\delta - \alpha^2}}{2} = \beta$$

$$\begin{aligned} P &= e^{\lambda t}(\cos(\omega t)I + \omega^{-1}\sin(\omega t)(A - \lambda I)) \\ &= e^{\alpha t}(\cos(\beta t)I + \beta^{-1}\sin(\beta t)(A - \alpha I)) \\ &= e^{\alpha t}\left(\begin{bmatrix} \cos(\beta t) & 0 \\ 0 & \cos(\beta t) \end{bmatrix} + \frac{\sin(\beta t)}{\beta}\begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}\right) \\ &= e^{\alpha t}\begin{bmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{bmatrix}. \end{aligned}$$

Thus, the solution is

$$\begin{aligned} x &= e^{\alpha t}(\cos(\beta t)x_0 + \sin(\beta t)y_0) \\ y &= e^{\alpha t}(-\sin(\beta t)x_0 + \cos(\beta t)y_0). \end{aligned}$$

Clockwise or anticlockwise?

Proposition: Suppose that the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has complex eigenvalues. Then $bc < 0$ (so b and c are nonzero and have opposite sign).

We will need this when we discuss whether the flow lines for A go clockwise or anticlockwise.

Proof.
Note that

$$\begin{aligned}\tau^2 - 4\delta &= (a+d)^2 - 4ad + 4bc = a^2 + 2ad + d^2 - 4ad + 4bc \\ &= a^2 - 2ad + d^2 + 4bc = (a-d)^2 + 4bc.\end{aligned}$$

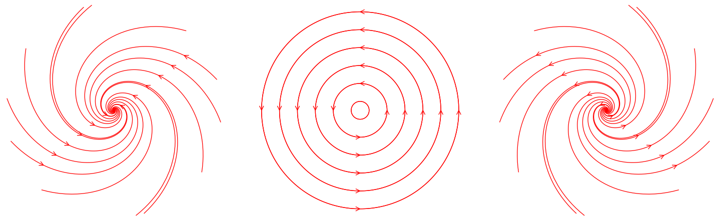
Thus,

$$bc = \frac{1}{4}\left((\tau^2 - 4\delta) - (a-d)^2\right).$$

As A has complex eigenvalues, we must have $\tau^2 - 4\delta < 0$.We also have $(a-d)^2 \geq 0$, so $bc < 0$ as claimed. □

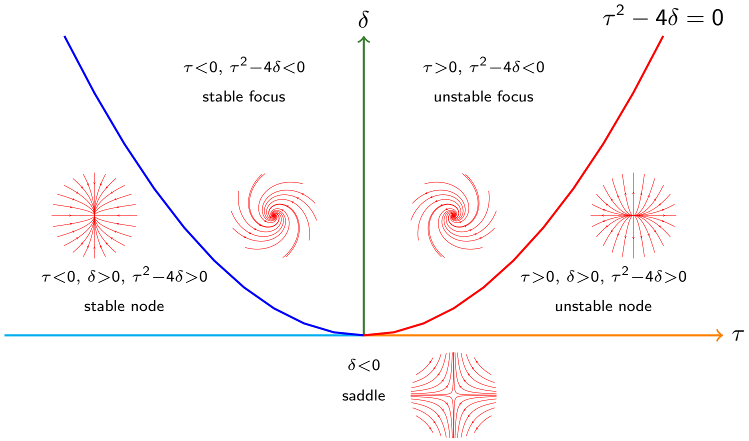
Complex eigenvalues — phase portraits

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \lambda = \frac{\tau}{2} = \frac{a+d}{2} \qquad \omega = \frac{\sqrt{4\delta - \tau^2}}{2} = \frac{\sqrt{2ad - a^2 - d^2 - 4bc}}{2}$$



$\lambda < 0,\ b < 0 < c$	$\lambda = 0,\ b < 0 < c$	$\lambda > 0,\ b < 0 < c$
anticlockwise	anticlockwise	anticlockwise
stable focus	centre	unstable focus

Map of the (τ, δ) plane



$$\lambda_1, \lambda_2 = \frac{1}{2}(\tau \mp \sqrt{\tau^2 - 4\delta}) \qquad \tau = \lambda_1 + \lambda_2 \qquad \delta = \lambda_1\lambda_2.$$

Repeated eigenvalues

Proposition: If A has only one eigenvalue, say λ then the matrix $P = e^{\lambda t}(I + t(A - \lambda I))$ satisfies $\dot{P} = AP$, and $P = I$ when $t = 0$.

Proof: Put $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so $\tau = a + d$ and $\delta = ad - bc$.
The eigenvalues $(\tau \pm \sqrt{\tau^2 - 4\delta})/2$ are the same, so we must have $\tau^2 = 4\delta$, and the eigenvalue is $\lambda = \tau/2 = a/2 + d/2$. Note that

$$\tau^2 - 4\delta = (a+d)^2 - 4ad + 4bc = a^2 + 2ad + d^2 - 4ad + 4bc = (a-d)^2 + 4bc,$$

So we see that $(a-d)^2 + 4bc = 0$, or $(a/2 - d/2)^2 + bc = 0$.
Now consider the matrix $B = A - \lambda I = A - \frac{1}{2}(a+d)I$, so $P = e^{\lambda t}(I + tB)$. In the simplest case, B would be zero. It is not always zero, but at least $B^2 = 0$:

$$\begin{aligned} B^2 &= \begin{bmatrix} a/2 - d/2 & b \\ c & d/2 - a/2 \end{bmatrix} \begin{bmatrix} a/2 - d/2 & b \\ c & d/2 - a/2 \end{bmatrix} \\ &= \begin{bmatrix} (a/2 - d/2)^2 + bc & (a/2 - d/2)b + b(d/2 - a/2) \\ c(a/2 - d/2) + (d/2 - a/2)c & cb + (d/2 - a/2)^2 \end{bmatrix} \\ &= \begin{bmatrix} (a/2 - d/2)^2 + bc & 0 \\ 0 & (a/2 - d/2)^2 + bc \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Repeated eigenvalues

Proposition: If A has only one eigenvalue, say λ then the matrix $P = e^{\lambda t}(I + t(A - \lambda I))$ satisfies $\dot{P} = AP$, and $P = I$ when $t = 0$.

The matrix $B = A - \lambda I$ satisfies $B^2 = 0$.

Next, we defined P to be $e^{\lambda t}(I + tB)$. This satisfies

$$\begin{aligned}\dot{P} &= \lambda e^{\lambda t}(I + tB) + e^{\lambda t}B = e^{\lambda t}(\lambda I + t\lambda B + B)\\ AP &= (\lambda I + B)P = e^{\lambda t}(\lambda I + B)(I + tB)\\ &= e^{\lambda t}(\lambda I + t\lambda B + B + tB^2) = e^{\lambda t}(\lambda I + t\lambda B + B) = \dot{P}\end{aligned}$$

as claimed. Also, at $t = 0$ we have $P = e^0(I + 0B) = I$.

Equilibrium points and stability

Consider a differential equation $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$.

An *equilibrium point* is a point (a, b) where $f(a, b) = 0$ and $g(a, b) = 0$. If (a, b) is an equilibrium point then we have a constant solution $(x, y) = (a, b)$ to the equation.

What happens if we start at a point (x_0, y_0) that is very close to (a, b)? Then $f(x_0, y_0)$ and $g(x_0, y_0)$ will be small, so the point will move slowly at first. If we wait longer, different things might happen.

(a) The point might move closer and closer to (a, b), and slow down even more, with $(x, y) \to (a, b)$ and $(\dot{x}, \dot{y}) \to (0, 0)$ as $t \to \infty$.

(b) The point might circle around (a, b), never moving very far away, but not slowing down.

(c) The point might eventually move far away from (a, b).

If (a) always happens, the equilibrium point is *asymptotically stable* .
If (b) can also happen (but not (c)), the equilibrium point is *stable* .
If (c) can happen then the equilibrium point is *unstable* .

Stability — precise definitions

More formal definitions are as follows.

- For any point $u \in \mathbb{R}^2$ and any $t \in \mathbb{R}$ we write $\phi(t, u)$ for the value at time t of the solution that passes through u at $t = 0$. Thus $\phi(0, u) = u$ and $\frac{d}{dt}\phi(t, u) = f(\phi(t, u))$.
- Example: for the system $\dot{x} = 2x$, $\dot{y} = 3y$ we have
$$\phi(t, (x_0, y_0)) = (e^{2t}x_0, e^{3t}y_0).$$
- Example: for the system $\dot{x} = y$, $\dot{y} = -x$ we have
$$\phi(t, (x_0, y_0)) = (\cos(t)x_0 + \sin(t)y_0,\ -\sin(t)x_0 + \cos(t)y_0).$$
- The equilibrium point a is *stable* if for all $\epsilon > 0$ there exists $\delta > 0$ such that whenever $\|u - a\| < \delta$ we have $\|\phi(t, u) - a\| < \epsilon$ for all $t \geq 0$.
- The equilibrium point a is *asymptotically stable* if it is stable, and there exists $\delta > 0$ such that whenever $\|u - a\| < \delta$ we have $\|\phi(t, u) - a\| \to 0$ as $t \to \infty$.
- If a is not stable, we say it is *unstable*.

Equilibrium points for linear systems

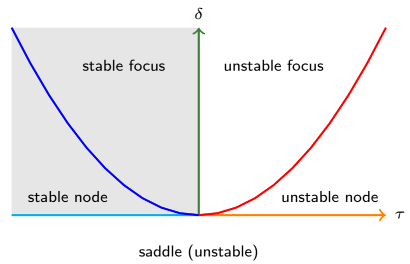
Consider a linear system $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$, with $\tau = a + d$ and $\delta = ad - bc$.

Then $(0, 0)$ is an equilibrium point (and is the only one, unless $ad - bc = 0$).

(a) If $\tau < 0$ and $\delta > 0$ then the system is a stable node or stable focus and the equilibrium point is asymptotically stable.

(b) If $\tau = 0$ and $\delta > 0$ then the system is a centre and the equilibrium point stable but not asymptotically stable.

(c) If $\tau > 0$ or $\delta \leq 0$ then the system is (usually) an unstable node or unstable focus or saddle, and the equilibrium point is unstable.

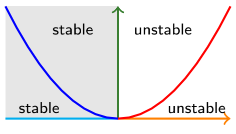


Equilibrium points for linear systems — eigenvalues

Another way to think about stability for linear systems is to use eigenvalues.

(a) Suppose that there are two real eigenvalues λ_1, λ_2. Then $\tau = \lambda_1 + \lambda_2$ and $\delta = \lambda_1\lambda_2$. The solutions involve $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$, so they will converge to zero if $\lambda_1, \lambda_2 < 0$, but will blow up to ∞ if $\lambda_1 > 0$ or $\lambda_2 > 0$.
If $\lambda_1\lambda_2 = \delta < 0$ then $\lambda_1 > 0$ or $\lambda_2 > 0$, so $(0,0)$ is unstable.
If $\lambda_1\lambda_2 = \delta > 0$ then λ_1 and λ_2 must both have the same sign.
If also $\lambda_1 + \lambda_2 = \tau > 0$ then $\lambda_1, \lambda_2 > 0$ so $(0,0)$ is unstable.
If $\lambda_1 + \lambda_2 = \tau < 0$ then $\lambda_1, \lambda_2 < 0$ so $(0,0)$ is asymptotically stable.

(b) Suppose that there are two complex eigenvalues, $\lambda \pm i\omega$, so $\tau = 2\lambda$ and $\delta = (\lambda + i\omega)(\lambda - i\omega) = \lambda^2 + \omega^2$. The solutions involve $e^{\lambda t}\sin(\omega t)$ and $e^{\lambda t}\cos(\omega t)$, so the overall size is like $e^{\lambda t}$.
If $\tau = 2\lambda > 0$ then $(0,0)$ is unstable.
If $\tau = 0$ then $(0,0)$ is stable but not asymptotically stable.
If $\tau = 2\lambda < 0$ then $(0,0)$ is asymptotically stable.



Linearisation (线性化)

Consider a system $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$.
Suppose that (a,b) is an equilibrium point, so $f(a,b) = g(a,b) = 0$.
We will study the behaviour of solutions (x,y) that are close to (a,b), so $(x,y) = (a+\alpha, b+\beta)$ with α and β small.
We write $f_x = \partial f/\partial x$ and $f_y = \partial f/\partial y$, so

$$f(x,y) = f(a+\alpha, b+\beta) \simeq f(a,b) + f_x(a,b)\alpha + f_y(a,b)\beta = f_x(a,b)\alpha + f_y(a,b)\beta.$$

Also, as $x = a + \alpha$ and a is constant, we have $\dot{\alpha} = \dot{x} = f(x,y)$.
We can do the same for $\dot{\beta}$, so we get

$$\dot{\alpha} = f_x(a,b)\alpha + f_y(a,b)\beta$$
$$\dot{\beta} = g_x(a,b)\alpha + g_y(a,b)\beta.$$

This is a linear system with matrix

$$J = \begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix},$$

called the *Jacobian*. We can classify it as before, using the trace and determinant, or the eigenvalues.
Usually the flow lines for the original nonlinear system will be similar to those for the linearised system, at least if we look close to (a,b).

Linearisation example

Consider the system $\dot{x} = 9y^2 - 1$, $\dot{y} = 9x^2 - 1$. There is an equilibrium point at $(1/3, 1/3)$. There we have

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 0 & 18y \\ 18x & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}.$$

It is easy to see that the vectors $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are eigenvectors, with eigenvalues $\lambda_1 = 6$ and $\lambda_2 = -6$. Solutions to the linear system $\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = J \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ are of the form

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = a_1 e^{6t} v_1 + a_2 e^{-6t} v_2 = \begin{bmatrix} a_1 e^{6t} + a_2 e^{-6t} \\ a_1 e^{6t} - a_2 e^{-6t} \end{bmatrix}.$$

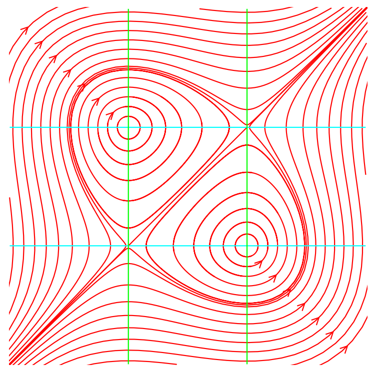
As $x = 1/3 + \alpha$ and $y = 1/3 + \beta$, the corresponding approximate solutions for the original system are

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} + a_1 e^{6t} v_1 + a_2 e^{-6t} v_2 = \begin{bmatrix} 1/3 + a_1 e^{6t} + a_2 e^{-6t} \\ 1/3 + a_1 e^{6t} - a_2 e^{-6t} \end{bmatrix}.$$

Linearisation example

$$\dot{x} = 9y^2 - 1 \qquad \dot{y} = 9x^2 - 1$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \simeq \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} + a_1 e^{6t} v_1 + a_2 e^{-6t} v_2 = \begin{bmatrix} 1/3 + a_1 e^{6t} + a_2 e^{-6t} \\ 1/3 + a_1 e^{6t} - a_2 e^{-6t} \end{bmatrix}.$$



These are solutions for the original system.

The eigenvectors, more slowly

We had $J = \begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}$. This has $\tau = 0$ and $\delta = -36$ so $\tau^2 - 4\delta = 144$.
This gives eigenvalues $(0 \pm \sqrt{144})/2$, so $\lambda_1 = -6$ and $\lambda_2 = 6$.

The eigenvector $v_1 = \begin{bmatrix} p \\ q \end{bmatrix}$ must satisfy $(J - \lambda_1 I)v_1 = 0$, or $\begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,
which means that $p + q = 0$. We can therefore take $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

The vector $v_2 = \begin{bmatrix} r \\ s \end{bmatrix}$ must satisfy $(J - \lambda_2 I)v_2 = 0$, or $\begin{bmatrix} -6 & 6 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,
which means that $p - q = 0$. We can therefore take $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Linearisation example

Consider again the system $\dot{x} = 9y^2 - 1$, $\dot{y} = 9x^2 - 1$.
There is another equilibrium point at $(-1/3, 1/3)$.
There we have

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 0 & 18y \\ 18x & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix},$$

giving equations $\dot{\alpha} = 6\beta$ and $\dot{\beta} = -6\alpha$.
Some solutions are

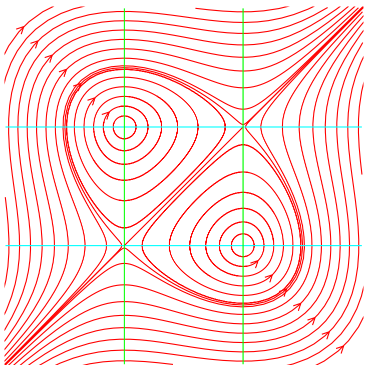
$$x = -1/3 + \alpha = -1/3 + R\cos(6t) \qquad y = 1/3 + \beta = 1/3 - R\sin(6t)$$

(with R constant).

This means that the solution curves are circles centred at $(-1/3, 1/3)$.

Linearisation example

$$\dot{x} = 9y^2 - 1 \qquad \dot{y} = 9x^2 - 1$$
$$\begin{bmatrix} x \\ y \end{bmatrix} \simeq \begin{bmatrix} -1/3 + R\cos(6t) \\ 1/3 - R\sin(6t) \end{bmatrix}$$



These are solutions for the original system.

The damped Duffing oscillator

The *damped Duffing oscillator* is given by $\dot{x} = y$ and $\dot{y} = 2x - x^3 - 0.1y$.
There is an equilibrium point at $(\sqrt{2}, 0)$. There we have

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 - 3x^2 & -0.1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -0.1 \end{bmatrix}.$$

This has $\tau = -0.1 < 0$ and $\delta = 4 > 0$ and $\tau^2 - 4\delta \simeq -16$. This gives a stable focus with growth rate $\lambda = \tau/2 = -0.05$ and angular frequency $\omega = \sqrt{4\delta - \tau^2}/2 \simeq \sqrt{16}/2 = 2$. Solutions of the linearised equations can be found as usual using the matrix

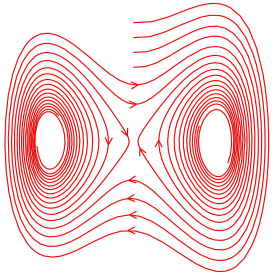
$$\begin{aligned} P &= e^{\lambda t}(\cos(\omega t)I + \omega^{-1}\sin(\omega t)(J - \lambda I)) \\ &\simeq e^{-0.05t}\left(\cos(2t)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0.5\sin(2t)\begin{bmatrix} 0.05 & 1 \\ -4 & -0.05 \end{bmatrix}\right) \end{aligned}$$

In particular, the solution with $\begin{bmatrix} \sqrt{2} + \alpha_0 \\ 0 \end{bmatrix}$ at time $t = 0$ is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} + e^{-0.05t}\alpha_0\begin{bmatrix} \cos(2t) + 0.025\sin(2t) \\ -2\sin(2t) \end{bmatrix}.$$

The damped Duffing oscillator

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= 2x - x^3 - 0.1y\end{aligned} \qquad \begin{bmatrix}x\\y\end{bmatrix} \simeq \begin{bmatrix}\sqrt{2}\\0\end{bmatrix} + e^{-0.05t}\alpha_0 \begin{bmatrix}\cos(2t)+0.025\sin(2t)\\ -2\sin(2t)\end{bmatrix}.$$



These are solutions for the original system.

Misleading linearisation

The flow lines for a nonlinear system do not always look like the flow lines for the linearisation. For example, consider the system

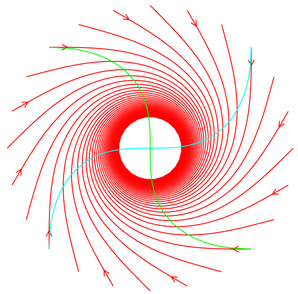
$$\begin{bmatrix}\dot{x}\\\dot{y}\end{bmatrix} = \begin{bmatrix}y\\-x\end{bmatrix} + (x^2+y^2)\begin{bmatrix}-x\\-y\end{bmatrix}$$

around the circle
towards the origin

There is an equilibrium point at $(0,0)$. There we have

$$J = \begin{bmatrix}\partial f/\partial x & \partial f/\partial y\\ \partial g/\partial x & \partial g/\partial y\end{bmatrix} = \begin{bmatrix}-3x^2-y^2 & 1-2xy\\ -1-2xy & -x^2-3y^2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$$

Linear system is $\dot{x}=y$, $\dot{y}=-x$; solutions are circles $(r\cos(t), r\sin(t))$.
However, the solution curves for the original system are not circles.
They spiral inwards, but very slowly.

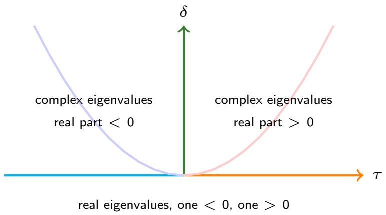


The Hartman-Grobman Theorem

In the last example:

- J was $\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, with eigenvalues $\lambda_1, \lambda_2 = \pm i$, so $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = 0$.
- The phase portrait for the linearisation had different properties from the phase portrait for the original system.

Theorem: Suppose that $e=(a,b)$ is an equilibrium point for a system $\dot{x}=f(x,y)$, $\dot{y}=g(x,y)$, and that the eigenvalues for the Jacobian matrix J satisfy $\mathrm{Re}(\lambda_1)\neq 0$ and $\mathrm{Re}(\lambda_2)\neq 0$. Then the original system is *locally topologically conjugate* to the linearised system.



The theorem applies unless $\delta=0$, or ($\tau=0$ and $\delta\geq 0$).

The Hartman-Grobman Theorem

Theorem: Suppose that $e=(a,b)$ is an equilibrium point for a system $\dot{x}=f(x,y)$, $\dot{y}=g(x,y)$, and that the eigenvalues for the Jacobian matrix J satisfy $\mathrm{Re}(\lambda_1)\neq 0$ and $\mathrm{Re}(\lambda_2)\neq 0$. Then the original system is *locally topologically conjugate* to the linearised system.

Explanation:

- Recall: there is a matrix $P(t)$ such that the solutions to $\dot{u}(t)=Ju(t)$ are $u(t)=P(t)u_0$.
- In the linearised system we have $x=e+u$ and $x_0=e+u_0$, so the solutions are $x(t)=e+P(t)(x_0-e)$.
- In other words, if we put $\varphi_0(x)=x-e$, then the solutions to the linearised system are $x(t)=\varphi_0^{-1}(P(t)\varphi_0(x_0))$.

Local topological conjugacy means that there is a function φ such that

- $\varphi(x)$ is defined and continuous for x sufficiently close to e, with $\varphi(e)=0$.
- $\varphi^{-1}(u)$ is defined and continuous for u close to 0, with $\varphi^{-1}(0)=e$.
- The solutions for the original system are $x(t)=\varphi^{-1}(P(t)\varphi(x_0))$.
- Thus, if we apply φ^{-1} to the phase portrait for the linear system, we get (part of) the phase portrait for the original system.

We will not prove this theorem.

Hartman-Grobman example

Here is an (unusual) example where we can find the map φ.
Consider the system $\dot{x} = -x + y + 3y^2$, $\dot{y} = y$.
The origin is an equilibrium, and the linearisation is $\dot{x} = -x + y$, $\dot{y} = y$, with solution

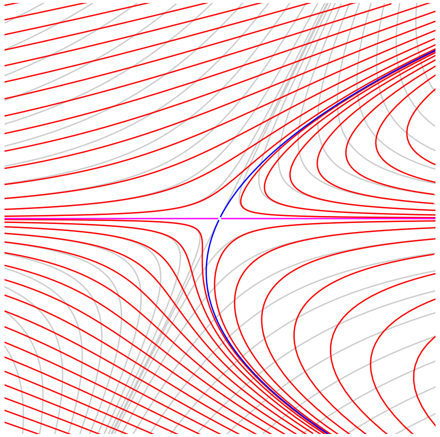
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (x_0 + \frac{1}{2}y_0)e^t - \frac{1}{2}y_0 e^{-t} \\ y_0 e^t \end{bmatrix}.$$

Suppose that x and y obey the linear equations, and we put $(X, Y) = (x + y^2, y)$. Then $\dot{Y} = Y$ and

$$\dot{X} = \dot{x} + 2y\dot{y} = -x + y + 2y^2 = -x - y^2 + y + 3y^2 = -X + Y + 3Y^2,$$

so X and Y obey the original nonlinear equations. This means that we can take $\varphi(x, y) = (x + y^2, y)$ in the Hartman-Grobman Theorem.

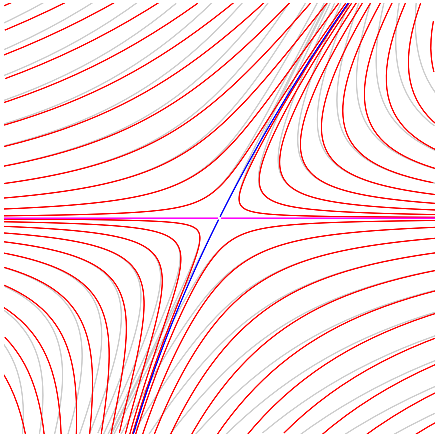
Hartman-Grobman example



This is the phase portrait for the original system $\dot{x} = -x + y + 3y^2$, $\dot{y} = y$.

Hartman-Grobman example — zoomed in

This is the same as the previous slide, but zoomed in by a factor of 10.



This is the phase portrait for the original system $\dot{x} = -x + y + 3y^2$, $\dot{y} = y$.

Conserved quantities

Consider a system $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$
A *conserved quantity* is a differentiable function $U(x, y)$ such that $\dot{U} = 0$.
This means that U is constant on each flow line.

For any function $U(x, y)$ we have

$$\dot{U}(x, y) = U_x(x, y)\dot{x} + U_y(x, y)\dot{y} = U_x(x, y)f(x, y) + U_y(x, y)g(x, y)$$

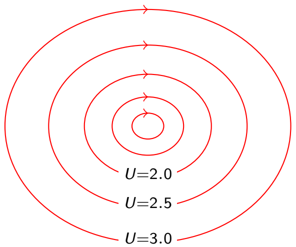
(where U_x and U_y are the partial derivatives of U).
Thus U is conserved if $U_x f + U_y g = 0$.

Conserved quantities — example

Suppose $\dot{x} = 3y$ and $\dot{y} = -2x$, and put $U = 2x^2 + 3y^2$. Then

$$\dot{U} = 2 \times 2x\dot{x} + 3 \times 2y\dot{y} = 4x \times 3y + 6y \times (-2x) = 0,$$

so U is a conserved quantity.



More generally, suppose $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ -c & -a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ and put $U = cx^2 + 2axy + by^2$.
Then

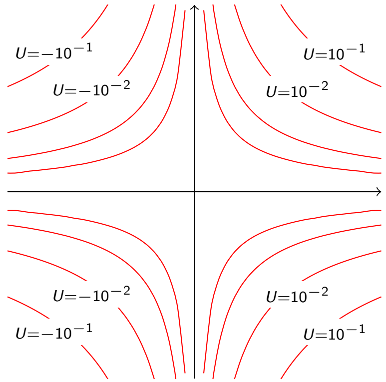
$$\dot{U} = U_x\dot{x} + U_y\dot{y} = (2cx + 2ay)(ax + by) + (2ax + 2by)(-cx - ay) = 0.$$

Conserved quantities — example

Suppose $\dot{x} = nx$ and $\dot{y} = -my$ (where n and m are integers). Put $U = x^m y^n$.
Then

$$\dot{U} = mx^{m-1}\dot{x}y^n + ny^{n-1}\dot{y}x^m = nmx^my^n - nmx^my^n = 0.$$

The picture shows the case $n = 4$, $m = 3$.



Conserved quantity for the Lotka-Volterra model

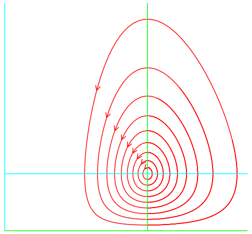
Recall the Lotka-Volterra model for populations of fish and sharks:
$\dot{F} = (\alpha - \beta S)F$, $\dot{S} = (\delta F - \gamma)S$. Put

$$U = \alpha\ln(S) + \gamma\ln(F) - \beta S - \delta F.$$

Then

$$\begin{aligned}\dot{U} &= \alpha S^{-1}\dot{S} + \gamma F^{-1}\dot{F} - \beta\dot{S} - \delta\dot{F} \\ &= \alpha(\delta F - \gamma) + \gamma(\alpha - \beta S) - \beta(\delta F - \gamma)S - \delta(\alpha - \beta S)F \\ &= \alpha\delta F - \alpha\gamma + \alpha\gamma - \beta\gamma S - \beta\delta SF + \beta\gamma S - \alpha\delta F + \beta\delta SF \\ &= 0,\end{aligned}$$

so U is a conserved quantity.



The pendulum conserves energy

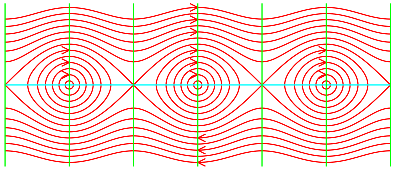
Recall the pendulum equations: $\dot{\theta} = \omega$, $\dot{\omega} = -\sin(\theta)$. Put

$$U = \tfrac{1}{2}\omega^2 - \cos(\theta).$$

Then

$$\begin{aligned}\dot{U} &= \tfrac{1}{2} \times 2\omega\dot{\omega} + \sin(\theta)\dot{\theta} \\ &= \omega \times (-\sin(\theta)) + \sin(\theta)\omega = 0.\end{aligned}$$

In this case, there is a clear physical interpretation: $\frac{1}{2}\omega^2$ is the rotational kinetic energy , $-\cos(\theta)$ is the gravitational potential energy , and U is the total energy .



Conserved quantity means no nodes or foci

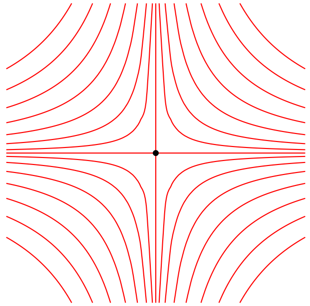
Proposition: If there is a conserved quantity U, there are no nodes or foci. (Unless there is a nonempty open region where U is constant.)

Proof.

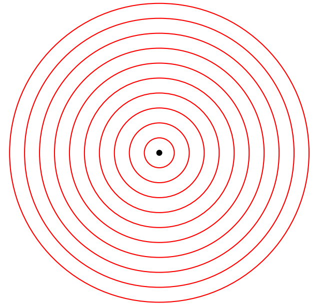
- Suppose that (a, b) is a stable node or focus.
- Consider a point (x_0, y_0) near (a, b).
- Then there is a solution $(x(t), y(t))$ with $(x(0), y(0)) = (x_0, y_0)$ and $(x(t), y(t)) \to (a, b)$ as $t \to \infty$.
- This means that $U(x(t), y(t)) \to U(a, b)$.
- However, $U(x(t), y(t))$ is constant because U is conserved.
- The only way this can happen is if $U(x(t), y(t)) = U(a, b)$ for all t.
- In particular, we can take $t = 0$ to get $U(x_0, y_0) = U(a, b)$.
- This means that for all points (x_0, y_0) close to (a, b) we have $U(x_0, y_0) = U(a, b)$, so U is constant on an open region.
- If there is an unstable node or focus, consider $t \to -\infty$ instead.

$\square$

Saddles and centres are possible



$\dot{x} = x, \ \dot{y} = -y$
$U = xy$ is conserved
The origin is a saddle



$\dot{x} = -y, \ \dot{y} = x$
$U = x^2 + y^2$ is conserved
The origin is a centre

Conserved quantity with arctan

Consider the linear system where $\dot{x} = -x - y$ and $\dot{y} = x - y$.

- The matrix is $\left[\begin{smallmatrix} -1 & -1 \\ 1 & -1 \end{smallmatrix}\right]$, with $\tau = -2, \ \delta = 2, \ \tau^2 - 4\delta = -4 < 0$.
  Eigenvalues are $\lambda \pm i\omega$ with $\lambda = -1$ and $\omega = 1$,
  so we have a stable focus.
- The fundamental solution is

$$P = e^{\lambda t}\left(\cos(\omega t)I + \omega^{-1}\sin(\omega t)(A - \lambda I)\right)$$
$$= e^{-t}\left(\cos(t)\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] + \sin(t)\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]\right) = e^{-t}\begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$

- Solution starting at $\left[\begin{smallmatrix} r \\ 0 \end{smallmatrix}\right]$ is $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = P\left[\begin{smallmatrix} r \\ 0 \end{smallmatrix}\right] = \begin{bmatrix} re^{-t}\cos(t) \\ re^{-t}\sin(t) \end{bmatrix}$.
- Put $V = \arctan(y/x)$ and $W = \frac{1}{2}\ln(x^2 + y^2)$ and $U = V + W$. Claim: U is conserved.

Conserved quantity with arctan

$$\dot{x} = -x - y \quad \dot{y} = x - y$$
$$U = V + W \quad V = \arctan(y/x) \quad W = \tfrac{1}{2}\ln(x^2 + y^2)$$

Recall that $\arctan'(z) = 1/(1 + z^2)$. Using this, we get

$$\dot{V} = \arctan'(y/x)\frac{d}{dt}(y/x) = \frac{1}{1 + y^2/x^2}\,\frac{\dot{y}x - y\dot{x}}{x^2}$$
$$= \frac{(x - y)x - y(-x - y)}{x^2 + y^2} = \frac{x^2 + y^2}{x^2 + y^2} = 1$$
$$\dot{W} = \frac{1}{2}\frac{1}{x^2 + y^2}\frac{d}{dt}(x^2 + y^2) = \frac{2x\dot{x} + 2y\dot{y}}{2(x^2 + y^2)}$$
$$= \frac{x(-x - y) + y(x - y)}{x^2 + y^2} = \frac{-x^2 - y^2}{x^2 + y^2} = -1$$
$$\dot{U} = \dot{V} + \dot{W} = 1 - 1 = 0.$$

Conserved quantity with arctan

$$\dot{x} = -x - y \quad \dot{y} = x - y$$
$$U = V + W \quad V = \arctan(y/x) \quad W = \tfrac{1}{2}\ln(x^2 + y^2)$$

We saw that the solution starting at $(r, 0)$ is $x = re^{-t}\cos(t)$ and $y = re^{-t}\sin(t)$. For this we have

$$y/x = \frac{re^{-t}\sin(t)}{re^{-t}\cos(t)} = \frac{\sin(t)}{\cos(t)} = \tan(t)$$
$$\arctan(y/x) = \arctan(\tan(t)) = t$$
$$x^2 + y^2 = r^2e^{-2t}(\cos^2(t) + \sin^2(t)) = r^2e^{-2t}$$
$$\tfrac{1}{2}\ln(x^2 + y^2) = \ln(r) - t$$
$$U = \arctan(y/x) + \tfrac{1}{2}\ln(x^2 + y^2) = t + (\ln(r) - t) = \ln(r).$$

As expected, this does not depend on t.

Conserved quantity with arctan

- We said that if there is a continuous, well-defined conserved quantity, then there can only be saddles and centres, not nodes or foci.
- In this example we have a conserved quantity and a stable focus. So what is wrong?
- The point is that U is not really well-defined (because you can add multiples of π to the arctan term). We can make try to make it well-defined by always taking the value of arctan that lies in $(-\pi/2, \pi/2]$. However, with this convention, U is discontinuous when $y = 0$. Also, U will always be discontinuous at the point $(0, 0)$, whatever convention we make. The theorem only covers the case where U is well-defined and continuous, so there is no contradiction.

Lyapunov functions

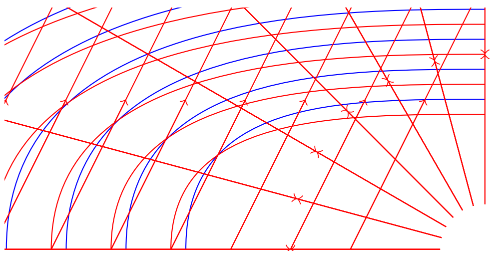
Definition: Consider a differentiable function $V(x, y)$, defined on some open region R containing a point (a, b).

- If $V(x, y) \geq 0$ for all (x, y), we say that V is *positive semi-definite*.
- If $V(a, b) = 0$ but $V(x, y) > 0$ for all $(x, y) \neq (a, b)$, we say that V is *positive definite* around (a, b).
- If $V(x, y) \leq 0$ for all (x, y), we say that V is *negative semi-definite*.
- If $V(a, b) = 0$ but $V(x, y) < 0$ for all $(x, y) \neq (a, b)$, we say that V is *negative definite* around (a, b).

Now suppose that x and y change following the equations $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$, so for any function $V(x, y)$ we have $\dot{V} = V_x f + V_y g$. Suppose also that $f(0, 0) = g(0, 0) = 0$, so $(0, 0)$ is an equilibrium point.

- If V is positive definite and $\dot{V}$ is negative semidefinite, we say that V is a *weak Lyapunov function*.
- If V is positive definite and $\dot{V}$ is negative definite, we say that V is a *strong Lyapunov function*.
- If there is a strong Lyapunov function, then the origin is an asymptotically stable equilibrium point.
- If there is a weak Lyapunov function, then the origin is an stable equilibrium point, but may not be asymptotically stable.
- Note: any positive definite conserved quantity is a weak Lyapunov function.

Contours of a Lyapunov function



The blue lines are the contours for a function $V(x, y)$.

These red lines show a flow that cuts across the contours going downwards, so V decreases as we move along this flow. The function V could be a Lyapunov function for this flow. These red lines show a flow that cuts across the contours going upwards, so V increases as we move along this flow. The function V could not be a Lyapunov function for this flow. These red lines show a flow that cuts across the contours sometimes going upwards and sometimes going downwards. As we move along the flow, V sometimes increases and sometimes decreases. The function V could not be a Lyapunov function for this flow. These red lines show a flow that cuts across the contours at a shallow angle. As we move along the flow, the function V decreases, but only slowly.

Definiteness for quadratic functions

Consider a quadratic function $Q = ax^2 + 2bxy + cy^2$.

(a) If $ac - b^2 > 0$ then a and c are nonzero and have the same sign.
(b) If $ac - b^2 > 0$ and $a, c > 0$ then Q is positive definite.
(c) If $ac - b^2 > 0$ and $a, c < 0$ then Q is negative definite.
(d) If $ac - b^2 \leq 0$ then Q is neither positive definite nor negative definite.

Proof:

(a) If $a = 0$ or $c = 0$ or a, c have opposite sign then $ac \leq 0$ so $ac - b^2 \leq 0$. Thus, if $ac - b^2 > 0$ then a and c must be nonzero with the same sign.

(b) Suppose that $ac - b^2 > 0$ with $a, c > 0$. We then find that

$$\begin{aligned} a^{-1}((ax+by)^2 + (ac-b^2)y^2) &= a^{-1}(a^2x^2 + 2abxy + b^2y^2 + acy^2 - b^2y^2) \\ &= ax^2 + 2bxy + cy^2 = Q. \end{aligned}$$

This representation makes it clear that $Q \geq 0$. Moreover Q can only be equal to 0 if $ax + by = 0$ and $y = 0$, which means that $x = y = 0$. Thus, Q is positive definite.

(c) Suppose instead that $ac - b^2 > 0$ with $a, c < 0$. We can then use (b) to show that $-Q$ is positive definite, and this means that Q is negative definite.

Definiteness for quadratic functions

Consider a quadratic function $Q = ax^2 + 2bxy + cy^2$.

(a) If $ac - b^2 > 0$ then a and c are nonzero and have the same sign.
(b) If $ac - b^2 > 0$ and $a, c > 0$ then Q is positive definite.
(c) If $ac - b^2 > 0$ and $a, c < 0$ then Q is negative definite.
(d) If $ac - b^2 \leq 0$ then Q is neither positive definite nor negative definite.

Proof continued:

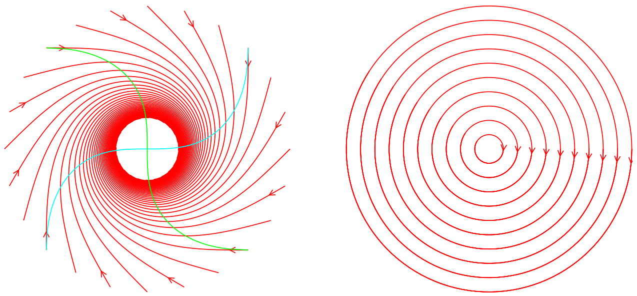
(d) Now suppose that $ac - b^2 \leq 0$. We need to show that Q is indefinite, so we need to find a point $(x, y) \neq (0, 0)$ where $Q = 0$. If $a \neq 0$ we note that $x = (-b + \sqrt{b^2 - ac})/a$ is a root of $ax^2 + 2bx + c = 0$, so $Q = 0$ at $(x, 1)$. Similarly, if $c \neq 0$ then $y = (-b + \sqrt{b^2 - ac})/c$ is a root of $a + 2by + cy^2 = 0$, so $Q = 0$ at $(1, y)$. This just leaves the case where $a = c = 0$ and $Q = 2bxy$, so $Q = 0$ at $(1, 0)$ or $(0, 1)$. $\square$

Lyapunov function for the slow spiral

Remember this system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} + (x^2 + y^2) \begin{bmatrix} -x \\ -y \end{bmatrix}$$

around the circle
towards the origin



The linearisation is $(\dot{x}, \dot{y}) = (y, -x)$, which has a centre, so it is not asymptotically stable. But the original system is asymptotically stable. We can prove this with a Lyapunov function.

Lyapunov function for the slow spiral

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} + (x^2 + y^2) \begin{bmatrix} -x \\ -y \end{bmatrix}$$

around the circle
towards the origin

Put $V = x^2 + y^2$.
Then $V > 0$ except $V = 0$ at $(0, 0)$, so V is positive definite.

$$\begin{aligned} \dot{V} &= V_x \dot{x} + V_y \dot{y} \\ &= 2x(y - (x^2 + y^2)x) + 2y(-x - (x^2 + y^2)y) \\ &= 2xy - 2x^2(x^2 + y^2) - 2xy - 2y^2(x^2 + y^2) \\ &= -2(x^2 + y^2)^2, \text{ which is negative definite.} \end{aligned}$$

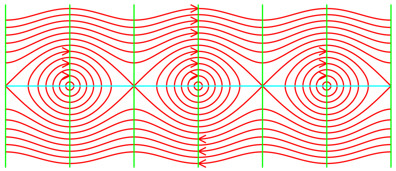
So V is a strong Lyapunov function around $(0, 0)$, so $(0, 0)$ is an asymptotically stable equilibrium point.
In fact $\dot{V} = -V^2$ gives $\frac{d}{dt}(V^{-1}) = -V^{-2}\dot{V} = 1$
so $V^{-1} = V_0^{-1} + t$ so $V = (V_0^{-1} + t)^{-1} = V_0/(1 + V_0 t)$.

Lyapunov function for the pendulum

Recall the pendulum equations: $\dot{\theta} = \omega$, $\dot{\omega} = -\sin(\theta)$.
The energy $U = \frac{1}{2}\omega^2 - \cos(\theta)$ is conserved.
Note that $U = -1$ when $(\omega, \theta) = (0, 0)$. However, we always have $1 - \cos(\theta) \geq 0$, so the function $V = U + 1 = \frac{1}{2}\omega^2 + 1 - \cos(\theta)$ is positive semi-definite. We only have $V = 0$ when $(\theta, \omega) = (2n\pi, 0)$ for some integer n.
If we consider only the region

$$R = \{(\theta, \omega) \mid -2\pi < \theta < 2\pi\}$$

then V is positive definite. It also has $\dot{V} = 0$, so it is a weak Lyapunov function. Flow lines near the origin do not converge to the origin, so the origin is not asymptotically stable, so there is no strong Lyapunov function.



Lyapunov function for a gradient flow

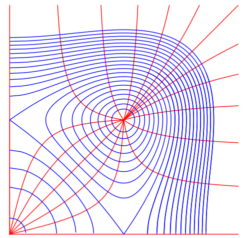
Consider the system $\dot{x} = x - x^3$, $\dot{y} = y - y^3$, and the function

$$V = (x^2 - 1)^2 + (y^2 - 1)^2 = x^4 - 2x^2 + y^4 - 2y^2 - 2.$$

This has $V \geq 0$ everywhere, and V is only equal to 0 at the points $(\pm 1, \pm 1)$.
It also satisfies

$$\dot{V} = V_x f + V_y g = (4x^3 - 4x)(x - x^3) + (4y^3 - 4y)(y - y^3) = -4((x - x^3)^2 + (y - y^3)^2).$$

This means that $\dot{V} \leq 0$ everywhere, and $\dot{V}$ is only equal to 0 if $x - x^3 = 0$ and $y - y^3 = 0$, which means that $x, y \in \{0, 1, -1\}$.
Now consider only the region $R = \{(x, y) \mid x > 0 \text{ and } y > 0\}$. In R we have $V > 0$ except at $(1, 1)$, and $\dot{V} < 0$ except at $(1, 1)$. Thus, V is a strong Lyapunov function for the equilibrium point $(1, 1)$.



Lyapunov function for the damped Duffing oscillator

The (damped) oscillator has $\dot{x} = y$ and $\dot{y} = 2x - x^3 - \epsilon y$ for some $\epsilon \geq 0$.
Previously we considered $\epsilon = 0$ (undamped) and $\epsilon = 0.1$ (damped).
Now consider the function

$$V = 2y^2 + x^4 - 4x^2 + 4 = 2y^2 + (x^2 - 2)^2.$$

This always has $V \geq 0$, with $V = 0$ only at $(\pm\sqrt{2}, 0)$.
Region $R = \{(x, y) \mid x > 0\}$: the function V is positive definite for $(\sqrt{2}, 0)$.
We also have

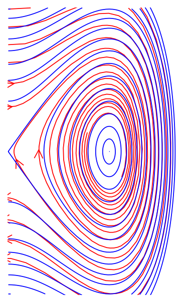
$$\dot{V} = 4y\dot{y} + 4x^3\dot{x} - 8x\dot{x} = 4y(2x - x^3 - \epsilon y) + 4x^3 y - 8xy = -4\epsilon y^2.$$

This means that $\dot{V} \leq 0$ everywhere, with $\dot{V} = 0$ only when $y = 0$. In particular, V is negative semidefinite on R, so it is a weak Lyapunov function. We deduce that $(\sqrt{2}, 0)$ is a stable equilibrium point. In fact, we can use more complicated properties of V to show that $(\sqrt{2}, 0)$ is even asymptotically stable. Note also that when $\epsilon = 0$ we have $\dot{V} = 0$, so V is a conserved quantity for the undamped Duffing oscillator.

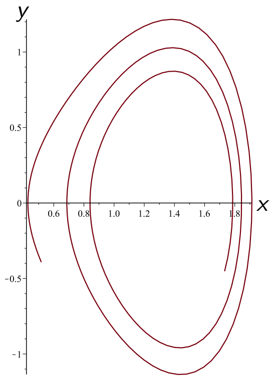
Lyapunov function for the damped Duffing oscillator

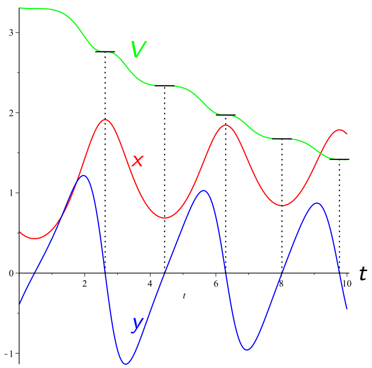
$$\dot{x} = y \qquad \dot{y} = 2x - x^3 - \epsilon y \qquad V = 2y^2 + (x^2 - 2)^2 \qquad \dot{V} = -4\epsilon y^2$$

In this example, $\dot{V}$ is quite small, so the flow lines cross the lines of constant V at a shallow angle, so it is hard to draw a clear picture.



Damped Duffing is asymptotically stable

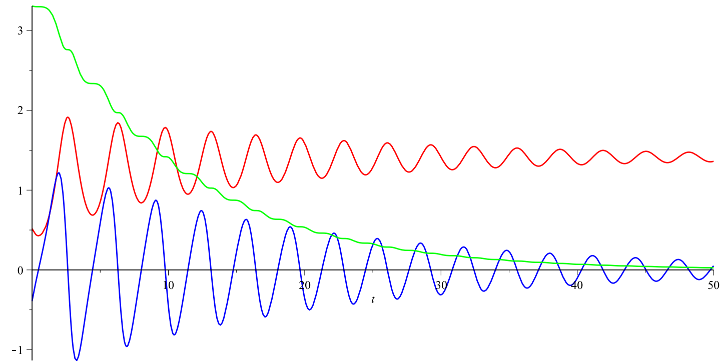




Recall that $\dot{V} = -4\epsilon y^2$, and this is the slope of the green graph of V against t.
The blue graph shows y against t.
When $y \neq 0$, we have $\dot{V} < 0$ and the green graph slopes downwards.
When $y = 0$ we have $\dot{V} = 0$ and the green graph is flat.
This only happens for an instant before y becomes nonzero again and the green graph continues to decrease.

Damped Duffing is asymptotically stable

Here is the same picture for a longer time:



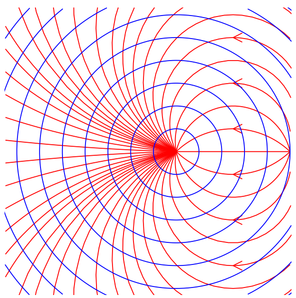
This shows that the flow line converges to the equilibrium point $(\sqrt{2}, 0)$ where $V = 0$.

Another Lyapunov example

Suppose that $\dot{x} = x^2 - y^2 - \frac{1}{4}$, $\dot{y} = 2xy$, so there are equilibria at $(\pm\frac{1}{2}, 0)$.
Put $V = (x + \frac{1}{2})^2 + y^2$, which is positive definite around $(-\frac{1}{2}, 0)$.
Then

$$\begin{aligned}\dot{V} &= 2(x+\tfrac{1}{2})\dot{x} + 2y\dot{y} = 2(x+\tfrac{1}{2})(x^2 - y^2 - \tfrac{1}{4}) + 4xy^2 \\ &= 2(x+\tfrac{1}{2})(x^2 - \tfrac{1}{4}) + (4x - 2(x+\tfrac{1}{2}))y^2 = 2(x+\tfrac{1}{2})^2(x - \tfrac{1}{2}) + (2x-1)y^2 \\ &= (2x-1)((x+\tfrac{1}{2})^2 + y^2).\end{aligned}$$

This is negative definite on the region $R = \{(x, y) \mid x < \frac{1}{2}\}$.
It follows that $(-\frac{1}{2}, 0)$ is asymptotically stable.



Finding a Lyapunov function

- Consider the system $\dot{x} = 80(y^{15} - x^9)$, $\dot{y} = -77(x^{13} + y^{11})$.
- There is an equilibrium point at the origin.
- How can we find a Lyapunov function? Guess the general form, and then adjust the coefficients.
- Try $V = \alpha x^{2n} + \beta y^{2m}$ with $\alpha, \beta, n, m > 0$, where n and m are integers. This is always positive definite, and we want to choose α, β, n, m to make sure that $\dot{V}$ is negative definite.

$$\begin{aligned}\dot{V} &= V_x\dot{x} + V_y\dot{y} \\ &= 2n\alpha x^{2n-1}.80(y^{15} - x^9) - 2m\beta y^{2m-1}.77(x^{13} + y^{11}) \\ &= -160n\alpha x^{2n+8} - 154\beta y^{2m+10} + 160n\alpha x^{2n-1}y^{15} - 154\beta x^{13}y^{2m-1}\end{aligned}$$

The first two terms give a negative definite function. The other two terms can be positive or negative depending on the signs of x and y. To make the whole thing negative definite, we need the last two terms to cancel.

Finding a Lyapunov function

$$\dot{x} = 80(y^{15} - x^9) \qquad \dot{y} = -77(x^{13} + y^{11}) \qquad V = \alpha x^{2n} + \beta y^{2m}$$
$$\dot{V} = -160 n\alpha x^{2n+8} - 154\beta y^{2m+10} + 160 n\alpha x^{2n-1} y^{15} - 154\beta x^{13} y^{2m-1}$$

The last two terms should cancel, so we want

$$160n\alpha = 154m\beta \qquad 2n - 1 = 13 \qquad 15 = 2m - 1$$

so $n = 7$ and $m = 8$.
Putting this in $160n\alpha = 154m\beta$ gives $\alpha = \frac{154\times 8}{160\times 7}\beta = \frac{11}{10}\beta$ so we can choose $\alpha = 11$ and $\beta = 10$.
We conclude that the function $V = 11x^{14} + 10y^{16}$ is a strong Lyapunov function.

Lyapunov method for instability

So far we have used Lyapunov functions to prove that points are stable.

We now give a similar method to prove that points are unstable.

Theorem: Let V be a differentiable function defined on some open region R containing an equilibrium point (a, b). Suppose that $\dot{V}$ is positive definite, and that for all $\epsilon > 0$ there is a point (x, y) with $\|(x, y) - (a, b)\| \leq \epsilon$ and $V(x, y) > 0$. Then (a, b) is unstable.

In particular:
If both V and $\dot{V}$ are positive definite around (a, b), then (a, b) is unstable.

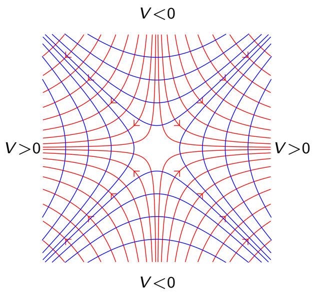
Instability for a linear saddle

If $\dot{V}$ is positive definite, and for all $\epsilon > 0$ there is a point where $\|(x, y) - (a, b)\| \leq \epsilon$ and $V > 0$, then (a, b) is unstable.

Consider the system $\dot{x} = x$, $\dot{y} = -y$, which has a saddle at $(0, 0)$.
Put $V = x^2 - y^2$. Then

$$\dot{V} = 2x\dot{x} - 2y\dot{y} = 2x^2 + 2y^2,$$

which is positive definite. Also, for any $\epsilon > 0$ there is a point $(0, \epsilon)$ with $\|(\epsilon, 0) - (0, 0)\| = \epsilon$ and $V = \epsilon^2 > 0$. Thus, $(0, 0)$ is unstable.



Instability for a gradient flow

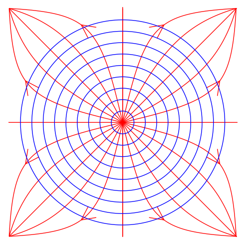
Consider the system $\dot{x} = x - x^3$, $\dot{y} = y - y^3$, and the function $V = x^2 + y^2$.
This is positive definite around $(0, 0)$. We also have

$$\dot{V} = 2x\dot{x} + 2y\dot{y} = 2x(x - x^3) + 2y(y - y^3) = 2x^2(1 - x^2) + 2y^2(1 - y^2).$$

This is positive definite on the region

$$R = \{(x, y) \mid -1 < x, y < 1\},$$

so the origin is an unstable equilibrium.



van der Pol instability

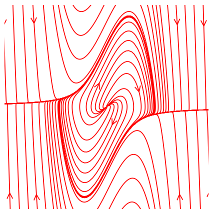
Consider the van der Pol oscillator, with $\dot{x} = y$ and $\dot{y} = 2(1-x^2)y - x$. Put

$$V = x^2 - xy + y^2 = \tfrac{3}{4}(x-y)^2 + \tfrac{1}{4}(x+y)^2,$$

so V is positive definite. We also have

$$\begin{aligned}\dot{V} &= (2x-y)\dot{x} + (2y-x)\dot{y} = (2x-y)y + (2y-x)(2y-2x^2y-x)\\ &= 2xy - y^2 + 4y^2 - 4x^2y^2 - 2xy - 2xy + 2x^3y + x^2\\ &= x^2 - 2xy + 3y^2 + 2x^3y - 4x^2y^2 = (x-y)^2 + 2y^2 + (2x^3y - 4x^2y^2).\end{aligned}$$

Now let R be a small square around $(0,0)$, say $R = \{(x,y) \mid |x|, |y| < 10^{-2}\}$. If $(x,y) \in R$ and $(x,y) \neq (0,0)$ then $(x-y)^2 + 2y^2$ will be strictly positive. The other term $2x^3y - 4x^2y^2$ might be negative, but it is smaller by a factor of about 10^{-4}, so it cannot cancel out the first term and we see that $\dot{V} > 0$. This shows that V and $\dot{V}$ are both positive definite on R, so $(0,0)$ is an unstable equilibrium.



van der Pol instability

$$\dot{x} = y,\ \dot{y} = 2(1-x^2)y - x \qquad V = x^2 - xy + y^2 \qquad \dot{V} = x^2 - 2xy + 3y^2 + 2x^3y - 4x^2y^2.$$

For a more careful argument, we can check by expanding everything out that

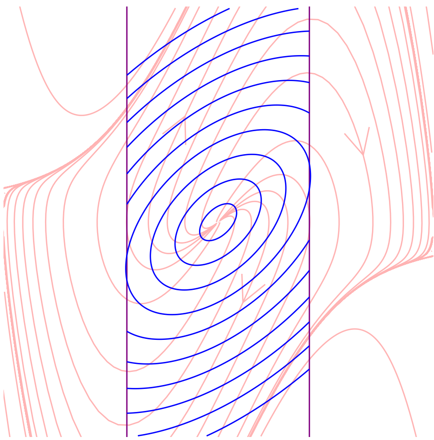
$$\dot{V} = (3-4x^2)^{-1}\left(((3-4x^2)y + (x^3-x))^2 + x^2(3-(1+x^2)^2)\right).$$

Suppose that $|x| < \sqrt{\sqrt{3}-1} \simeq 0.855$.

Then $3 - 4x^2 > 3 - 4(\sqrt{3}-1) \simeq 0.072 > 0$,
and $1 + x^2 < \sqrt{3}$ so $3 - (1+x^2)^2 > 0$.
Moreover, squares are always nonnegative, so $((3-4x^2)y + (x - x^3))^2 \geq 0$.

Putting this together, we see that $\dot{V} \geq 0$. After examining the above equation more closely, we also see that $\dot{V}$ can only be zero if $(x,y) = (0,0)$, so $\dot{V}$ (as well as V) is positive definite on the region $R' = \{(x,y) \mid |x| < \sqrt{\sqrt{3}-1}\}$. We again see that the origin is an unstable equilibrium.

van der Pol instability — phase portrait



Second order linear differential equations

Second order linear equations

We will consider differential equations of the form

$$Ay'' + By' + Cy = 0,$$

where A, B, C and y are functions of x, and y' means dy/dx. Examples:

- If A, B and C are constant then the solutions are like $y = Pe^{\lambda x} + Qe^{\mu x}$ or $y = e^{\lambda x}(P\cos(\omega x) + Q\sin(\omega x))$.
- Bessel's equation $x^2y'' + xy' + (x^2 - n^2)y = 0$ (where n is constant). (This is relevant for many problems with circular symmetry, such as vibrations of a drum, or signals in an optic fibre.)
- The Legendre equation $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$.
- The Airy equation $y'' - (x - \lambda)y = 0$, which is related to the optics of rainbows.
- The Hermite equation $y'' - 2xy' + 2ny = 0$, which is related to the normal distribution in statistics.

We will use

- Power series methods.
- Sturm-Liouville theory: eigenvalues of self-adjoint differential operators.

Reminder of the constant coefficient case

Consider the equation $y'' + Py' + Qy = 0$, where P and Q are *constant*. We look for solutions of the form $y = e^{\lambda x}$. Then $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$ so

$$y'' + Py' + Qy = \lambda^2 e^{\lambda x} + P\lambda e^{\lambda x} + Qe^{\lambda x} = p(\lambda)e^{\lambda x},$$

where $p(t) = t^2 + Pt + Q$ is the *auxiliary polynomial*.

(a) If $P^2 - 4Q > 0$ then there are two distinct real roots, say λ and μ. We then have solutions $y = Ae^{\lambda x} + Be^{\mu x}$, where A and B can be any constants.

(b) If $P^2 - 4Q < 0$ then there are two distinct complex roots, say $\lambda + i\omega$ and $\lambda - i\omega$. This gives solutions $u = e^{(\lambda+i\omega)x}$ and $v = e^{(\lambda-i\omega)x}$. However, it is more convenient to use the combinations

$$(u+v)/2 = e^{\lambda x}\cos(\omega x) \qquad (u-v)/(2i) = e^{\lambda x}\sin(\omega x).$$

Any solution can be written as $y = e^{\lambda x}(A\cos(\omega x) + B\sin(\omega x))$ for some constants A and B.

Reminder of the constant coefficient case

Consider the equation $y'' + Py' + Qy = 0$, where P and Q are *constant*. We look for solutions of the form $y = e^{\lambda x}$. Then $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$ so

$$y'' + Py' + Qy = \lambda^2 e^{\lambda x} + P\lambda e^{\lambda x} + Qe^{\lambda x} = p(\lambda)e^{\lambda x},$$

where $p(t) = t^2 + Pt + Q$ is the *auxiliary polynomial*.

(c) If $P^2 - 4Q = 0$ then there is only one root $\lambda = -P/2$, and the differential equation is $y'' - 2\lambda y' + \lambda^2 y = 0$. To understand this equation, put $y = e^{\lambda x}z$. We then have

$$y' = \lambda e^{\lambda x}z + e^{\lambda x}z' = e^{\lambda x}(z' + \lambda z)$$
$$y'' = \lambda e^{\lambda x}(z' + \lambda z) + e^{\lambda x}(z'' + \lambda z') = e^{\lambda x}(z'' + 2\lambda z' + \lambda^2 z),$$

so

$$y'' + Py' + Qy = e^{\lambda x}(z'' + (2\lambda + P)z' + (\lambda^2 + P\lambda + Q)z).$$

However, $2\lambda + P = 0$ and $\lambda^2 + P\lambda + Q = 0$ so $y'' + Py' + Qy = e^{\lambda x}z''$, so the differential equation is equivalent to $z'' = 0$. This means that $z = Ax + B$ and $y = e^{\lambda x}(Ax + B)$ for some constants A and B.

Power series for constant coefficient case

Consider again $y'' + Py' + Qy = 0$, and suppose that the auxiliary polynomial $p(t) = t^2 + Pt + Q$ has two distinct roots λ and μ.
Any solution has the form $y = Ae^{\lambda x} + Be^{\mu x}$.
Using $e^x = \sum_k x^k/k!$, this becomes

$$y = \sum_k \left(A\frac{\lambda^k x^k}{k!} + B\frac{\mu^k x^k}{k!}\right) = \sum_k \frac{A\lambda^k + B\mu^k}{k!}x^k.$$

We can also find similar formulae for the case when $p(t)$ has two complex roots, or one repeated root. Later we will explain how to find power series solutions even when P is not constant.

Questions: standard power series

$$1 + x + x^2 + x^3 + \cdots =$$

$$1 + 10x + \frac{100x^2}{2} + \frac{1000x^3}{6} + \cdots =$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots =$$

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots =$$

$$1 - \frac{\pi^2 x^2}{2!} + \frac{\pi^4 x^4}{4!} - \frac{\pi^6 x^6}{6!} + \cdots =$$

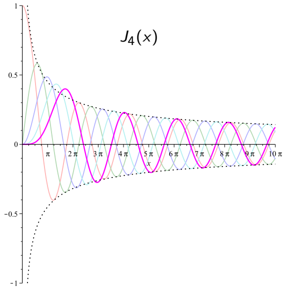
$$1 - \pi x + \pi^2 x^2 - \pi^3 x^3 + \pi^4 x^4 + \cdots =$$

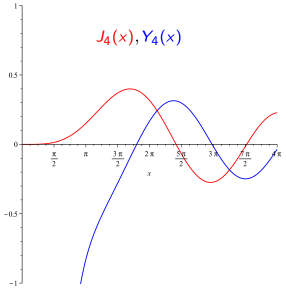
$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots =$$

$$\frac{1}{10} + \frac{x}{100} + \frac{x^2}{1000} + \frac{x^3}{10000} + \cdots =$$

Bessel's equation

Consider $x^2 y'' + xy' + (x^2 - n^2)y = 0$ where n is a natural number.
We will see that there are two basic solutions, $J_n(x)$ and $Y_n(x)$.





$$J_4(x) \simeq \tfrac{1}{384}x^4 - \tfrac{1}{7680}x^6 + \tfrac{1}{368640}x^8 - \cdots$$
$$\simeq \sqrt{\frac{2}{\pi x}} \cos\left(x - 9\frac{\pi}{4}\right)$$

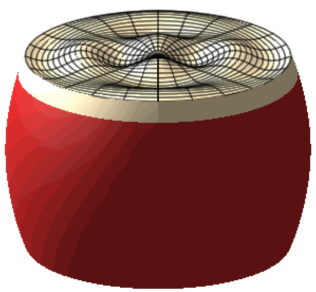
Every solution has the form $y = AJ_n(x) + BY_n(x)$ for constants A and B.

Drum

Modes of vibration of a drum of radius 1 are given by

$$z = A\sin(t)\cos(n\theta)J_n(a_{nk}r),$$

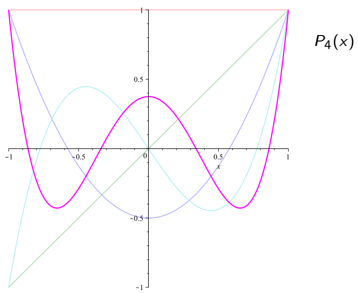
where (r, θ) are polar coordinates and a_{nk} is the k'th root of $J_n(x)$.



The movie shows the case where $n = 2$ and $k = 3$, so
$z = A\sin(t)\cos(3\theta)J_3(a_{32}r)$.

Legendre's equation

Consider $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$ where n is a natural number.
We will see that there are two basic solutions, $P_n(x)$ and $Q_n(x)$.

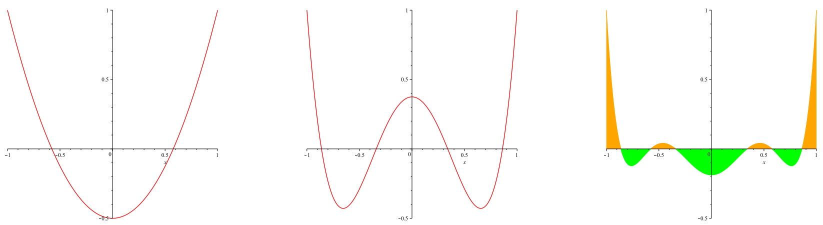


$$P_4(x) = \tfrac{35}{8}x^4 - \tfrac{15}{4}x^2 + \tfrac{3}{8} \qquad Q_4(x) = \left(\tfrac{35}{16}x^4 - \tfrac{15}{8}x^2 + \tfrac{3}{16}\right)\ln\left(\frac{x+1}{x-1}\right) - \tfrac{35}{8}x^3 + \tfrac{55}{24}x$$

Legendre polynomials — orthogonality

Whenever n and m are different we have $\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0$.

Example:



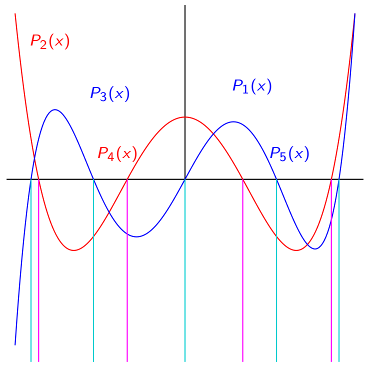
$P_2(x)$ $P_4(x)$ $P_2(x)P_4(x)$

This is similar to the fact that $\int_0^{2\pi} \sin(nx)\sin(mx)\,dx = 0$ for $n \neq m$, which is the basis of Fourier theory.

We will show that solutions of many other linear second order differential equations have similar orthogonality properties.

Alternating roots

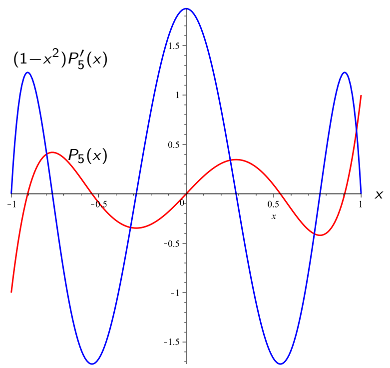
The roots of $P_k(x)$ alternate with the roots of $P_{k+1}(x)$.

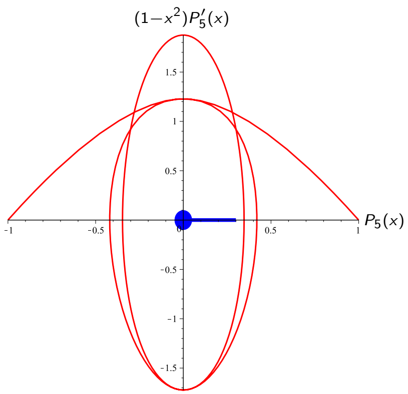


This is not just a special property of Legendre functions; it is a fairly general feature of linear second order differential equations.

Rotation

As x runs from -1 to 1, the point $(P_5(x), (1-x^2)P_5'(x))$ rotates around the origin through an angle of 5π.

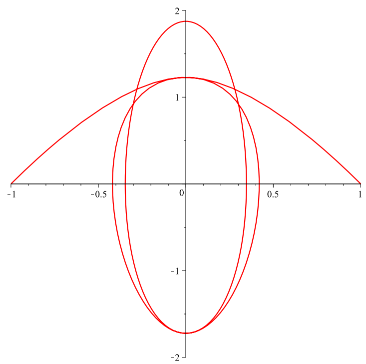




$P_5(x) = \frac{1}{8}\left(15x - 70x^3 + 63x^5\right)$ $\qquad (1-x^2)P_5'(x) = \frac{1}{8}\left(15 - 225x^2 + 525x^4 - 315x^6\right)$

Rotation around the origin

As x runs from -1 to 1, the point $(P_k(x), (1-x^2)P_k'(x))$ rotates around the origin through an angle of $k\pi$.

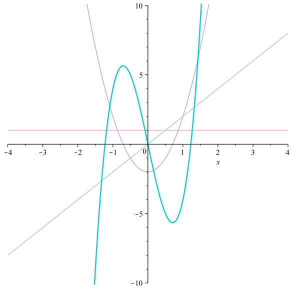


$(P_5(x), (1-x^2)P_5'(x))$ rotates through an angle of 5π.

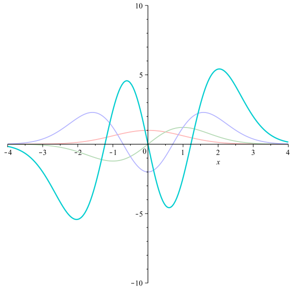
This is not just a special property of Legendre functions; it is a fairly general feature of linear second order differential equations.

The Hermite equation

Consider $y'' - 2xy' + 2ny = 0$ where n is a natural number.
We will see that there is a polynomial solution $H_n(x)$.
The function $e^{-x^2/2} H_n(x)$ is also important.



$H_3(x)$



$e^{-x^2/2} H_3(x)$

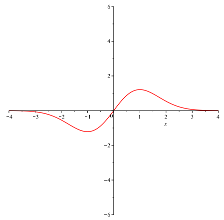
$H_0(x) = 1$ $\qquad$ $H_1(x) = 2x$

$H_2(x) = 4x^2 - 2$ $\qquad$ $H_3(x) = 8x^3 - 12x$

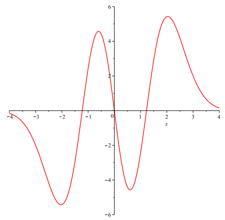
Hermite polynomials — orthogonality

Whenever n and m are different we have $\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 0$.

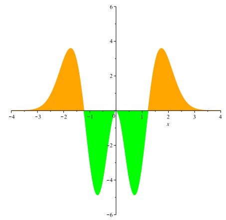
Example:



$e^{-x^2/2} H_1(x)$



$e^{-x^2/2} H_3(x)$



$e^{-x^2} H_1(x) H_3(x)$

This is similar to the fact that $\int_0^{2\pi} \sin(nx)\sin(mx)\, dx = 0$ for $n \neq m$, which is the basis of Fourier theory.

We will show that solutions of many other linear second order differential equations have similar orthogonality properties.

Power series solutions — first few terms

Consider an equation $y'' + Py' + Qy = 0$.
Suppose that y, P and Q are given by convergent power series:

$$y = a_0 + a_1 x + a_2 x^2 + \cdots \qquad P = p_0 + p_1 x + p_2 x^2 + \cdots \qquad Q = q_0 + q_1 x + q_2 x^2 + \cdots$$

Then

$$y'' = 2a_2 + 6a_3 x + \cdots$$
$$Py' = (p_0 + p_1 x + p_2 x^2)(a_1 + 2a_2 x) + \cdots = p_0 a_1 + (p_1 a_1 + 2p_0 a_2)x + \cdots$$
$$Qy = (q_0 + q_1 x + q_2 x^2)(a_0 + a_1 x + a_2 x^2) + \cdots = q_0 a_0 + (q_0 a_1 + q_1 a_0)x + \cdots$$

$$y'' + Py' + Qy = (2a_2 + p_0 a_1 + q_0 a_0) + (6a_3 + p_1 a_1 + 2p_0 a_2 + q_0 a_1 + q_1 a_0)x + \cdots.$$

Thus, for the differential equation to hold we must have

$$2a_2 + p_0 a_1 + q_0 a_0 = 0$$
$$6a_3 + p_1 a_1 + 2p_0 a_2 + q_0 a_1 + q_1 a_0 = 0 \qquad \text{or}$$

$$a_2 = -\tfrac{1}{2}(p_0 a_1 + q_0 a_0)$$
$$a_3 = -\tfrac{1}{6}(p_1 a_1 + 2p_0 a_2 + q_0 a_1 + q_1 a_0).$$

Here a_0 and a_1 are arbitrary, and they determine a_2, a_3 and so on.

Multiplication of power series

$$Q = q_0 + q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4 + \cdots$$
$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$Qy = \begin{array}{lllll} q_0 a_0 & +q_0 a_1 x & +q_0 a_2 x^2 & +q_0 a_3 x^3 & +q_0 a_4 x^4 \\ +q_1 a_0 x & +q_1 a_1 x^2 & +q_1 a_2 x^3 & +q_1 a_3 x^4 & +q_1 a_4 x^5 \\ +q_2 a_0 x^2 & +q_2 a_1 x^3 & +q_2 a_2 x^4 & +q_2 a_3 x^5 & +q_2 a_4 x^6 \\ +q_3 a_0 x^3 & +q_3 a_1 x^4 & +q_3 a_2 x^5 & +q_3 a_3 x^6 & +q_3 a_4 x^7 \\ +q_4 a_0 x^4 & +q_4 a_1 x^5 & +q_4 a_2 x^6 & +q_4 a_3 x^7 & +q_4 a_4 x^8 \end{array} + \cdots$$

$$= q_0 a_0 + (q_0 a_1 + q_1 a_0)x + (q_0 a_2 + q_1 a_1 + q_2 a_0)x^2 + (q_0 a_3 + q_1 a_2 + q_2 a_1 + q_3 a_0)x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} q_i a_{n-i} \right) x^n.$$

Power series solutions — all terms

Consider an equation $y'' + Py' + Qy = 0$.
Suppose that y, P and Q are given by convergent power series:

$$y = \sum_{k=0}^{\infty} a_k x^k \qquad P = \sum_{k=0}^{\infty} p_k x^k \qquad Q = \sum_{k=0}^{\infty} q_k x^k.$$

Then we have convergent power series for all terms in the differential equation:

$$y' = \sum_{k=0}^{\infty} k\, a_k\, x^{k-1} = \sum_{j=0}^{\infty} (j+1)\, a_{j+1}\, x^j \qquad \text{(reindexing with } j = k-1\text{)}$$

$$y'' = \sum_{k=0}^{\infty} (k-1)k\, a_k\, x^{k-2} = \sum_{j=0}^{\infty} (j+1)(j+2)\, a_{j+2}\, x^j$$

$$Qy = \sum_{n,m=0}^{\infty} q_n a_m x^{n+m} = \sum_{j=0}^{\infty} \left(\sum_{n=0}^{j} q_n a_{j-n} \right) x^j$$

$$Py' = \sum_{n,m=0}^{\infty} p_n (m+1) a_{m+1} x^{n+m} = \sum_{j=0}^{\infty} \left(\sum_{n=0}^{j} (j-n+1) p_n a_{j-n+1} \right) x^j.$$

Power series solutions — all terms

$$y'' = \textstyle\sum_{j=0}^{\infty} (j+1)(j+2)\, a_{j+2}\, x^j$$
$$Py' = \textstyle\sum_{j=0}^{\infty} \left(\sum_{n=0}^{j} (j-n+1) p_n a_{j-n+1} \right) x^j$$
$$Qy = \textstyle\sum_{j=0}^{\infty} \left(\sum_{n=0}^{j} q_n a_{j-n} \right) x^j.$$

Consider the coefficient of x^j in $y'' + Py' + Qy = 0$:

$$(j+1)(j+2) a_{j+2} + \left(\sum_{n=0}^{j} (j-n+1) p_n a_{j-n+1} \right) + \left(\sum_{n=0}^{j} q_n a_{j-n} \right) = 0$$

so

$$a_{j+2} = \frac{-1}{(j+1)(j+2)} \left(\sum_{n=0}^{j} (j-n+1) p_n a_{j-n+1} + \sum_{n=0}^{j} q_n a_{j-n} \right)$$

Note that only $a_0, \dots, a_{j+1}$ appear on the right hand side.
Thus a_0 and a_1 are arbitrary,
but $a_2, a_3, a_4, \dots$ are determined inductively by the above formula.

A note on indexing

Suppose we have a series like

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots.$$

We usually define $a_k = 0$ for $k < 0$, so

$$a_{-1} = a_{-2} = a_{-3} = \cdots = 0.$$

With this convention, we have

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=-\infty}^{\infty} a_k x^k = \cdots + a_{-2}x^{-2} + a_{-1}x^{-1} + a_0 + a_1 x + a_2 x^2 + \cdots.$$

(When $k < 0$ the terms $a_k x^k$ are zero, so it does not matter whether we include them or not.)

This will simplify various formulae, because we do not need to remember where the series starts.

Power series solutions — simple example

Consider the equation $y'' + y = 0$ with $y = 1$, $y' = 0$ at $x = 0$.

$$y = \sum_i a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$y' = \sum_i i\, a_i x^{i-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$y'' = \sum_i (i-1)i\, a_i x^{i-2} = 2a_2 + 6a_3 x + 12 a_4 x^2 + \cdots$$

$$= \sum_j (j+1)(j+2) a_{j+2} x^j \qquad \text{(reindexing with } j = i-2\text{)}$$

$$y'' + y = \sum_j (a_j + (j+1)(j+2) a_{j+2}) x^j.$$

At $x = 0$ we have $y = a_0$ and $y' = a_1$, so $a_0 = 1$ and $a_1 = 0$.

For the differential equation $y'' + y = 0$ to hold, we must have $a_j + (j+1)(j+2)a_{j+2} = 0$, so

$$a_{j+2} = \frac{-1}{(j+1)(j+2)} a_j.$$

Power series solutions — simple example

$$y'' + y = 0, \qquad y = \sum_i a_i x^i, \qquad a_0 = 1,\ a_1 = 0, \qquad a_{j+2} = \frac{-a_j}{(j+1)(j+2)}.$$

$$a_0 = 1 \qquad\qquad a_1 = 0$$

$$a_2 = \frac{-a_0}{1\times 2} = -\frac{1}{2} = \frac{-1}{2!} \qquad\qquad a_3 = \frac{-a_1}{2\times 3} = 0$$

$$a_4 = \frac{-a_2}{3\times 4} = +\frac{1}{24} = \frac{+1}{4!} \qquad\qquad a_5 = \frac{-a_3}{4\times 5} = 0$$

$$a_6 = \frac{-a_4}{5\times 6} = -\frac{1}{720} = \frac{-1}{6!} \qquad\qquad a_7 = \frac{-a_5}{6\times 7} = 0$$

$$a_{2p} = \frac{(-1)^p}{(2p)!} \qquad\qquad a_{2p+1} = 0$$

So

$$y = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots = \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p)!} x^{2p} = \cos(x).$$

Power series solution — another example

Consider $(x-1)y'' + 2y' = 0$, with $y = y' = 1$ when $x = 0$.
Rewrite as $y'' - 2y'/(1-x) = 0$.

$$y = \sum_i a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y' = \sum_i i\, a_i x^{i-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_j (j+1) a_{j+1} x^j$$

$$y'' = \sum_i i(i-1)\, a_i x^{i-2} = 2a_2 + 6a_3 x + \cdots = \sum_n (n+1)(n+2) a_{n+2} x^n$$

$$1/(1-x) = \sum_i x^i \qquad \text{(geometric progression)}$$

$$2y'/(1-x) = 2\sum_i x^i \sum_j (j+1)\, a_{j+1} x^j = 2\sum_{i,j} (j+1)\, a_{j+1} x^{i+j}$$

$$= \sum_n x^n \sum_{j=0}^{n} 2(j+1)\, a_{j+1}.$$

For the equation $y'' - y'/(1-x)$ to hold, we must have

$$(n+1)(n+2)a_{n+2} = \sum_{j=0}^{n} 2(j+1)\, a_{j+1}.$$

Power series solution — another example

Consider $(x-1)y'' + 2y' = 0$, with $y = y' = 1$ when $x = 0$.
$y = \sum_i a_i x^i$ with $(n+1)(n+2)a_{n+2} = \sum_{j=0}^{n} 2(j+1)\, a_{j+1}$.

At $x = 0$ we have $y = a_0$ and $y' = a_1$, but also $y = y' = 1$, so $a_0 = a_1 = 1$.
Take $n = 0$ in $(n+1)(n+2)a_{n+2} = \sum_{j=0}^{n} 2(j+1)\, a_{j+1}$ to get

$$2a_2 = \sum_{j=0}^{0} 2(j+1)a_{j+1} = 2a_1 = 2 \qquad a_2 = 1$$

$$6a_3 = \sum_{j=0}^{1} 2(j+1)a_{j+1} = 2a_1 + 4a_2 = 6 \qquad a_3 = 1$$

$$12a_4 = \sum_{j=0}^{2} 2(j+1)a_{j+1} = 2a_1 + 4a_2 + 6a_3 = 12 \qquad a_4 = 1$$

$$20a_5 = \sum_{j=0}^{3} 2(j+1)a_{j+1} = 2a_1 + 4a_2 + 6a_3 + 8a_4 = 20 \qquad a_5 = 1.$$

It looks like $a_k = 1$ for all k.

Power series solution — another example

$(x-1)y'' + 2y' = 0$, $y = \sum_i a_i x^i$ with $a_0 = a_1 = 1$ and
$(n+1)(n+2)a_{n+2} = \sum_{j=0}^{n} 2(j+1)\, a_{j+1}$.

Claim: $a_k = 1$ for all k.

Proof by induction : We are given that $a_0 = a_1 = 1$.
Suppose we already know that $a_0 = \cdots = a_{n+1} = 1$. Then

$$(n+1)(n+2)a_{n+2} = \sum_{j=0}^{n} 2(j+1)a_{j+1} = \sum_{j=0}^{n} 2(j+1).$$

This is an arithmetic progression .
There are $n+1$ terms, from 2 to $2(n+1)$.
The average is $\frac{1}{2}(2 + 2(n+1)) = n+2$, so the total is $(n+1)(n+2)$.
We therefore have $(n+1)(n+2)a_{n+2} = (n+1)(n+2)$, so $a_{n+2} = 1$. □

This gives $y = \sum_k a_k x^k = \sum_k x^k = 1/(1-x)$.

Check: $y' = 1/(1-x)^2$, $y'' = 2/(1-x)^3$,

$$(x-1)y'' + 2y' = (x-1)\frac{2}{(1-x)^3} + \frac{2}{(1-x)^2} = \frac{-2}{(1-x)^2} + \frac{2}{(1-x)^2} = 0.$$

Radius of convergence

Consider a series $f(x) = \sum_{k=0}^{\infty} c_k x^k$.

- There is a number R with $0 \leq R \leq \infty$, called the *radius of convergence*.
- If $|x| < R$ then the series $\sum_{k=0}^{\infty} c_k x^k$ converges.
- If $|x| > R$ then the series $\sum_{k=0}^{\infty} c_k x^k$ does not converge.
- If $|x| = R$ then the series $\sum_{k=0}^{\infty} c_k x^k$ may or may not converge.
- The derivative is $f'(x) = \sum_{k=0}^{\infty} (k+1)\, c_{k+1} x^k$, and this has the same radius of convergence as $f(x)$.

Most common ways to find R:

(a) If the sequence $|a_k|/|a_{k+1}|$ has a limit, then that limit is R. (only meaningful if $a_k \neq 0$ for all $k > k_0$).

(b) If $a_{2k+1} = 0$ and $|a_{2k}|/|a_{2k+2}|$ has a limit, then that limit is R^2.

(c) If $a_{2k} = 0$ and $|a_{2k+1}|/|a_{2k+3}|$ has a limit, then that limit is R^2.

Examples:

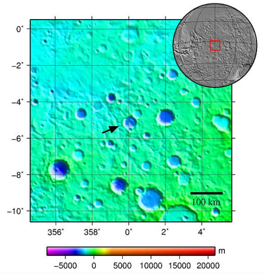
(a) $e^x = \sum_k \frac{x^k}{k!}$, $a_k = \frac{1}{k!}$, $\frac{|a_k|}{|a_{k+1}|} = \frac{(k+1)!}{k!} = k+1 \to \infty$, so $R = \infty$.

(b) $\frac{1}{1+2x^2} = \sum_k (-2x^2)^k$, $a_{2k} = (-2)^k$, $a_{2k+1} = 0$, $\frac{|a_{2k}|}{|a_{2k+2}|} = \frac{1}{2}$, so $R = \frac{1}{\sqrt{2}}$.

(c) $\ln\left(\frac{1+x}{1-x}\right) = \sum_k \frac{2}{2k+1} x^{2k+1}$, $a_{2k} = 0$, $a_{2k+1} = \frac{2}{2k+1}$,
$\frac{|a_{2k+1}|}{|a_{2k+3}|} = \frac{2k+3}{2k+1} = 1 + \frac{2}{2k+1} \to 1$, so $R = 1$.

Airy's equation

Airy's equation is $y'' - xy = 0$.



This is George Biddell Airy. He was the British Astronomer Royal from 1835 to 1881. Among his many achievements, he measured the mass of Jupiter, and found small corrections to the orbit of Venus, and to the theory of the rainbow. He established the Prime Meridian (zero degrees of latitude) through the Royal Greenwich Observatory in London. This crater on Mars is named after him.

Airy's equation

Airy's equation is $y'' - xy = 0$.

$$y = \sum_i a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$xy = \sum_i a_i x^{i+1} = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots = \sum_n a_{n-1} x^n$$

$$y'' = \sum_i i(i-1)\, a_i x^{i-2} = 2a_2 + 6a_3 x + \cdots = \sum_n (n+1)(n+2) a_{n+2} x^n.$$

For the equation $y'' - xy = 0$ to hold, we must have $(n+1)(n+2)a_{n+2} = a_{n-1}$, or equivalently $(m+2)(m+3)a_{m+3} = a_m$, or $a_{m+3} = a_m/((m+2)(m+3))$. (Special case: the constant term in $y'' - xy = 0$ gives $a_2 = 0$.)

$$a_3 = \frac{a_0}{2.3} \qquad a_4 = \frac{a_1}{3.4} \qquad a_5 = \frac{a_2}{4.5} = 0$$

$$a_6 = \frac{a_3}{5.6} = \frac{a_0}{2.3.5.6} \qquad a_7 = \frac{a_4}{6.7} = \frac{a_1}{3.4.6.7} \qquad a_8 = \frac{a_5}{7.8} = 0$$

$$a_9 = \frac{a_6}{8.9} = \frac{a_0}{2.3.5.6.8.9} \qquad a_{10} = \frac{a_7}{9.10} = \frac{a_1}{3.4.6.7.9.10} \qquad a_{11} = \frac{a_8}{10.11} = 0$$

Airy's equation

Airy's equation is $y'' - xy = 0$, where $y = \sum_k a_k x^k$.

$$a_3 = \frac{a_0}{2.3} \qquad a_4 = \frac{a_1}{1.3.4} \qquad a_5 = 0$$

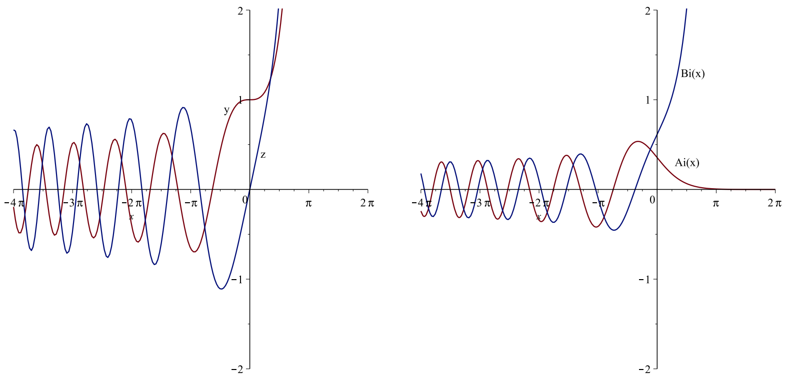
$$a_6 = \frac{a_0}{2.3.5.6} \qquad a_7 = \frac{a_1}{1.3.4.6.7} \qquad a_8 = 0$$

$$a_9 = \frac{a_0}{2.3.5.6.8.9} \qquad a_{10} = \frac{a_1}{1.3.4.6.7.9.10} \qquad a_{11} = 0$$

$$\frac{a_0}{9!} = \frac{a_0}{1.2.3.4.5.6.7.8.9} \qquad \frac{a_1}{10!} = \frac{a_0}{1.2.3.4.5.6.7.8.9.10}$$

- a_{3k} is like $a_0/(3k)!$, except that terms like $3i+1$ are missing from the factorial.
- a_{3k+1} is like $a_1/(3k+1)!$, except that terms like $3i+2$ are missing.
- The radius of convergence is infinite.

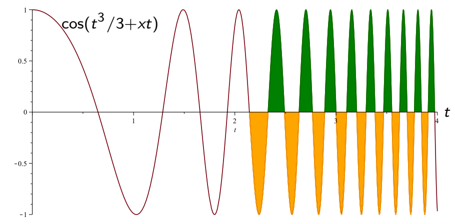
Four solutions for Airy's equation

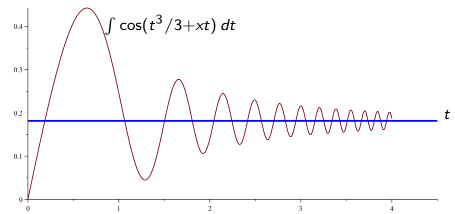


- $y = 1 + O(x^2)$ and $z = x + O(x^2)$
- It is more traditional to use $Ai(x)$ and $Bi(x)$.
- These are $Ai(x) = \alpha y + \beta z$ and $Bi(x) = \gamma y + \delta z$ for some (complicated) constants $\alpha, \beta, \gamma, \delta$.

The Airy integral

Another method (relevant for the rainbow): consider the function $A(x) = \int_{t=0}^{\infty} \cos(t^3/3 + xt)\, dt$ (which only converges because of cancellation.)





The pictures show $x = 2$; we see that $A(2) \simeq 0.1818604914$.

The Airy integral

Consider the function

$$A(x) = \int_{t=0}^{\infty} \cos(t^3/3 + xt)\, dt$$

$$A'(x) = \int_{t=0}^{\infty} \frac{d}{dx} \cos(t^3/3 + xt)\, dt = -\int_{t=0}^{\infty} t \sin(t^3/3 + xt)\, dt$$

$$A''(x) = -\int_{t=0}^{\infty} \frac{d}{dx} \left(t \sin(t^3/3 + xt) \right) dt = -\int_{t=0}^{\infty} t^2 \cos(t^3/3 + xt)\, dt$$

$$A''(x) - x\,A(x) = -\int_{t=0}^{\infty} (t^2 + x) \cos(t^3/3 + xt)\, dt.$$

We can integrate this by substituting $u = t^3/3 + xt$, so $(t^2 + x)\, dt = du$. Moreover, we have $u = 0$ when $t = 0$, and $u = \infty$ when $t = \infty$. This gives

$$A''(x) - x\,A(x) = \int_{u=0}^{\infty} \cos(u)\, du.$$

This integral does not really converge, but it is natural to think that the value should be zero, because the graph of $\cos(u)$ is symmetrical about the u-axis. However, some difficult arguments are needed to justify this. Anyway, the conclusion is that $A''(x) - x\,A(x) = 0$, so $A(x)$ is a solution for the Airy equation.

Singular points

Consider the equation $4x^2 y'' + y = 0$, or $y'' + \frac{1}{4}x^{-2} y = 0$.

The function $\frac{1}{4}x^{-2}$ is not a power series, so our previous method does not work.

Maybe we need to let y have negative powers of x as well? Still does not work.

If the first term in y is ax^n, then
the first term in y'' is $n(n-1)ax^{n-2}$, and the first term in $\frac{1}{4}x^{-2}y$ is $\frac{1}{4}ax^{n-2}$.
If $y'' + \frac{1}{4}x^{-2}y = 0$ then these must cancel, so $n(n-1) + \frac{1}{4} = 0$.
There are no integers n with this property.
However, there is a fractional solution, namely $n = 1/2$.

If $y = x^{1/2}$ then $y' = \frac{1}{2}x^{-1/2}$ and $y'' = -\frac{1}{4}x^{-3/2}$ so $4x^2 y'' + y = 0$.

If $y = \ln(x) x^{1/2}$ then

$$y' = x^{-1}x^{1/2} + \ln(x).\tfrac{1}{2}x^{-1/2} = (1 + \tfrac{1}{2}\ln(x))x^{-1/2}$$

$$y'' = \tfrac{1}{2}x^{-1}x^{-1/2} + (1 + \tfrac{1}{2}\ln(x)).(-\tfrac{1}{2})x^{-3/2} = -\tfrac{1}{4}\ln(x)x^{-3/2}$$

so again $4x^2 y'' + y = 0$.

We will see that many equations of the form $y'' + Py' + Qy = 0$ have similar properties.

Regular singular points

Consider an equation $y'' + Py' + Qy = 0$.

- If $P = \sum_{k=0}^{\infty} p_k x^k$ and $Q = \sum_{k=0}^{\infty} q_k x^k$ (ordinary power series), we say that $x = 0$ is an *ordinary point*. We studied this case already.
- Suppose instead that

$$P = p_0 x^{-1} + p_1 + p_2 x + p_3 x^2 + \cdots = x^{-1} \sum_{k=0}^{\infty} p_k x^k$$

$$Q = q_0 x^{-2} + q_1 x^{-1} + q_2 + q_3 x + q_4 x^2 + \cdots = x^{-2} \sum_{k=0}^{\infty} q_k x^k,$$

  where p_0, q_0 and q_1 are not all zero. Then we say that $x = 0$ is a *regular singular point*. In this case, the *indicial polynomial* is defined to be

$$\chi(\alpha) = \alpha(\alpha - 1) + p_0 \alpha + q_0.$$

- In any other case, we say that $x = 0$ is an *irregular singular point*.

Regular singular point — simplest case

Consider an equation $y'' + Py' + Qy = 0$, where $P = p_0 x^{-1}$ and $Q = q_0 x^{-2}$ for some constants p_0 and q_0. We look for solutions of the form $y = x^{\alpha}$. We have

$$\begin{aligned}
y'' &= \alpha(\alpha - 1)x^{\alpha-2} \\
Py' &= p_0 x^{-1}.\alpha x^{\alpha-1} = p_0 \alpha x^{\alpha-2} \\
Qy &= q_0 x^{-2}.x^{\alpha} = q_0 x^{\alpha-2} \\
y'' + Py' + Qy &= (\alpha(\alpha - 1) + p_0\alpha + q_0)x^{\alpha-2} = \chi(\alpha)x^{\alpha-2}.
\end{aligned}$$

Thus $y = x^{\alpha}$ is a solution if and only if $\chi(\alpha) = 0$; in other words, α should be a root of the indicial polynomial.

Series solution at a regular singular point

Consider an equation $y'' + Py' + Qy$ with a regular singular point at $x = 0$. Let $\chi(t)$ be the indicial polynomial, with roots α and β where $\mathrm{Re}(\alpha) \geq \mathrm{Re}(\beta)$.

Theorem: Suppose that $\alpha - \beta$ is not an integer. Then there is a unique solution of the form $y = \sum_{k=0}^{\infty} a_k x^{\alpha+k}$ with $a_0 = 1$, and there is a unique solution of the form $z = \sum_{k=0}^{\infty} b_k x^{\beta+k}$ with $b_0 = 1$.

Theorem: Suppose that $\alpha - \beta$ is a nonzero integer. Then there is a unique solution of the form $y = \sum_{k=0}^{\infty} a_k x^{\alpha+k}$ with $a_0 = 1$, and there is another solution of the form $z = cy\ln(x) + \sum_{k=0}^{\infty} b_k x^{\beta+k}$ with $b_0 = 1$.
(Sometimes $c = 0$, so the end result is the same as the first theorem.)

Theorem: Suppose that $\alpha = \beta$. Then there is a unique solution of the form $y = \sum_{k=0}^{\infty} a_k x^{\alpha+k}$ with $a_0 = 1$, and there is a unique solution of the form $z = y\ln(x) + \sum_{k=0}^{\infty} b_k x^{\alpha+k}$ with $b_0 = 1$.

In all three cases, every solution is $Ay + Bz$ for some constants A and B.

Series solution at a regular singular point

- If the indicial polynomial is $(t - 1/2)(t - 1/3)$ then there are solutions $y = x^{1/2}(1 + O(x))$ and $z = x^{1/3}(1 + O(x))$.
- If the indicial polynomial is $(t - 8)(t - 9)$ then there are solutions $y = x^9(1 + O(x))$ and $z = x^8(1 + O(x)) + cy\ln(x)$.
  Here c might be zero, in which case $z = x^8(1 + O(x))$.
- If the indicial polynomial is $(t - 1/2)^2$ then there are solutions $y = \sqrt{x}(1 + O(x))$ and $z = \sqrt{x}(1 + O(x)) + y\ln(x)$.

Series solution at a regular singular point

We have not stated these theorems very precisely.

- There are various problems about convergence of series and domains of solutions.
- It is hard to interpret $x^{\alpha+k}$ if $x < 0$ or if α is complex.

We will discuss some of these problems later.
We will prove the first theorem but not the other two.

Series solution at a regular singular point

Consider again an equation $y'' + Py' + Qy = 0$, where

$$y = \sum_{k=0}^{\infty} a_k x^{\alpha+k} \qquad P = \sum_{k=0}^{\infty} p_k x^{k-1} \qquad Q = \sum_{k=0}^{\infty} q_k x^{k-2}.$$

$$y'' = \sum_{n=0}^{\infty} (\alpha+n-1)(\alpha+n)\, a_n\, x^{\alpha+n-2}$$

$$Py' = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} p_j(\alpha+k)a_k x^{\alpha+j+k-2}$$

$$Qy = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} q_j a_k x^{\alpha+j+k-2}$$

$$(\alpha+n-1)(\alpha+n)a_n + \sum_{j=0}^{n} p_j(\alpha+n-j)a_{n-j} + \sum_{j=0}^{n} q_j a_{n-j} = 0$$

$$\begin{aligned}\chi(\alpha+n)a_n &= ((\alpha+n-1)(\alpha+n) + p_0(\alpha+n) + q_0)\, a_n \\ &= -\sum_{j=1}^{n} p_j(\alpha+n-j)a_{n-j} - \sum_{j=1}^{n} q_j a_{n-j}\end{aligned}$$

Series solution at a regular singular point

$$y = \sum_{k=0}^{\infty} a_k x^{\alpha+k} \qquad P = \sum_{k=0}^{\infty} p_k x^{k-1} \qquad Q = \sum_{k=0}^{\infty} q_k x^{k-2} \qquad \chi(t) = t(t-1)+p_0t+q_0.$$

$$\chi(\alpha+n)a_n = -\textstyle\sum_{j=1}^{n} p_j(\alpha+n-j)a_{n-j} - \sum_{j=1}^{n} q_j a_{n-j}.$$

- For $n = 0$ we have $\chi(\alpha)a_0 = 0$; so for a solution with $a_0 = 1$, we must have $\chi(\alpha) = 0$.
- If $\chi(\alpha+n) \neq 0$ for all $n > 0$ then we can define a_n recursively by

$$a_n = -\chi(n+\alpha)^{-1}\left(\sum_{j=1}^{n} p_j(\alpha+n-j)a_{n-j} + \sum_{j=1}^{n} q_j a_{n-j}\right),$$

  and this will give a solution $y = \sum_k a_k x^{\alpha+k}$.
- Usually $\chi(t)$ will have two different roots α and β such that $\alpha - \beta$ is not an integer, so $\chi(\alpha+n)$ and $\chi(\beta+n)$ are nonzero for all $n > 0$. We then have one solution $y = \sum_k a_k x^{\alpha+k}$ and another solution $y = \sum b_k x^{\beta+k}$.
- If $\chi(t)$ has a repeated root, or two roots separated by an integer, then the situation is more complicated.

Regular singular point example — non-integer gap

Consider the equation
$2x^2y'' + xy' - (x+1)y = 0$ or $y'' + \frac{1}{2}x^{-1}y' + (-\frac{1}{2}x^{-1} - \frac{1}{2}x^{-2})y = 0$.

There is a regular singular point at $x = 0$, with $p_0 = \frac{1}{2}$ and $q_0 = -\frac{1}{2}$,
so the indicial polynomial is $\alpha(\alpha-1) + \frac{1}{2}\alpha - \frac{1}{2} = 0$ or $\alpha^2 - \frac{1}{2}\alpha - \frac{1}{2} = 0$.

The roots are $-\frac{1}{2}$ and 1; the difference is not an integer.

Regular singular point example — non-integer gap

Consider the equation $2x^2y'' + xy' - (x+1)y = 0$; indicial roots $\alpha = -\frac{1}{2}, 1$.

There is a solution $y = \sum_{k=0}^{\infty} a_k x^{1+k}$ with $a_0 = 1$.

$$\begin{aligned} 2x^2y'' &= \sum_{k=0}^{\infty} 2(1+k)ka_k x^{1+k} \\ xy' &= \sum_{k=0}^{\infty} (1+k)a_k x^{1+k} \\ -(x+1)y &= \sum_{k=0}^{\infty} (-a_k - a_{k-1})x^{1+k}, \end{aligned}$$

so we need $2(1+k)ka_k + (1+k)a_k - a_k - a_{k-1} = 0$.
This gives $(2k^2+3k)a_k = a_{k-1}$, so $a_k = a_{k-1}/(2k^2+3k)$.
The first few terms are

$$a_0 = 1 \qquad a_1 = \frac{1}{5} \qquad a_2 = \frac{1}{5\times 14} = \frac{1}{70} \qquad a_3 = \frac{1}{70 \times 27} = \frac{1}{1890}$$

so

$$y = x + \frac{1}{5}x^2 + \frac{1}{70}x^3 + \frac{1}{1890}x^4 + \cdots$$

Regular singular point example — non-integer gap

Consider the equation $2x^2y'' + xy' - (x+1)y = 0$; indicial roots $\alpha = -\frac{1}{2}, 1$.

There is another solution $z = \sum_{k=0}^{\infty} b_k x^{-\frac{1}{2}+k}$ with $b_0 = 1$.

$$\begin{aligned} 2x^2z'' &= \sum_{k=0}^{\infty} 2(-\tfrac{1}{2}+k)(-\tfrac{3}{2}+k)b_k x^{-\frac{1}{2}+k} \\ xz' &= \sum_{k=0}^{\infty} (-\tfrac{1}{2}+k)b_k x^{-\frac{1}{2}+k} \\ -(x+1)z &= \sum_{k=0}^{\infty} (-b_k - b_{k-1})x^{-\frac{1}{2}+k}, \end{aligned}$$

so we need $2(-\frac{1}{2}+k)(-\frac{3}{2}+k)b_k + (-\frac{1}{2}+k)b_k - b_k - b_{k-1} = 0$.
This gives $(2k^2-3k)b_k = b_{k-1}$, so $b_k = b_{k-1}/(2k^2-3k)$.
The first few terms are

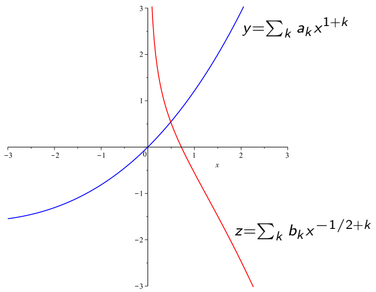
$$b_0 = 1 \qquad b_1 = -1 \qquad b_2 = -\frac{1}{2} \qquad b_3 = -\frac{1}{2\times 9} = -\frac{1}{18}$$

so

$$z = x^{-1/2} - x^{1/2} - \frac{1}{2}x^{3/2} - \frac{1}{18}x^{5/2} + \cdots$$

Regular singular point example — non-integer gap

Consider the equation $2x^2y'' + xy' - (x+1)y = 0$; indicial roots $\alpha = -\frac{1}{2}, 1$.



In fact, in this case it is possible to find exact solutions:

$$u = e^{\sqrt{2x}}(1 - 1/\sqrt{2x}) \qquad v = e^{-\sqrt{2x}}(1 + 1/\sqrt{2x})$$

$$y = \frac{3}{4}(u+v) = \frac{3}{2}\left(\cosh(\sqrt{2x}) - \frac{\sinh(\sqrt{2x})}{\sqrt{2x}}\right)$$

$$z = \frac{1}{\sqrt{2}}(v-u) = \sqrt{2}\left(\frac{\cosh(\sqrt{2x})}{\sqrt{2x}} - \sinh(\sqrt{2x})\right)$$

Regular singular point example — repeated root

Now consider instead the equation $y'' + (x^{-1}+1)y' + 2x^{-1}y = 0$.
This has a regular singular point at $x = 0$, with $p_0 = 1$ and $q_0 = 0$.
The indicial polynomial is

$$\alpha(\alpha-1) + p_0\alpha + q_0 = \alpha^2 - \alpha + \alpha = \alpha^2,$$

so there is a repeated root $\alpha = 0$.
Thus, there is a unique solution $y = \sum_{k=0}^{\infty} a_k x^k$ with $a_0 = 1$,
and there is a unique solution $z = \ln(x)y + \sum_{k=0}^{\infty} b_k x^k$ with $b_0 = 1$.

$$\begin{aligned} y'' &= \sum_k k(k-1)a_k x^{k-2} && = \sum_j (j+2)(j+1)a_{j+2}x^j \\ (x^{-1}+1)y' &= \sum_k ka_k x^{k-2} + \sum_k ka_k x^{k-1} && = \sum_j ((j+2)a_{j+2} + (j+1)a_{j+1})x^j \\ 2x^{-1}y &= \sum_k 2a_k x^{k-1} && = \sum_j 2a_{j+1}x^j \end{aligned}$$

We need $(j+2)(j+1)a_{j+2} + (j+2)a_{j+2} + (j+1)a_{j+1} + 2a_{j+1} = 0$,
which simplifies to $(j+2)^2a_{j+2} + (j+3)a_{j+1} = 0$.
Put $m = j+2$ to get $a_m = -(m+1)m^{-2}a_{m-1}$ for $m > 0$.

Regular singular point example — repeated root

$$y''+(x^{-1}+1)y'+2x^{-1}y=0 \qquad y=\sum_{k=0}^{\infty} a_k x^k \qquad a_k=-(k+1)k^{-2}a_{k-1}$$

$$a_0 = 1 \qquad a_1 = -\frac{2}{1^2} \qquad a_2 = +\frac{2.3}{1^2 2^2}$$

$$a_3 = -\frac{2.3.4}{1^2 2^2 3^2} = -\frac{4}{1.2.3} = -\frac{4}{3!} \qquad a_4 = +\frac{2.3.4.5}{1^2 2^2 3^2 4^2} = +\frac{5}{1.2.3.4} = +\frac{5}{4!} \qquad a_5 = -\frac{2.3.4.5.6}{1^2 2^2 3^2 4^2 5^2} = -\frac{6}{1.2.3.4.5} = -\frac{6}{5!}$$

In general, $a_k = (-1)^k \frac{k+1}{k!} = (-1)^k(\frac{1}{k!} + \frac{k}{k!}) = (-1)^k(\frac{1}{k!} + \frac{1}{(k-1)!})$. Thus

$$y = \sum_k \frac{(-x)^k}{k!} + \sum_k \frac{(-x)^k}{(k-1)!} = e^{-x} - x\sum_k \frac{(-x)^{k-1}}{(k-1)!} = e^{-x} - xe^{-x} = (1-x)e^{-x}.$$

Regular singular point example — repeated root

$$y''+(x^{-1}+1)y'+2x^{-1}y=0 \qquad \text{first solution: } y=(1-x)e^{-x}=\sum \frac{k+1}{k!}(-x)^k$$

Second solution: $z = y\ln(x) + u$, where $u = \sum_k b_k x^k$ with $b_0 = 1$.

$$\begin{aligned}
z &= y\ln(x) + u \\
z' &= y'\ln(x) + yx^{-1} + u' \\
z'' &= y''\ln(x) + y'x^{-1} + y'x^{-1} - yx^{-2} + u'' \\
&= y''\ln(x) + 2y'x^{-1} - yx^{-2} + u''
\end{aligned}$$

$$\begin{aligned}
z'' + (x^{-1}+1)z' + 2x^{-1}z = {} & (y'' + (x^{-1}+1)y' + 2x^{-1}y)\ln(x) + \\
& u'' + (x^{-1}+1)u' + 2x^{-1}u + \\
& 2y'x^{-1} - yx^{-2} + (x^{-1}+1)yx^{-1}
\end{aligned}$$

$$= u'' + (x^{-1}+1)u' + 2x^{-1}u + x^{-1}(2y'+y).$$

We need to find u such that this last expression is zero.

Regular singular point example — repeated root

$$y''+(x^{-1}+1)y'+2x^{-1}y=0 \qquad \text{solutions: } y=(1-x)e^{-x}, \quad z = y\ln(x)+u$$

$$u'' + (x^{-1}+1)u' + 2x^{-1}u + x^{-1}(2y'+y) = 0$$

$$u = \sum_{k=0}^{\infty} b_k x^k, \qquad b_0 = 1$$

$$u'' + (x^{-1}+1)u' + 2x^{-1}u = \sum_k ((k+2)^2 b_{k+2} + (k+3)b_{k+1})x^k$$

$$y' = -e^{-x} + (1-x)(-e^{-x}) = (x-2)e^{-x}$$

$$\begin{aligned}
x^{-1}(2y'+y) &= x^{-1}(2(x-2)+(1-x))e^{-x} = (1-3x^{-1})e^{-x} \\
&= \sum_k \frac{(-1)^k}{k!}x^k - \sum_k 3\frac{(-1)^k}{k!}x^{k-1} \\
\text{(warning: limits)} &= \sum_k (-1)^k \left(\frac{1}{k!} + 3\frac{1}{(k+1)!}\right)x^k = \sum_k (-1)^k \frac{k+4}{(k+1)!}x^k
\end{aligned}$$

We therefore need $(k+2)^2 b_{k+2} + (k+3)b_{k+1} + (-1)^k(k+4)/(k+1)! = 0$.

Regular singular point example — repeated root

$$y''+(x^{-1}+1)y'+2x^{-1}y=0 \qquad \text{solutions: } y=(1-x)e^{-x}, \quad z=y\ln(x)+u$$

$$u = \sum_k b_k x^k \qquad (k+2)^2 b_{k+2} + (k+3)b_{k+1} + (-1)^k(k+4)/(k+1)! = 0$$

The above equation for b_{k+1} and b_{k+2} is valid when $k \geq 0$. It gives

$$b_m = -\frac{1}{m^2}\left((m+1)b_{m-1} + (-1)^m \frac{m+2}{(m-1)!}\right) \qquad \text{for } m \geq 2.$$

If we look more carefully at the first few terms, we get

$$u'' + (x^{-1}+1)u' + 2x^{-1}u = (b_1+2)x^{-1} + \text{ terms in } x^0 \text{ and above}$$

$$x^{-1}(2y'+y) = -3x^{-1} + \text{ terms in } x^0 \text{ and above.}$$

As $u'' + (x^{-1}+1)u' + 2x^{-1}u + x^{-1}(2y'+y) = 0$, we must have $(b_1+2)+(-3)=0$, or in other words $b_1 = 1$. The recurrence relation now gives

$$b_0 = 1 \qquad b_1 = 1 \qquad b_2 = -\frac{7}{4}$$

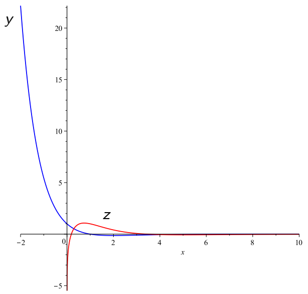
$$b_3 = \frac{19}{18} \qquad b_4 = -\frac{113}{288} \qquad b_5 = \frac{127}{1200}$$

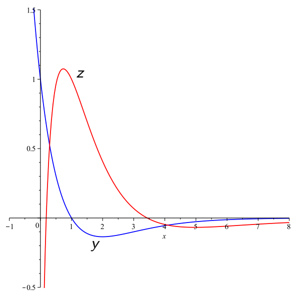
Regular singular point example — repeated root

Solutions for $y'' + (x^{-1}+1)y' + 2x^{-1}y = 0$ are

$$y = (1-x)e^{-x}$$

$$z = \ln(x)(1-x)e^{-x} + 1 + x - \frac{7}{4}x^2 + \frac{19}{18}x^3 - \frac{113}{288}x^4 + \frac{127}{1200}x^5 + \cdots$$





Reduction of order

There is a more general method similar to the method used in the last example. Suppose we have already found a function y satisfying $y'' + Py' + Qy = 0$, and we want to find another linearly independent solution z.

Proposition: If we put $v = \int P\,dx$ (so $v' = P$) and $u = \int y^{-2}e^{-v}\,dx$ and $z = uy$, then $z'' + Pz' + Qz = 0$.

Proof:

$$\begin{aligned}
z' &= u'y + uy' \\
z'' &= u''y + 2u'y' + uy'' \\
z'' + Pz' + Qz &= u''y + 2u'y + uy'' + Pu'y + Puy' + Quy \\
&= u(y'' + Py' + Qy) + (u''y + 2u'y' + Pu'y) \\
&= yu'' + (2y' + Py)u' \\
u' &= y^{-2}e^{-v} \\
u'' &= -2y^{-3}y'e^{-v} - y^{-2}e^{-v}v' = e^{-v}(-2y^{-3}y' - y^{-2}P) \\
yu'' + (2y' + Py)u' &= e^{-v}(-2y^{-2}y' - y^{-1}P) + (2y' + Py)y^{-2}e^{-v} \\
&= 0. \square
\end{aligned}$$

This method is called *reduction of order*.

Reduction of order example

Consider the equation $y'' - 2(1+x^{-1})y' + (1+2x^{-1})y = 0$.
One solution is $y = e^x$ (because then $y'' = y' = y$ and everything cancels).
We use reduction of order to find another solution.

$$\begin{aligned}
P &= -2(1+x^{-1}) \\
v &= \int P\,dx = -2(x + \ln(x)) \\
y^{-2}e^{-v} &= e^{-2x}e^{2x+2\ln(x)} = e^{2\ln(x)} = (e^{\ln(x)})^2 = x^2 \\
u &= \int y^{-2}e^{-v}\,dx = \int x^2\,dx = \frac{1}{3}x^3 \\
z &= uy = \frac{1}{3}x^3e^x.
\end{aligned}$$

Check:

$$\begin{aligned}
z' &= (x^2 + \tfrac{1}{3}x^3)e^x \\
z'' &= (2x + x^2)e^x + (x^2 + \tfrac{1}{3}x^3)e^x = (2x + 2x^2 + \tfrac{1}{3}x^3)e^x \\
-2(1+x^{-1})z' &= (-2x - \tfrac{8}{3}x^2 - \tfrac{2}{3}x^3)e^x \\
(1+2x^{-1})z &= (\tfrac{2}{3}x^2 + \tfrac{1}{3}x^3)e^x
\end{aligned}$$

So $z'' - 2(1+x^{-1})z' + (1+2x^{-1})z = 0$.

Another reduction of order example

Consider the operator $Ly = \sin^2(x)y'' - 3\sin(x)\cos(x)y' + (3 - 2\sin^2(x))y$.
Claim: one solution for $Ly = 0$ is $y = \sin(x)$.
To check, write $s = \sin(x)$ and $c = \cos(x)$, so $y = s$ and $y' = c$ and $y'' = -s$:

$$\begin{aligned}
Ly &= s^2.(-s) - 3sc.c + (3 - 2s^2)s = -s^3 - 3sc^2 + 3s - 2s^3 \\
&= -s^3 - 3s(1 - s^2) + 3s - 2s^3 = 0.
\end{aligned}$$

We use reduction of order to find another solution.

$$\begin{aligned}
P &= -3\cos(x)/\sin(x) \\
v &= \int P\,dx = -3\ln(\sin(x)) \\
y^{-2}e^{-v} &= y^{-2}\sin^3(x) = \sin(x) \\
u &= \int y^{-2}e^{-v}\,dx = \int \sin(x)\,dx = -\cos(x) \\
z &= uy = -\sin(x)\cos(x) = -\tfrac{1}{2}\sin(2x)
\end{aligned}$$

Series solutions for the Bessel equation

The Bessel equation is $x^2y'' + xy' + (x^2 - n^2)y = 0$.
Here we will assume n is real and nonnegative but not necessarily an integer.
The equation is equivalent to $y'' + x^{-1}y' + (1 - n^2x^{-2})y = 0$,
so there is a regular singular point at $x = 0$ with $p_0 = 1$ and $q_0 = -n^2$.
The indicial polynomial is

$$\alpha(\alpha - 1) + p_0\alpha + q_0 = \alpha^2 - n^2 = (\alpha - n)(\alpha + n),$$

so the roots are $\pm n$. There is a solution $y = j_n(x) = \sum_{k=0}^{\infty} a_k x^{n+k}$ with $a_0 = 1$.

$$x^2y'' = \sum_k (n + k)(n + k - 1)a_k x^{n+k}$$

$$xy' = \sum_k (n + k)a_k x^{n+k}$$

$$(x^2 - n^2)x = \sum_k (a_{k-2} - n^2 a_k)x^{n+k}$$

so we need $(n + k)(n + k - 1)a_k + (n + k)a_k - n^2a_k + a_{k-2} = 0$.
This gives $a_{k-2} = (n^2 - (n + k)^2)a_k = -k(2n + k)a_k$.

Series solutions for the Bessel equation

$x^2y''+xy'+(x^2-n^2)y = 0 \qquad y = \sum_{k=0}^{\infty} a_k x^{n+k} \qquad a_0 = 1 \qquad a_{k-2} = -k(2n+k)a_k$

Here $n \geq 0$, so for $k > 0$ we have $k(2n + k) \neq 0$ and $a_k = -a_{k-2}/(k(2n + k))$.

For odd k: $a_{-1} = 0$, so $a_1 = 0$, so $a_3 = 0$ and so on; so $a_k = 0$ when k is odd.

For even k:

$$a_0 = 1 \qquad a_2 = \frac{-1}{2(2n+2)} \qquad a_4 = \frac{1}{2.4.(2n+2)(2n+4)} \qquad a_6 = \frac{-1}{2.4.6.(2n+2)(2n+4)(2n+6)}$$

and so on. It is convenient to write this using the Pochhammer symbol:

$$(a)_p = a(a + 1)(a + 2)\cdots(a + p - 1).$$

With this notation we have

$$a_{2p} = \frac{(-1)^p}{2^p p! 2^p (n + 1)_p} = \frac{(-1)^p}{4^p p!(n + 1)_p},$$

so

$$y = j_n(x) = \sum_{p=0}^{\infty} a_{2p}x^{2p+n} = \sum_{p=0}^{\infty} \frac{(-1/4)^p}{p!(n + 1)_p} x^{n+2p}.$$

Series solutions for the Bessel equation

$x^2y''+xy'+(x^2-n^2)y = 0 \qquad$ One solution: $\qquad y = j_n(x) = \sum_{p=0}^{\infty} \frac{(-1/4)^p}{p!(n + 1)_p} x^{n+2p}.$

You will more often see a different function $J_n(x)$, called the Bessel function, which is a certain constant C_n times $j_n(x)$. We will not give a formula for C_n here.

If n is not an integer then there is another solution

$$z = j_{-n}(x) = \sum_{p=0}^{\infty} \frac{(-1/4)^p}{p!(-n + 1)_p} x^{-n+2p}.$$

However, if n is a positive integer then the n'th term involves division by $(-n + 1)_n$ which is zero; so $j_{-n}(x)$ cannot be defined. Instead, the second solution is $c\ln(x)j_n(x) + u(x)$ with $u(x) = \sum_k b_k x^{-n+k}$ say. We will not give the formula for b_k here.

Bessel functions of order zero

The case $n = 0$ of the Bessel equation is $x^2y'' + xy' + x^2y = 0$. One solution is $j_0(x)$. The formula for $j_n(x)$ involves $(n + 1)_p$ but $(1)_p = p!$ so the formula is

$$y = j_0(x) = \sum_{p=0}^{\infty} \frac{(-1/4)^p}{p!^2} x^{2p}.$$

Here we will give the formula for the other solution, but we will not check it.
We first need the function

$$\phi(p) = 1 + \frac{1}{2} + \cdots + \frac{1}{p} = \sum_{k=1}^{p} \frac{1}{k}.$$

The second solution is

$$z = \sum_{p=0}^{\infty} \frac{(-1/4)^p}{p!^2} x^{2p}(\log(x) - \phi(p)).$$

Sturm-Liouville form

If p, q, r are smooth functions of x, we can define

$$L(y) = ((py')' + qy)/r.$$

A *Sturm-Liouville operator* is an operator L of the above form.

Proposition: Consider an operator $L(y) = Ay'' + By' + Cy$, where A, B and C are functions of x, with $A > 0$ at all points of interest. Then L can be rewritten in Sturm-Liouville form: $L(y) = ((py')' + qy)/r$, where

$$p = \exp(\int B/A\,dx) \qquad q = pC/A \qquad r = p/A$$

Proof: Put $v = \int B/A\,dx$, so $v' = B/A$. Then $p = e^v$, so $p' = v'e^v = (B/A)e^v = pB/A$. Now

$$((py')' + qy)/r = (py'' + p'y' + qy)/r = \frac{p}{r}y'' + \frac{p'}{r}y' + \frac{q}{r}y$$
$$= \frac{p}{p/A}y'' + \frac{pB/A}{p/A}y' + \frac{pC/A}{p/A}y = Ay'' + By' + Cy.\square$$

The Bessel equation in Sturm-Liouville form

$$Ay'' + By' + Cy = ((py')' + qy)/r, \qquad \text{where}$$
$$p = \exp(\int B/A\,dx) \qquad q = pC/A \qquad r = p/A.$$

Recall the Bessel equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0.$$

We can write this as $Ly = n^2y$, where $Ly = x^2y'' + xy' + x^2y$.
Here $A = x^2$ and $B = x$ and $C = x^2$. Thus
$B/A = x^{-1}$, so $\int B/A\,dx = \ln(x)$, so $p = \exp(\int B/A\,dx) = \exp(\ln(x)) = x$.
This gives $q = pC/A = x.x^2/x^2 = x$ and $r = p/A = x/x^2 = x^{-1}$.

In conclusion:

$$L(y) = ((xy')' + xy)/x^{-1} = ((xy')' + xy)x.$$

The Legendre equation in Sturm-Liouville form

$$Ay'' + By' + Cy = ((py')' + qy)/r, \qquad \text{where}$$
$$p = \exp(\int B/A\,dx) \qquad q = pC/A \qquad r = p/A.$$

Recall the Legendre equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0.$$

We can write this as $Ly = -n(n+1)y$, where $Ly = (1 - x^2)y'' - 2xy'$.
Here $A = 1 - x^2$ and $B = -2x$ and $C = 0$. This gives

$$\frac{B}{A} = \frac{-2x}{1-x^2} = \frac{(1-x)-(1+x)}{(1-x)(1+x)} = \frac{1}{1+x} - \frac{1}{1-x},$$

so

$$\int B/A\,dx = \ln(1+x) + \ln(1-x) = \ln((1+x)(1-x)) = \ln(1-x^2),$$

so $p = \exp(\int B/A\,dx) = 1 - x^2$.
This in turn gives $q = pC/A = 0$ and $r = p/A = 1$. In conclusion:

$$L(y) = ((1 - x^2)y')'.$$

The Hermite equation in Sturm-Liouville form

$$Ay'' + By' + Cy = ((py')' + qy)/r, \qquad \text{where}$$
$$p = \exp(\int B/A\,dx) \qquad q = pC/A \qquad r = p/A.$$

Recall the Hermite equation

$$y'' - 2xy' + 2ny = 0.$$

We can write this as $Ly = -2ny$, where $Ly = y'' - 2xy'$.
Here $A = 1$ and $B = -2x$ and $C = 0$.
This gives $p = \exp(\int B/A\,dx) = e^{-x^2}$, so $q = pC/A = 0$ and $r = p/A = e^{-x^2}$.

In conclusion:

$$L(y) = e^{x^2}(e^{-x^2}y')'.$$

Reminder about matrices and eigenvectors

- Let A be an $n \times n$ matrix, and let u and v be vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$.
- The *transpose* A^T has entries $(A^T)_{ij} = A_{ji}$.
- We say that A is *symmetric* if $A^T = A$.
- The *inner product* $\langle u, v\rangle$ is $\sum_{i=1}^n u_i v_i$.
- The transpose and inner product are related by $\langle Au, v\rangle = \langle u, A^T v\rangle$.
- A number $\lambda \in \mathbb{C}$ is an *eigenvalue* of A if there is a nonzero vector $u \in \mathbb{C}^n$ with $Au = \lambda u$. Any such vector is called an *eigenvector*.
- If A is symmetric, then all eigenvalues are real. Moreover, if u and v are eigenvectors with different eigenvalues, then $\langle u, v\rangle = 0$.

Now let L be a differential operator, like $L_{\text{Bessel}}(y) = x^2y'' + xy' + x^2y$.
A number $\lambda \in \mathbb{C}$ is an *eigenvalue* if there is a nonzero function y with $L(y) = \lambda y$. Any such function is called an *eigenfunction*.
eg: $L_{\text{Bessel}}(J_n) = n^2 J_n$, so J_n is an eigenfunction of L_{Bessel} with eigenvalue n^2.

We will see that Sturm-Liouville operators behave like symmetric matrices: all eigenvalues are real, and eigenfunctions with distinct eigenvalues have an orthogonality property.

The Wronskian identity

Let L be a Sturm-Liouville operator, say $L(y) = ((py')' + qy)/r$.
For any two functions f and g, we put

$$W(f,g) = pfg' - pf'g.$$

This is called the *modified Wronskian*.

Proposition: $r.(f\,L(g) - L(f)\,g) = W(f,g)'$.

Proof:

$$\begin{aligned}
rL(g) &= (pg')' + qg = pg'' + p'g' + qg \\
rfL(g) &= pfg'' + p'fg' + qfg \\
r.(fL(g) - L(f)g) &= pfg'' + p'fg' + qfg - pf''g - p'f'g - qfg \\
&= p.(fg'' - f''g) + p'.(fg' - f'g) \\
W(f,g)' &= (pfg' - pf'g)' = p'fg' + pf'g' + pfg'' - p'f'g - pf''g - pf'g' \\
&= p.(fg'' - f''g) + p'.(fg' - f'g). \square
\end{aligned}$$

Boundary conditions

In applications, we often want to solve differential equations with boundary conditions. For example, consider again the vibration of a drum of radius R. The height y is a function of the distance r from the centre. The edge of the drum cannot move, so $y = 0$ when $r = R$; this is a boundary condition.

Consider a Sturm-Liouville operator $Ly = ((py')' + qy)/r$,
where p, q, r and y are all defined on some interval $[a, b]$.
We might need to use boundary conditions of the following kinds:

- Dirichlet conditions: $y = 0$ when $x = a$ or $x = b$.
- Neumann conditions: $y' = 0$ when $x = a$ or $x = b$.
- Periodic conditions: $y(a) = y(b)$ and $y'(a) = y'(b)$.

We will only discuss Dirichlet conditions. Other cases are similar.

Eigenfunction example

We consider the the operator $L(y) = y''$ on the interval $[0, 1]$.
Suppose we have a function f with $f'' = L(f) = \lambda f$,
and that f also satisfies Dirichlet conditions $f(0) = f(1) = 0$.

- Suppose that $\lambda > 0$, so $\lambda = \mu^2$ for some $\mu > 0$. We have seen before that every solution of $f'' = \mu^2 f$ has the form $f = Ae^{\mu x} + Be^{-\mu x}$ for some constants A and B. The boundary condition $f(0) = 0$ gives $A + B = 0$, and the condition $f(1) = 0$ gives $Ae^{\mu} + Be^{-\mu} = 0$. As $\mu > 0$ we have $e^{\mu} \neq e^{-\mu}$ and it follows easily that $A = B = 0$, so $f = 0$. Thus, there are no eigenfunctions with $\lambda < 0$.
- Suppose instead that $\lambda = 0$, so the equation $L(f) = \lambda f$ just gives $f'' = 0$. It is easy to see that the only solutions are $f = Ax + B$ with A and B constant. The conditions $f(0) = f(1) = 0$ give $B = A + B = 0$, so $A = B = 0$, so again $f = 0$. Thus, there are no eigenfunctions with $\lambda = 0$.
- Suppose that $\lambda < 0$, so $\lambda = -\omega^2$ for some $\omega > 0$. The equation $Lf = \lambda f$ says $f'' + \omega^2 f = 0$, which has solutions $f = A\sin(\omega x) + B\cos(\omega x)$. The condition $f(0) = 0$ gives $B = 0$. The condition $f(1) = 0$ becomes $A\sin(\omega) = 0$, which gives $A = 0$ unless $\omega = n\pi$ for some integer $n > 0$.

Conclusion:
the only real eigenvalues are $\lambda = -n^2\pi^2$; eigenfunctions are $\sin(n\pi x)$.

Self-adjointness under Dirichlet conditions

Consider a Sturm-Liouville operator $Ly = ((py')' + qy)/r$,
where p, q, r and y are all defined on some interval $[a, b]$.
Suppose also that $r(x) > 0$ for all $x \in [a, b]$.
For smooth functions f and g on $[a, b]$ we define

$$\langle f, g\rangle = \int_a^b r(x)f(x)g(x)\,dx.$$

Proposition: If $f(a) = g(a) = 0$ and $f(b) = g(b) = 0$ then $\langle Lf, g\rangle = \langle f, Lg\rangle$.

Proof:

$$\begin{aligned}\langle f, Lg\rangle - \langle Lf, g\rangle &= \int_a^b r.(f\,L(g) - L(f)\,g)\,dx && \text{(definition of } \langle,\rangle\text{)}\\ &= \int_a^b W(f, g)'(x)\,dx && \text{(Wronskian identity)}\\ &= W(f, g)(b) - W(f, g)(a).\end{aligned}$$

Here $W(f, g) = p.(f'g - fg')$ but $f = g = 0$ at $x = a$ so $W(f, g)(a) = 0$.
Similarly $W(f, g)(b) = 0$, so $\langle f, Lg\rangle - \langle Lf, g\rangle = 0$. □

Self-adjointness under Dirichlet conditions

Corollary: If $Lf = \lambda f$ and $Lg = \mu g$ with $\lambda \neq \mu$, and
$f(a) = g(a) = f(b) = g(b) = 0$, then $\langle f, g\rangle = 0$.

Proof: By the proposition, $\langle Lf, g\rangle = \langle f, Lg\rangle$. As $Lf = \lambda f$, the left hand side is $\lambda\langle f, g\rangle$. As $Lg = \mu g$, the right hand side is $\mu\langle f, g\rangle$. As both sides are the same, $(\lambda - \mu)\langle f, g\rangle = 0$, but $\lambda - \mu \neq 0$ so $\langle f, g\rangle = 0$. □

Corollary: All eigenvalues of L (subject to Dirichlet conditions) are real.

Proof: Suppose we have a complex eigenfunction $f = g + ih \neq 0$, with a complex eigenvalue $\lambda = \mu + i\nu$. Now

$$Lg + iLh = Lf = \lambda f = (\mu + i\nu)(g + ih) = (\mu g - \nu h) + i(\mu h + \nu g),$$

so $Lg = \mu g - \nu h$ and $Lh = \mu h + \nu g$. We assume that f is 0 at a and b, so g and h are also 0 at a and b, so the Proposition gives $\langle Lg, h\rangle - \langle g, Lh\rangle = 0$.
This gives
$0 = \langle \mu g - \nu h, h\rangle - \langle g, \mu h + \nu g\rangle = -\nu(\langle g, g\rangle + \langle h, h\rangle) = -\nu \int_a^b r(g^2 + h^2)\,dx.$

Now $r > 0$ everywhere and $f = g + ih$ is not the zero function so
$\int_a^b r(g^2 + h^2)\,dx > 0$. This means that $\nu = 0$, so λ is real.
We also see that $Lg = \lambda g$ and $Lh = \lambda h$, so g and h are real eigenfunctions. □

Discreteness of eigenvalues

Consider again a Sturm-Liouville operator $Ly = ((py')' + qy)/r$,
where p, q, r and y are all defined on some interval $[a, b]$,
and p and r are everywhere positive.
Consider eigenvalues and eigenfunctions subject to Dirichlet conditions.

Theorem: The eigenvalues can be listed as $\lambda_0, \lambda_1, \lambda_2, \ldots$ with $|\lambda_k| \leq |\lambda_{k+1}|$ and $|\lambda_k| \to \infty$. Moreover, for any given k, the space of eigenfunctions of eigenvalue λ_k has dimension one or two.

We will not prove this theorem.

Normal form

Sometimes it is easier to work with equations like $y'' + Ry = 0$, where there is no term involving y'. This is called *normal form*.

For any equation $y'' + Py' + Qy = 0$, there is an equivalent equation in normal form.
In more detail: put $v = \int P\,dx$ and $m = e^{-v/2}$. Then put $R = Q - \frac{1}{2}P' - \frac{1}{4}P^2$.

Proposition: If z satisfies $z'' + Rz = 0$, then the function $y = zm$ satisfies the original equation $y'' + Py' + Qy = 0$.

Proof: First note that $m' = -\frac{1}{2}Pm$. We can differentiate this again to get

$$m'' = -\tfrac{1}{2}P'm - \tfrac{1}{2}Pm' = -\tfrac{1}{2}P'm - \tfrac{1}{2}P(-\tfrac{1}{2}Pm) = -\tfrac{1}{2}P'm + \tfrac{1}{4}P^2m.$$

$$\begin{aligned}y'' &= z''m + 2z'm' + zm'' = z''m - Pz'm - \tfrac{1}{2}P'zm + \tfrac{1}{4}P^2zm\\ Py' &= Pz'm + Pzm' = Pz'm - \tfrac{1}{2}P^2zm\\ Qy &= Qzm\end{aligned}$$

$$y'' + Py' + Qy = z''m - \tfrac{1}{2}P'zm - \tfrac{1}{4}P^2zm + Qzm = (z'' + Rz)m.$$

Thus, if $z'' + Rz = 0$ then $y'' + Pz' + Qz = 0$. □

Normal form — constant coefficients

$$y'' + Py' + Qy = 0 \qquad m = \exp(-\tfrac{1}{2}\int P\,dx) \qquad R = Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2$$

$$y = mz \qquad z'' + Rz = 0$$

Suppose P and Q are constant. The auxiliary polynomial is $t^2 + Pt + Q$; the roots are $(-P \pm \sqrt{D})/2$ where $D = P^2 - 4Q$. The solutions are

$$y = Ae^{(-P+\sqrt{D})x/2} + Be^{(-P-\sqrt{D})x/2} = e^{-Px/2}(Ae^{\sqrt{D}x/2} + Be^{-\sqrt{D}x/2}).$$

(If $D < 0$: use $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ to rewrite this in terms of sin and cos.)

Normal form: $\int P\,dx = Px$ so $m = e^{Px/2}$.
Also $P' = 0$ so $R = Q - \frac{1}{4}P^2 = -D/4$.
Equation for z is $z'' - \frac{1}{4}Dz = 0$, with solutions $z = Ae^{\sqrt{D}x/2} + Be^{-\sqrt{D}x/2}$.
This gives

$$y = mz = e^{-Px/2}(Ae^{\sqrt{D}x/2} + Be^{-\sqrt{D}x/2}),$$

which is the same as before.

Normal form example

$$y'' + Py' + Qy = 0 \qquad m = \exp(-\tfrac{1}{2}\int P\,dx) \qquad R = Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2$$

$$y = mz \qquad z'' + Rz = 0$$

Consider the equation $x^2y'' - 2\mu xy' + (\mu(\mu+1) + x^2)y = 0$.
We can divide by x^2 to get $y'' - 2\mu x^{-1}y' + (\mu(\mu+1)x^{-2} + 1)y = 0$.
This is $y'' + Py' + Qy = 0$, where $P = -2\mu x^{-1}$ and $Q = \mu(\mu+1)x^{-2} + 1$.
We now have

$$m = \exp(-\tfrac{1}{2}\int P\,dx) = \exp(\mu\ln(x)) = x^{\mu}$$

$$R = Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2 = \mu(\mu+1)x^{-2} + 1 - \tfrac{1}{2}\times(2\mu x^{-2}) - \tfrac{1}{4}\times 4\mu^2x^{-2} = 1.$$

We thus have $y = x^{\mu}z$ with $z'' + z = 0$, which means that $z = A\cos(x) + B\sin(x)$ for some constants A and B.

Conclusion: the solution for $x^2y'' - 2\mu xy' + (\mu(\mu+1) + x^2)y = 0$ is

$$y = (A\cos(x) + B\sin(x))x^{\mu}.$$

Normal form for the Bessel equation

$$y'' + Py' + Qy = 0 \qquad m = \exp(-\tfrac{1}{2}\int P\,dx) \qquad R = Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2$$

$$y = mz \qquad z'' + Rz = 0$$

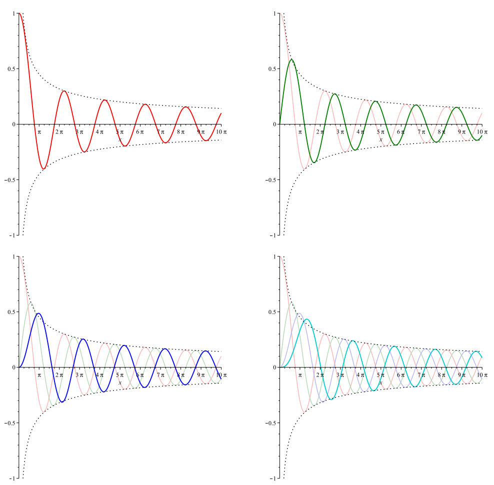
Consider again the Bessel equation $x^2y'' + xy' + (x^2 - n^2)y = 0$.
We can divide by x^2 to get $y'' + x^{-1}y' + (1 - n^2x^{-2})y = 0$.
This is $y'' + Py' + Qy = 0$, where $P = x^{-1}$ and $Q = 1 - n^2x^{-2}$.
We now have

$$m = \exp(-\tfrac{1}{2}\int P\,dx) = \exp(-\tfrac{1}{2}\ln(x)) = x^{-1/2}$$

$$R = Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2 = 1 - n^2x^{-2} + \tfrac{1}{2}x^{-2} - \tfrac{1}{4}x^{-2} = 1 + \frac{1-4n^2}{4x^2}$$

Conclusion: the solutions for $x^2y'' + xy' + (x^2 - n^2)y = 0$ have the form $y = x^{-1/2}z$, where $z'' + \left(1 + \frac{1-4n^2}{4x^2}\right)z = 0$.
For large x, this is approximately $z'' + z = 0$, so z is like $A\cos(x+\phi)$, so y is like $Ax^{-1/2}\cos(x+\phi)$.

Normal form for the Bessel equation



For large x, this is approximately $z'' + z = 0$, so z is like $A\cos(x+\phi)$, so y is like $Ax^{-1/2}\cos(x+\phi)$

Prüfer angles for the Legendre equation

- Suppose u solves the Legendre equation $((1-x^2)u')' + n(n+1)u = 0$.
- Put $v = (1-x^2)u'$, so $v' = -n(n+1)u$.
- We explained before that the point (u,v) rotates through $n\pi$ about the origin as x goes from -1 to 1.
- (A similar thing works for other Sturm-Liouville equations as well.)
- To understand this, put

$$\begin{aligned} \theta &= \text{angle from the } y\text{-axis to } (u,v) &= \arctan(u/v) \\ \rho &= \text{distance from } (0,0) \text{ to } (u,v) &= \sqrt{u^2+v^2} \end{aligned}$$

so $\quad u = \rho\sin(\theta) \quad$ and $\quad v = \rho\cos(\theta)$.

- We will show that

$$\begin{aligned} \theta' &= n(n+1)\sin^2(\theta) + \cos^2(\theta)/(1-x^2) \\ \rho' &= \tfrac{1}{2}\rho\sin(2\theta)(1/(1-x^2) - n(n+1)) \\ \rho &= \exp\left(\tfrac{1}{2}\int \sin(2\theta)(1/(1-x^2) - n(n+1))\,dx\right). \end{aligned}$$

- Note that this is a first order nonlinear equation for θ, not involving ρ.

Prüfer angles for the Legendre equation

$$v = (1-x^2)u' \qquad v' = -n(n+1)u \qquad u = \rho\sin(\theta) \qquad v = \rho\cos(\theta).$$

(a) $v = (1-x^2)u'$ and $v' = -n(n+1)u$ gives

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} v/(1-x^2) \\ -n(n+1)u \end{bmatrix} = \begin{bmatrix} \rho\cos(\theta)/(1-x^2) \\ -n(n+1)\rho\sin(\theta) \end{bmatrix}$$

(b) $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \rho\sin(\theta) \\ \rho\cos(\theta) \end{bmatrix}$ gives

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \rho'\sin(\theta) + \rho\cos(\theta)\theta' \\ \rho'\cos(\theta) - \rho\sin(\theta)\theta' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \rho\theta' \\ \rho' \end{bmatrix}.$$

(c) Using $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}^{-1} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ and (a) and (b) we get

$$\begin{bmatrix} \rho\theta' \\ \rho' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \rho\cos(\theta)/(1-x^2) \\ -n(n+1)\rho\sin(\theta) \end{bmatrix}$$

Prüfer angles for the Legendre equation

$$\begin{bmatrix} \rho\theta' \\ \rho' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \rho\cos(\theta)/(1-x^2) \\ -n(n+1)\rho\sin(\theta) \end{bmatrix}$$

Expand and divide by ρ to get

$$\begin{aligned} \theta' &= \cos^2(\theta)/(1-x^2) + n(n+1)\sin^2(\theta) \geq 0 \\ \rho'/\rho &= \sin(\theta)\cos(\theta)/(1-x^2) - n(n+1)\sin(\theta)\cos(\theta) \\ &= \tfrac{1}{2}\sin(2\theta)(1/(1-x^2) - n(n+1)). \end{aligned}$$

Note that $\rho'/\rho = \ln(\rho)'$, so

$$\begin{aligned} \ln(\rho) &= \int \tfrac{1}{2}\sin(2\theta)(1/(1-x^2) - n(n+1))\,dx \\ \rho &= \exp\left(\int \tfrac{1}{2}\sin(2\theta)(1/(1-x^2) - n(n+1))\,dx\right). \end{aligned}$$

When $x = \pm 1$ we have $v = (1-x^2)u' = 0$, so (u,v) starts and ends on the (positive or negative) x-axis. It must therefore rotate through an angle $m\pi$ for some integer $m \geq 0$. This means that there must be m times where (u,v) passes through the y-axis, ie m roots of u. (In fact $m = n$, but this is harder.)

Transformations

Suppose that $x = e^t$, and write $u' = du/dx$ and $\dot{u} = du/dt$.
Claim: the Bessel equation $x^2y'' + xy' + (x^2 - n^2)y = 0$
is equivalent to $\ddot{y} + (e^{2t} - n^2)y = 0$.
Proof:

$$\begin{aligned} \dot{x} &= \frac{dx}{dt} = \frac{d}{dt}e^t = e^t = x \\ \dot{y} &= \frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx} = xy' \\ \frac{d}{dx}(\dot{y}) &= \frac{d}{dx}(xy') = y' + xy'' \\ \ddot{y} &= \frac{d}{dt}(\dot{y}) = \frac{dx}{dt}\frac{d}{dx}(\dot{y}) = x(y' + xy'') = x^2y'' + xy' \end{aligned}$$

so

$$\ddot{y} + (e^{2t} - n^2)y = x^2y'' + xy' + (x^2 - n^2)y$$

so solutions to $\ddot{y} + (e^{2t} - n^2)y = 0$ are $y = AJ_n(e^t) + BY_n(e^t)$.

Transformations

Suppose that $x = t^2$ and $z = y/x^3$.
Claim: Legendre equation $(1-x^2)y'' - 2xy' + 12y = 0$
is equivalent to $(t^6 - t^2)\ddot{z} + (15t^5 - 11t)\dot{z} - 24z = 0$.
Proof:

$$\begin{aligned}
z' &= x^{-3}y' - 3x^{-4}y = t^{-6}y' - 3t^{-8}y \\
z'' &= x^{-3}y'' - 6x^{-4}y' + 12x^{-5}y = t^{-6}y'' - 6t^{-8}y' + 12t^{-10}y \\
\dot{z} &= \frac{dz}{dt} = \frac{dx}{dt}\frac{dz}{dx} = 2tz' = 2t^{-5}y' - 6t^{-7}y \\
\ddot{z} &= 2z' + 2t\frac{d}{dt}(z') = 2z' + 2t\frac{dx}{dt}\frac{d}{dx}(z') \\
&= 2z' + 2t.2tz'' = 2z' + 4t^2z'' \\
&= 2(t^{-6}y' - 3t^{-8}y) + 4t^2(t^{-6}y'' - 6t^{-8}y' + 12t^{-10}y) \\
&= 4t^{-4}y'' - 22t^{-6}y' + 42t^{-8}y
\end{aligned}$$

Transformations

$x = t^2 \qquad z = y/x^3 = y/t^6$
$\dot{z} = 2t^{-5}y' - 6t^{-7}y \qquad \ddot{z} = 4t^{-4}y'' - 22t^{-6}y' + 42t^{-8}y$

$$\begin{aligned}
&(t^6 - t^2)\ddot{z} + (15t^5 - 11)\dot{z} - 24z \\
=&(t^6 - t^2)(4t^{-4}y'' - 22t^{-6}y' + 42t^{-8}y) + \\
&(15t^5 - 11t)(2t^{-5}y' - 6t^{-7}y) - 24t^{-6}y \\
=&4(t^2 - t^{-2})y'' + (-22 + 22t^{-4} + 30 - 22t^{-4})y' + \\
&(42t^{-2} - 42t^{-6} - 90t^{-2} + 66t^{-6} - 24t^{-6})y \\
=&4(t^2 - t^{-2})y'' + 8y' - 48t^{-2}y \\
=&-4t^{-2}((1-t^4)y'' - 2t^2y' + 12y) = -4x^{-1}((1-x^2)y'' - 2xy' + 12y)
\end{aligned}$$

So $(1-x^2)y'' - 2xy' + 12y = 0$ is equivalent to
$(t^6 - t^2)\ddot{z} + (15t^5 - 11)\dot{z} - 24z = 0$. So solutions to
$(t^6 - t^2)\ddot{z} + (15t^5 - 11)\dot{z} - 24z = 0$ are $(AP_n(t^2) + BQ_n(t^2))/t^6$.