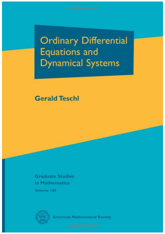
Ordinary Differential Equations

A recommended book



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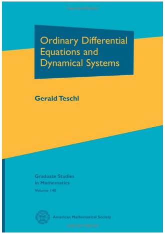

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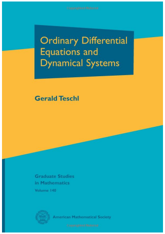

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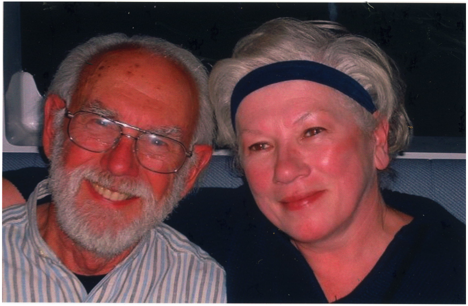
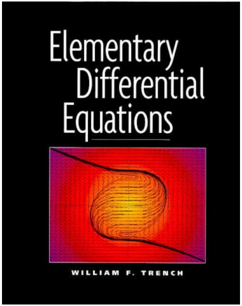
- Most of the book is too advanced for this course, but still it should be useful.

Another recommended book





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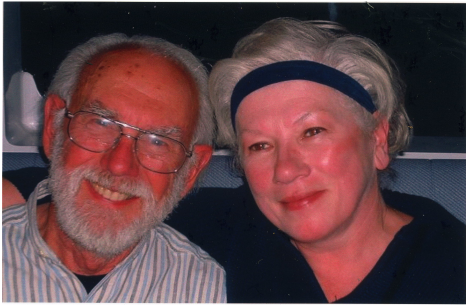

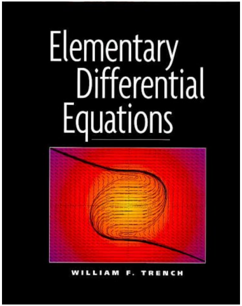

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Planar Ordinary Differential Equations

Introduction

The first half of this course is about planar differential equations, like this example:

$$\dot{x} = \frac{dx}{dt} = -\frac{9}{40}x^2 + \frac{3}{10}y^2 - \frac{3}{40} \qquad \dot{y} = \frac{dy}{dt} = \frac{5}{8}x^2 + \frac{23}{20}y^2 - \frac{71}{40}$$

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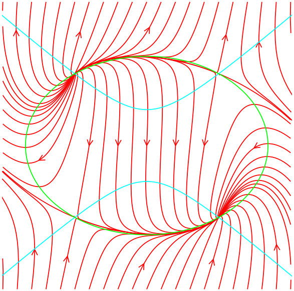

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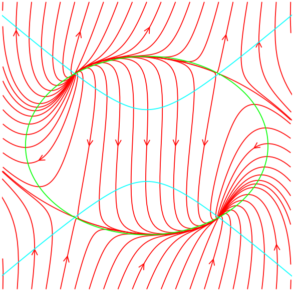

We will start by looking quickly at some examples. Later we will develop some mathematical theory, then look at the examples again.

The Lotka-Volterra model

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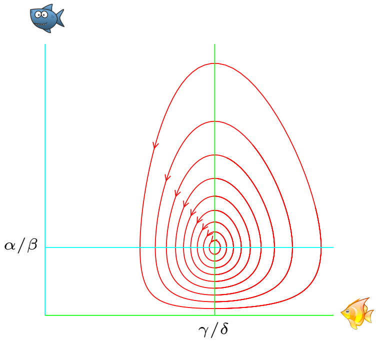
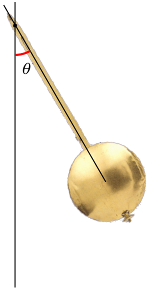
The *phase portrait* shows how the point (F, S) moves over time:



$$\dot{F} = -\beta F(S - \alpha/\beta)$$
$$\dot{S} = \delta S(F - \gamma/\delta)$$

Motion of a pendulum

Consider a swinging pendulum, hanging at angle θ.



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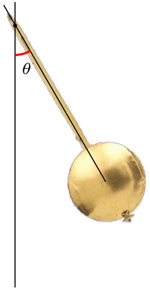

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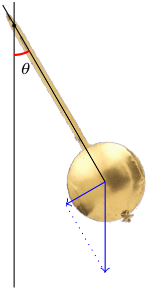

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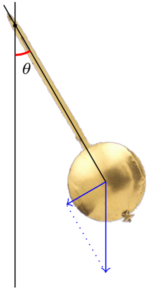

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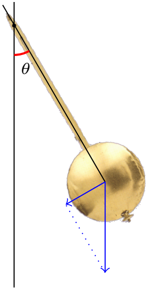

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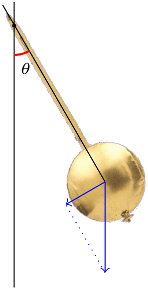

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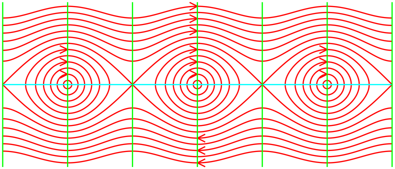
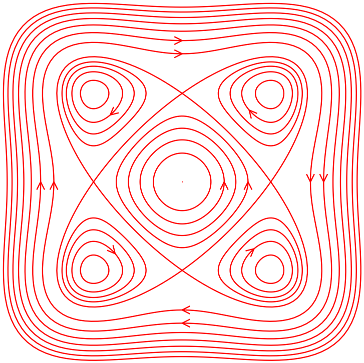


A contour flow

This system has equations $\dot{x} = y^3 - y$ and $\dot{y} = x - x^3$.



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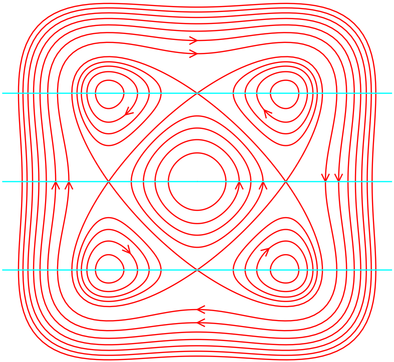

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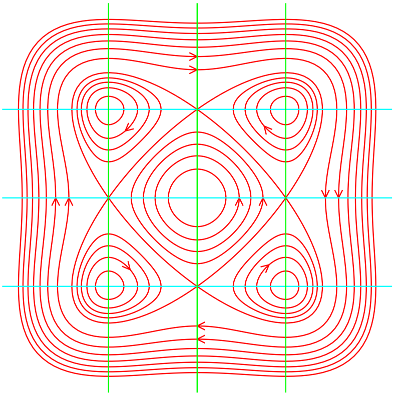

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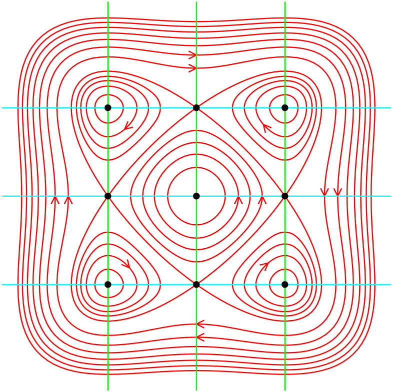

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- The black dots (*equilibrium points*) show where $\dot{x} = \dot{y} = 0$.

This is a contour map. The height is $h(x, y)$. If you stay on one of the brown lines (contours), then you stay at the same height. That is a contour flow.

A contour map

When the contours are close together, the ground is steep.
When the contours are far apart, the ground is not steep.

A gradient flow

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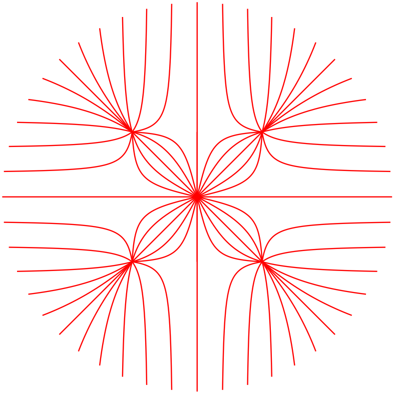

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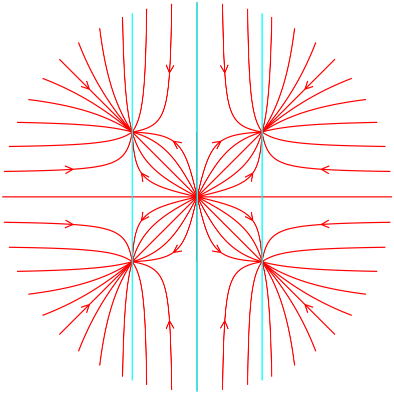

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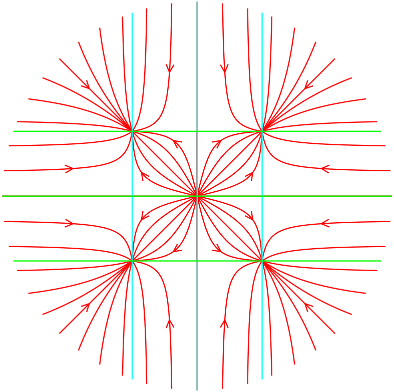

- The blue lines (x-*nullclines*) show where $x - x^3 = 0$ and so $\dot{x} = 0$.
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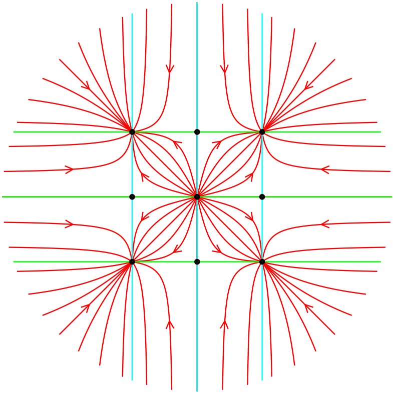

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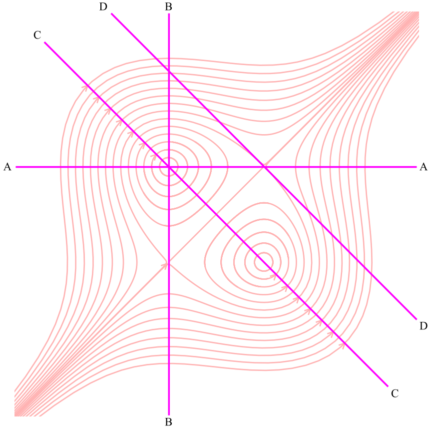
Question: which is the y-nullcline?

Which of the curves below is the y-nullcline?



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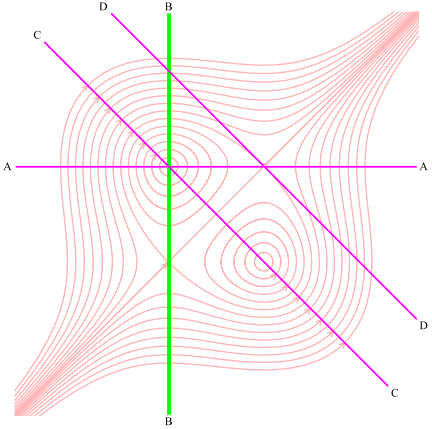
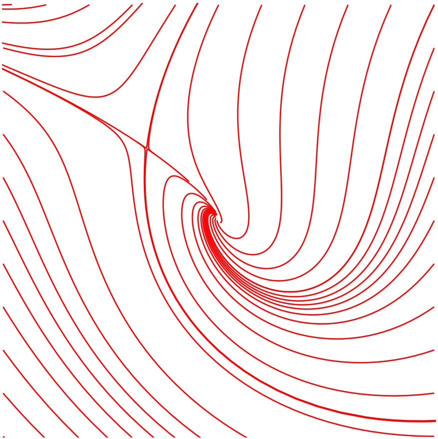


Curve B is the y-nullcline; the flow lines cross it horizontally, with $\dot{y} = 0$.

Question: how many equilibria?

How many equilibrium points are there in this phase portrait?



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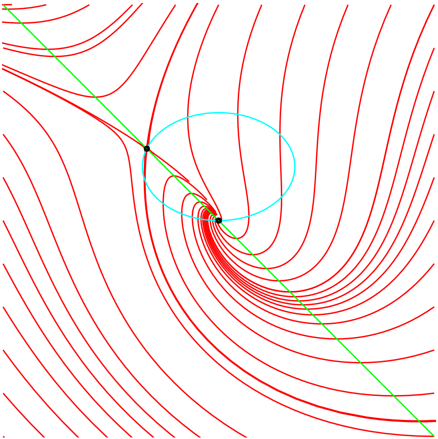
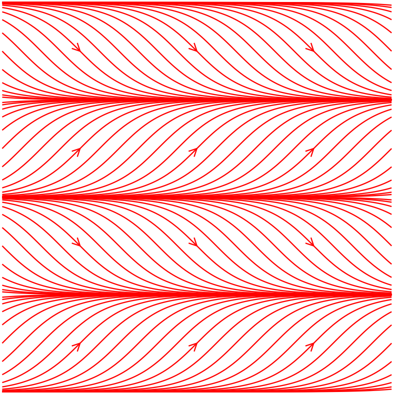


There are two equilibrium points, as shown.

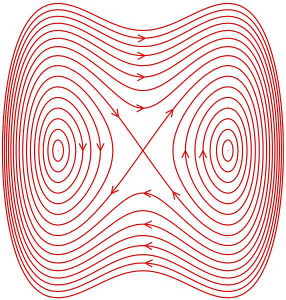
This system has equations $\dot{x} = 1$ and $\dot{y} = \sin(\pi y)$.



The solutions move steadily to the right, and converge to one of the lines where y is an odd integer.

Duffing oscillator

The Duffing oscillator has $\dot{x} = y$ and $\dot{y} = 2x - x^3$.



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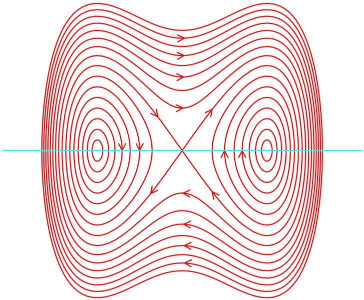

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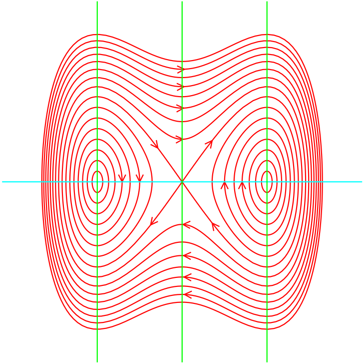

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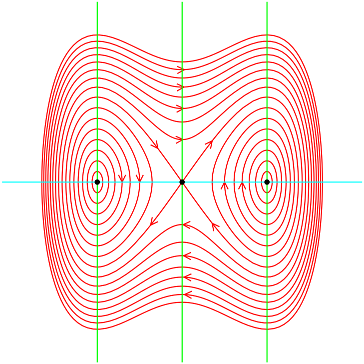

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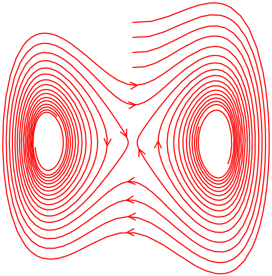

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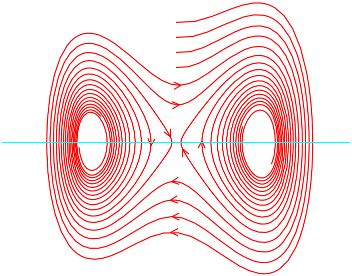

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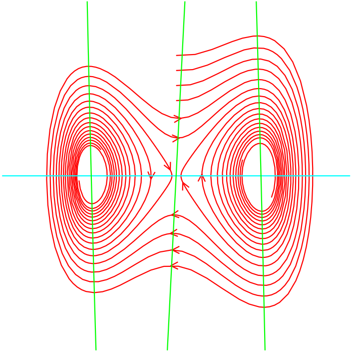

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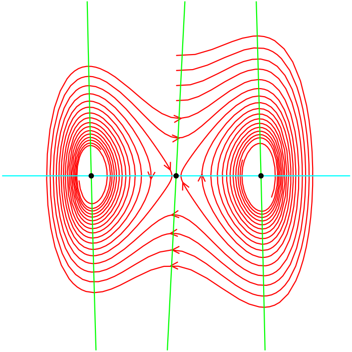

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- The x-nullcline is the same as before
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- The equilibrium points are unchanged.

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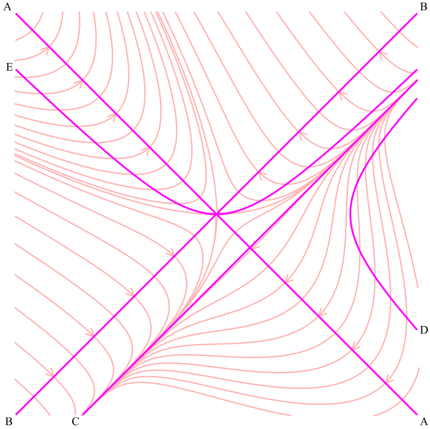

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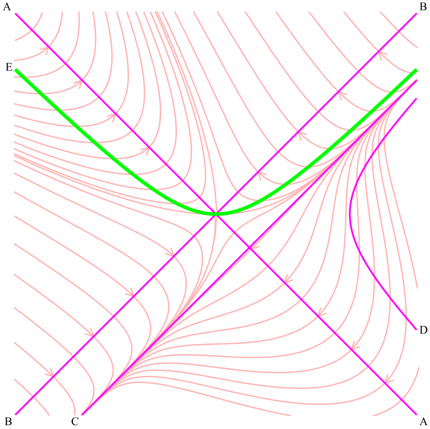

Curve E is the y-nullcline; the flow lines cross it horizontally, with $\dot{y} = 0$.

van der Pol oscillator

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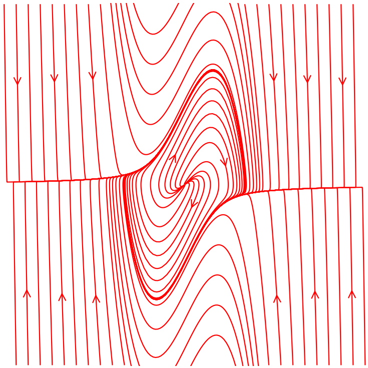

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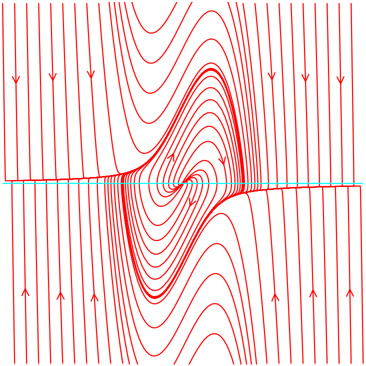

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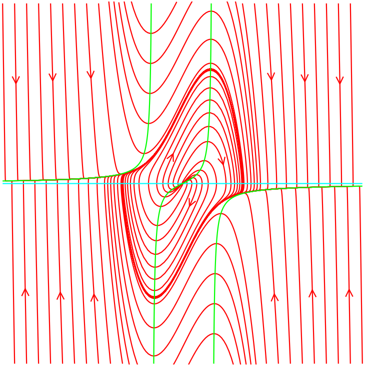
van der Pol oscillator

This system has $\dot{x} = y$ and $\dot{y} = 2(1 - x^2)y - x$.



- The blue line (*x-nullcline*) shows where $y = 0$ and so $\dot{x} = 0$.
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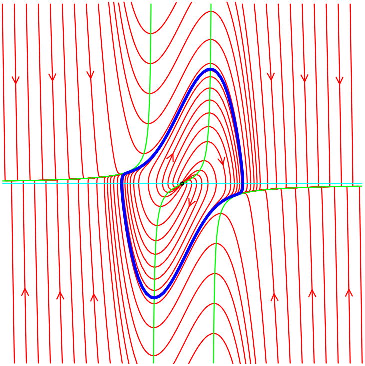

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- The green lines (*y-nullclines*) show where $2(1 - x^2)y - x = 0$ and so $\dot{y} = 0$.
- There is only one equilibrium point, but there is also a *limit cycle*, shown in blue. All non-constant solutions converge to the limit cycle.

Sketching a phase portrait

We will sketch the phase portrait for the system
$\dot{x} = x(3 - x - 2y), \ \dot{y} = y(x - 1)$.

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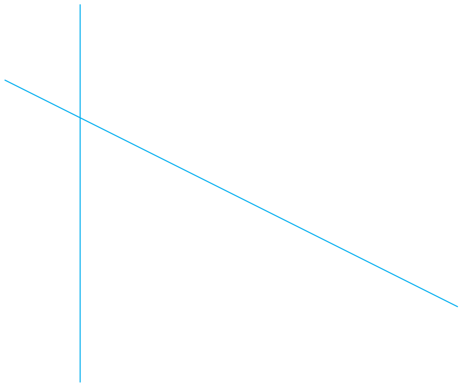

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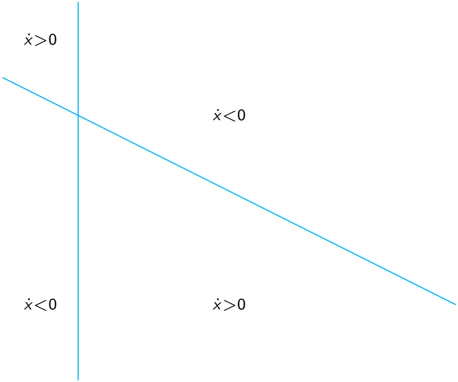

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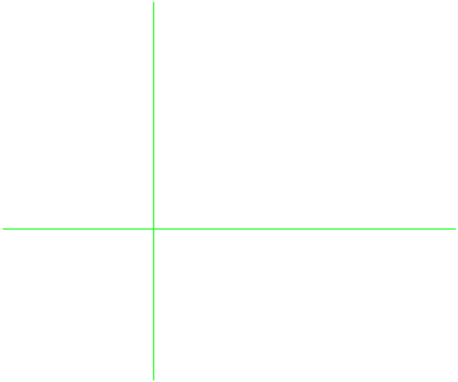

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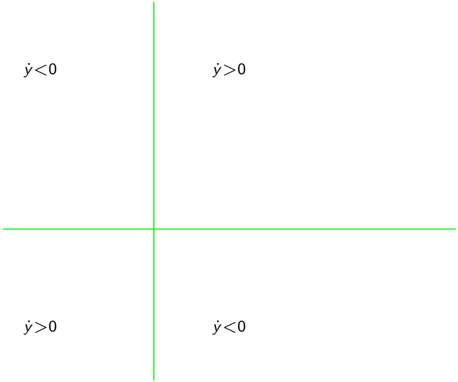

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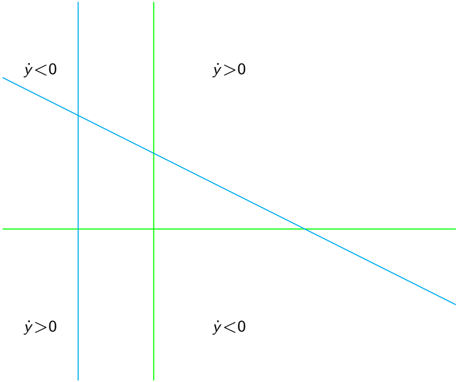

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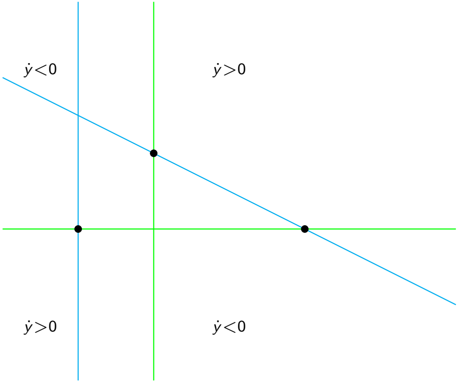

We now draw both nullclines. The equilibrium points appear where the x-nullcline meets the y-nullcline.

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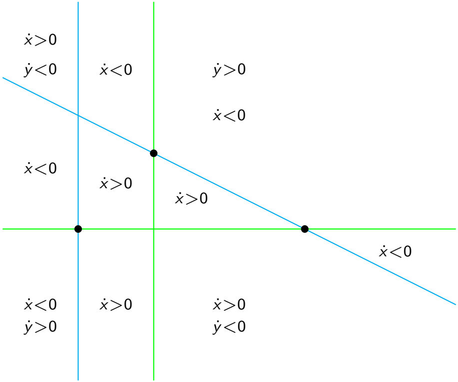

The nullclines divide the plane into ten regions. In each region, we can determine the signs of $\dot{x}$ and $\dot{y}$.

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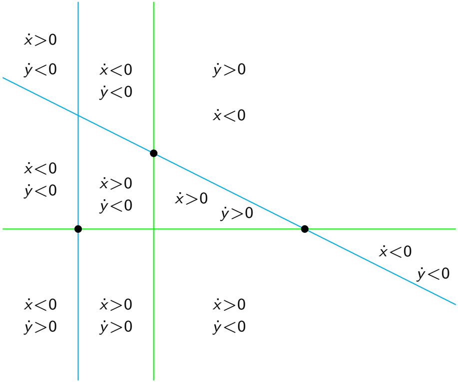

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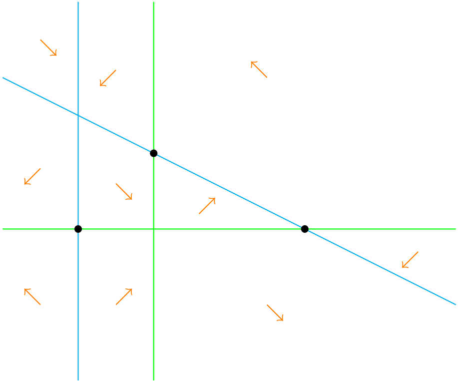

It is more convenient to display the signs of $\dot{x}$ and $\dot{y}$ by drawing arrows.

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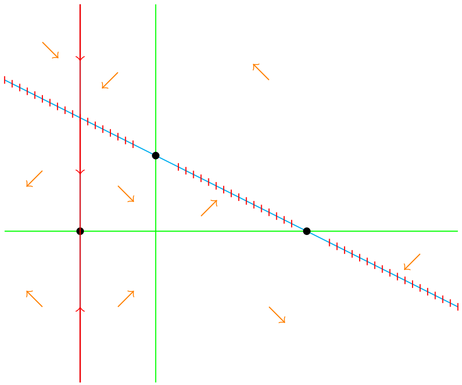

The flow lines cross the x-nullcline vertically.

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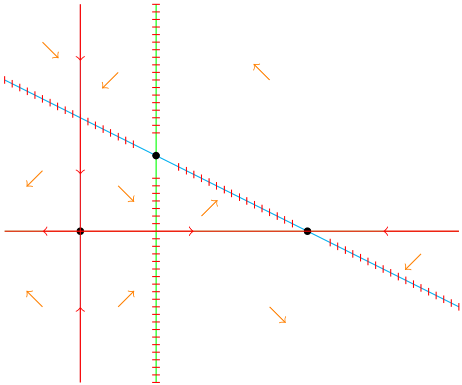

The flow lines cross the y-nullcline horizontally.

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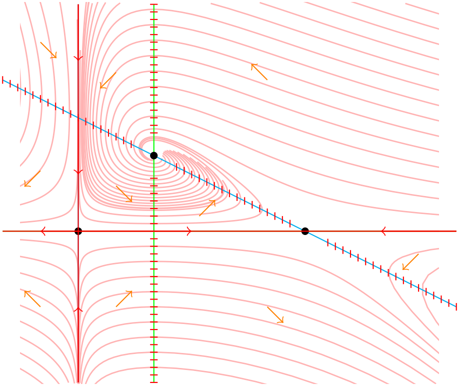

The actual flow lines are as shown above.

A problem which we will mostly ignore

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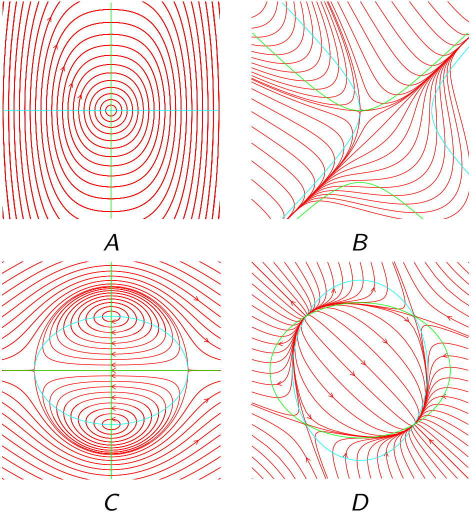
We mostly ignore this problem and consider only equations where $x(t)$ and $y(t)$ are defined for all $t \in \mathbb{R}$.

Question: which portrait is which?



Which picture shows $\dot{x} = x^2 + 2y^2 - \frac{1}{2}$, $\dot{y} = -\sqrt{2}xy$?

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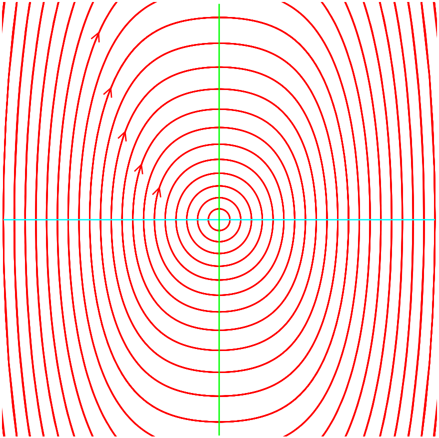
A : 

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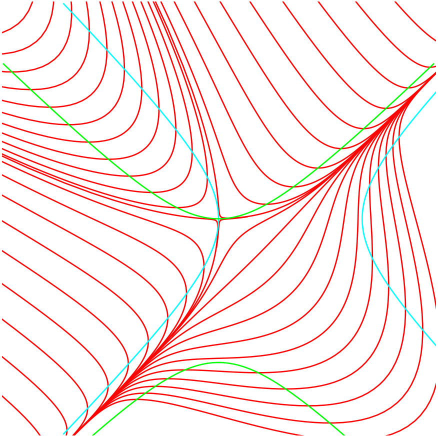
B : 

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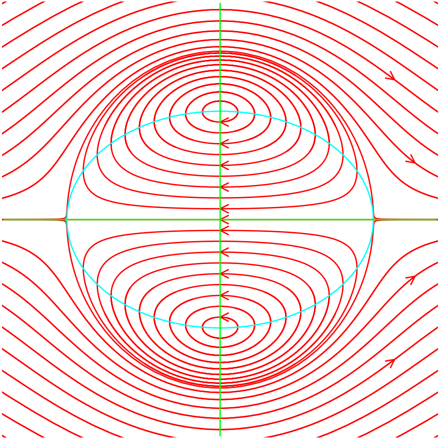
C : 

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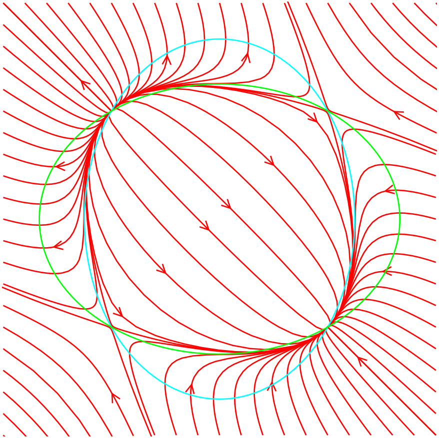
D : 

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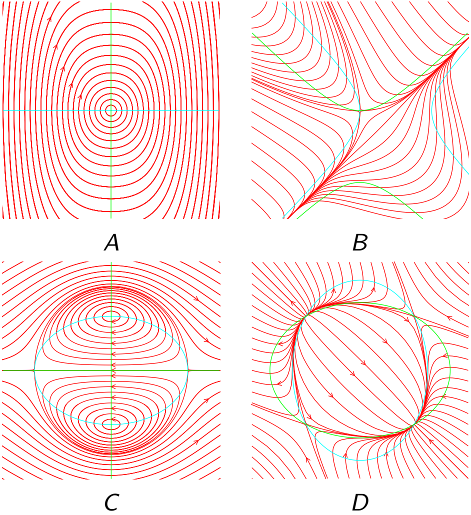

Picture C shows $\dot{x} = x^2 + 2y^2 - \frac{1}{2}$, $\dot{y} = -\sqrt{2}xy$.

Linear systems

A *(first order, autonomous) linear system* has the form

$$\dot{x} = \frac{dx}{dt} = ax+by \qquad \dot{y} = \frac{dy}{dt} = cx+dy \qquad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

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Suppose $b = c = 0$, so
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Reminder of simple harmonic motion

Proposition: Suppose that x is a function of t such that $\ddot{x} = -\omega^2 x$; then $x(t) = A\cos(\omega t) + B\sin(\omega t)$ for some constants A and B.

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$$\dot{u}(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$

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Now put $E(t) = \omega^2 v(t)^2 + \dot{v}(t)^2$

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This means that E is constant

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Question

A *(first order, autonomous) linear system* has the form

$$\dot{x} = \frac{dx}{dt} = ax+by \qquad \dot{y} = \frac{dy}{dt} = cx+dy \qquad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Which of the following is a first order, autonomous linear system?

(a) $\dot{x} = 3x + t$, $\dot{y} = 4y - t$

(b) $\dot{x} = 3x - y$, $\dot{y} = x + 9y$

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(c) is of first order and autonomous but not linear

(d) is of second order, autonomous and linear.

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What is the solution to $\dot{x} = y$ and $\dot{y} = -x$ with $x = 0$ and $y = 5$ when $t = 0$?

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Here $x_0 = 0$ and $y_0 = 5$, so we just get

$$x = 5 \sin(t) \qquad\qquad y = 5 \cos(t).$$

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We put $u = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so $\dot{u} = Au$.

Linear systems

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The eigenvalues are the roots of $\chi_A(t)$, which are

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$$\lambda_1 + \lambda_2 = \tau \qquad \lambda_1\lambda_2 = \delta$$

Question

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{aligned} \tau &= a + b \\ \delta &= ad - bc \end{aligned} \qquad \begin{aligned} \lambda_1 &= \tfrac{1}{2}(\tau - \sqrt{\tau^2 - 4\delta}) \\ \lambda_2 &= \tfrac{1}{2}(\tau + \sqrt{\tau^2 - 4\delta}). \end{aligned}$$

Which of the following has real eigenvalues?

(a) $A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 7 & -2 \\ 7 & 0 \end{bmatrix}$

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(a) $A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$; $\tau = 0,\ \delta = 2$; $\tau^2 - 4\delta = -8 < 0$

(b) $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$; $\tau = 2,\ \delta = 2$; $\tau^2 - 4\delta = -4 < 0$

(c) $A = \begin{bmatrix} 7 & -2 \\ 7 & 0 \end{bmatrix}$; $\tau = 7,\ \delta = 14$; $\tau^2 - 4\delta = -7 < 0$

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Eigenvalues are real if $\tau^2 - 4\delta \geq 0$. Only (d) has real eigenvalues.

Linear systems with real eigenvalues

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Suppose for the moment that $\tau^2 > 4\delta$, so λ_1 and λ_2 are real, and $\lambda_1 < \lambda_2$.

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If $0 < \lambda_1 < \lambda_2$ then both terms will be very large when t is large, but the term $c_2 e^{\lambda_2 t} v_2$ will still grow much more quickly than $c_1 e^{\lambda_1 t} v_1$.

Linear systems with real eigenvalues — example

Consider the system

$$\begin{aligned}\dot{x} &= 2y \\ \dot{y} &= x + y\end{aligned} \qquad \begin{bmatrix}\dot{x} \\ \dot{y}\end{bmatrix} = A \begin{bmatrix}x \\ y\end{bmatrix}, \text{ where } A = \begin{bmatrix}0 & 2 \\ 1 & 1\end{bmatrix}$$

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Eigenvector $v_1 = \begin{bmatrix} p \\ q \end{bmatrix}$ should satisfy $(A - \lambda_1 I)v_1 = 0$, or $(A + I)v_1 = 0$, or $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, or $p + 2q = 0$.

Consider the system

$$\begin{aligned}\dot{x} &= 2y \\ \dot{y} &= x + y\end{aligned} \qquad\qquad \begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = A\begin{bmatrix}x\\ y\end{bmatrix}, \text{ where } A = \begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix}$$

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Linear systems with real eigenvalues — example

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$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{eigenvectors } \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ with eigenvalues } -1, 2.$$

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Linear systems with real eigenvalues — example

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$$u = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} e^{-t} + \frac{1}{3} e^{2t} \\ \frac{2}{3} e^{-t} + \frac{1}{3} e^{2t} \end{bmatrix}.$$

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To check this, note that when $t = 0$ it gives
$u = \begin{bmatrix} -4/3 + 1/3 \\ 2/3 + 1/3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ as expected.

Linear systems with real eigenvalues — example

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2c_1e^{-t} + c_2e^{2t} \\ c_1e^{-t} + c_2e^{2t} \end{bmatrix} \qquad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0/3 - x_0/3 \\ 2y_0/3 + x_0/3 \end{bmatrix}.$$

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To check this, note that when $t = 0$ it gives
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$$Au = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{4}{3}e^{-t} + \frac{1}{3}e^{2t} \\ \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t} \end{bmatrix} = \begin{bmatrix} \frac{4}{3}e^{-t} + \frac{2}{3}e^{2t} \\ -\frac{2}{3}e^{-t} + \frac{2}{3}e^{2t} \end{bmatrix},$$

so $\dot{u} = Au$ as expected.

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What are the eigenvalues of the matrix

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Linear systems with real eigenvalues — reformulation

Consider again a system $\dot{u} = Au$, where A has real eigenvalues $\lambda_1 < \lambda_2$ and corresponding eigenvectors v_1, v_2.

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Also, the solution to $\dot{u} = Au$ with $u = u_0$ at $t = 0$ is $u = Pu_0$.

Linear systems with real eigenvalues — reformulation

$$V = \left[\begin{array}{c|c} v_1 & v_2 \end{array}\right] \qquad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad E = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \qquad P = VEV^{-1}$$

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Now suppose we have a vector u_0, and we put $u = Pu_0$. When $t = 0$ we have $P = I$ so $u = u_0$. We also have $\dot{u} = \dot{P}u_0 = APu_0 = Au$ as required.

Linear systems with real eigenvalues — reformulated example

As before, take $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ $\quad \begin{aligned} \lambda_1 &= -1 \\ \lambda_2 &= 2 \end{aligned}$ $\quad v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

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$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \qquad E = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

$$V = \left[\begin{array}{c|c} v_1 & v_2 \end{array}\right] = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \qquad V^{-1} = \frac{1}{-3}\begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}.$$

Linear systems with real eigenvalues — reformulated example

As before, take $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ $\quad \begin{aligned} \lambda_1 &= -1 \\ \lambda_2 &= 2 \end{aligned}$ $\quad v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

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$$P = VEV^{-1}$$

Linear systems with real eigenvalues — reformulated example

As before, take $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ $\quad \begin{aligned} \lambda_1 &= -1 \\ \lambda_2 &= 2 \end{aligned}$ $\quad v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

Then

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \qquad E = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

$$V = \left[\begin{array}{c|c} v_1 & v_2 \end{array} \right] = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \qquad V^{-1} = \frac{1}{-3} \begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}.$$

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Linear systems with real eigenvalues — reformulated example

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Linear systems with real eigenvalues — reformulated example

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Linear systems with real eigenvalues — reformulated example

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If $u_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ then

Linear systems with real eigenvalues — reformulated example

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If $u_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ then

$$u = Pu_0$$

Linear systems with real eigenvalues — reformulated example

As before, take $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ $\qquad \begin{aligned} \lambda_1 &= -1 \\ \lambda_2 &= 2 \end{aligned}$ $\qquad v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

Then

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \qquad E = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

$$V = \left[\begin{array}{c|c} v_1 & v_2 \end{array}\right] = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \qquad V^{-1} = \frac{1}{-3} \begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}.$$

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If $u_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ then

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Then

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \qquad E = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

$$V = \left[\begin{array}{c|c} v_1 & v_2 \end{array}\right] = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \qquad V^{-1} = \frac{1}{-3} \begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}.$$

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Linear systems with real eigenvalues — reformulated example

As before, take $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ $\qquad \begin{aligned} \lambda_1 &= -1 \\ \lambda_2 &= 2 \end{aligned}$ $\qquad v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

Then

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \qquad E = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

$$V = \left[\begin{array}{c|c} v_1 & v_2 \end{array}\right] = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \qquad V^{-1} = \frac{1}{-3} \begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}.$$

$$\begin{aligned} P = VEV^{-1} &= \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \\ &= \begin{bmatrix} -2e^{-t} & e^{2t} \\ e^{-t} & e^{2t} \end{bmatrix} \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} e^{2t} + 2e^{-t} & 2e^{2t} - 2e^{-t} \\ e^{2t} - e^{-t} & 2e^{2t} + e^{-t} \end{bmatrix} \end{aligned}$$

If $u_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ then

$$u = Pu_0 = \frac{1}{3} \begin{bmatrix} e^{2t} + 2e^{-t} & 2e^{2t} - 2e^{-t} \\ e^{2t} - e^{-t} & 2e^{2t} + e^{-t} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} e^{2t} - 4e^{-t} \\ e^{2t} + 2e^{-t} \end{bmatrix}.$$

This is the same answer as before.

Linear systems with real eigenvalues — another example

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Linear systems with real eigenvalues — another example

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{aligned} \tau &= 1 + 1 = 2 \\ \delta &= 0 \end{aligned}$$

Linear systems with real eigenvalues — another example

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{aligned} \tau &= 1 + 1 = 2 \\ \delta &= 0 \end{aligned} \qquad \begin{aligned} \lambda_1 &= \tfrac{1}{2}(\tau - \sqrt{\tau^2 - 4\delta}) = 0 \\ \lambda_2 &= \tfrac{1}{2}(\tau + \sqrt{\tau^2 - 4\delta}) = 2. \end{aligned}$$

Linear systems with real eigenvalues — another example

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{aligned} \tau &= 1 + 1 = 2 \\ \delta &= 0 \end{aligned} \qquad \begin{aligned} \lambda_1 &= \tfrac{1}{2}(\tau - \sqrt{\tau^2 - 4\delta}) = 0 \\ \lambda_2 &= \tfrac{1}{2}(\tau + \sqrt{\tau^2 - 4\delta}) = 2. \end{aligned}$$

$$v_1 = \begin{bmatrix} p \\ q \end{bmatrix} \text{ with } (A - \lambda_1)v_1 = 0, \text{ so } \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ so } v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Linear systems with real eigenvalues — another example

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{aligned} \tau &= 1 + 1 = 2 \\ \delta &= 0 \end{aligned} \qquad \begin{aligned} \lambda_1 &= \tfrac{1}{2}(\tau - \sqrt{\tau^2 - 4\delta}) = 0 \\ \lambda_2 &= \tfrac{1}{2}(\tau + \sqrt{\tau^2 - 4\delta}) = 2. \end{aligned}$$

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$$v_2 = \begin{bmatrix} p \\ q \end{bmatrix} \text{ with } (A - \lambda_2)v_2 = 0, \text{ so } \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ so } v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Linear systems with real eigenvalues — another example

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{aligned} \tau &= 1+1=2 \\ \delta &= 0 \end{aligned} \qquad \begin{aligned} \lambda_1 &= \tfrac{1}{2}(\tau - \sqrt{\tau^2-4\delta}) = 0 \\ \lambda_2 &= \tfrac{1}{2}(\tau + \sqrt{\tau^2-4\delta}) = 2. \end{aligned}$$

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$$v_2 = \begin{bmatrix} p \\ q \end{bmatrix} \text{ with } (A-\lambda_2)v_2 = 0, \text{ so } \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ so } v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

Linear systems with real eigenvalues — another example

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{aligned} \tau &= 1 + 1 = 2 \\ \delta &= 0 \end{aligned} \qquad \begin{aligned} \lambda_1 &= \tfrac{1}{2}(\tau - \sqrt{\tau^2 - 4\delta}) = 0 \\ \lambda_2 &= \tfrac{1}{2}(\tau + \sqrt{\tau^2 - 4\delta}) = 2. \end{aligned}$$

$$v_1 = \begin{bmatrix} p \\ q \end{bmatrix} \text{ with } (A - \lambda_1)v_1 = 0, \text{ so } \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ so } v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

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$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \qquad E = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{2t} \end{bmatrix}.$$

Linear systems with real eigenvalues — another example

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$$v_2 = \begin{bmatrix} p \\ q \end{bmatrix} \text{ with } (A-\lambda_2)v_2 = 0, \text{ so } \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ so } v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \qquad E = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{2t} \end{bmatrix}.$$

$$V = \left[\begin{array}{c|c} v_1 & v_2 \end{array}\right] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Linear systems with real eigenvalues — another example

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{aligned} \tau &= 1+1=2 \\ \delta &= 0 \end{aligned} \qquad \begin{aligned} \lambda_1 &= \tfrac{1}{2}(\tau - \sqrt{\tau^2-4\delta}) = 0 \\ \lambda_2 &= \tfrac{1}{2}(\tau + \sqrt{\tau^2-4\delta}) = 2. \end{aligned}$$

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$$v_2 = \begin{bmatrix} p \\ q \end{bmatrix} \text{ with } (A-\lambda_2)v_2 = 0, \text{ so } \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ so } v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \qquad E = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{2t} \end{bmatrix}.$$

$$V = \left[\begin{array}{c|c} v_1 & v_2 \end{array}\right] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad V^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Linear systems with real eigenvalues — another example

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Linear systems with real eigenvalues — another example

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We have also seen that $\lambda_1 + \lambda_2 = \tau = \text{trace}(A)$, so $\det(P) = e^{\text{trace}(A)t}$.

Questions

The system $\dot{u} = Au$ has solution $u = \begin{bmatrix} 10e^{11t} - 9e^{111t} \\ 99e^{111t} - e^{11t} \end{bmatrix}$.

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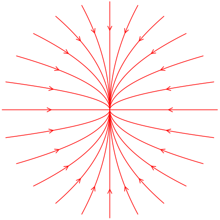
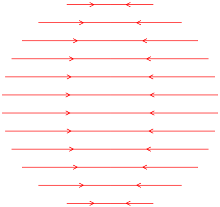
- z is a solution for $\dot{x} = (3x + y)/2$, $\dot{y} = (y - x)/2$ (which has $\lambda = 1$ as a repeated eigenvalue).

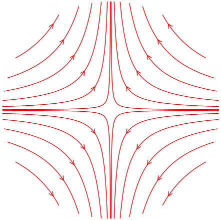
Linear systems with real eigenvalues — phase portraits



$\lambda_1 < \lambda_2 < 0$
stable node



$\lambda_1 < \lambda_2 = 0$
semistable node



$\lambda_1 < 0 < \lambda_2$
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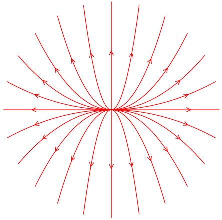
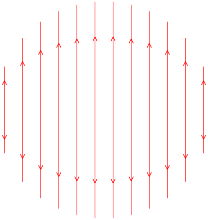


$0 < \lambda_1 < \lambda_2$
unstable node



$0 = \lambda_1 < \lambda_2$
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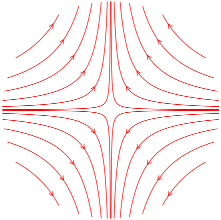

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&= \begin{bmatrix} \lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} & 0 \\ 0 & \lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} \end{bmatrix} + \begin{bmatrix} \lambda_1 (e^{\lambda_2 t} - e^{\lambda_1 t}) & 0 \\ 0 & \lambda_2 (e^{\lambda_2 t} - e^{\lambda_1 t}) \end{bmatrix} \\
&= \begin{bmatrix} (\lambda_2 - \lambda_1) e^{\lambda_1 t} & 0 \\ 0 & (\lambda_2 - \lambda_1) e^{\lambda_2 t} \end{bmatrix} = (\lambda_2 - \lambda_1)E.
\end{aligned}
$$

It follows that $P = VEV^{-1} = (\lambda_2 - \lambda_1)^{-1} VFV^{-1}$. However,

$$
VFV^{-1} = (\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})VIV^{-1} + (e^{\lambda_2 t} - e^{\lambda_1 t})VDV^{-1}
$$

A different formula for P

The solution is $u = Pu_0$, where $P = VEV^{-1}$. To find P we need V, and to find V we need the eigenvectors. However, there is another formula which is easier.

Proposition: $P = (\lambda_2 - \lambda_1)^{-1}((\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})I + (e^{\lambda_2 t} - e^{\lambda_1 t})A)$.

Proof: First put

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F &= (\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})I + (e^{\lambda_2 t} - e^{\lambda_1 t})D \\
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VFV^{-1} &= (\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})VIV^{-1} + (e^{\lambda_2 t} - e^{\lambda_1 t})VDV^{-1} \\
&= (\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})I + (e^{\lambda_2 t} - e^{\lambda_1 t})A.
\end{aligned}$$

After multiplying by $(\lambda_2 - \lambda_1)^{-1}$ we get the claimed formula for P.

A different formula for P — example

$$P = (\lambda_2 - \lambda_1)^{-1}((\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})I + (e^{\lambda_2 t} - e^{\lambda_1 t})A).$$

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Consider again $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$, so $\lambda_1 = -1$ and $\lambda_2 = 2$.

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Consider again $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$, so $\lambda_1 = -1$ and $\lambda_2 = 2$. Then

$$P = \frac{1}{3}((2e^{-t} + e^{2t})I + (e^{2t} - e^{-t})A)$$

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$$\begin{aligned} P &= \frac{1}{3}((2e^{-t} + e^{2t})I + (e^{2t} - e^{-t})A) \\ &= \frac{1}{3}\left(\begin{bmatrix} 2e^{-t} + e^{2t} & 0 \\ 0 & 2e^{-t} + e^{2t} \end{bmatrix} + \begin{bmatrix} 0 & 2e^{2t} - 2e^{-t} \\ e^{2t} - e^{-t} & e^{2t} - e^{-t} \end{bmatrix} \right) \end{aligned}$$

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&= \frac{1}{3}\begin{bmatrix} 2e^{-t} + e^{2t} & 2e^{2t} - 2e^{-t} \\ e^{2t} - e^{-t} & e^{-t} + 2e^{2t} \end{bmatrix}
\end{aligned}$$

This is the same answer as we found previously.

A different formula for P — another example

$$P = (\lambda_2 - \lambda_1)^{-1}((\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})I + (e^{\lambda_2 t} - e^{\lambda_1 t})A).$$

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Consider again $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, so $\lambda_1 = 0$ and $\lambda_2 = 2$. Then

$$P = \frac{1}{2}((2e^{0t} - 0e^{2t})I + (e^{2t} - e^{0t})A)$$

A different formula for P — another example

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Complex eigenvalues

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\lambda_1, \lambda_2 = \frac{1}{2}(\tau \mp \sqrt{\tau^2 - 4\delta})$, where $\begin{aligned} \tau &= a + d \\ \delta &= ad - bc \end{aligned}$.

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Now suppose that $\tau^2 - 4\delta < 0$, so λ_1 and λ_2 are complex numbers.

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We can use the same method as before, remembering that

$$e^{(\lambda \mp i\omega)t} = e^{\lambda t} e^{\mp i\omega t} = e^{\lambda t}(\cos(\omega t) \mp i \sin(\omega t))$$

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Some complex numbers appear, but in the end the imaginary parts cancel.

Proposition: The solution to $\dot{u} = Au$ with $u = u_0$ at $t = 0$ is $u = Pu_0$, where

$$P = e^{\lambda t}(\cos(\omega t)I + \omega^{-1} \sin(\omega t)(A - \lambda I))$$

Complex eigenvalues — formula for P

$$\lambda_1 = \lambda - i\omega \qquad \lambda_2 = \lambda + i\omega \qquad \text{Solution: } u = Pu_0$$

Proposition: $P = e^{\lambda t}(\cos(\omega t)I + \omega^{-1}\sin(\omega t)(A - \lambda I))$

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e^{\lambda_2 t} - e^{\lambda_1 t} &= e^{\lambda t}e^{i\omega t} - e^{\lambda t}e^{-i\omega t} = 2ie^{\lambda t}\sin(\omega t) \\
\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} &= (\lambda + i\omega)e^{\lambda t}e^{-i\omega t} - (\lambda - i\omega)e^{\lambda t}e^{i\omega t}
\end{aligned}$$

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&= e^{\lambda t}\left(i\omega(e^{i\omega t} + e^{-i\omega t}) - \lambda(e^{i\omega t} - e^{-i\omega t})\right)
\end{aligned}$$

Complex eigenvalues — formula for P

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\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} &= (\lambda + i\omega)e^{\lambda t}e^{-i\omega t} - (\lambda - i\omega)e^{\lambda t}e^{i\omega t} \\
&= e^{\lambda t}\left(i\omega(e^{i\omega t} + e^{-i\omega t}) - \lambda(e^{i\omega t} - e^{-i\omega t})\right) \\
&= e^{\lambda t}(2i\omega\cos(\omega t) - 2i\lambda\sin(\omega t))
\end{aligned}$$

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\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} &= (\lambda + i\omega)e^{\lambda t}e^{-i\omega t} - (\lambda - i\omega)e^{\lambda t}e^{i\omega t} \\
&= e^{\lambda t}\left(i\omega(e^{i\omega t} + e^{-i\omega t}) - \lambda(e^{i\omega t} - e^{-i\omega t})\right) \\
&= e^{\lambda t}(2i\omega\cos(\omega t) - 2i\lambda\sin(\omega t)) \\
P &= (2i\omega)^{-1}e^{\lambda t}\,(2i\omega\cos(\omega t)I - 2i\lambda\sin(\omega t)I + 2i\sin(\omega t)A)
\end{aligned}$$

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&= e^{\lambda t}\left(i\omega(e^{i\omega t} + e^{-i\omega t}) - \lambda(e^{i\omega t} - e^{-i\omega t})\right) \\
&= e^{\lambda t}(2i\omega\cos(\omega t) - 2i\lambda\sin(\omega t)) \\
P &= (2i\omega)^{-1}e^{\lambda t}\left(2i\omega\cos(\omega t)I - 2i\lambda\sin(\omega t)I + 2i\sin(\omega t)A\right) \\
&= e^{\lambda t}(\cos(\omega t)I + \omega^{-1}\sin(\omega t)(A - \lambda I)) \qquad \square
\end{aligned}$$

Complex eigenvalues — example

Suppose that $\begin{aligned} \dot{x} &= \alpha x + \beta y \\ \dot{y} &= -\beta x + \alpha y \end{aligned}$

Complex eigenvalues — example

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Thus, the solution is

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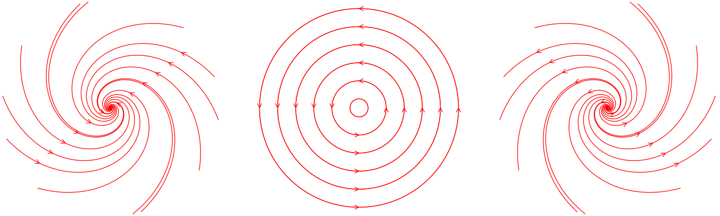
Complex eigenvalues — phase portraits

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \lambda = \frac{\tau}{2} = \frac{a+d}{2} \qquad \omega = \frac{\sqrt{4\delta - \tau^2}}{2} = \frac{\sqrt{2ad - a^2 - d^2 - 4bc}}{2}$$



$\lambda < 0,\ b < 0 < c$
anticlockwise
stable focus

$\lambda = 0,\ b < 0 < c$
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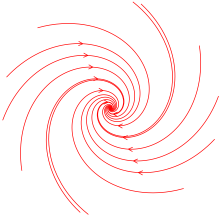

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clockwise
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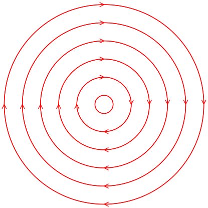

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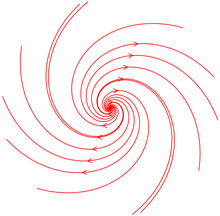

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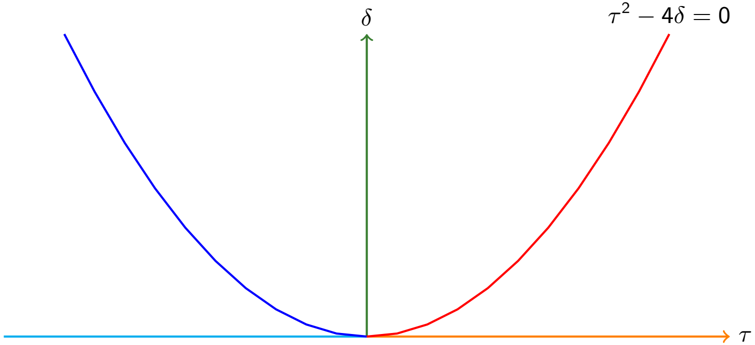
Map of the (τ, δ) plane



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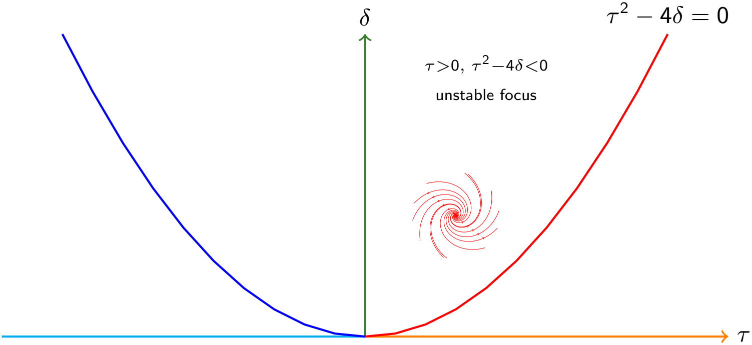

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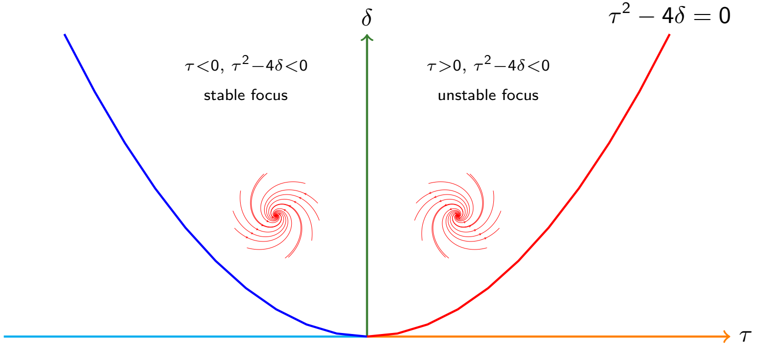

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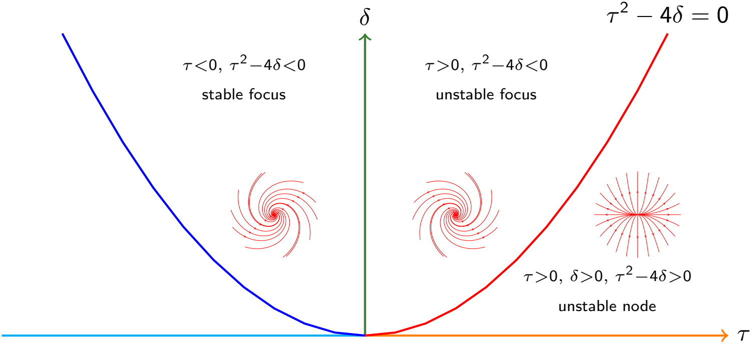

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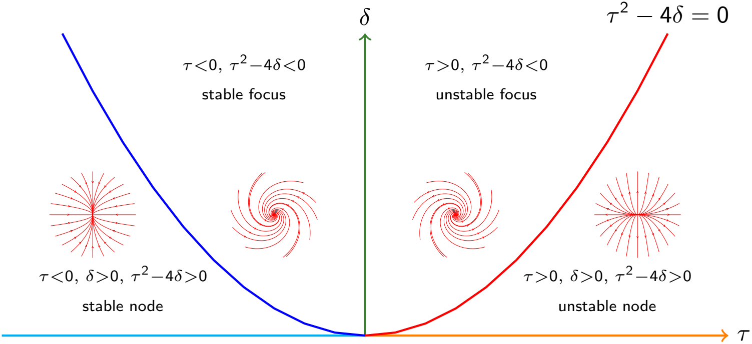

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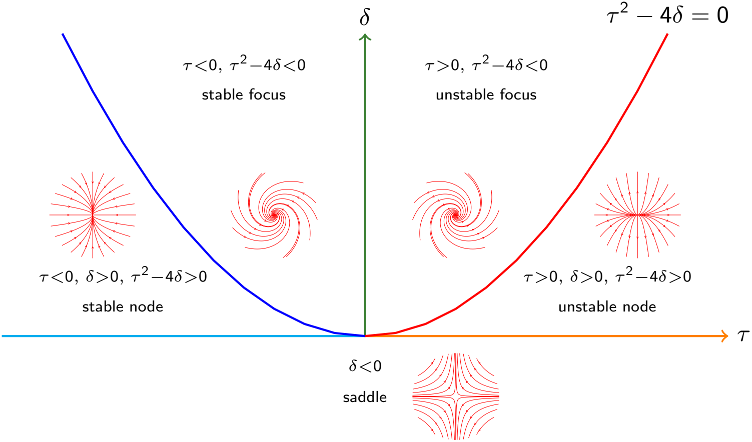

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Now consider the matrix $B = A - \lambda I = A - \frac{1}{2}(a+d)I$, so $P = e^{\lambda t}(I + tB)$. In the simplest case, B would be zero. It is not always zero, but at least $B^2 = 0$:

$$B^2 = \begin{bmatrix} a/2 - d/2 & b \\ c & d/2 - a/2 \end{bmatrix} \begin{bmatrix} a/2 - d/2 & b \\ c & d/2 - a/2 \end{bmatrix}$$

Repeated eigenvalues

Proposition: If A has only one eigenvalue, say λ then the matrix $P = e^{\lambda t}(I + t(A - \lambda I))$ satisfies $\dot{P} = AP$, and $P = I$ when $t = 0$.

Proof: Put $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so $\tau = a + d$ and $\delta = ad - bc$.

The eigenvalues $(\tau \pm \sqrt{\tau^2 - 4\delta})/2$ are the same, so we must have $\tau^2 = 4\delta$, and the eigenvalue is $\lambda = \tau/2 = a/2 + d/2$. Note that

$$\tau^2 - 4\delta = (a+d)^2 - 4ad + 4bc = a^2 + 2ad + d^2 - 4ad + 4bc = (a-d)^2 + 4bc,$$

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Repeated eigenvalues

Proposition: If A has only one eigenvalue, say λ then the matrix $P = e^{\lambda t}(I + t(A - \lambda I))$ satisfies $\dot{P} = AP$, and $P = I$ when $t = 0$.

The matrix $B = A - \lambda I$ satisfies $B^2 = 0$.

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$$\dot{P} = \lambda e^{\lambda t}(I + tB) + e^{\lambda t} B$$

Repeated eigenvalues

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$$\dot{P} = \lambda e^{\lambda t}(I + tB) + e^{\lambda t}B = e^{\lambda t}(\lambda I + t\lambda B + B)$$

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$$\begin{aligned} \dot{P} &= \lambda e^{\lambda t}(I + tB) + e^{\lambda t} B = e^{\lambda t}(\lambda I + t\lambda B + B) \\ AP &= (\lambda I + B)P \end{aligned}$$

Repeated eigenvalues

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$$\dot{P} = \lambda e^{\lambda t}(I + tB) + e^{\lambda t}B = e^{\lambda t}(\lambda I + t\lambda B + B)$$

$$AP = (\lambda I + B)P = e^{\lambda t}(\lambda I + B)(I + tB)$$

Repeated eigenvalues

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The matrix $B = A - \lambda I$ satisfies $B^2 = 0$.

Next, we defined P to be $e^{\lambda t}(I + tB)$. This satisfies

$$\begin{aligned} \dot{P} &= \lambda e^{\lambda t}(I + tB) + e^{\lambda t}B = e^{\lambda t}(\lambda I + t\lambda B + B) \\ AP &= (\lambda I + B)P = e^{\lambda t}(\lambda I + B)(I + tB) \\ &= e^{\lambda t}(\lambda I + t\lambda B + B + tB^2) \end{aligned}$$

Repeated eigenvalues

Proposition: If A has only one eigenvalue, say λ then the matrix $P = e^{\lambda t}(I + t(A - \lambda I))$ satisfies $\dot{P} = AP$, and $P = I$ when $t = 0$.

The matrix $B = A - \lambda I$ satisfies $B^2 = 0$.

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$$\begin{aligned}
\dot{P} &= \lambda e^{\lambda t}(I + tB) + e^{\lambda t}B = e^{\lambda t}(\lambda I + t\lambda B + B) \\
AP &= (\lambda I + B)P = e^{\lambda t}(\lambda I + B)(I + tB) \\
&= e^{\lambda t}(\lambda I + t\lambda B + B + tB^2) = e^{\lambda t}(\lambda I + t\lambda B + B)
\end{aligned}$$

Proposition: If A has only one eigenvalue, say λ then the matrix $P = e^{\lambda t}(I + t(A - \lambda I))$ satisfies $\dot{P} = AP$, and $P = I$ when $t = 0$.

The matrix $B = A - \lambda I$ satisfies $B^2 = 0$.

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$$\begin{aligned}\dot{P} &= \lambda e^{\lambda t}(I + tB) + e^{\lambda t}B = e^{\lambda t}(\lambda I + t\lambda B + B)\\ AP &= (\lambda I + B)P = e^{\lambda t}(\lambda I + B)(I + tB)\\ &= e^{\lambda t}(\lambda I + t\lambda B + B + tB^2) = e^{\lambda t}(\lambda I + t\lambda B + B) = \dot{P}\end{aligned}$$

as claimed.

Repeated eigenvalues

Proposition: If A has only one eigenvalue, say λ then the matrix $P = e^{\lambda t}(I + t(A - \lambda I))$ satisfies $\dot{P} = AP$, and $P = I$ when $t = 0$.

The matrix $B = A - \lambda I$ satisfies $B^2 = 0$.

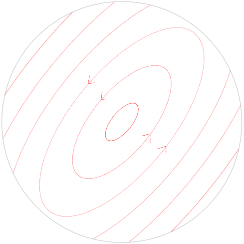
Next, we defined P to be $e^{\lambda t}(I + tB)$. This satisfies

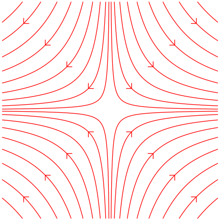
$$\begin{aligned}
\dot{P} &= \lambda e^{\lambda t}(I + tB) + e^{\lambda t}B = e^{\lambda t}(\lambda I + t\lambda B + B) \\
AP &= (\lambda I + B)P = e^{\lambda t}(\lambda I + B)(I + tB) \\
&= e^{\lambda t}(\lambda I + t\lambda B + B + tB^2) = e^{\lambda t}(\lambda I + t\lambda B + B) = \dot{P}
\end{aligned}$$

as claimed. Also, at $t = 0$ we have $P = e^0(I + 0B) = I$.

Questions

Which types are shown here?





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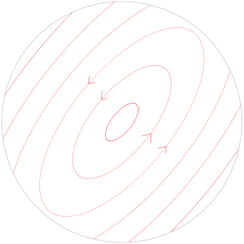
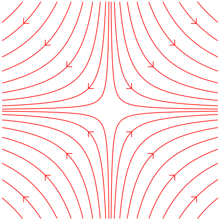
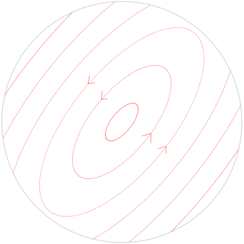




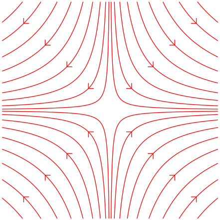
Anticlockwise centre

Questions

Which types are shown here?



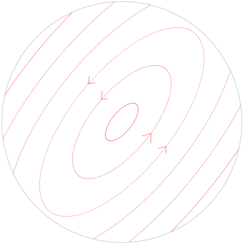
Anticlockwise centre



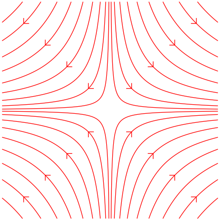
Saddle

Questions

Which types are shown here?



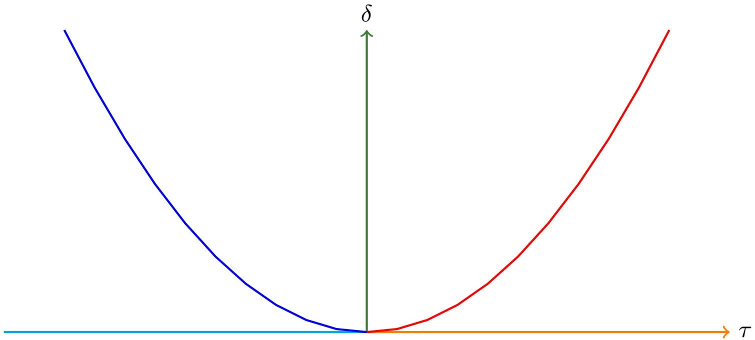
Anticlockwise centre



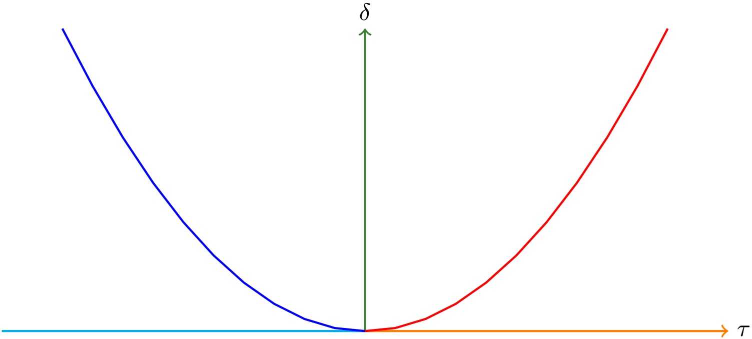
Saddle

Stable node

Questions

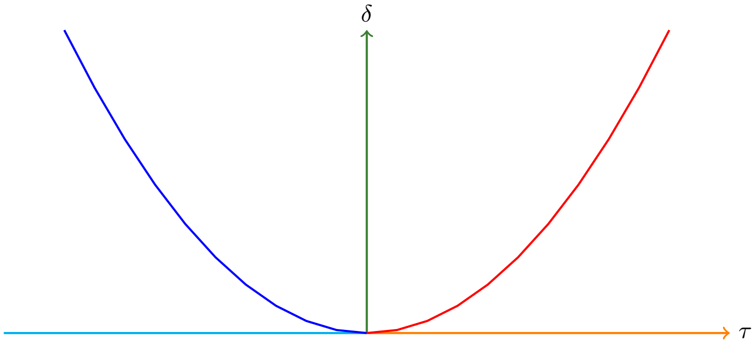


Questions



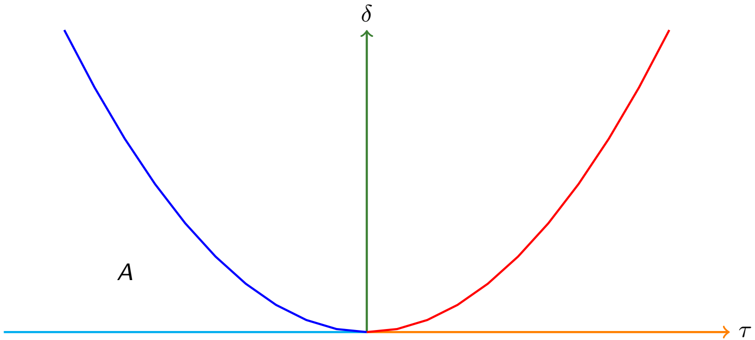
- What is the equation of the red curve?

Questions



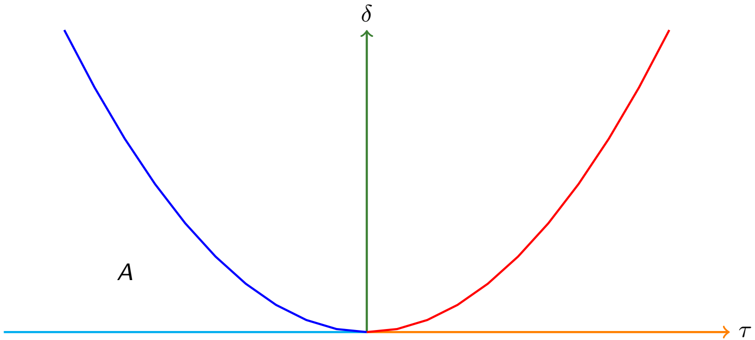
- What is the equation of the red curve? $\tau^2 - 4\delta = 0$

Questions



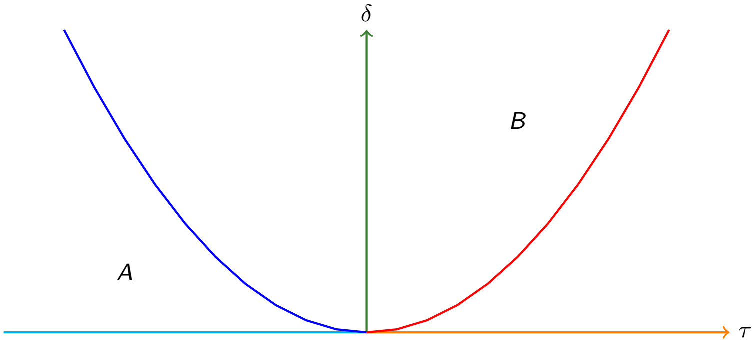
- What is the equation of the red curve? $\tau^2 - 4\delta = 0$
- What type is region A?

Questions



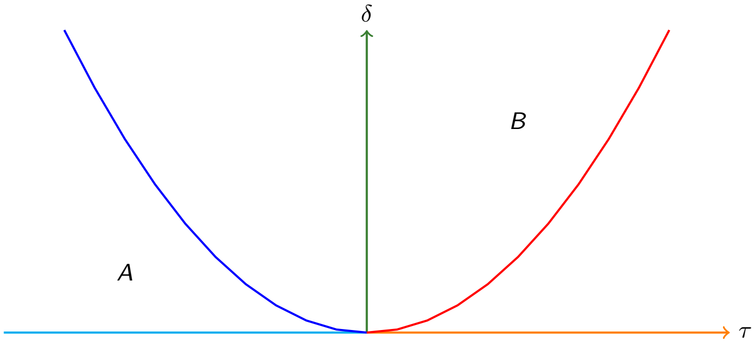
- What is the equation of the red curve? $\tau^2 - 4\delta = 0$
- What type is region A? Stable node.

Questions



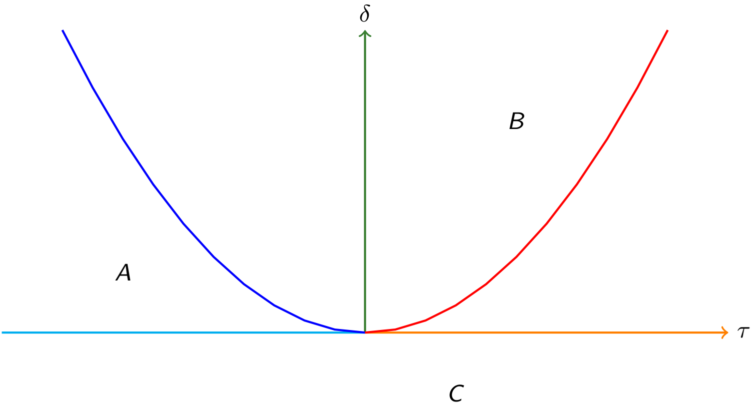
- What is the equation of the red curve? $\tau^2 - 4\delta = 0$
- What type is region A? Stable node.
- What are the equations for region B?

Questions



- What is the equation of the red curve? $\tau^2 - 4\delta = 0$
- What type is region A? Stable node.
- What are the equations for region B? $4\delta > \tau^2$, $\tau > 0$.

Questions



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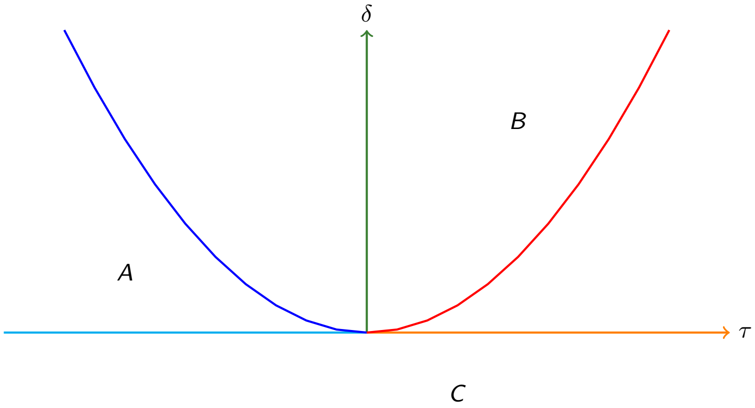
Questions



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- What are the equations for region B? $4\delta > \tau^2$, $\tau > 0$.
- What can we say about the eigenvalues in region C?
  Both real, one positive, one negative.

Equilibrium points and stability

Consider a differential equation $\dot{x} = f(x, y), \ \dot{y} = g(x, y)$.

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What happens if we start at a point (x_0, y_0) that is very close to (a, b)?

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What happens if we start at a point (x_0, y_0) that is very close to (a, b)? Then $f(x_0, y_0)$ and $g(x_0, y_0)$ will be small, so the point will move slowly at first.

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(a) The point might move closer and closer to (a, b), and slow down even more, with $(x, y) \to (a, b)$ and $(\dot{x}, \dot{y}) \to (0, 0)$ as $t \to \infty$.

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(b) The point might circle around (a,b), never moving very far away, but not slowing down.

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(b) The point might circle around (a, b), never moving very far away, but not slowing down.

(c) The point might eventually move far away from (a, b).

Equilibrium points and stability

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If (a) always happens, the equilibrium point is *asymptotically stable* .

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(c) The point might eventually move far away from (a, b).

If (a) always happens, the equilibrium point is *asymptotically stable* .
If (b) can also happen (but not (c)), the equilibrium point is *stable* .

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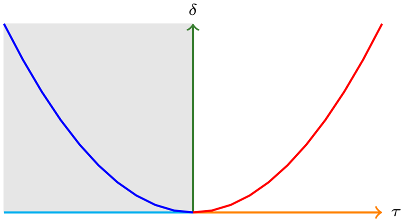
Equilibrium points for linear systems

Consider a linear system $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, with $\tau = a + d$ and $\delta = ad - bc$.



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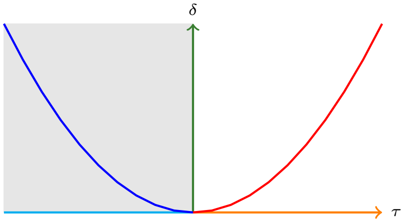

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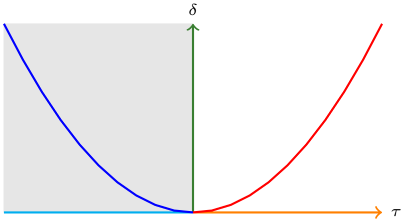

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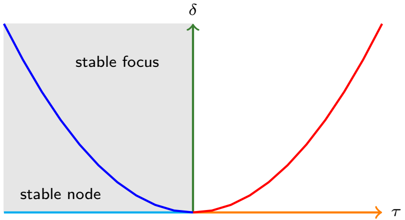

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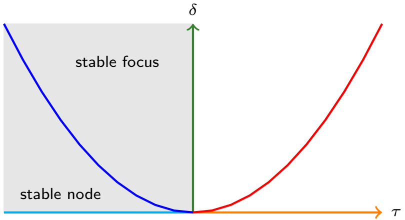

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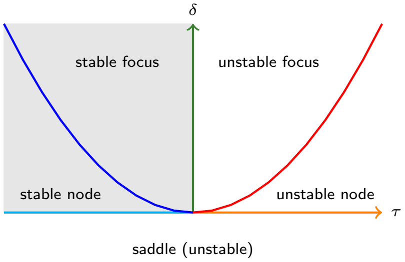

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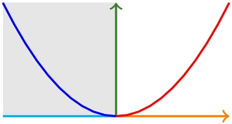

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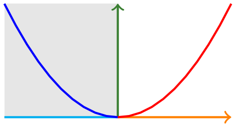

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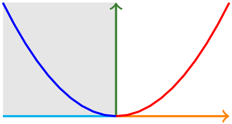

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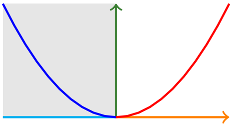

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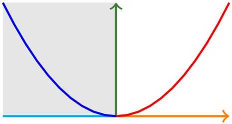

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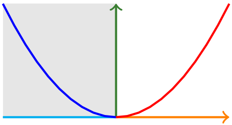

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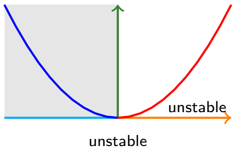

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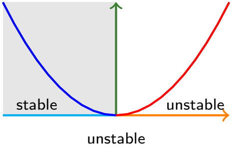

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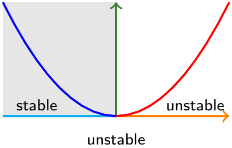

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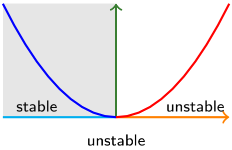

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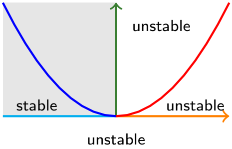

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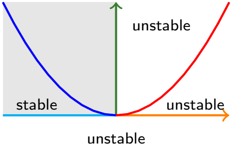

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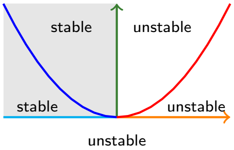

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This is a linear system with matrix

$$J = \begin{bmatrix} f_x(a, b) & f_y(a, b) \\ g_x(a, b) & g_y(a, b) \end{bmatrix},$$

called the *Jacobian*.

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Usually the flow lines for the original nonlinear system will be similar to those for the linearised system, at least if we look close to (a, b).

Linearisation example

Consider the system $\dot{x} = 9y^2 - 1$, $\dot{y} = 9x^2 - 1$.

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It is easy to see that the vectors $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are eigenvectors, with eigenvalues $\lambda_1 = 6$ and $\lambda_2 = -6$.

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$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = a_1 e^{6t} v_1 + a_2 e^{-6t} v_2 = \begin{bmatrix} a_1 e^{6t} + a_2 e^{-6t} \\ a_1 e^{6t} - a_2 e^{-6t} \end{bmatrix}.$$

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As $x = 1/3 + \alpha$ and $y = 1/3 + \beta$, the corresponding approximate solutions for the original system are

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} + a_1 e^{6t} v_1 + a_2 e^{-6t} v_2 = \begin{bmatrix} 1/3 + a_1 e^{6t} + a_2 e^{-6t} \\ 1/3 + a_1 e^{6t} - a_2 e^{-6t} \end{bmatrix}.$$

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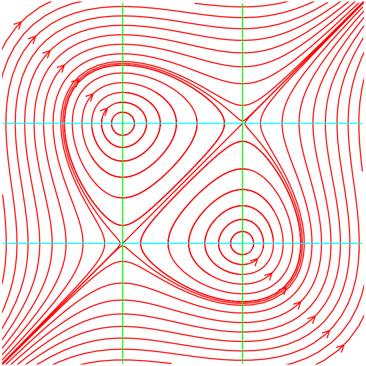

These are solutions for the original system.

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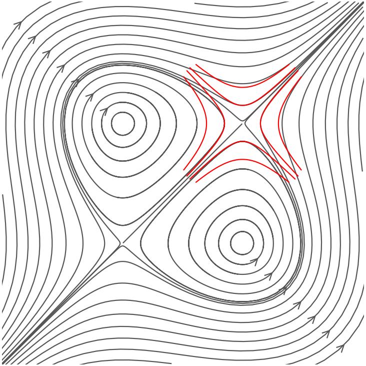

These are solutions for the linearised system.

The eigenvectors, more slowly

We had $J = \begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}$. This has $\tau = 0$ and $\delta = -36$ so $\tau^2 - 4\delta = 144$.
This gives eigenvalues $(0 \pm \sqrt{144})/2$, so $\lambda_1 = -6$ and $\lambda_2 = 6$.

The eigenvector $v_1 = \begin{bmatrix} p \\ q \end{bmatrix}$ must satisfy $(J - \lambda_1 I)v_1 = 0$, or $\begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,
which means that $p + q = 0$. We can therefore take $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

The vector $v_2 = \begin{bmatrix} r \\ s \end{bmatrix}$ must satisfy $(J - \lambda_2 I)v_2 = 0$, or $\begin{bmatrix} -6 & 6 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,
which means that $p - q = 0$. We can therefore take $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

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Some solutions are

$$x = -1/3 + \alpha = -1/3 + R\cos(6t) \qquad y = 1/3 + \beta = 1/3 - R\sin(6t)$$

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This means that the solution curves are circles centred at $(-1/3, 1/3)$.

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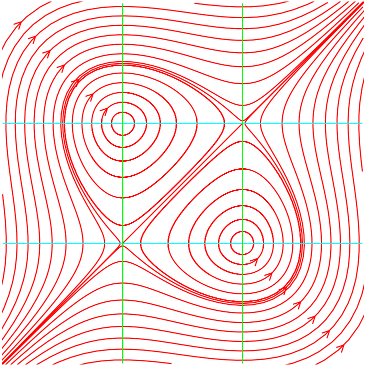

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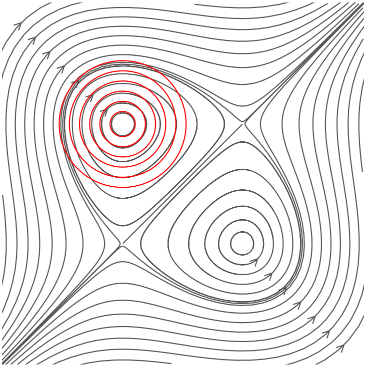

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The damped Duffing oscillator

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In particular, the solution with $\begin{bmatrix} \sqrt{2} + \alpha_0 \\ 0 \end{bmatrix}$ at time $t = 0$ is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} + e^{-0.05t}\alpha_0 \begin{bmatrix} \cos(2t) + 0.025\sin(2t) \\ -2\sin(2t) \end{bmatrix}.$$

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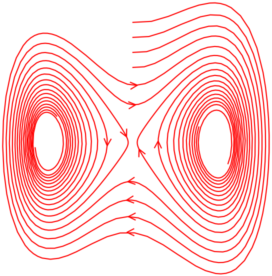

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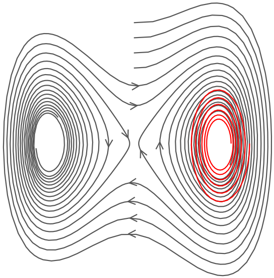

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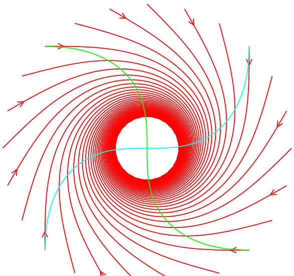

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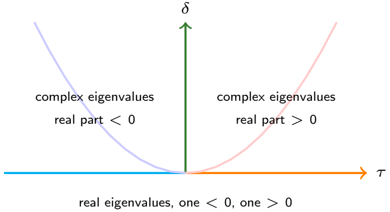

The theorem applies unless $\delta = 0$, or ($\tau = 0$ and $\delta \geq 0$).

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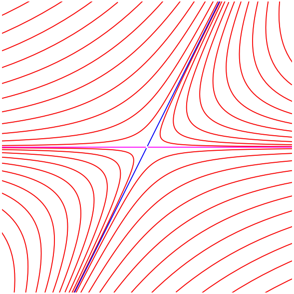

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Consider the system $\dot{x} = -x + y + 3y^2$, $\dot{y} = y$.
The origin is an equilibrium, and the linearisation is $\dot{x} = -x + y$, $\dot{y} = y$, with solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (x_0 + \frac{1}{2}y_0)e^t - \frac{1}{2}y_0 e^{-t} \\ y_0 e^t \end{bmatrix}.$$

Suppose that x and y obey the linear equations, and we put $(X, Y) = (x + y^2, y)$. Then $\dot{Y} = Y$ and

$$\dot{X} = \dot{x} + 2y\dot{y} = -x + y + 2y^2 = -x - y^2 + y + 3y^2$$

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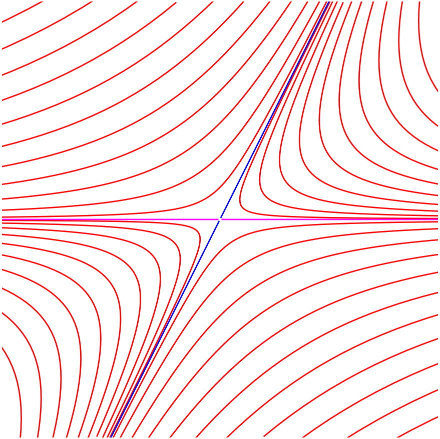
Hartman-Grobman example



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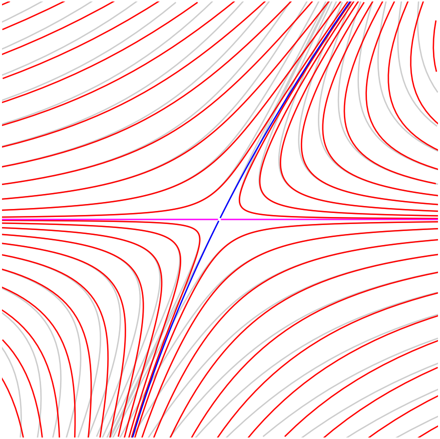

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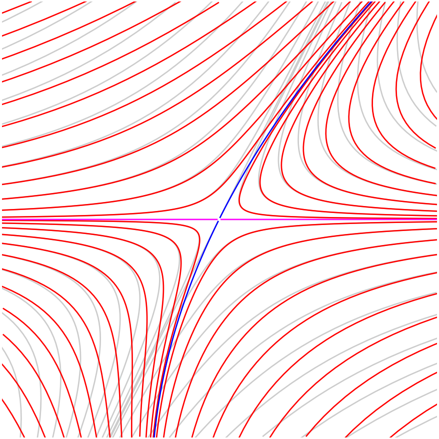

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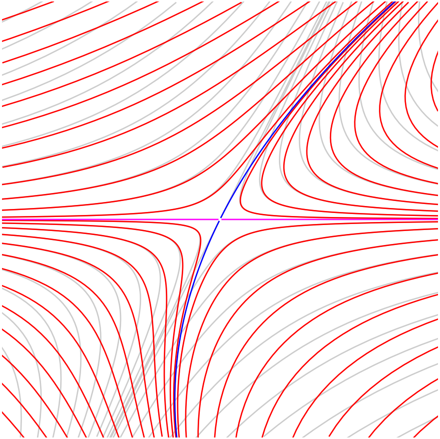

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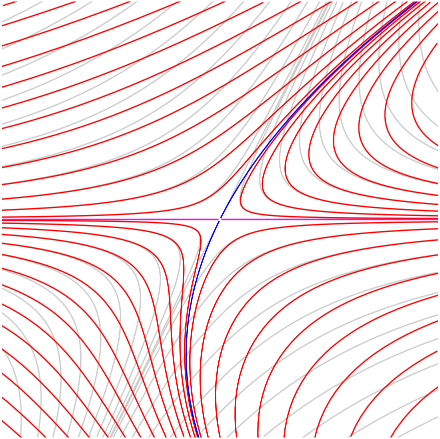

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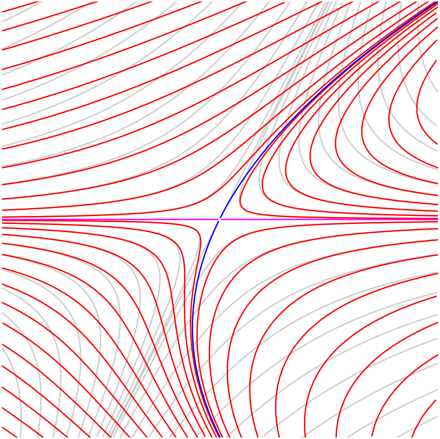

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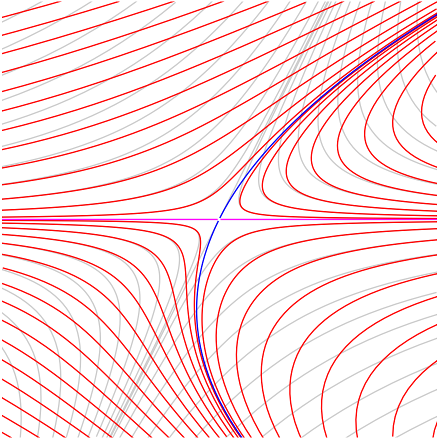

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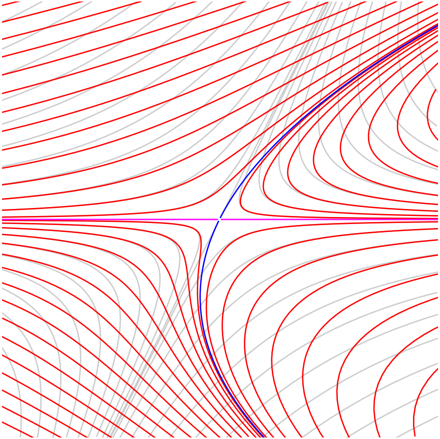

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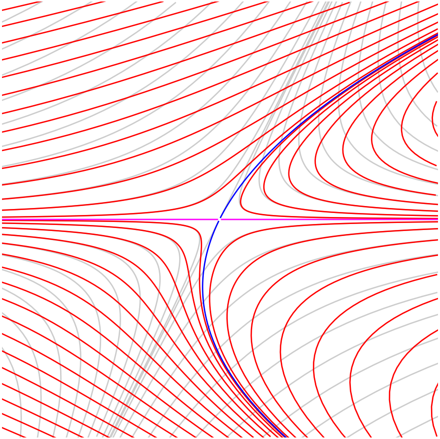

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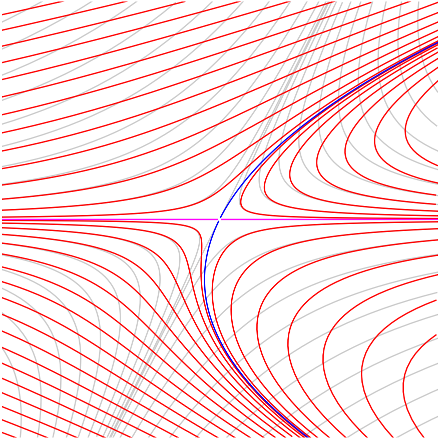

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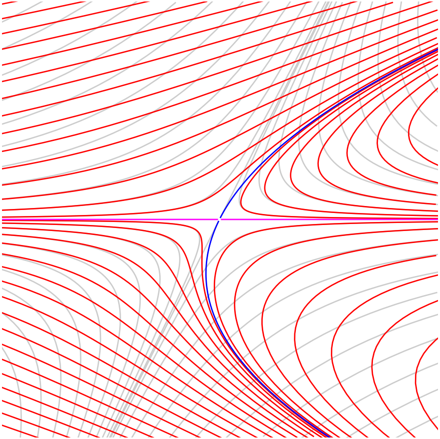

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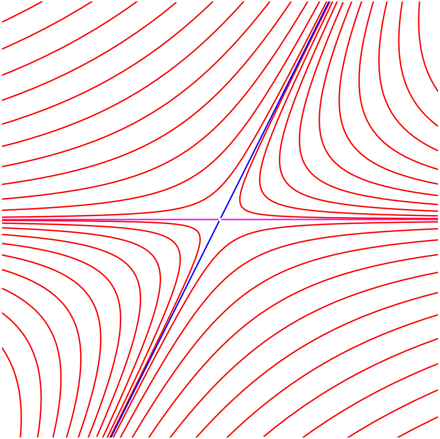

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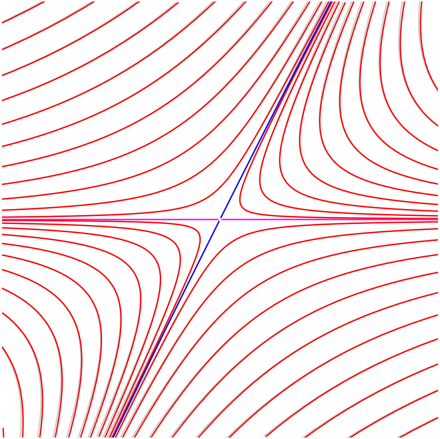

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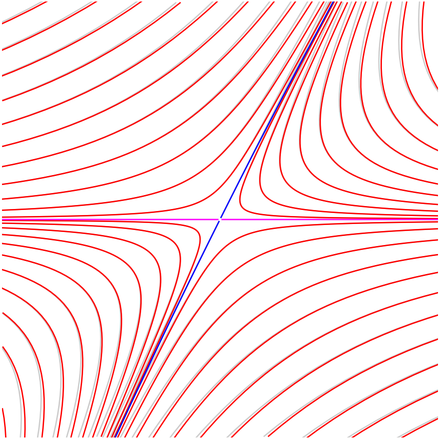

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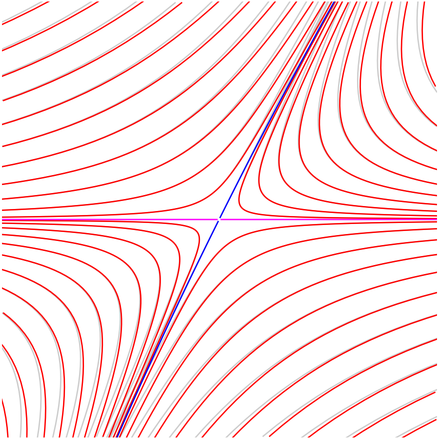

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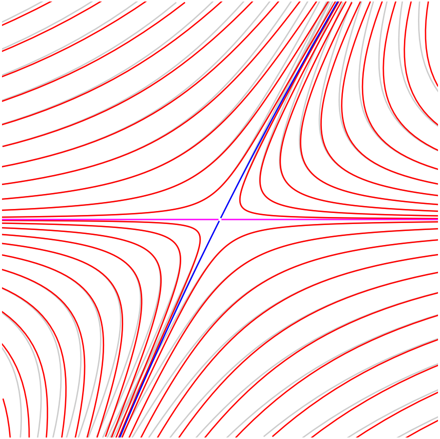

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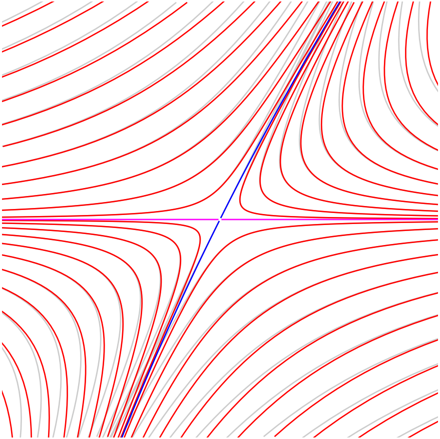

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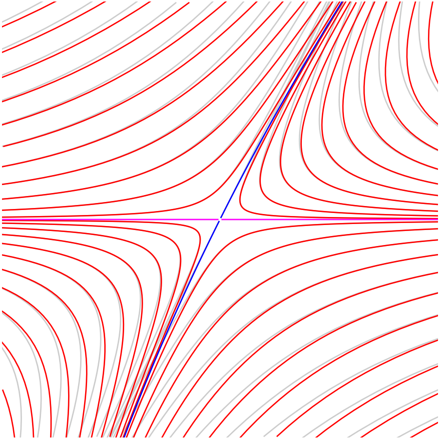

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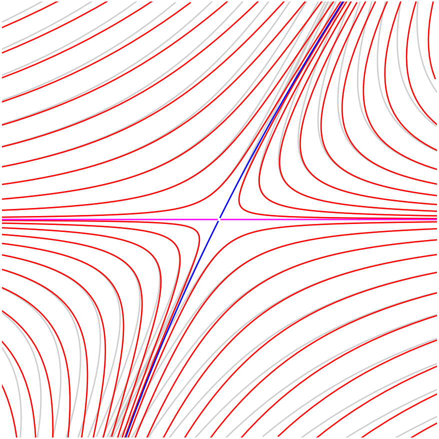

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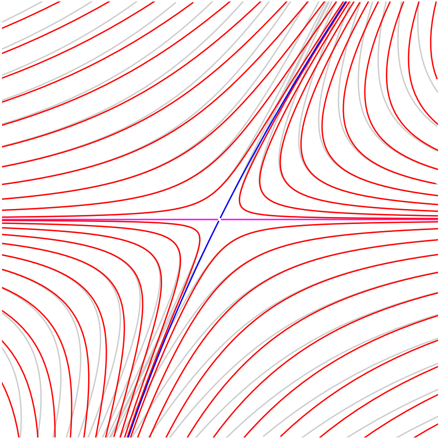

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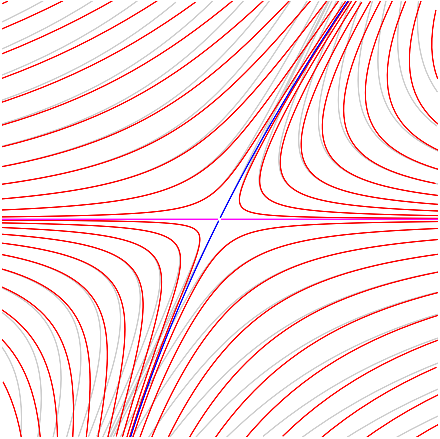

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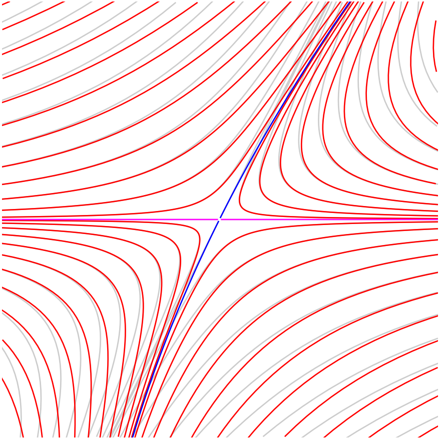

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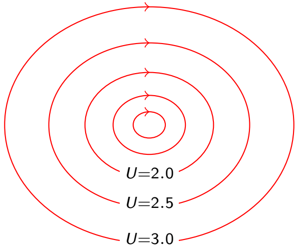

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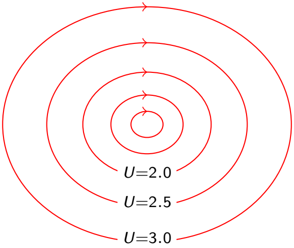

More generally, suppose $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ -c & -a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

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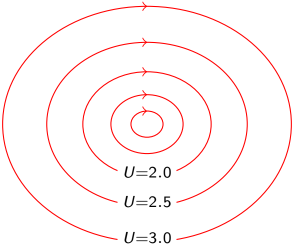

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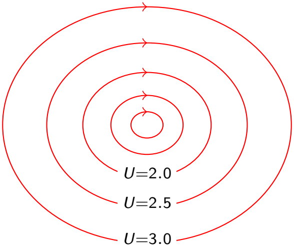

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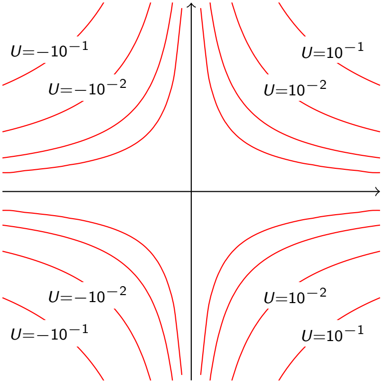
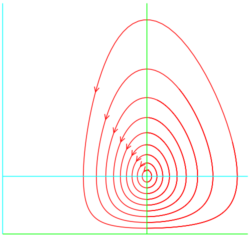
The picture shows the case $n = 4$, $m = 3$.



Conserved quantity for the Lotka-Volterra model

Recall the Lotka-Volterra model for populations of fish and sharks:
$\dot{F} = (\alpha - \beta S)F$, $\dot{S} = (\delta F - \gamma)S$.



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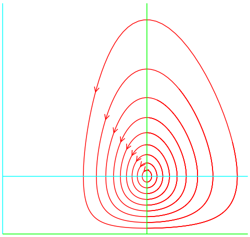

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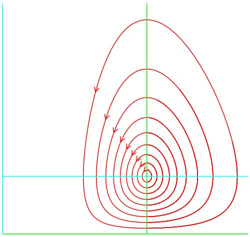

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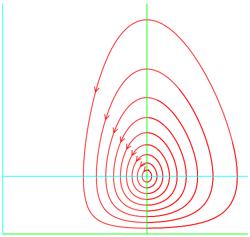

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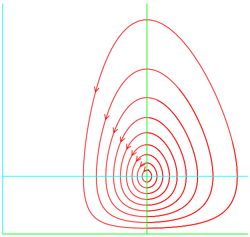

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&= \alpha(\delta F - \gamma) + \gamma(\alpha - \beta S) - \beta(\delta F - \gamma)S - \delta(\alpha - \beta S)F \\
&= \alpha\delta F - \alpha\gamma + \alpha\gamma - \beta\gamma S - \beta\delta SF + \beta\gamma S - \alpha\delta F + \beta\delta SF \\
&= 0,
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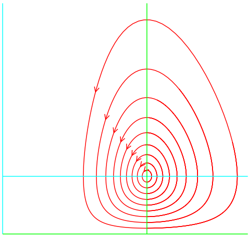

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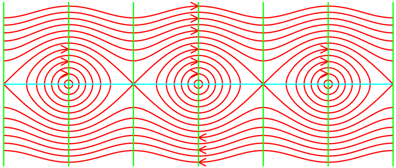

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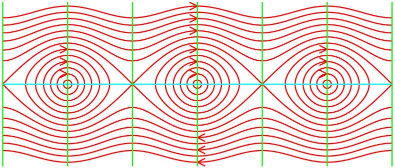

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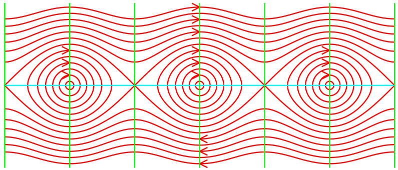

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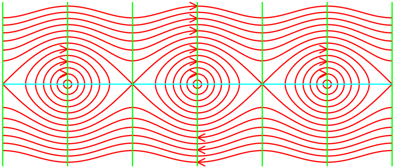

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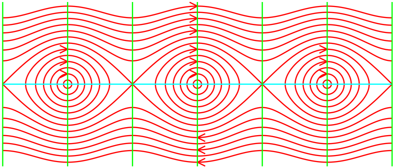

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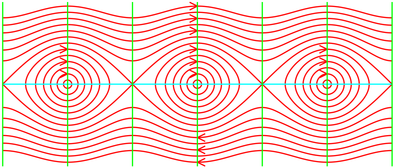
In this case, there is a clear physical interpretation: $\frac{1}{2}\omega^2$ is the rotational kinetic energy , $-\cos(\theta)$ is the gravitational potential energy , and U is the total energy .



Conserved quantity means no nodes or foci

Proposition: If there is a conserved quantity U, there are no nodes or foci. (Unless there is a nonempty open region where U is constant.)

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- This means that $U(x(t), y(t)) \to U(a, b)$.

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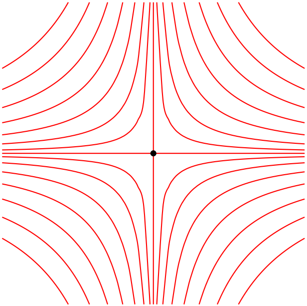
- If there is an unstable node or focus, consider $t \to -\infty$ instead.

$\square$

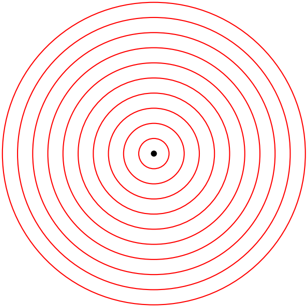
Saddles and centres are possible



$\dot{x} = x, \ \dot{y} = -y$

$U = xy$ is conserved

The origin is a saddle



$\dot{x} = -y, \ \dot{y} = x$

$U = x^2 + y^2$ is conserved

The origin is a centre

Conserved quantity with arctan

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- Put $V = \arctan(y/x)$ and $W = \frac{1}{2}\ln(x^2 + y^2)$ and $U = V + W$. Claim: U is conserved.

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$$\dot{x} = -x - y \quad \dot{y} = x - y$$

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Recall that $\arctan'(z) = 1/(1 + z^2)$. Using this, we get

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\dot{V} &= \arctan'(y/x)\frac{d}{dt}(y/x) = \frac{1}{1 + y^2/x^2}\,\frac{\dot{y}x - y\dot{x}}{x^2} \\
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As expected, this does not depend on t.

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- If V is positive definite and $\dot{V}$ is negative semidefinite, we say that V is a *weak Lyapunov function*.

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- If V is positive definite and $\dot{V}$ is negative semidefinite, we say that V is a *weak Lyapunov function*.
- If V is positive definite and $\dot{V}$ is negative definite, we say that V is a *strong Lyapunov function*.

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- If V is positive definite and $\dot{V}$ is negative definite, we say that V is a *strong Lyapunov function*.
- If there is a strong Lyapunov function, then the origin is an asymptotically stable equilibrium point.

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- If V is positive definite and $\dot{V}$ is negative semidefinite, we say that V is a *weak Lyapunov function*.
- If V is positive definite and $\dot{V}$ is negative definite, we say that V is a *strong Lyapunov function*.
- If there is a strong Lyapunov function, then the origin is an asymptotically stable equilibrium point.
- If there is a weak Lyapunov function, then the origin is an stable equilibrium point, but may not be asymptotically stable.

Lyapunov functions

Definition: Consider a differentiable function $V(x, y)$, defined on some open region R containing a point (a, b).

- If $V(x, y) \geq 0$ for all (x, y), we say that V is *positive semi-definite*.
- If $V(a, b) = 0$ but $V(x, y) > 0$ for all $(x, y) \neq (a, b)$, we say that V is *positive definite* around (a, b).
- If $V(x, y) \leq 0$ for all (x, y), we say that V is *negative semi-definite*.
- If $V(a, b) = 0$ but $V(x, y) < 0$ for all $(x, y) \neq (a, b)$, we say that V is *negative definite* around (a, b).

Now suppose that x and y change following the equations $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$, so for any function $V(x, y)$ we have $\dot{V} = V_x f + V_y g$. Suppose also that $f(0, 0) = g(0, 0) = 0$, so $(0, 0)$ is an equilibrium point.

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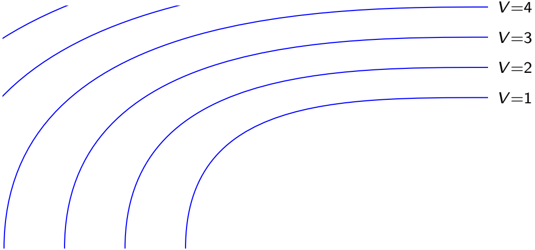
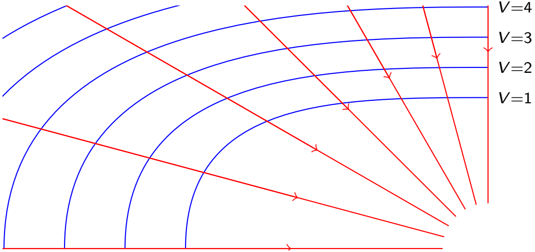
- If V is positive definite and $\dot{V}$ is negative definite, we say that V is a *strong Lyapunov function*.
- If there is a strong Lyapunov function, then the origin is an asymptotically stable equilibrium point.
- If there is a weak Lyapunov function, then the origin is an stable equilibrium point, but may not be asymptotically stable.
- Note: any positive definite conserved quantity is a weak Lyapunov function.

Contours of a Lyapunov function



The blue lines are the contours for a function $V(x, y)$.

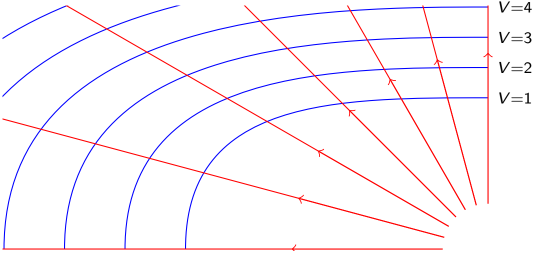
Contours of a Lyapunov function



The blue lines are the contours for a function $V(x, y)$.

These red lines show a flow that cuts across the contours going downwards, so V decreases as we move along this flow. The function V could be a Lyapunov function for this flow.

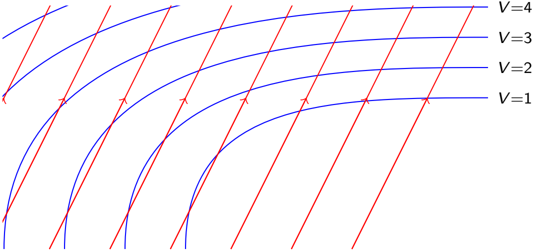
Contours of a Lyapunov function



The blue lines are the contours for a function $V(x, y)$.

These red lines show a flow that cuts across the contours going upwards, so V increases as we move along this flow. The function V could not be a Lyapunov function for this flow.

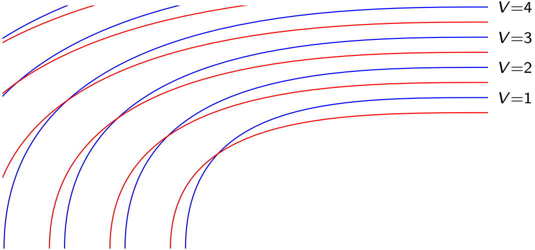
Contours of a Lyapunov function



The blue lines are the contours for a function $V(x, y)$.

These red lines show a flow that cuts across the contours sometimes going upwards and sometimes going downwards. As we move along the flow, V sometimes increases and sometimes decreases. The function V could not be a Lyapunov function for this flow.

Contours of a Lyapunov function



The blue lines are the contours for a function $V(x, y)$.

These red lines show a flow that cuts across the contours at a shallow angle. As we move along the flow, the function V decreases, but only slowly.

Definiteness for quadratic functions

Consider a quadratic function $Q = ax^2 + 2bxy + cy^2$.

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Proof:

(a) If $a = 0$ or $c = 0$ or a, c have opposite sign then $ac \leq 0$

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Proof:

(a) If $a = 0$ or $c = 0$ or a, c have opposite sign then $ac \leq 0$ so $ac - b^2 \leq 0$. Thus, if $ac - b^2 > 0$ then a and c must be nonzero with the same sign.

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Proof:

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(b) Suppose that $ac - b^2 > 0$ with $a, c > 0$. We then find that

$$a^{-1}((ax + by)^2 + (ac - b^2)y^2) = a^{-1}(a^2x^2 + 2abxy + b^2y^2 + acy^2 - b^2y^2)$$

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Proof:

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(b) Suppose that $ac - b^2 > 0$ with $a, c > 0$. We then find that

$$\begin{aligned} a^{-1}((ax+by)^2 + (ac-b^2)y^2) &= a^{-1}(a^2x^2 + 2abxy + b^2y^2 + acy^2 - b^2y^2) \\ &= ax^2 + 2bxy + cy^2 = Q. \end{aligned}$$

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Proof:

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(b) Suppose that $ac - b^2 > 0$ with $a, c > 0$. We then find that

$$\begin{aligned} a^{-1}((ax+by)^2 + (ac-b^2)y^2) &= a^{-1}(a^2x^2 + 2abxy + b^2y^2 + acy^2 - b^2y^2) \\ &= ax^2 + 2bxy + cy^2 = Q. \end{aligned}$$

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Proof:

(a) If $a = 0$ or $c = 0$ or a, c have opposite sign then $ac \leq 0$ so $ac - b^2 \leq 0$. Thus, if $ac - b^2 > 0$ then a and c must be nonzero with the same sign.

(b) Suppose that $ac - b^2 > 0$ with $a, c > 0$. We then find that

$$\begin{aligned} a^{-1}((ax+by)^2 + (ac-b^2)y^2) &= a^{-1}(a^2x^2 + 2abxy + b^2y^2 + acy^2 - b^2y^2) \\ &= ax^2 + 2bxy + cy^2 = Q. \end{aligned}$$

This representation makes it clear that $Q \geq 0$. Moreover Q can only be equal to 0 if $ax + by = 0$ and $y = 0$, which means that $x = y = 0$. Thus, Q is positive definite.

(c) Suppose instead that $ac - b^2 > 0$ with $a, c < 0$.

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Proof:

(a) If $a = 0$ or $c = 0$ or a, c have opposite sign then $ac \leq 0$ so $ac - b^2 \leq 0$. Thus, if $ac - b^2 > 0$ then a and c must be nonzero with the same sign.

(b) Suppose that $ac - b^2 > 0$ with $a, c > 0$. We then find that

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This representation makes it clear that $Q \geq 0$. Moreover Q can only be equal to 0 if $ax + by = 0$ and $y = 0$, which means that $x = y = 0$. Thus, Q is positive definite.

(c) Suppose instead that $ac - b^2 > 0$ with $a, c < 0$. We can then use (b) to show that $-Q$ is positive definite

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Proof:

(a) If $a = 0$ or $c = 0$ or a, c have opposite sign then $ac \leq 0$ so $ac - b^2 \leq 0$. Thus, if $ac - b^2 > 0$ then a and c must be nonzero with the same sign.

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This representation makes it clear that $Q \geq 0$. Moreover Q can only be equal to 0 if $ax + by = 0$ and $y = 0$, which means that $x = y = 0$. Thus, Q is positive definite.

(c) Suppose instead that $ac - b^2 > 0$ with $a, c < 0$. We can then use (b) to show that $-Q$ is positive definite, and this means that Q is negative definite.

Definiteness for quadratic functions

Consider a quadratic function $Q = ax^2 + 2bxy + cy^2$.

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Proof continued:

(d) Now suppose that $ac - b^2 \leq 0$.

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Proof continued:

(d) Now suppose that $ac - b^2 \leq 0$. We need to show that Q is indefinite, so we need to find a point $(x, y) \neq (0, 0)$ where $Q = 0$.

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(d) Now suppose that $ac - b^2 \leq 0$. We need to show that Q is indefinite, so we need to find a point $(x, y) \neq (0, 0)$ where $Q = 0$. If $a \neq 0$ we note that $x = (-b + \sqrt{b^2 - ac})/a$ is a root of $ax^2 + 2bx + c = 0$

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(d) Now suppose that $ac - b^2 \leq 0$. We need to show that Q is indefinite, so we need to find a point $(x, y) \neq (0, 0)$ where $Q = 0$. If $a \neq 0$ we note that $x = (-b + \sqrt{b^2 - ac})/a$ is a root of $ax^2 + 2bx + c = 0$, so $Q = 0$ at $(x, 1)$. Similarly, if $c \neq 0$ then $y = (-b + \sqrt{b^2 - ac})/c$ is a root of $a + 2by + cy^2 = 0$

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Proof continued:

(d) Now suppose that $ac - b^2 \leq 0$. We need to show that Q is indefinite, so we need to find a point $(x, y) \neq (0, 0)$ where $Q = 0$. If $a \neq 0$ we note that $x = (-b + \sqrt{b^2 - ac})/a$ is a root of $ax^2 + 2bx + c = 0$, so $Q = 0$ at $(x, 1)$. Similarly, if $c \neq 0$ then $y = (-b + \sqrt{b^2 - ac})/c$ is a root of $a + 2by + cy^2 = 0$, so $Q = 0$ at $(1, y)$.

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Proof continued:

(d) Now suppose that $ac - b^2 \leq 0$. We need to show that Q is indefinite, so we need to find a point $(x, y) \neq (0, 0)$ where $Q = 0$. If $a \neq 0$ we note that $x = (-b + \sqrt{b^2 - ac})/a$ is a root of $ax^2 + 2bx + c = 0$, so $Q = 0$ at $(x, 1)$. Similarly, if $c \neq 0$ then $y = (-b + \sqrt{b^2 - ac})/c$ is a root of $a + 2by + cy^2 = 0$, so $Q = 0$ at $(1, y)$. This just leaves the case where $a = c = 0$ and $Q = 2bxy$

Definiteness for quadratic functions

Consider a quadratic function $Q = ax^2 + 2bxy + cy^2$.

(a) If $ac - b^2 > 0$ then a and c are nonzero and have the same sign.

(b) If $ac - b^2 > 0$ and $a, c > 0$ then Q is positive definite.

(c) If $ac - b^2 > 0$ and $a, c < 0$ then Q is negative definite.

(d) If $ac - b^2 \leq 0$ then Q is neither positive definite nor negative definite.

Proof continued:

(d) Now suppose that $ac - b^2 \leq 0$. We need to show that Q is indefinite, so we need to find a point $(x, y) \neq (0, 0)$ where $Q = 0$. If $a \neq 0$ we note that $x = (-b + \sqrt{b^2 - ac})/a$ is a root of $ax^2 + 2bx + c = 0$, so $Q = 0$ at $(x, 1)$. Similarly, if $c \neq 0$ then $y = (-b + \sqrt{b^2 - ac})/c$ is a root of $a + 2by + cy^2 = 0$, so $Q = 0$ at $(1, y)$. This just leaves the case where $a = c = 0$ and $Q = 2bxy$, so $Q = 0$ at $(1, 0)$ or $(0, 1)$. $\square$

Question: Definiteness of quadratic forms

Are these positive (semi)definite or negative (semi)definite?

$$Q_1 = 2x^2 + 3y^2 \qquad Q_2 = x^2 + 2xy + y^2 \qquad Q_3 = (x+y)(x+2y)$$

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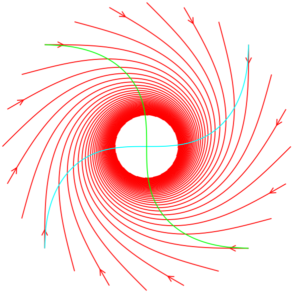
- $Q_2 = (x+y)^2 \geq 0$, but $Q_2 = 0$ at $(1, -1)$. So Q_2 is positive semidefinite, but not positive definite.
- Q_3 can be positive (at $(1, 0)$, for example) or negative (at $(3, -2)$, for example). Thus, it is indefinite.

Lyapunov function for the slow spiral

Remember this system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} + (x^2 + y^2) \begin{bmatrix} -x \\ -y \end{bmatrix}$$

around the circle
towards the origin



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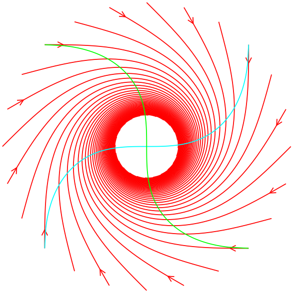

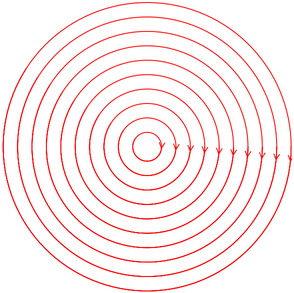

The linearisation is $(\dot{x}, \dot{y}) = (y, -x)$, which has a centre, so it is not asymptotically stable.

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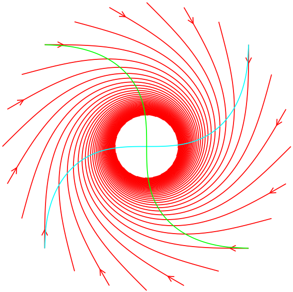

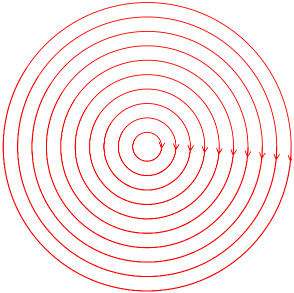

The linearisation is $(\dot{x}, \dot{y}) = (y, -x)$, which has a centre, so it is not asymptotically stable. But the original system is asymptotically stable.

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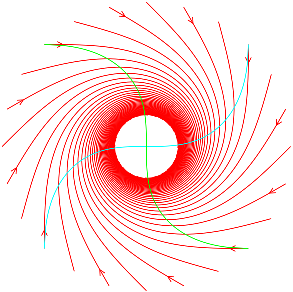

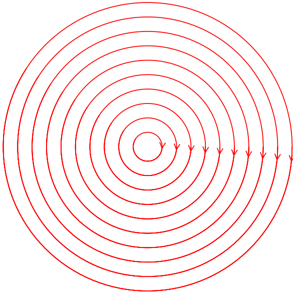

The linearisation is $(\dot{x}, \dot{y}) = (y, -x)$, which has a centre, so it is not asymptotically stable. But the original system is asymptotically stable. We can prove this with a Lyapunov function.

Lyapunov function for the slow spiral

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Put $V = x^2 + y^2$.

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around the circle
towards the origin

Put $V = x^2 + y^2$.
Then $V > 0$ except $V = 0$ at $(0, 0)$, so V is positive definite.

Lyapunov function for the slow spiral

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$$\dot{V} = V_x \dot{x} + V_y \dot{y}$$

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$$\begin{aligned} \dot{V} &= V_x \dot{x} + V_y \dot{y} \\ &= 2x(y - (x^2 + y^2)x) + 2y(-x - (x^2 + y^2)y) \end{aligned}$$

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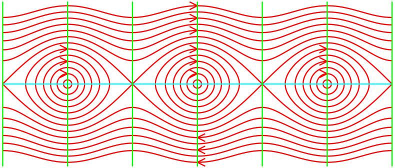
In fact $\dot{V} = -V^2$ gives $\frac{d}{dt}(V^{-1}) = -V^{-2}\dot{V} = 1$
so $V^{-1} = V_0^{-1} + t$ so $V = (V_0^{-1}+t)^{-1} = V_0/(1+V_0t)$.

Lyapunov function for the pendulum

Recall the pendulum equations: $\dot{\theta} = \omega$, $\dot{\omega} = -\sin(\theta)$.



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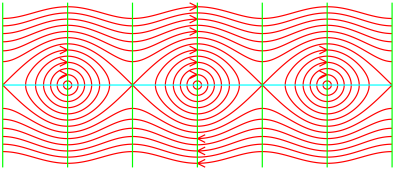

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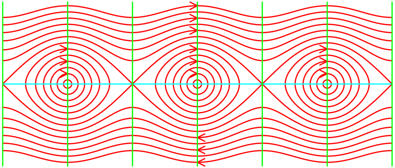

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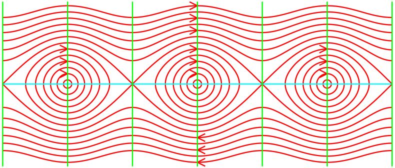

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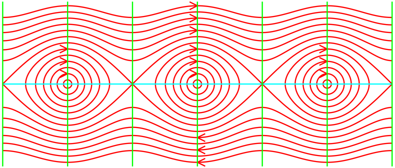

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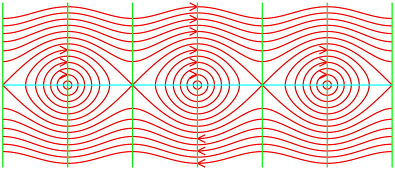
If we consider only the region

$$R = \{(\theta, \omega) \mid -2\pi < \theta < 2\pi\}$$

then V is positive definite.



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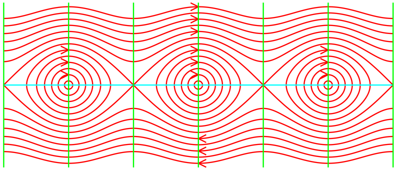

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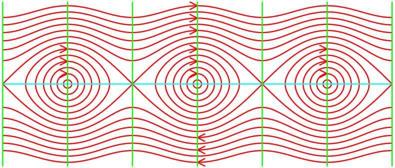
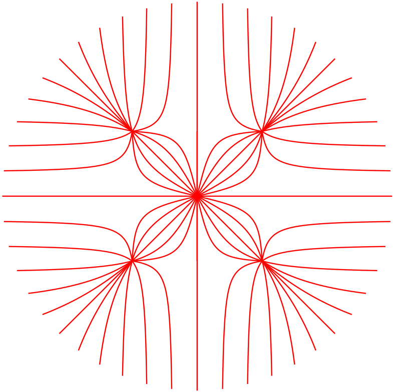


Lyapunov function for a gradient flow

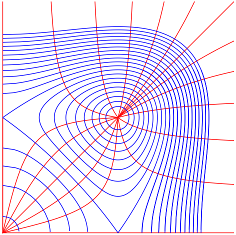
Consider the system $\dot{x} = x - x^3$, $\dot{y} = y - y^3$



Lyapunov function for a gradient flow

Consider the system $\dot{x} = x - x^3$, $\dot{y} = y - y^3$, and the function

$$V = (x^2 - 1)^2 + (y^2 - 1)^2 = x^4 - 2x^2 + y^4 - 2y^2 - 2.$$



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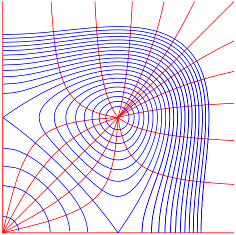
This has $V \geq 0$ everywhere



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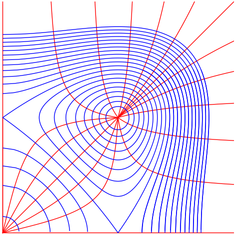
This has $V \geq 0$ everywhere, and V is only equal to 0 at the points $(\pm 1, \pm 1)$.



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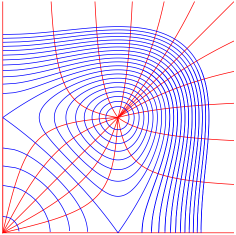
This has $V \geq 0$ everywhere, and V is only equal to 0 at the points $(\pm 1, \pm 1)$. It also satisfies

$\dot{V} = V_x f + V_y g$



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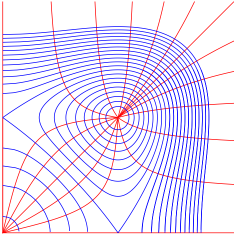

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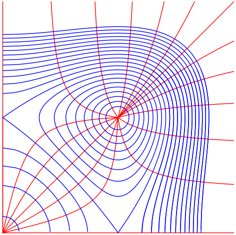

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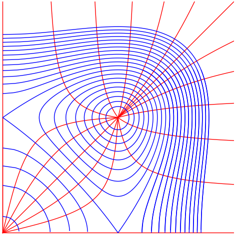

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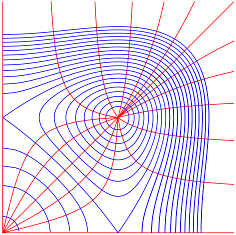

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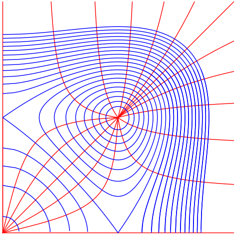

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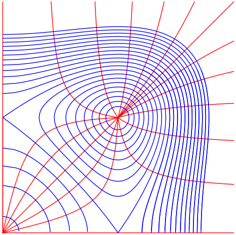

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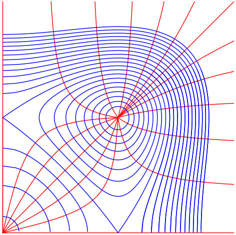

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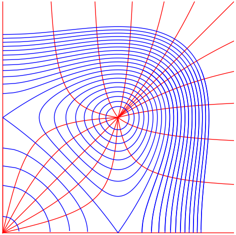

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The (damped) oscillator has $\dot{x} = y$ and $\dot{y} = 2x - x^3 - \epsilon y$ for some $\epsilon \geq 0$.

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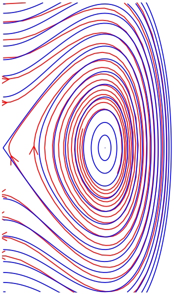
This means that $\dot{V} \leq 0$ everywhere, with $\dot{V} = 0$ only when $y = 0$. In particular, V is negative semidefinite on R, so it is a weak Lyapunov function. We deduce that $(\sqrt{2}, 0)$ is a stable equilibrium point. In fact, we can use more complicated properties of V to show that $(\sqrt{2}, 0)$ is even asymptotically stable. Note also that when $\epsilon = 0$ we have $\dot{V} = 0$, so V is a conserved quantity for the undamped Duffing oscillator.

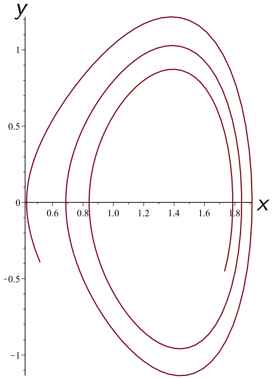
Lyapunov function for the damped Duffing oscillator

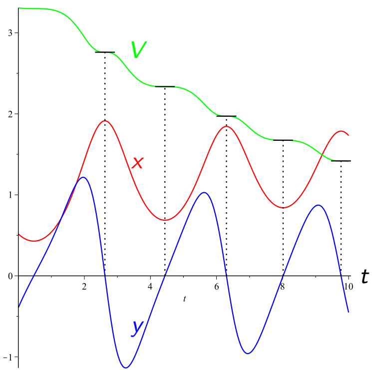
$$\dot{x} = y \qquad \dot{y} = 2x - x^3 - \epsilon y \qquad V = 2y^2 + (x^2 - 2)^2 \qquad \dot{V} = -4\epsilon y^2$$

In this example, $\dot{V}$ is quite small, so the flow lines cross the lines of constant V at a shallow angle, so it is hard to draw a clear picture.



Damped Duffing is asymptotically stable





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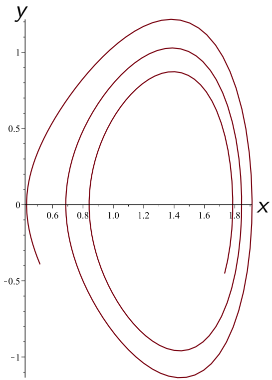

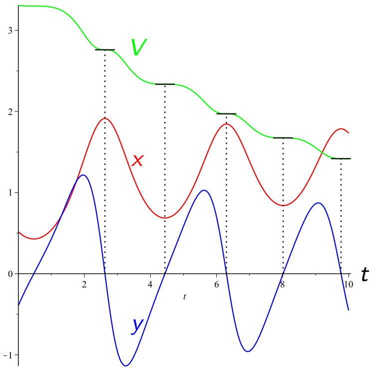

Recall that $\dot{V} = -4\epsilon y^2$, and this is the slope of the green graph of V against t.

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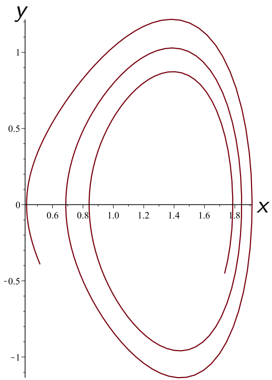

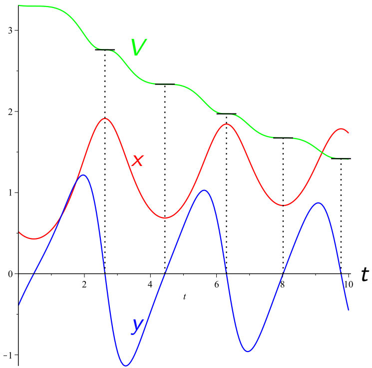

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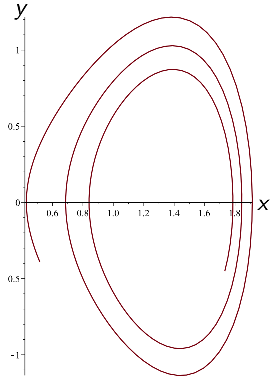

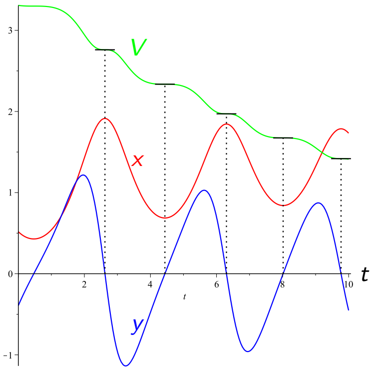

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When $y \neq 0$, we have $\dot{V} < 0$ and the green graph slopes downwards.

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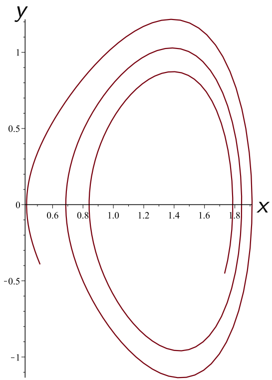

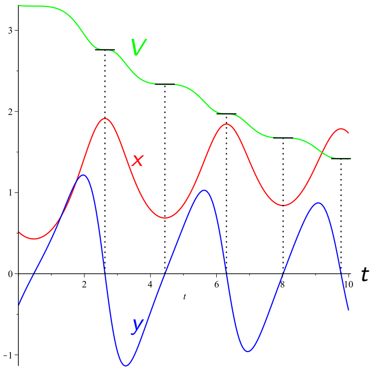

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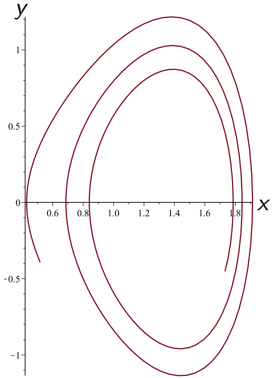

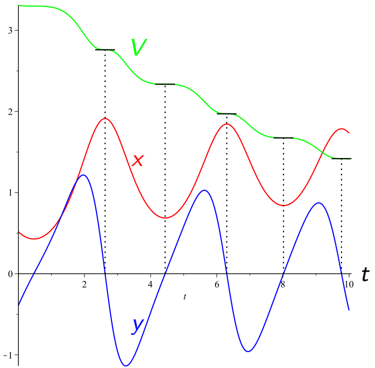

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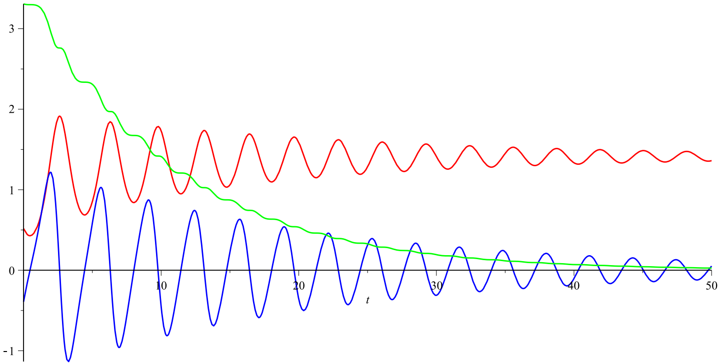
This only happens for an instant before y becomes nonzero again and the green graph continues to decrease.

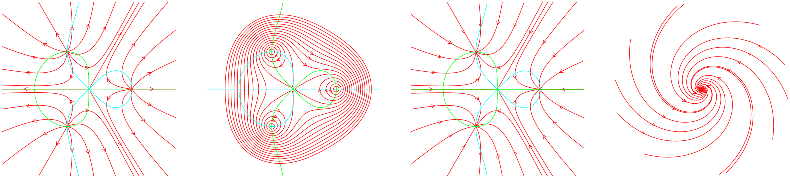
Damped Duffing is asymptotically stable

Here is the same picture for a longer time:



This shows that the flow line converges to the equilibrium point $(\sqrt{2}, 0)$ where $V = 0$.

Questions about Lyapunov functions



A *B* *C* *D*

(p) One system has a strong Lyapunov function on the whole plane.

(q) One system has a strong Lyapunov function on a region R, but not on the the whole plane.

(r) One system has a weak Lyapunov function but not a strong Lyapunov function.

(s) One system does not even have a weak Lyapunov function.

Which is which?

Another Lyapunov example

Suppose that $\dot{x} = x^2 - y^2 - \frac{1}{4}$, $\dot{y} = 2xy$

Another Lyapunov example

Suppose that $\dot{x} = x^2 - y^2 - \frac{1}{4}$, $\dot{y} = 2xy$, so there are equilibria at $(\pm\frac{1}{2}, 0)$.

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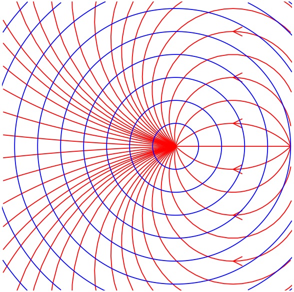

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We conclude that the function $V = 11x^{14} + 10y^{16}$ is a strong Lyapunov function.

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In particular:
If both V and $\dot{V}$ are positive definite around (a, b), then (a, b) is unstable.

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If $\dot{V}$ is positive definite, and for all $\epsilon > 0$ there is a point where $\|(x, y) - (a, b)\| \leq \epsilon$ and $V > 0$, then (a, b) is unstable.

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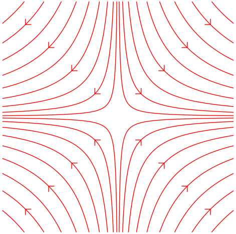

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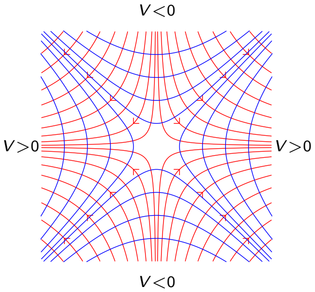

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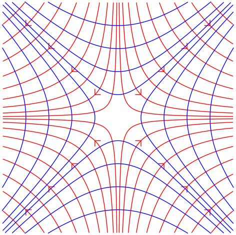

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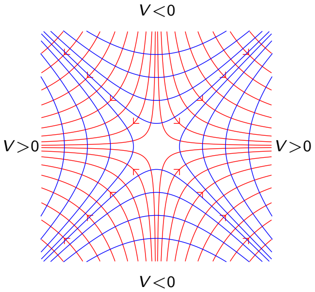

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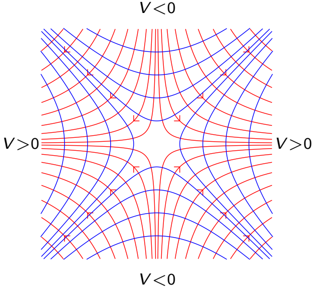

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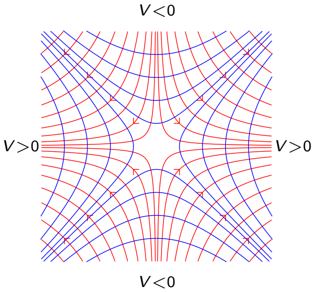

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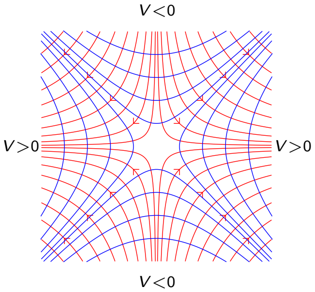
which is positive definite. Also, for any $\epsilon > 0$ there is a point $(0, \epsilon)$ with $\|(\epsilon, 0) - (0, 0)\| = \epsilon$ and $V = \epsilon^2 > 0$. Thus, $(0, 0)$ is unstable.



Instability for a gradient flow

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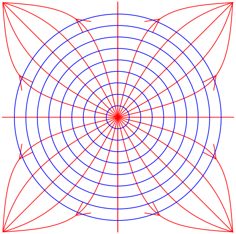

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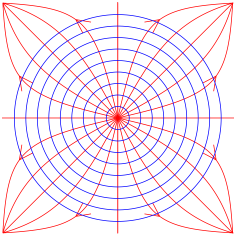

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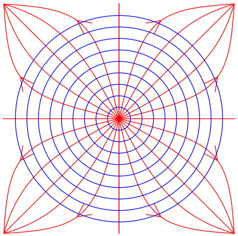

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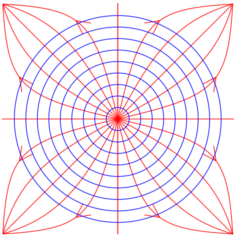

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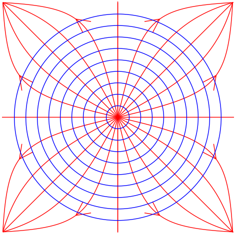

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$$R = \{(x, y) \mid -1 < x, y < 1\},$$

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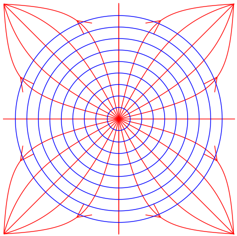

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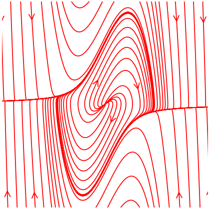

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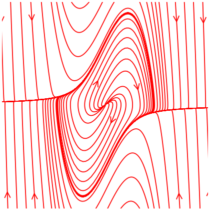

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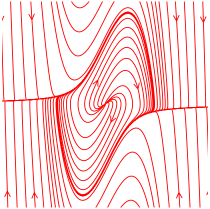

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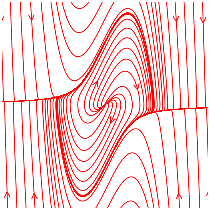

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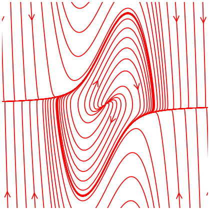

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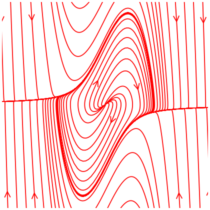

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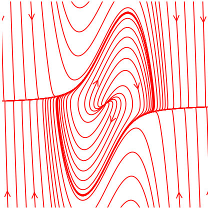

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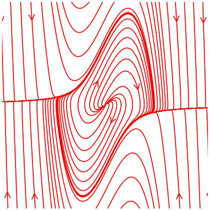

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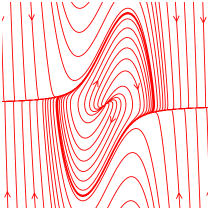

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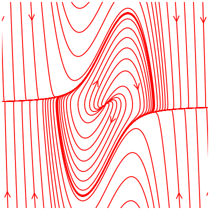

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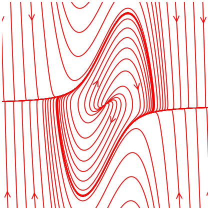

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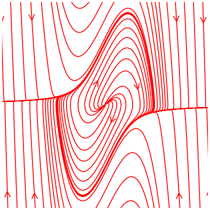

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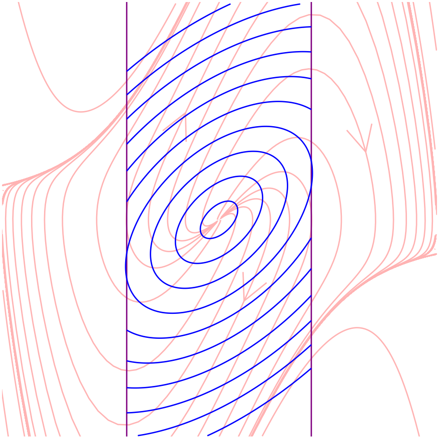
van der Pol instability — phase portrait



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Suppose we have a system with an equilibrium at the origin, and that V is defined on all of $\mathbb{R}^2$. Are the following true or false?

(A) If V is a strong Lyapunov function then $\dot{V}$ is positive definite.

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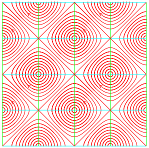
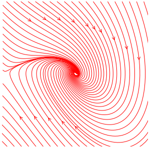
- (C) is True: if V is a weak Lyapunov function then the origin is stable, but saddles are not stable, so the origin cannot be a saddle.

Questions about conserved quantities

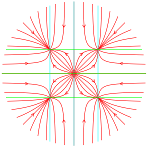
Which system has a (continuous, non-constant) conserved quantity?



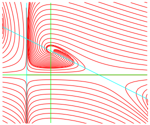
A



B



C



D

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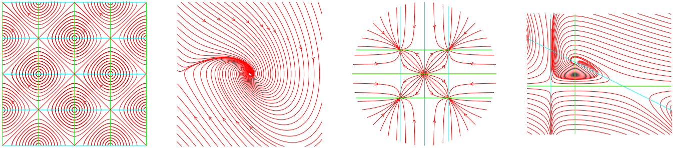

A *B* *C* *D*

Systems (B) and (D) have foci, and (C) has nodes, so these cannot have an interesting conserved quantity. System (A) has only saddles and centres, so it can have a conserved quantity.

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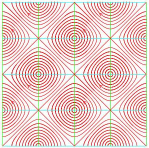

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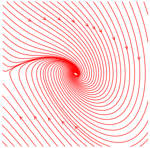

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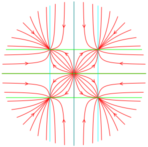

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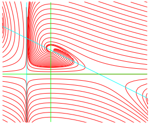

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In fact, system (A) is on the problem sheet. The equations are $\dot{x} = \sin(\pi y)$ and $\dot{y} = \sin(\pi x)$, and the function $U = \cos(\pi x) - \cos(\pi y)$ is a conserved quantity.

Second order linear differential equations

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$$Ay'' + By' + Cy = 0,$$

where A, B, C and y are functions of x, and y' means dy/dx.

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- Sturm-Liouville theory: eigenvalues of self-adjoint differential operators.

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Any solution can be written as $y = e^{\lambda x}(A\cos(\omega x) + B\sin(\omega x))$ for some constants A and B.

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Reminder of the constant coefficient case

Consider the equation $y'' + Py' + Qy = 0$, where P and Q are *constant*.
We look for solutions of the form $y = e^{\lambda x}$. Then $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$ so

$$y'' + Py' + Qy = \lambda^2 e^{\lambda x} + P\lambda e^{\lambda x} + Qe^{\lambda x} = p(\lambda)e^{\lambda x},$$

where $p(t) = t^2 + Pt + Q$ is the *auxiliary polynomial*.

(c) If $P^2 - 4Q = 0$ then there is only one root $\lambda = -P/2$, and the differential equation is $y'' - 2\lambda y' + \lambda^2 y = 0$. To understand this equation, put $y = e^{\lambda x} z$. We then have

$$\begin{aligned} y' &= \lambda e^{\lambda x} z + e^{\lambda x} z' = e^{\lambda x}(z' + \lambda z) \\ y'' &= \lambda e^{\lambda x}(z' + \lambda z) + e^{\lambda x}(z'' + \lambda z') = e^{\lambda x}(z'' + 2\lambda z' + \lambda^2 z), \end{aligned}$$

so

$$y'' + Py' + Qy = e^{\lambda x}(z'' + (2\lambda + P)z' + (\lambda^2 + P\lambda + Q)z).$$

However, $2\lambda + P = 0$ and $\lambda^2 + P\lambda + Q = 0$ so $y'' + Py' + Qy = e^{\lambda x} z''$, so the differential equation is equivalent to $z'' = 0$. This means that $z = Ax + B$ and $y = e^{\lambda x}(Ax + B)$ for some constants A and B.

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We can also find similar formulae for the case when $p(t)$ has two complex roots, or one repeated root. Later we will explain how to find power series solutions even when P is not constant.

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$$1 + 10x + \frac{100x^2}{2} + \frac{1000x^3}{6} + \cdots =$$

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1 + 10x + \frac{100x^2}{2} + \frac{1000x^3}{6} + \cdots &= e^{10x} \\
x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots &= \sin(x) \\
x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots &= \ln(1+x) \\
1 - \frac{\pi^2 x^2}{2!} + \frac{\pi^4 x^4}{4!} - \frac{\pi^6 x^6}{6!} + \cdots &= \cos(\pi x) \\
1 - \pi x + \pi^2 x^2 - \pi^3 x^3 + \pi^4 x^4 + \cdots &= 1/(1+\pi x) \\
1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots &= \\
\frac{1}{10} + \frac{x}{100} + \frac{x^2}{1000} + \frac{x^3}{10000} + \cdots &=
\end{aligned}$$

Questions: standard power series

$$1 + x + x^2 + x^3 + \cdots = 1/(1-x)$$

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Bessel's equation

Consider $x^2y'' + xy' + (x^2 - n^2)y = 0$ where n is a natural number.

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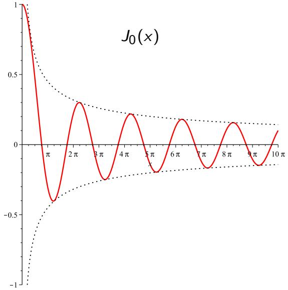

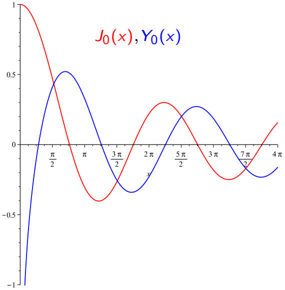

$$J_0(x) \simeq 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \cdots$$

$$\simeq \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right)$$

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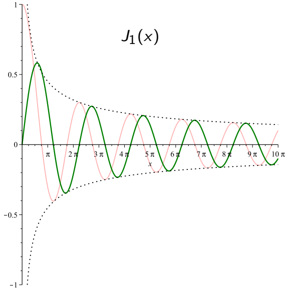

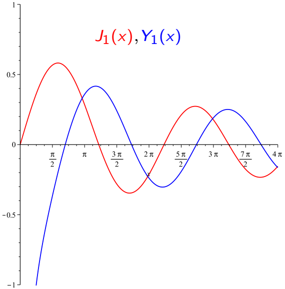

$$\begin{aligned} J_1(x) &\simeq \tfrac{1}{2}x - \tfrac{1}{16}x^3 + \tfrac{1}{384}x^5 - \cdots \\ &\simeq \sqrt{\frac{2}{\pi x}} \cos\left(x - 3\frac{\pi}{4}\right) \end{aligned}$$

Bessel's equation

Consider $x^2y'' + xy' + (x^2 - n^2)y = 0$ where n is a natural number.
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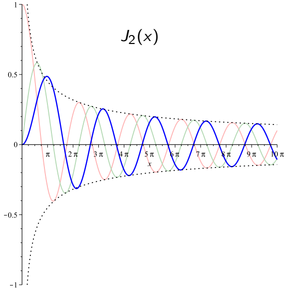

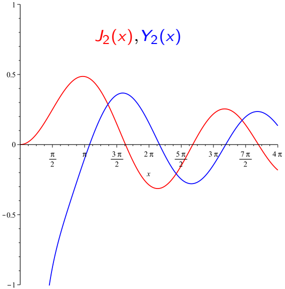

$$
\begin{aligned}
J_2(x) &\simeq \tfrac{1}{8}x^2 - \tfrac{1}{96}x^4 + \tfrac{1}{3072}x^6 - \cdots \\
&\simeq \sqrt{\frac{2}{\pi x}} \cos\left(x - 5\frac{\pi}{4}\right)
\end{aligned}
$$

Bessel's equation

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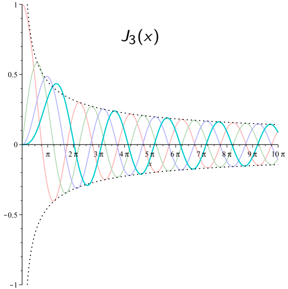

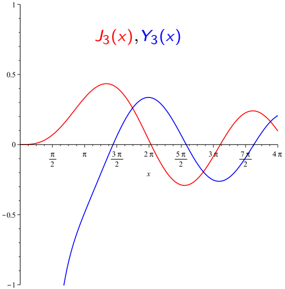

$$J_3(x) \simeq \tfrac{1}{48}x^3 - \tfrac{1}{768}x^5 + \tfrac{1}{30720}x^7 - \cdots$$
$$\simeq \sqrt{\frac{2}{\pi x}} \cos\left(x - 7\frac{\pi}{4}\right)$$

Bessel's equation

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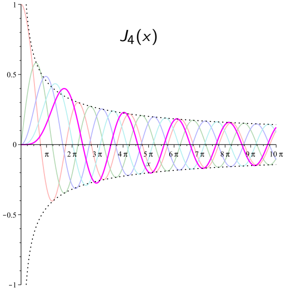

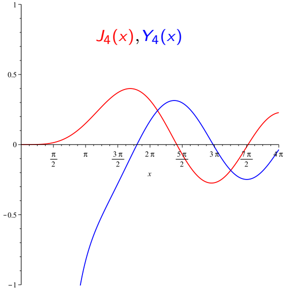

$$J_4(x) \simeq \tfrac{1}{384}x^4 - \tfrac{1}{7680}x^6 + \tfrac{1}{368640}x^8 - \cdots$$
$$\simeq \sqrt{\frac{2}{\pi x}} \cos\left(x - 9\frac{\pi}{4}\right)$$

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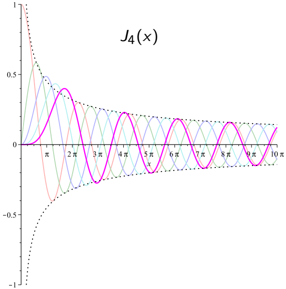

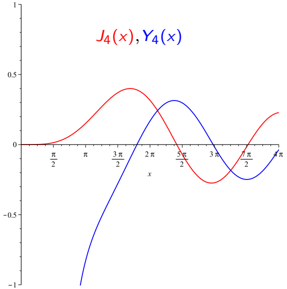

$$J_4(x) \simeq \frac{1}{384}x^4 - \frac{1}{7680}x^6 + \frac{1}{368640}x^8 - \cdots$$
$$\simeq \sqrt{\frac{2}{\pi x}}\cos\left(x - 9\frac{\pi}{4}\right)$$

Every solution has the form $y = AJ_n(x) + BY_n(x)$ for constants A and B.

Drum

Modes of vibration of a drum of radius 1 are given by

$$z = A\sin(t)\cos(n\theta)J_n(a_{nk}r),$$

where (r, θ) are polar coordinates and a_{nk} is the k'th root of $J_n(x)$.

The movie shows the case where $n = 2$ and $k = 3$, so
$z = A\sin(t)\cos(2\theta)J_2(a_{23}r)$.

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Consider $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$ where n is a natural number.

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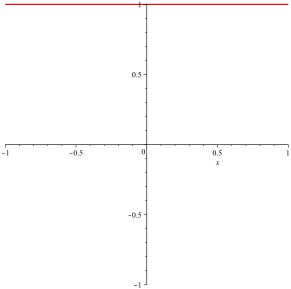

$P_0(x)$

$P_0(x) = 1$

$Q_0(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$

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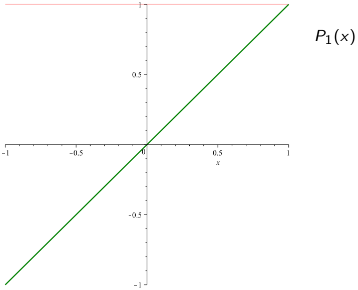

$P_1(x) = x$ $\qquad$ $Q_1(x) = \frac{1}{2}x \ln\left(\frac{x+1}{x-1}\right) - 1$

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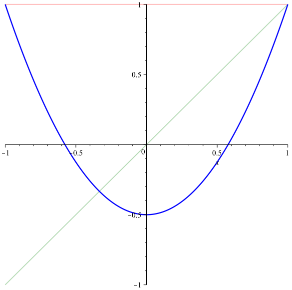

$P_2(x)$

$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$

$$Q_2(x) = \left(\frac{3}{4}x^2 - \frac{1}{4}\right) \ln\left(\frac{x+1}{x-1}\right) - \frac{3}{2}x$$

Legendre's equation

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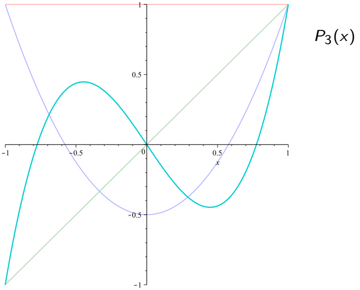

$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$ $\qquad$ $Q_3(x) = \left(\frac{5}{4}x^3 - \frac{3}{4}x\right) \ln\left(\frac{x+1}{x-1}\right) - \frac{5}{2}x^2 + \frac{2}{3}$

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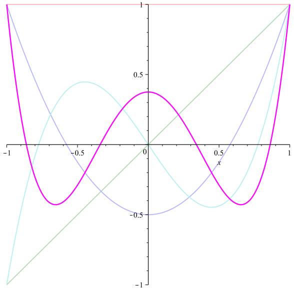

$P_4(x)$

$$P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8} \qquad Q_4(x) = \left(\frac{35}{16}x^4 - \frac{15}{8}x^2 + \frac{3}{16}\right)\ln\left(\frac{x+1}{x-1}\right) - \frac{35}{8}x^3 + \frac{55}{24}x$$

Legendre polynomials — orthogonality

Whenever n and m are different we have $\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0$.

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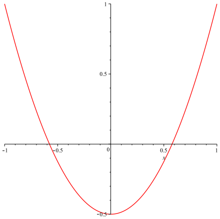
Example:



$P_2(x)$

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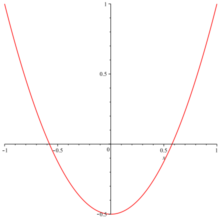

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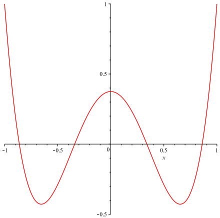

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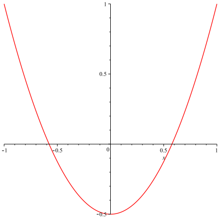

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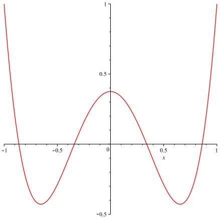

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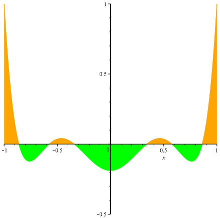

$P_2(x)P_4(x)$

Legendre polynomials — orthogonality

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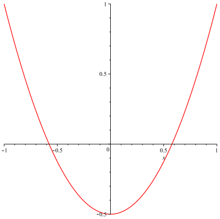
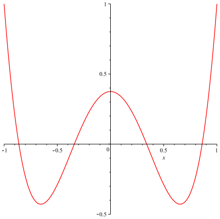


$P_2(x)$



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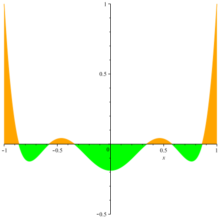

$P_2(x)P_4(x)$

This is similar to the fact that $\int_0^{2\pi} \sin(nx)\sin(mx)\,dx = 0$ for $n \neq m$, which is the basis of Fourier theory.

Legendre polynomials — orthogonality

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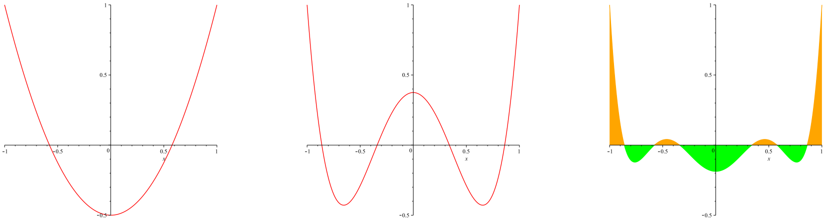

$P_2(x)$ $\quad$ $P_4(x)$ $\quad$ $P_2(x)P_4(x)$

This is similar to the fact that $\int_0^{2\pi} \sin(nx)\sin(mx)\,dx = 0$ for $n \neq m$, which is the basis of Fourier theory.

We will show that solutions of many other linear second order differential equations have similar orthogonality properties.

Alternating roots

The roots of $P_k(x)$ alternate with the roots of $P_{k+1}(x)$.

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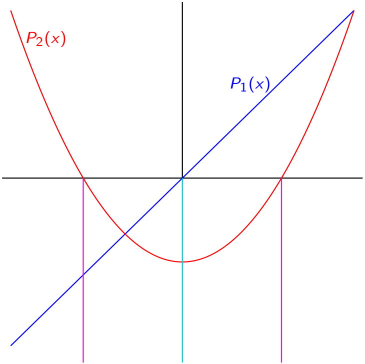

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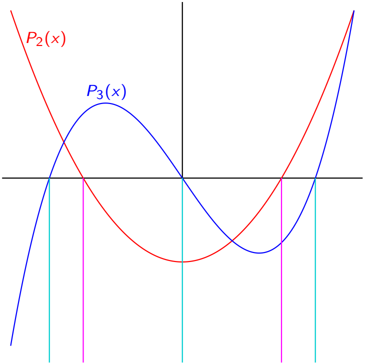

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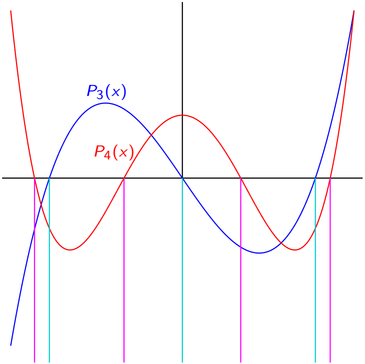

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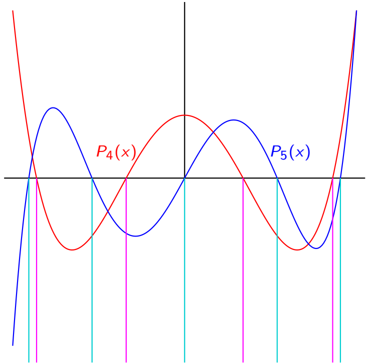

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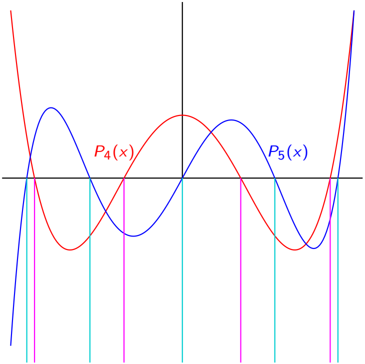
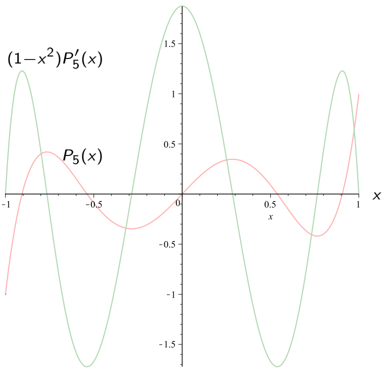
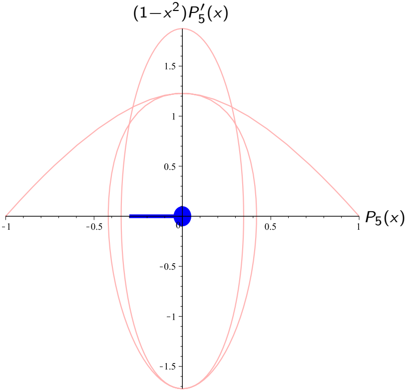


This is not just a special property of Legendre functions; it is a fairly general feature of linear second order differential equations.

Rotation

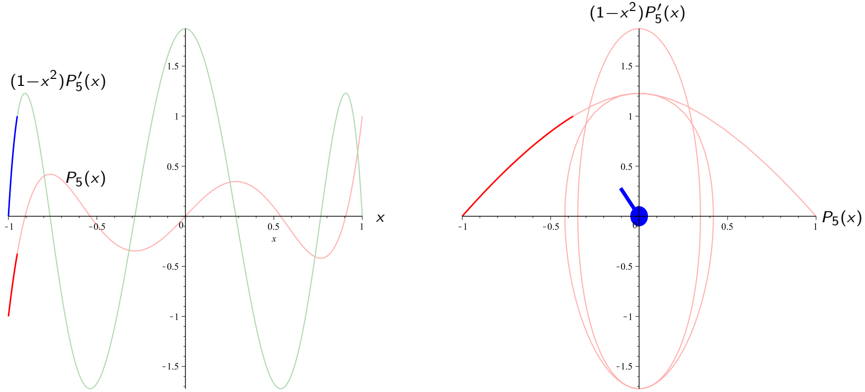
As x runs from -1 to 1, the point $(P_5(x), (1-x^2)P_5'(x))$ rotates around the origin through an angle of 5π.





$$P_5(x) = \frac{1}{8}\left(15x - 70x^3 + 63x^5\right) \qquad (1-x^2)P_5'(x) = \frac{1}{8}\left(15 - 225x^2 + 525x^4 - 315x^6\right)$$

As x runs from -1 to 1, the point $(P_5(x), (1-x^2)P_5'(x))$ rotates around the origin through an angle of 5π.



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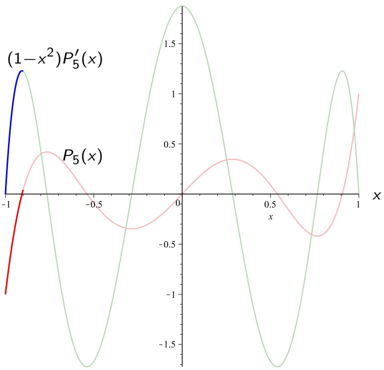

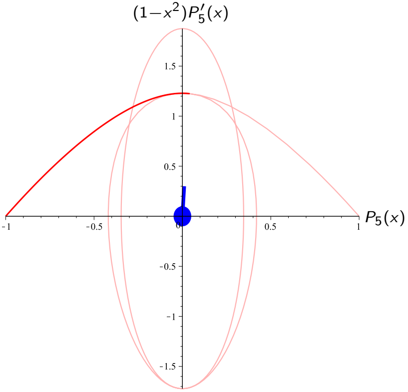

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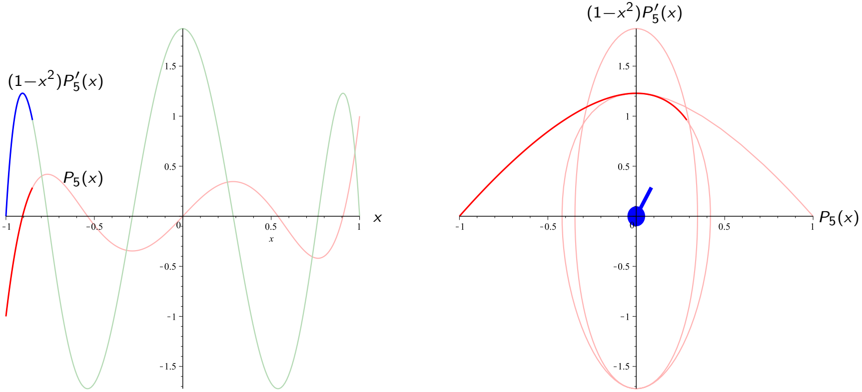

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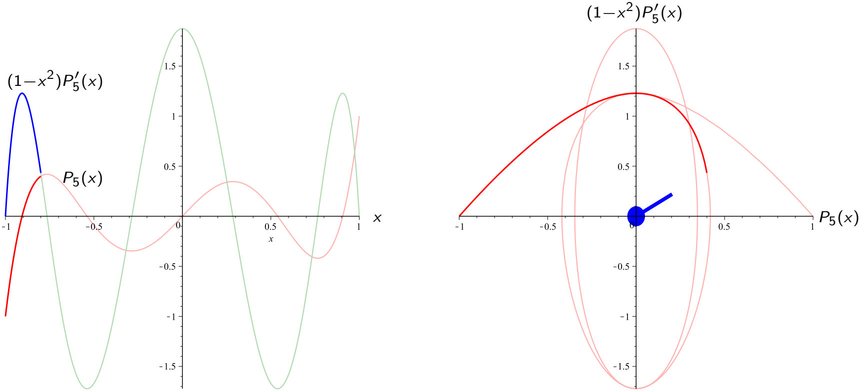

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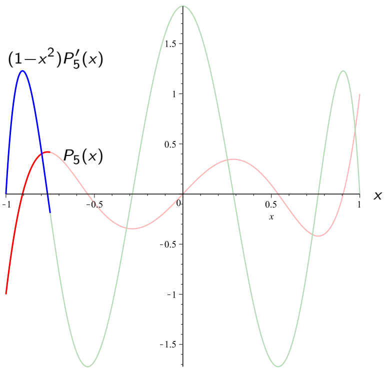

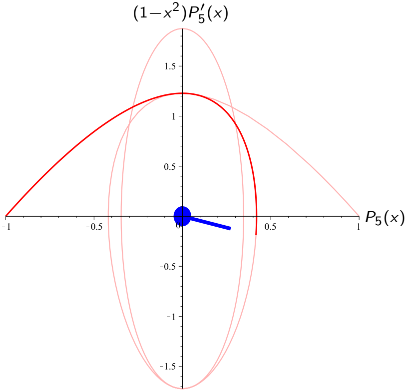

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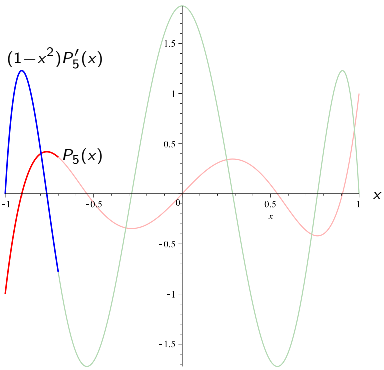

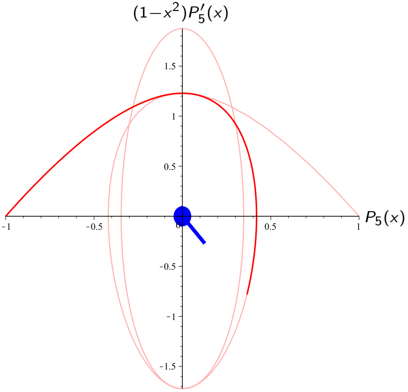

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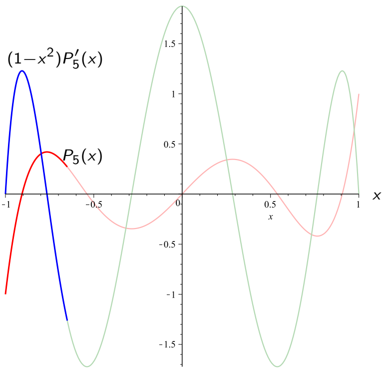

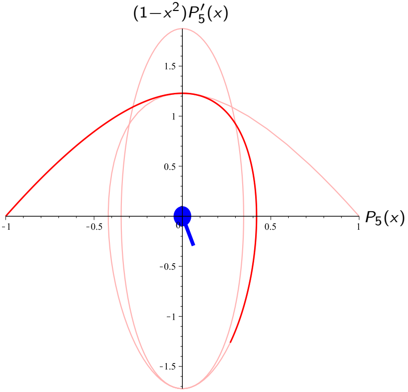

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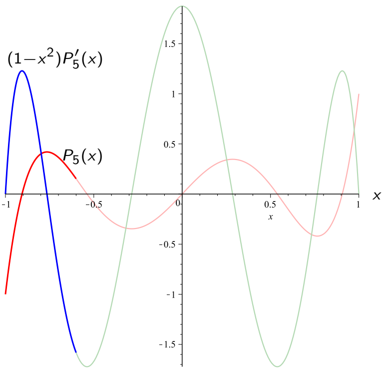

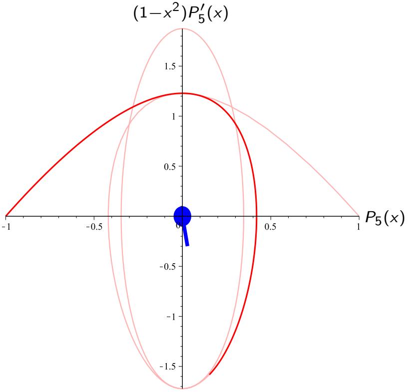

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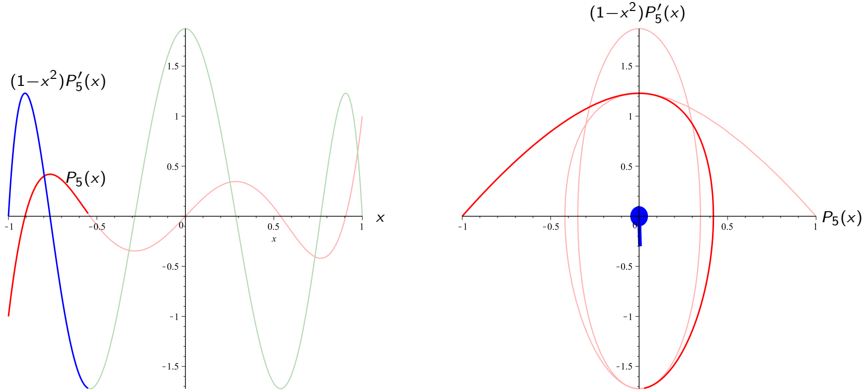

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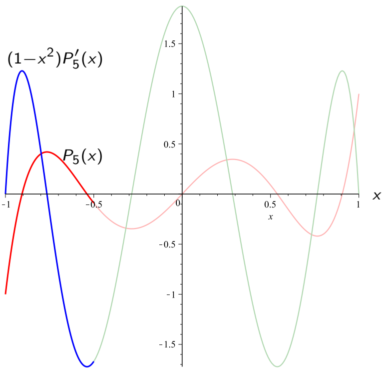

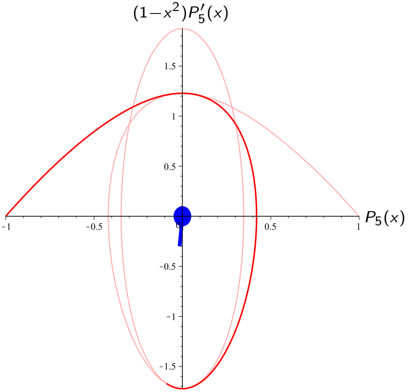

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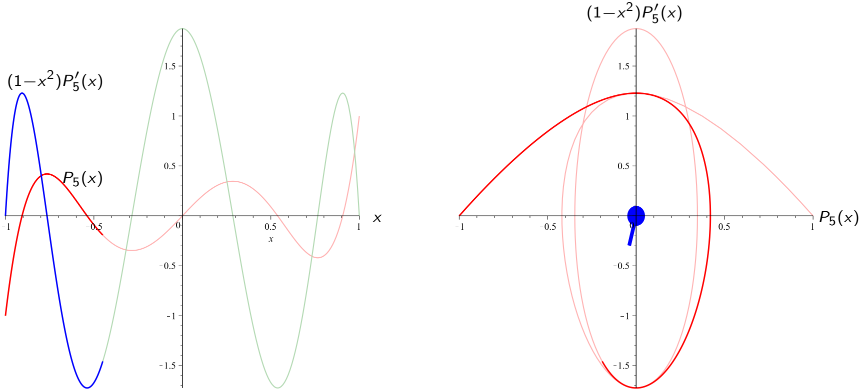

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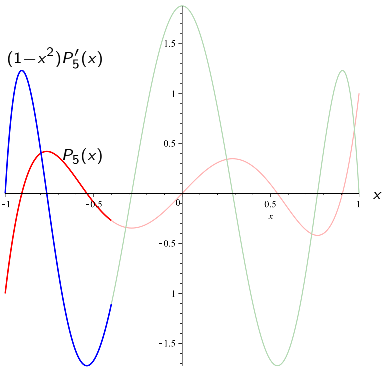

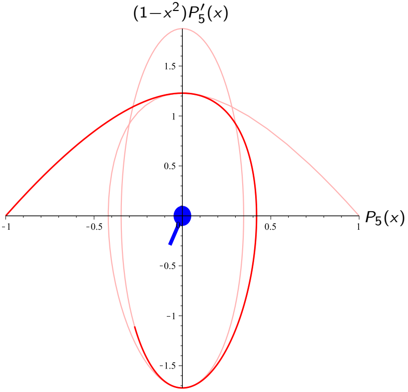

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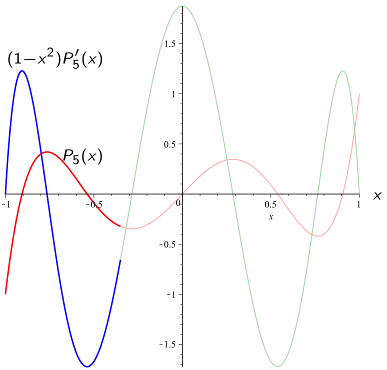

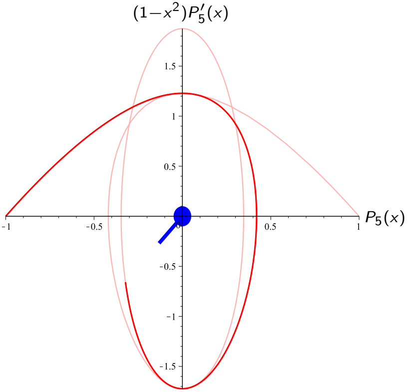

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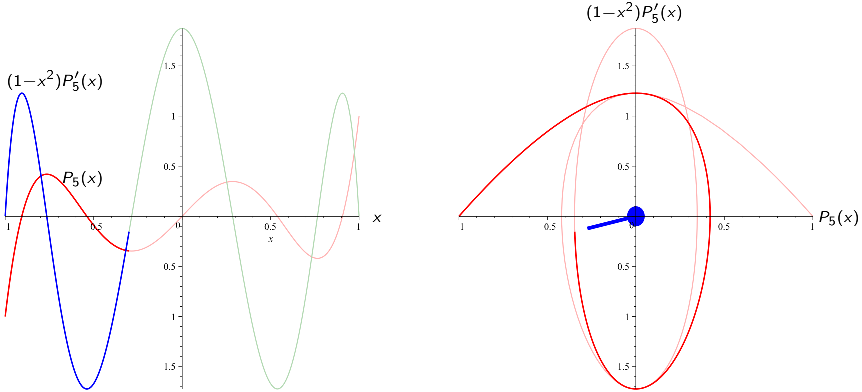

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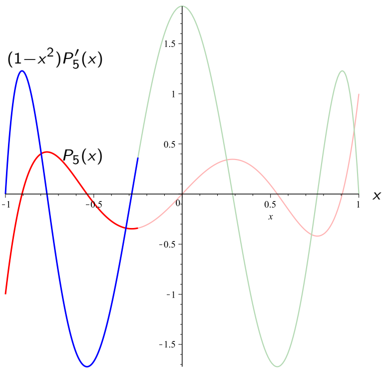

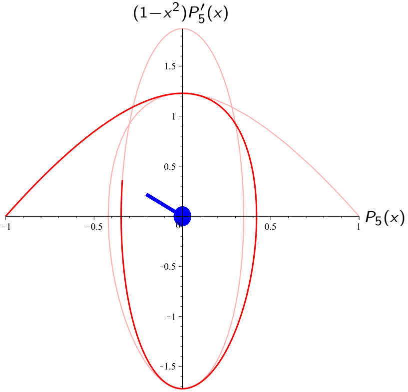

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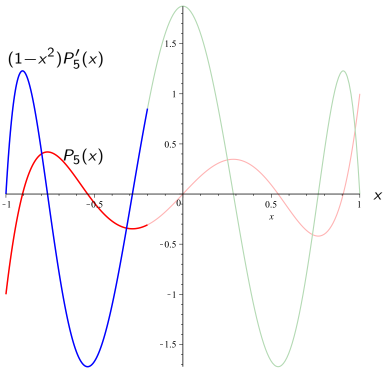

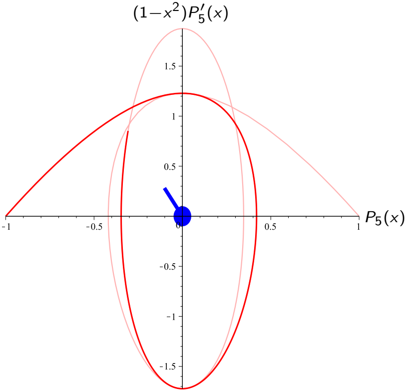

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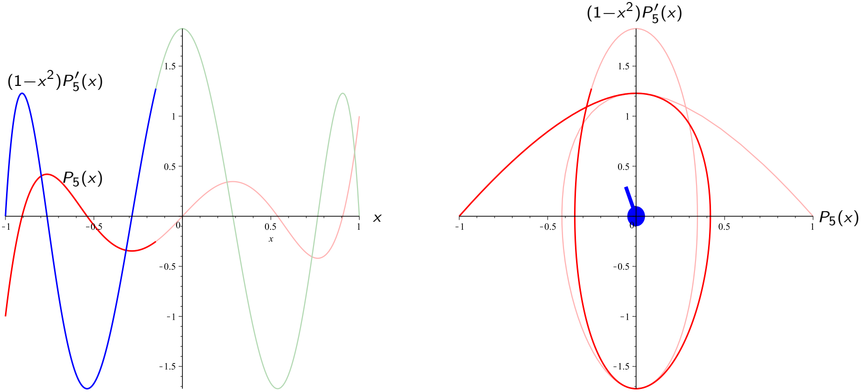

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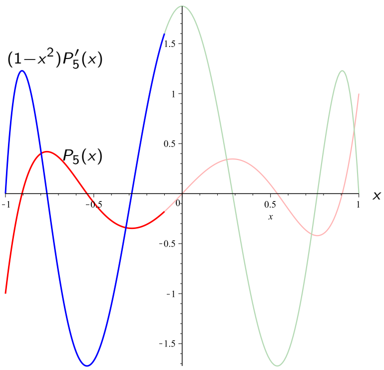

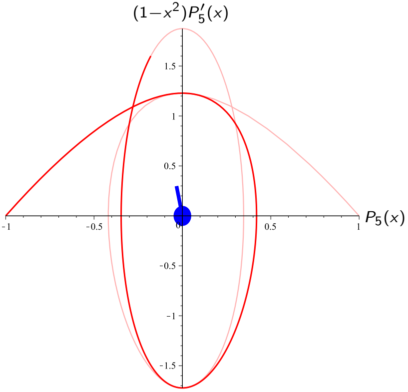

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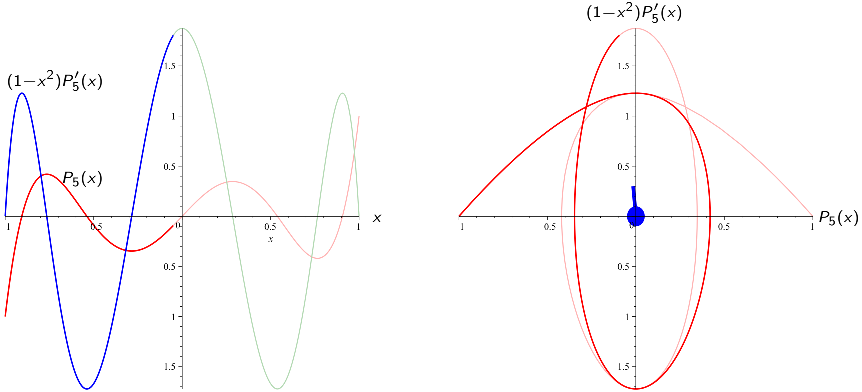

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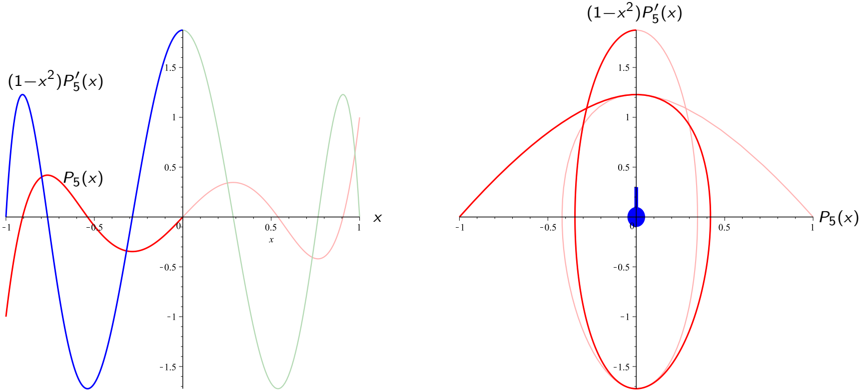

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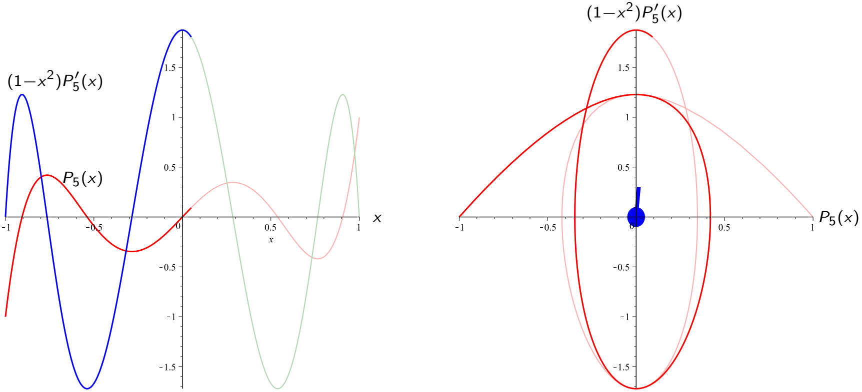

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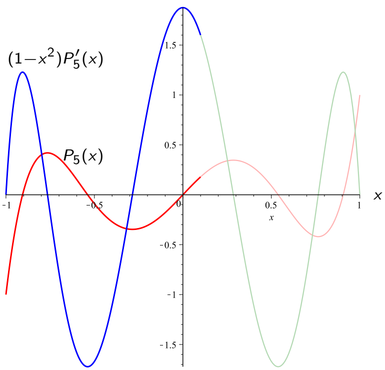

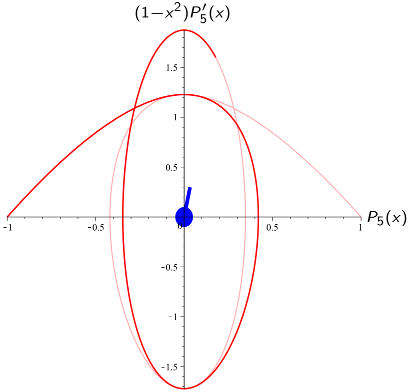

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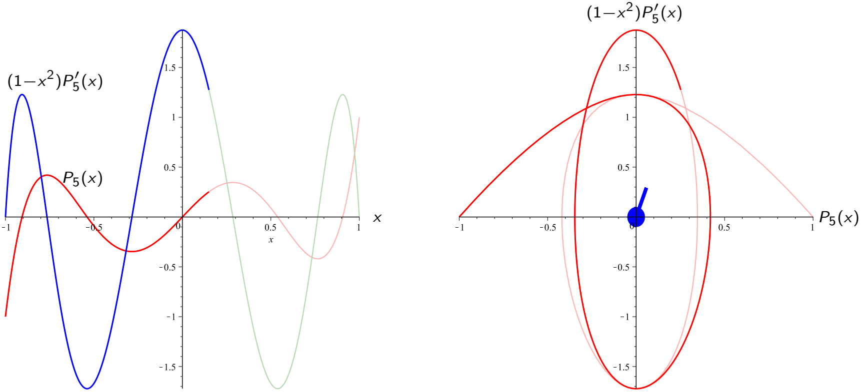

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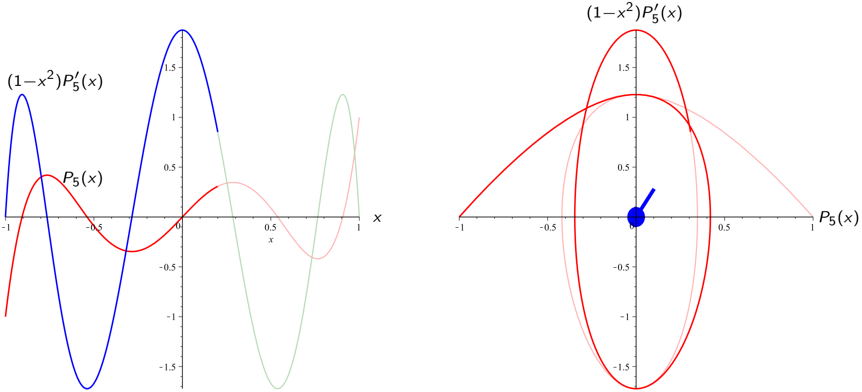

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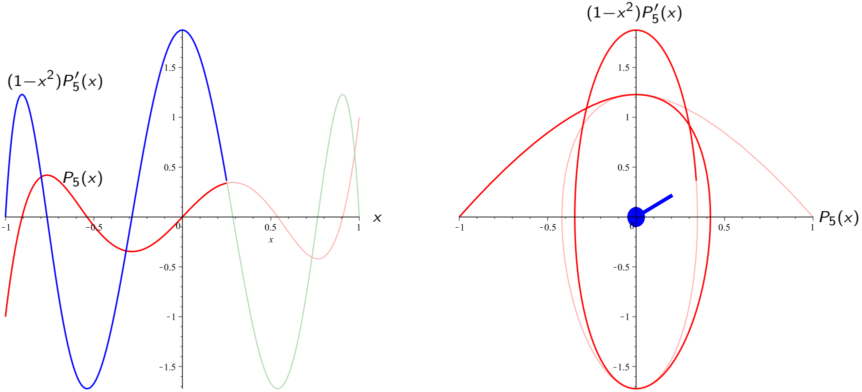

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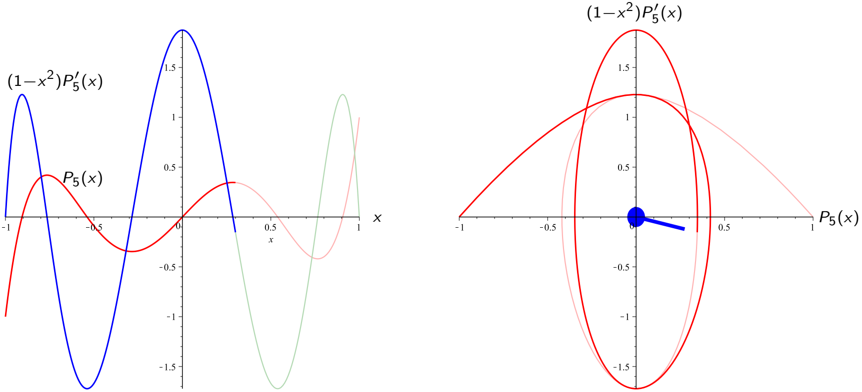

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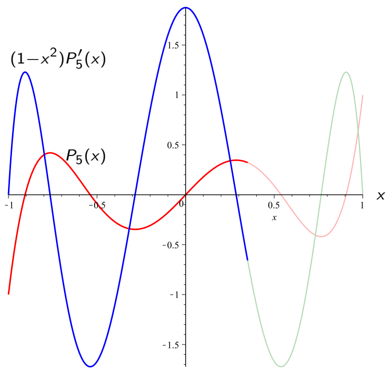

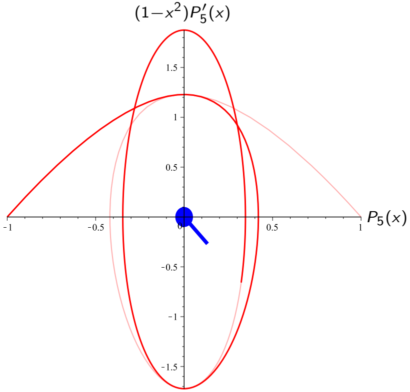

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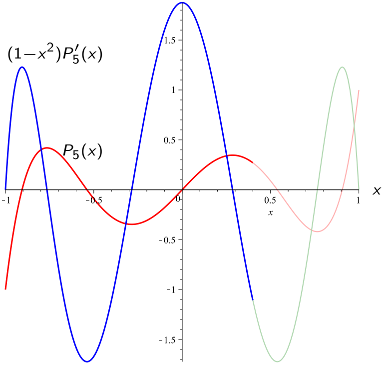

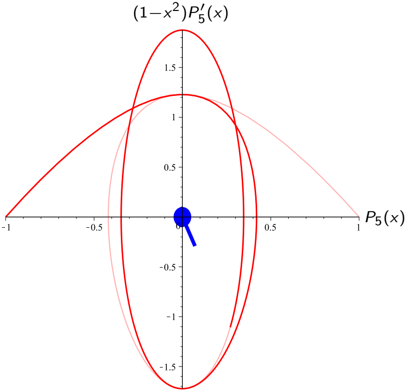

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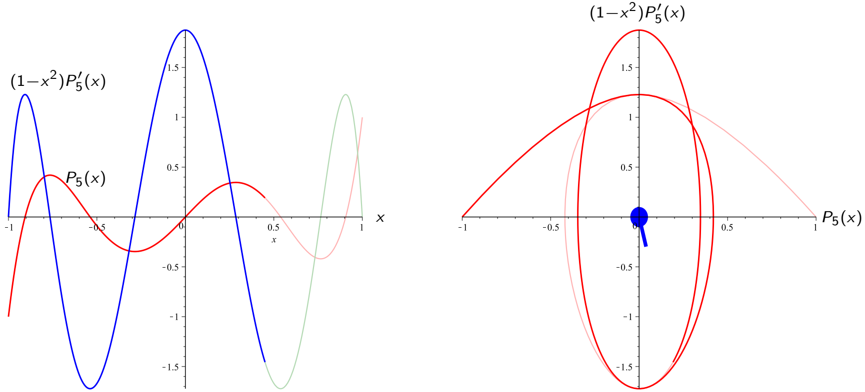

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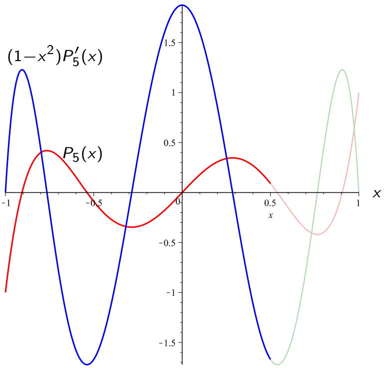

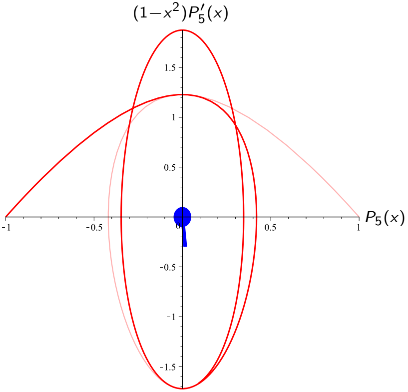

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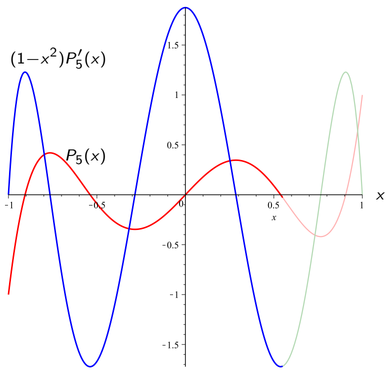

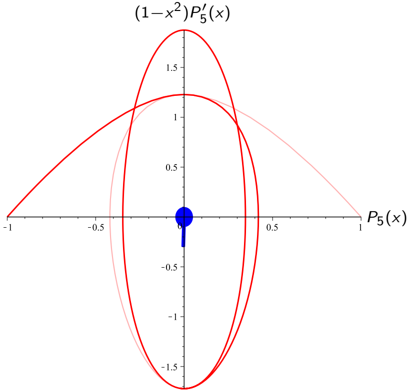

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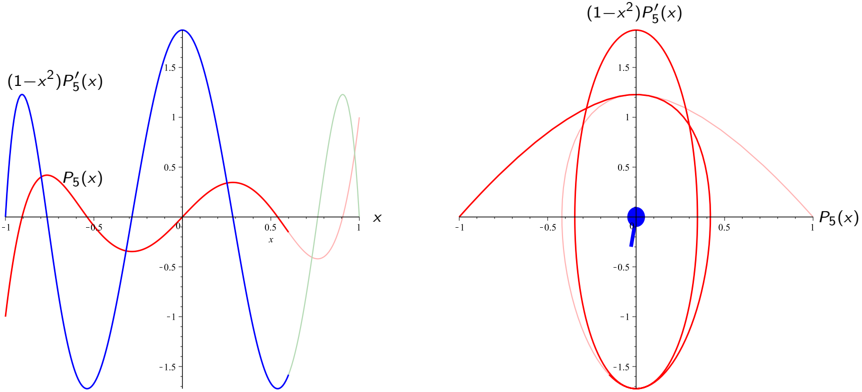

$P_5(x) = \frac{1}{8}\left(15x - 70x^3 + 63x^5\right)$ $\qquad$ $(1-x^2)P_5'(x) = \frac{1}{8}\left(15 - 225x^2 + 525x^4 - 315x^6\right)$

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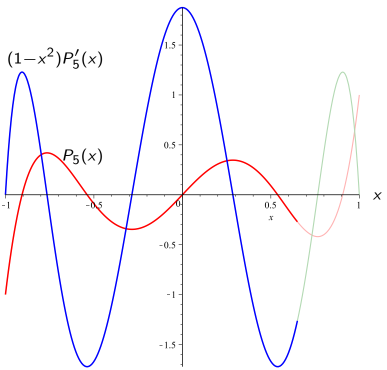

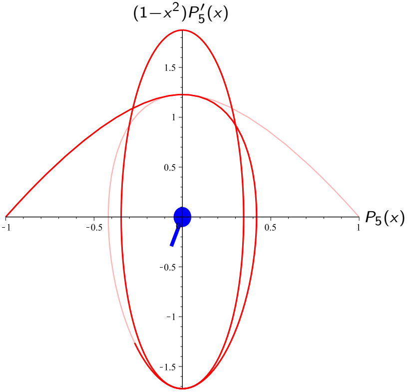

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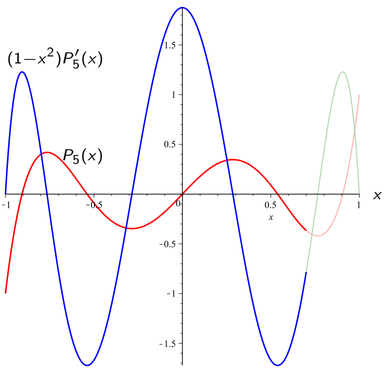

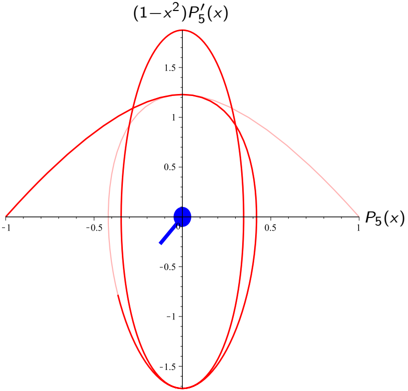

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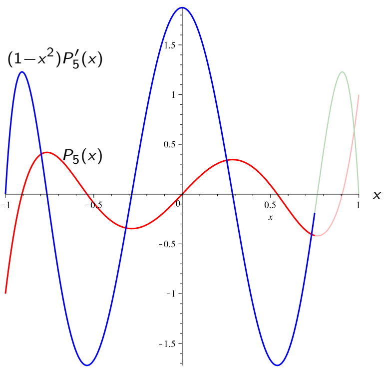

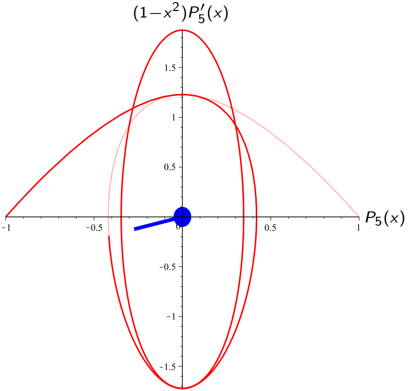

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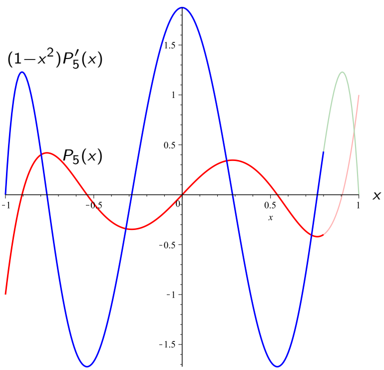

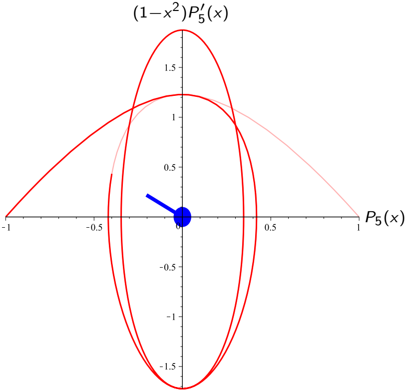

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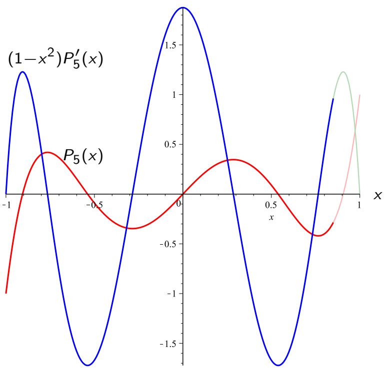

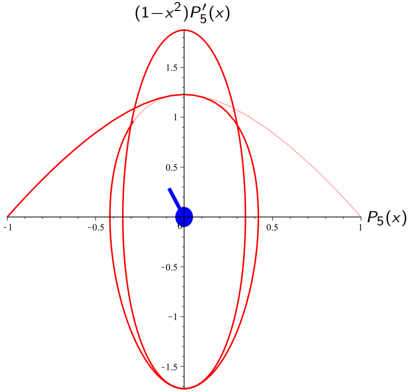

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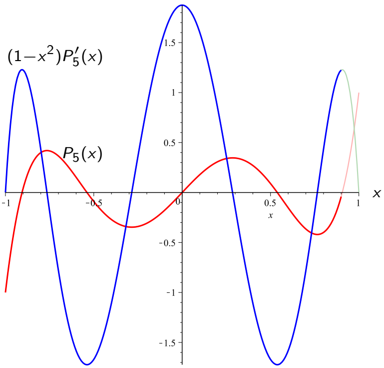

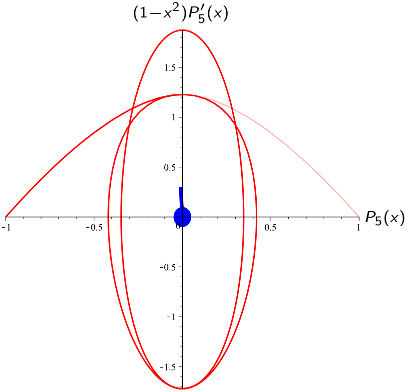

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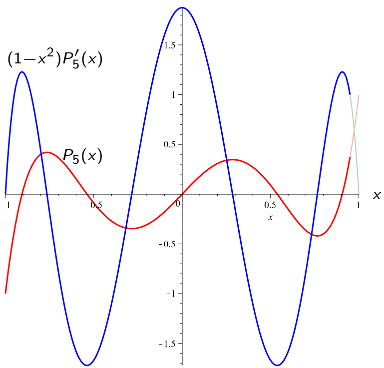

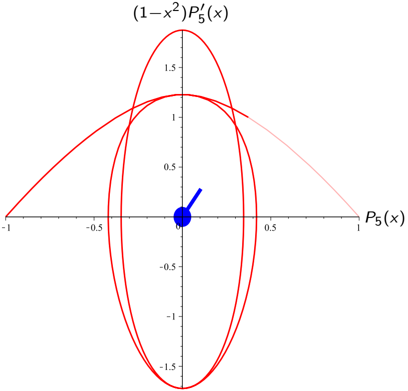

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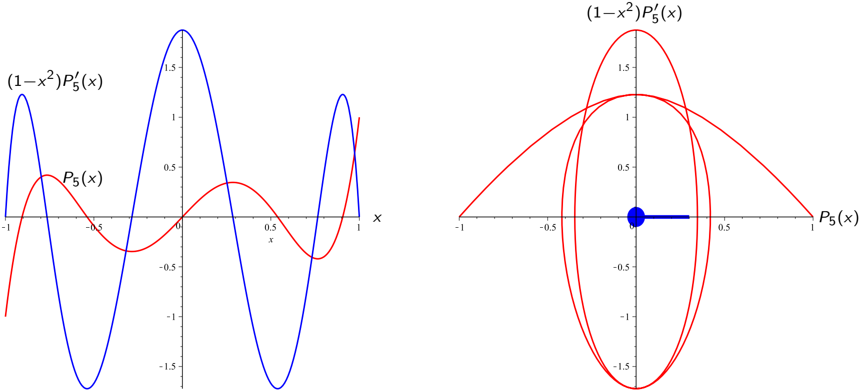

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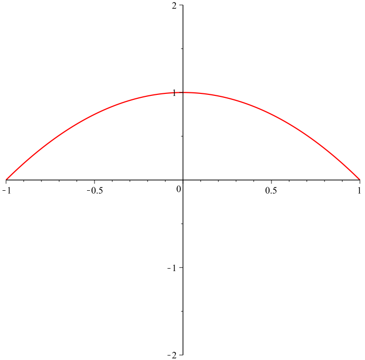

$(P_1(x), (1-x^2)P_1'(x))$ rotates through an angle of π.

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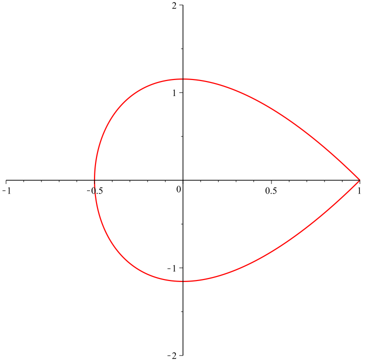
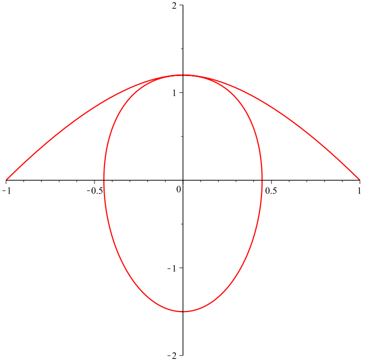


$(P_2(x), (1-x^2)P_2'(x))$ rotates through an angle of 2π.

Rotation around the origin

As x runs from -1 to 1, the point $(P_k(x), (1-x^2)P_k'(x))$ rotates around the origin through an angle of $k\pi$.



$(P_3(x), (1-x^2)P_3'(x))$ rotates through an angle of 3π.

Rotation around the origin

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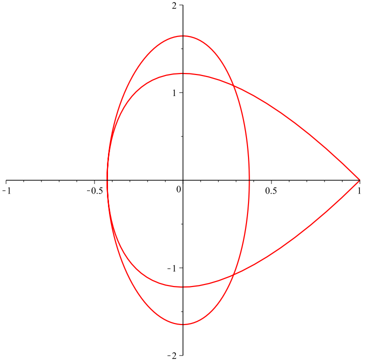

$(P_4(x), (1-x^2)P_4'(x))$ rotates through an angle of 4π.

Rotation around the origin

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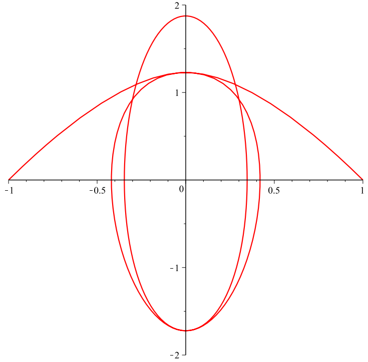

$(P_5(x), (1-x^2)P_5'(x))$ rotates through an angle of 5π.

Rotation around the origin

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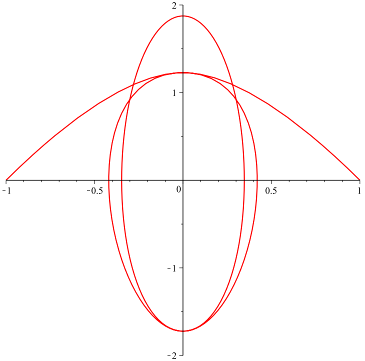

$(P_5(x), (1 - x^2)P_5'(x))$ rotates through an angle of 5π.

This is not just a special property of Legendre functions; it is a fairly general feature of linear second order differential equations.

The Hermite equation

Consider $y'' - 2xy' + 2ny = 0$ where n is a natural number.

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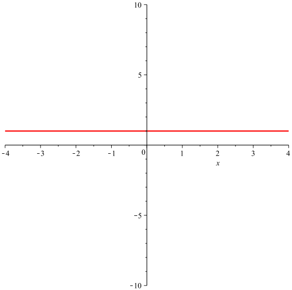
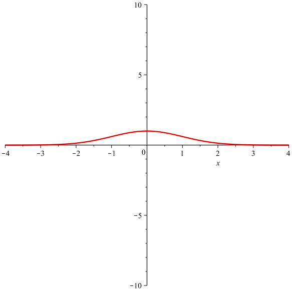


$H_0(x)$



$e^{-x^2/2}H_0(x)$

$$H_0(x) = 1$$

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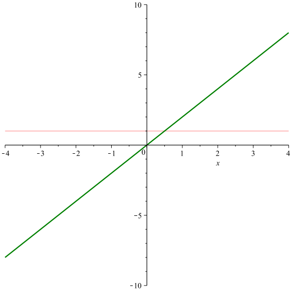
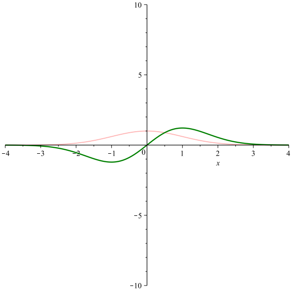


$H_1(x)$



$e^{-x^2/2}H_1(x)$

$$H_0(x) = 1 \qquad H_1(x) = 2x$$

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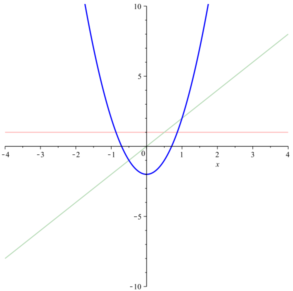
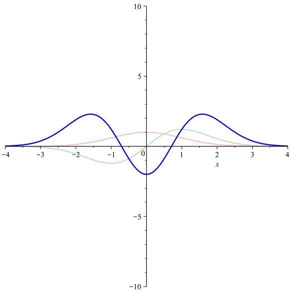


$H_2(x)$



$e^{-x^2/2}H_2(x)$

$H_0(x) = 1$

$H_1(x) = 2x$

$H_2(x) = 4x^2 - 2$

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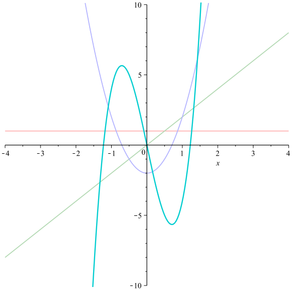
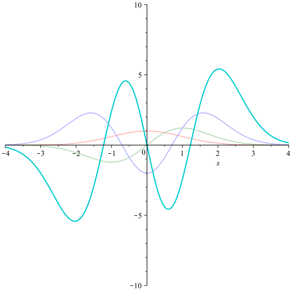


$H_3(x)$



$e^{-x^2/2}H_3(x)$

$H_0(x) = 1$ $\qquad$ $H_1(x) = 2x$

$H_2(x) = 4x^2 - 2$ $\qquad$ $H_3(x) = 8x^3 - 12x$

Hermite polynomials — orthogonality

Whenever n and m are different we have $\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 0$.

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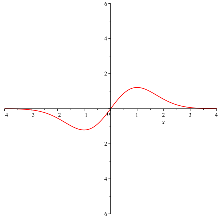
Example:



$e^{-x^2/2} H_1(x)$

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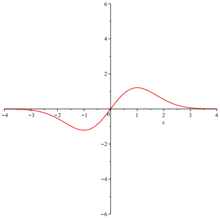

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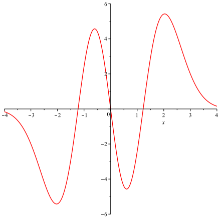

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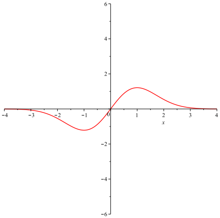

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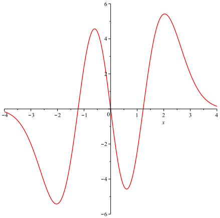

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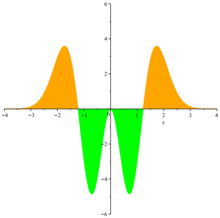

$e^{-x^2}H_1(x)H_3(x)$

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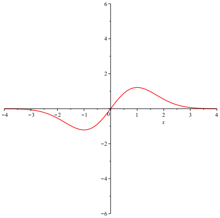

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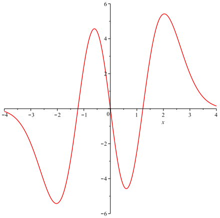

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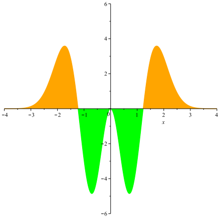

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This is similar to the fact that $\displaystyle\int_0^{2\pi} \sin(nx) \sin(mx)\, dx = 0$ for $n \neq m$, which is the basis of Fourier theory.

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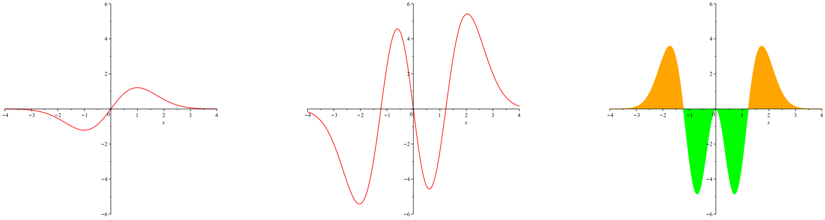


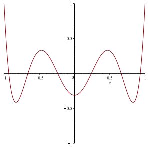
$e^{-x^2/2} H_1(x)$ $\quad$ $e^{-x^2/2} H_3(x)$ $\quad$ $e^{-x^2} H_1(x) H_3(x)$

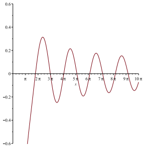
This is similar to the fact that $\int_0^{2\pi} \sin(nx) \sin(mx)\, dx = 0$ for $n \neq m$, which is the basis of Fourier theory.

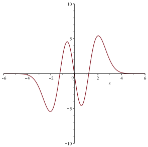
We will show that solutions of many other linear second order differential equations have similar orthogonality properties.

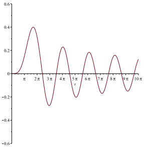
Question: which plot is which?

These are $J_1(x)$, $Y_1(x)$, $P_6(x)$ and $e^{-x/2}H_3(x)$. Which is which?









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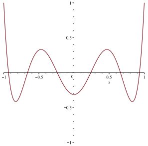
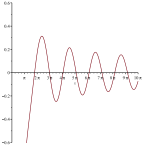
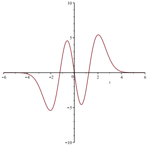
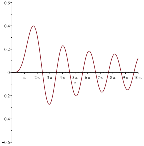


$P_6(x)$

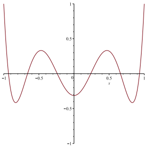




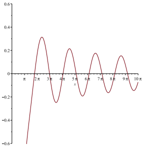


Question: which plot is which?

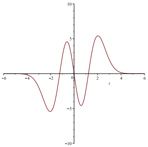
These are $J_1(x)$, $Y_1(x)$, $P_6(x)$ and $e^{-x/2}H_3(x)$. Which is which?

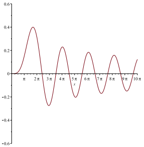


$P_6(x)$



$Y_1(x)$





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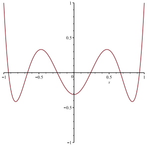
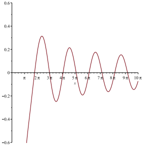
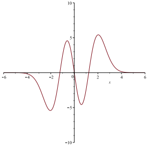


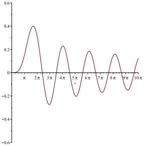
$P_6(x)$



$Y_1(x)$



$e^{-x/2}H_3(x)$



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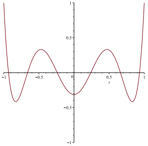

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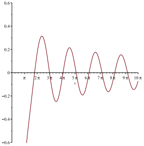
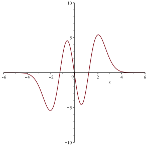


$Y_1(x)$



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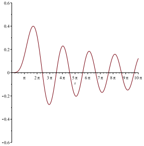

$J_1(x)$

Power series solutions — first few terms

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$$y'' + Py' + Qy = (2a_2 + p_0a_1 + q_0a_0) + (6a_3 + p_1a_1 + 2p_0a_2 + q_0a_1 + q_1a_0)x + \cdots .$$

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Thus, for the differential equation to hold we must have

$$2a_2 + p_0 a_1 + q_0 a_0 = 0$$

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$$y'' + Py' + Qy = (2a_2 + p_0a_1 + q_0a_0) + (6a_3 + p_1a_1 + 2p_0a_2 + q_0a_1 + q_1a_0)x + \cdots .$$

Thus, for the differential equation to hold we must have

$$2a_2 + p_0a_1 + q_0a_0 = 0$$
$$6a_3 + p_1a_1 + 2p_0a_2 + q_0a_1 + q_1a_0 = 0 \qquad \text{or}$$

$$a_2 = -\tfrac{1}{2}(p_0a_1 + q_0a_0)$$
$$a_3 = -\tfrac{1}{6}(p_1a_1 + 2p_0a_2 + q_0a_1 + q_1a_0).$$

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Suppose that y, P and Q are given by convergent power series:

$$y = a_0+a_1x+a_2x^2+\cdots \qquad P = p_0+p_1x+p_2x^2+\cdots \qquad Q = q_0+q_1x+q_2x^2+\cdots$$

Then

$$
\begin{aligned}
y'' &= 2a_2 + 6a_3x + \cdots \\
Py' &= (p_0 + p_1x + p_2x^2)(a_1 + 2a_2x) + \cdots = p_0a_1 + (p_1a_1 + 2p_0a_2)x + \cdots \\
Qy &= (q_0 + q_1x + q_2x^2)(a_0 + a_1x + a_2x^2) + \cdots = q_0a_0 + (q_0a_1 + q_1a_0)x + \cdots
\end{aligned}
$$

$$y''+Py'+Qy = (2a_2 + p_0a_1 + q_0a_0)+(6a_3 + p_1a_1 + 2p_0a_2 + q_0a_1 + q_1a_0)x+\cdots .$$

Thus, for the differential equation to hold we must have

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\begin{aligned}
2a_2 + p_0a_1 + q_0a_0 &= 0 \\
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a_2 &= -\tfrac{1}{2}(p_0a_1 + q_0a_0) \\
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\end{aligned}
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Here a_0 and a_1 are arbitrary, and they determine a_2, a_3 and so on.

Multiplication of power series

$$Q = q_0 + q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4 + \cdots$$
$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Multiplication of power series

$$Q = q_0 + q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4 + \cdots$$
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$$Qy = \begin{array}{lllll} q_0 a_0 & +q_0 a_1 x & +q_0 a_2 x^2 & +q_0 a_3 x^3 & +q_0 a_4 x^4 \\ +q_1 a_0 x & +q_1 a_1 x^2 & +q_1 a_2 x^3 & +q_1 a_3 x^4 & +q_1 a_4 x^5 \\ +q_2 a_0 x^2 & +q_2 a_1 x^3 & +q_2 a_2 x^4 & +q_2 a_3 x^5 & +q_2 a_4 x^6 \\ +q_3 a_0 x^3 & +q_3 a_1 x^4 & +q_3 a_2 x^5 & +q_3 a_3 x^6 & +q_3 a_4 x^7 \\ +q_4 a_0 x^4 & +q_4 a_1 x^5 & +q_4 a_2 x^6 & +q_4 a_3 x^7 & +q_4 a_4 x^8 \end{array} + \cdots$$

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$$= q_0 a_0 + (q_0 a_1 + q_1 a_0) x + (q_0 a_2 + q_1 a_1 + q_2 a_0) x^2 + (q_0 a_3 + q_1 a_2 + q_2 a_1 + q_3 a_0) x^3 + \cdots$$

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$$= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} q_i a_{n-i} \right) x^n.$$

Power series solutions — all terms

Consider an equation $y'' + Py' + Qy = 0$.
Suppose that y, P and Q are given by convergent power series:

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Then we have convergent power series for all terms in the differential equation:

$$y' = \sum_{k=0}^{\infty} k\, a_k\, x^{k-1}$$

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Power series solutions — all terms

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Power series solutions — all terms

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$$(j+1)(j+2)a_{j+2} + \left(\sum_{n=0}^{j} (j-n+1) p_n a_{j-n+1} \right) + \left(\sum_{n=0}^{j} q_n a_{j-n} \right) = 0$$

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Note that only $a_0, \ldots, a_{j+1}$ appear on the right hand side.

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Note that only $a_0, \ldots, a_{j+1}$ appear on the right hand side.
Thus a_0 and a_1 are arbitrary,
but $a_2, a_3, a_4, \ldots$ are determined inductively by the above formula.

A note on indexing

Suppose we have a series like

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots .$$

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With this convention, we have

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=-\infty}^{\infty} a_k x^k = \cdots + a_{-2} x^{-2} + a_{-1} x^{-1} + a_0 + a_1 x + a_2 x^2 + \cdots .$$

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(When $k < 0$ the terms $a_k x^k$ are zero, so it does not matter whether we include them or not.)

This will simplify various formulae, because we do not need to remember where the series starts.

Power series solutions — simple example

Consider the equation $y'' + y = 0$ with $y = 1$, $y' = 0$ at $x = 0$.

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At $x = 0$ we have $y = a_0$ and $y' = a_1$, so $a_0 = 1$ and $a_1 = 0$.

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For the differential equation $y'' + y = 0$ to hold, we must have $a_j + (j+1)(j+2)a_{j+2} = 0$

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For the differential equation $y'' + y = 0$ to hold, we must have $a_j + (j+1)(j+2)a_{j+2} = 0$, so

$$
a_{j+2} = \frac{-1}{(j+1)(j+2)} a_j .
$$

Power series solutions — simple example

$$y'' + y = 0, \qquad y = \sum_i a_i x^i, \qquad a_0 = 1,\ a_1 = 0, \qquad a_{j+2} = \frac{-a_j}{(j+1)(j+2)}.$$

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a_{2p} &= \frac{(-1)^p}{(2p)!}
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a_{2p+1} &= 0
\end{aligned}$$

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a_{2p} &= \frac{(-1)^p}{(2p)!}
\end{aligned}
\qquad
\begin{aligned}
a_1 &= 0 \\
a_3 &= \frac{-a_1}{2\times 3} = 0 \\
a_5 &= \frac{-a_3}{4\times 5} = 0 \\
a_7 &= \frac{-a_5}{6\times 7} = 0 \\
a_{2p+1} &= 0
\end{aligned}$$

So

$$y = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots$$

Power series solutions — simple example

$$y'' + y = 0, \qquad y = \sum_i a_i x^i, \qquad a_0 = 1,\ a_1 = 0, \qquad a_{j+2} = \frac{-a_j}{(j+1)(j+2)}.$$

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$$y = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots = \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p)!} x^{2p}$$

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So

$$y = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots = \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p)!} x^{2p} = \cos(x).$$

Power series solution — another example

Consider $(x-1)y'' + 2y' = 0$, with $y = y' = 1$ when $x = 0$.
Rewrite as $y'' - 2y'/(1-x) = 0$.

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We can always rewrite $Ny'' + Py' + Qy = 0$ as $y'' + (P/N)y' + (Q/N)y = 0$

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Note:
We can always rewrite $Ny'' + Py' + Qy = 0$ as $y'' + (P/N)y' + (Q/N)y = 0$, but this will cause trouble at places where $N = 0$. Here there is no problem because we are looking for a power series at $x = 0$, and $1 - x \neq 0$ when $x = 0$.

Power series solution — another example

Consider $(x-1)y'' + 2y' = 0$, with $y = y' = 1$ when $x = 0$.
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$$y'' = \sum_i i(i-1)\, a_i x^{i-2} = 2a_2 + 6a_3 x + \cdots = \sum_n (n+1)(n+2) a_{n+2} x^n$$

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$$2y'/(1-x) = 2\sum_i x^i \sum_j (j+1)\, a_{j+1} x^j = 2\sum_{i,j} (j+1)\, a_{j+1} x^{i+j}$$

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1/(1-x) &= \sum_i x^i \qquad\qquad \text{(geometric progression)} \\
2y'/(1-x) &= 2 \sum_i x^i \sum_j (j+1)\, a_{j+1} x^j = 2 \sum_{i,j} (j+1)\, a_{j+1} x^{i+j} \\
&= \sum_n x^n \sum_{j=0}^{n} 2(j+1)\, a_{j+1}.
\end{aligned}
$$

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$$y'' = \sum_i i(i-1)\, a_i x^{i-2} = 2a_2 + 6a_3 x + \cdots = \sum_n (n+1)(n+2) a_{n+2} x^n$$

$$1/(1-x) = \sum_i x^i \qquad \text{(geometric progression)}$$

$$2y'/(1-x) = 2 \sum_i x^i \sum_j (j+1)\, a_{j+1} x^j = 2 \sum_{i,j} (j+1)\, a_{j+1} x^{i+j}$$

$$= \sum_n x^n \sum_{j=0}^{n} 2(j+1)\, a_{j+1}.$$

For the equation $y'' - y'/(1-x)$ to hold, we must have

$$(n+1)(n+2) a_{n+2} = \sum_{j=0}^{n} 2(j+1)\, a_{j+1}.$$

Power series solution — another example

Consider $(x-1)y'' + 2y' = 0$, with $y = y' = 1$ when $x = 0$.
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Take $n = 0$ in $(n+1)(n+2)a_{n+2} = \sum_{j=0}^{n} 2(j+1)\, a_{j+1}$ to get

$$2a_2 = \sum_{j=0}^{0} 2(j+1)a_{j+1}$$

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$$2a_2 = \sum_{j=0}^{0} 2(j+1)a_{j+1} = 2a_1 = 2$$

Power series solution — another example

Consider $(x-1)y'' + 2y' = 0$, with $y = y' = 1$ when $x = 0$.
$y = \sum_i a_i x^i$ with $(n+1)(n+2)a_{n+2} = \sum_{j=0}^{n} 2(j+1)\, a_{j+1}$.

At $x = 0$ we have $y = a_0$ and $y' = a_1$, but also $y = y' = 1$, so $a_0 = a_1 = 1$.
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$$2a_2 = \sum_{j=0}^{0} 2(j+1)a_{j+1} = 2a_1 = 2 \qquad\qquad a_2 = 1$$

$$6a_3 = \sum_{j=0}^{1} 2(j+1)a_{j+1}$$

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Power series solution — another example

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$$12a_4 = \sum_{j=0}^{2} 2(j+1)a_{j+1}$$

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Power series solution — another example

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$y = \sum_i a_i x^i$ with $(n+1)(n+2)a_{n+2} = \sum_{j=0}^{n} 2(j+1)\, a_{j+1}$.

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Power series solution — another example

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At $x = 0$ we have $y = a_0$ and $y' = a_1$, but also $y = y' = 1$, so $a_0 = a_1 = 1$.
Take $n = 0$ in $(n+1)(n+2)a_{n+2} = \sum_{j=0}^{n} 2(j+1)\, a_{j+1}$ to get

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$$20a_5 = \sum_{j=0}^{3} 2(j+1)a_{j+1} = 2a_1 + 4a_2 + 6a_3 + 8a_4 = 20 \qquad a_5 = 1.$$

It looks like $a_k = 1$ for all k.

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$(x-1)y'' + 2y' = 0$, $y = \sum_i a_i x^i$ with $a_0 = a_1 = 1$ and
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Claim: $a_k = 1$ for all k.

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Proof by induction :

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Claim: $a_k = 1$ for all k.

Proof by induction : We are given that $a_0 = a_1 = 1$.
Suppose we already know that $a_0 = \cdots = a_{n+1} = 1$.

Power series solution — another example

$(x-1)y'' + 2y' = 0$, $y = \sum_i a_i x^i$ with $a_0 = a_1 = 1$ and
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Suppose we already know that $a_0 = \cdots = a_{n+1} = 1$. Then

$$(n+1)(n+2)a_{n+2} = \sum_{j=0}^{n} 2(j+1)a_{j+1}$$

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$(x-1)y'' + 2y' = 0$, $y = \sum_i a_i x^i$ with $a_0 = a_1 = 1$ and
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$$(n+1)(n+2)a_{n+2} = \sum_{j=0}^{n} 2(j+1)a_{j+1} = \sum_{j=0}^{n} 2(j+1).$$

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Suppose we already know that $a_0 = \cdots = a_{n+1} = 1$. Then

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This is an arithmetic progression .

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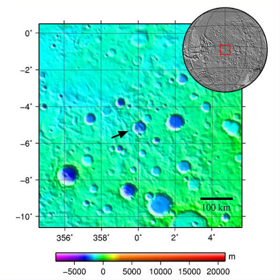

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For the equation $y'' - xy = 0$ to hold, we must have $(n+1)(n+2)a_{n+2} = a_{n-1}$, or equivalently $(m+2)(m+3)a_{m+3} = a_m$, or $a_{m+3} = a_m/((m+2)(m+3))$. (Special case: the constant term in $y'' - xy = 0$ gives $a_2 = 0$.)

Airy's equation

Airy's equation is $y'' - xy = 0$.

$$y = \sum_i a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$xy = \sum_i a_i x^{i+1} = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots = \sum_n a_{n-1} x^n$$

$$y'' = \sum_i i(i-1)\, a_i x^{i-2} = 2a_2 + 6a_3 x + \cdots = \sum_n (n+1)(n+2) a_{n+2} x^n.$$

For the equation $y'' - xy = 0$ to hold, we must have $(n+1)(n+2)a_{n+2} = a_{n-1}$, or equivalently $(m+2)(m+3)a_{m+3} = a_m$, or $a_{m+3} = a_m/((m+2)(m+3))$. (Special case: the constant term in $y'' - xy = 0$ gives $a_2 = 0$.)

$$a_3 = \frac{a_0}{2.3}$$

Airy's equation

Airy's equation is $y'' - xy = 0$.

$$
\begin{aligned}
y &= \sum_i a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\
xy &= \sum_i a_i x^{i+1} = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots = \sum_n a_{n-1} x^n \\
y'' &= \sum_i i(i-1)\, a_i x^{i-2} = 2a_2 + 6a_3 x + \cdots = \sum_n (n+1)(n+2) a_{n+2} x^n.
\end{aligned}
$$

For the equation $y'' - xy = 0$ to hold, we must have $(n+1)(n+2)a_{n+2} = a_{n-1}$, or equivalently $(m+2)(m+3)a_{m+3} = a_m$, or $a_{m+3} = a_m/((m+2)(m+3))$. (Special case: the constant term in $y'' - xy = 0$ gives $a_2 = 0$.)

$$a_3 = \frac{a_0}{2.3} \qquad\qquad a_4 = \frac{a_1}{3.4}$$

Airy's equation

Airy's equation is $y'' - xy = 0$.

$$y = \sum_i a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$xy = \sum_i a_i x^{i+1} = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots = \sum_n a_{n-1} x^n$$

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For the equation $y'' - xy = 0$ to hold, we must have $(n+1)(n+2)a_{n+2} = a_{n-1}$, or equivalently $(m+2)(m+3)a_{m+3} = a_m$, or $a_{m+3} = a_m/((m+2)(m+3))$. (Special case: the constant term in $y'' - xy = 0$ gives $a_2 = 0$.)

$$a_3 = \frac{a_0}{2.3} \qquad a_4 = \frac{a_1}{3.4} \qquad a_5 = \frac{a_2}{4.5} = 0$$

Airy's equation

Airy's equation is $y'' - xy = 0$.

$$
\begin{aligned}
y &= \sum_i a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\
xy &= \sum_i a_i x^{i+1} = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots = \sum_n a_{n-1} x^n \\
y'' &= \sum_i i(i-1)\, a_i x^{i-2} = 2a_2 + 6a_3 x + \cdots = \sum_n (n+1)(n+2) a_{n+2} x^n.
\end{aligned}
$$

For the equation $y'' - xy = 0$ to hold, we must have $(n+1)(n+2)a_{n+2} = a_{n-1}$, or equivalently $(m+2)(m+3)a_{m+3} = a_m$, or $a_{m+3} = a_m/((m+2)(m+3))$. (Special case: the constant term in $y'' - xy = 0$ gives $a_2 = 0$.)

$$a_3 = \frac{a_0}{2.3} \qquad a_4 = \frac{a_1}{3.4} \qquad a_5 = \frac{a_2}{4.5} = 0$$

$$a_6 = \frac{a_3}{5.6} = \frac{a_0}{2.3.5.6}$$

Airy's equation

Airy's equation is $y'' - xy = 0$.

$$y = \sum_i a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$xy = \sum_i a_i x^{i+1} = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots = \sum_n a_{n-1} x^n$$

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For the equation $y'' - xy = 0$ to hold, we must have $(n+1)(n+2)a_{n+2} = a_{n-1}$, or equivalently $(m+2)(m+3)a_{m+3} = a_m$, or $a_{m+3} = a_m/((m+2)(m+3))$. (Special case: the constant term in $y'' - xy = 0$ gives $a_2 = 0$.)

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$$a_6 = \frac{a_3}{5.6} = \frac{a_0}{2.3.5.6} \qquad a_7 = \frac{a_4}{6.7} = \frac{a_1}{3.4.6.7}$$

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$$y'' = \sum_i i(i-1)\, a_i x^{i-2} = 2a_2 + 6a_3 x + \cdots = \sum_n (n+1)(n+2) a_{n+2} x^n.$$

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$$a_3 = \frac{a_0}{2.3} \qquad a_4 = \frac{a_1}{3.4} \qquad a_5 = \frac{a_2}{4.5} = 0$$

$$a_6 = \frac{a_3}{5.6} = \frac{a_0}{2.3.5.6} \qquad a_7 = \frac{a_4}{6.7} = \frac{a_1}{3.4.6.7} \qquad a_8 = \frac{a_5}{7.8} = 0$$

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For the equation $y'' - xy = 0$ to hold, we must have $(n+1)(n+2)a_{n+2} = a_{n-1}$, or equivalently $(m+2)(m+3)a_{m+3} = a_m$, or $a_{m+3} = a_m/((m+2)(m+3))$. (Special case: the constant term in $y'' - xy = 0$ gives $a_2 = 0$.)

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$$a_6 = \frac{a_3}{5.6} = \frac{a_0}{2.3.5.6} \qquad a_7 = \frac{a_4}{6.7} = \frac{a_1}{3.4.6.7} \qquad a_8 = \frac{a_5}{7.8} = 0$$

$$a_9 = \frac{a_6}{8.9} = \frac{a_0}{2.3.5.6.8.9}$$

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$$a_3 = \frac{a_0}{2.3} \qquad a_4 = \frac{a_1}{3.4} \qquad a_5 = \frac{a_2}{4.5} = 0$$

$$a_6 = \frac{a_3}{5.6} = \frac{a_0}{2.3.5.6} \qquad a_7 = \frac{a_4}{6.7} = \frac{a_1}{3.4.6.7} \qquad a_8 = \frac{a_5}{7.8} = 0$$

$$a_9 = \frac{a_6}{8.9} = \frac{a_0}{2.3.5.6.8.9} \qquad a_{10} = \frac{a_7}{9.10} = \frac{a_1}{3.4.6.7.9.10}$$

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Airy's equation is $y'' - xy = 0$.

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For the equation $y'' - xy = 0$ to hold, we must have $(n+1)(n+2)a_{n+2} = a_{n-1}$, or equivalently $(m+2)(m+3)a_{m+3} = a_m$, or $a_{m+3} = a_m/((m+2)(m+3))$. (Special case: the constant term in $y'' - xy = 0$ gives $a_2 = 0$.)

$$a_3 = \frac{a_0}{2.3} \qquad a_4 = \frac{a_1}{3.4} \qquad a_5 = \frac{a_2}{4.5} = 0$$

$$a_6 = \frac{a_3}{5.6} = \frac{a_0}{2.3.5.6} \qquad a_7 = \frac{a_4}{6.7} = \frac{a_1}{3.4.6.7} \qquad a_8 = \frac{a_5}{7.8} = 0$$

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Airy's equation

Airy's equation is $y'' - xy = 0$, where $y = \sum_k a_k x^k$.

$$a_3 = \frac{a_0}{2.3} \qquad a_4 = \frac{a_1}{1.3.4} \qquad a_5 = 0$$

$$a_6 = \frac{a_0}{2.3.5.6} \qquad a_7 = \frac{a_1}{1.3.4.6.7} \qquad a_8 = 0$$

$$a_9 = \frac{a_0}{2.3.5.6.8.9} \qquad a_{10} = \frac{a_1}{1.3.4.6.7.9.10} \qquad a_{11} = 0$$

Airy's equation

Airy's equation is $y'' - xy = 0$, where $y = \sum_k a_k x^k$.

$$
\begin{array}{lll}
a_3 = \dfrac{a_0}{2.3} & a_4 = \dfrac{a_1}{1.3.4} & a_5 = 0 \\
a_6 = \dfrac{a_0}{2.3.5.6} & a_7 = \dfrac{a_1}{1.3.4.6.7} & a_8 = 0 \\
a_9 = \dfrac{a_0}{2.3.5.6.8.9} & a_{10} = \dfrac{a_1}{1.3.4.6.7.9.10} & a_{11} = 0 \\
\dfrac{a_0}{9!} = \dfrac{a_0}{\color{red}{1}.2.3.\color{red}{4}.5.6.\color{red}{7}.8.9} & &
\end{array}
$$

- a_{3k} is like $a_0/(3k)!$, except that terms like $3i+1$ are missing from the factorial.

Airy's equation

Airy's equation is $y'' - xy = 0$, where $y = \sum_k a_k x^k$.

$$
\begin{array}{lll}
a_3 = \dfrac{a_0}{2.3} & a_4 = \dfrac{a_1}{1.3.4} & a_5 = 0 \\
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\dfrac{a_0}{9!} = \dfrac{a_0}{1.2.3.4.5.6.7.8.9} & \dfrac{a_1}{10!} = \dfrac{a_0}{1.2.3.4.5.6.7.8.9.10} &
\end{array}
$$

- a_{3k} is like $a_0/(3k)!$, except that terms like $3i+1$ are missing from the factorial.
- a_{3k+1} is like $a_1/(3k+1)!$, except that terms like $3i+2$ are missing.

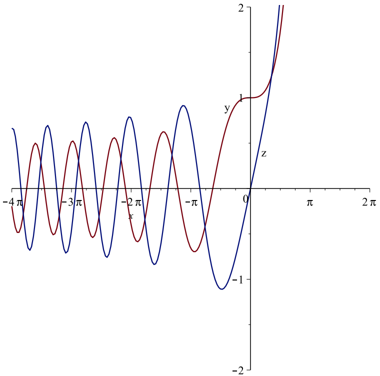
Airy's equation

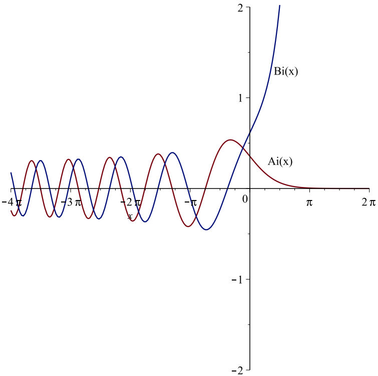
Airy's equation is $y'' - xy = 0$, where $y = \sum_k a_k x^k$.

$$
\begin{array}{lll}
a_3 = \dfrac{a_0}{2.3} & a_4 = \dfrac{a_1}{1.3.4} & a_5 = 0 \\
a_6 = \dfrac{a_0}{2.3.5.6} & a_7 = \dfrac{a_1}{1.3.4.6.7} & a_8 = 0 \\
a_9 = \dfrac{a_0}{2.3.5.6.8.9} & a_{10} = \dfrac{a_1}{1.3.4.6.7.9.10} & a_{11} = 0 \\
\dfrac{a_0}{9!} = \dfrac{a_0}{\color{red}{1}.2.3.\color{red}{4}.5.6.\color{red}{7}.8.9} & \dfrac{a_1}{10!} = \dfrac{a_0}{1.\color{red}{2}.3.4.\color{red}{5}.6.7.\color{red}{8}.9.10} &
\end{array}
$$

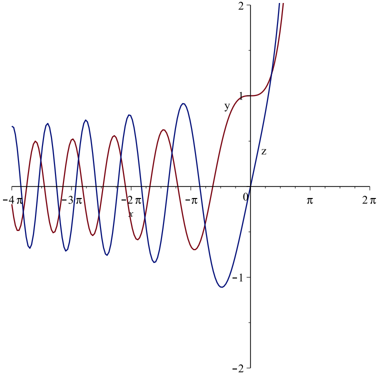
- a_{3k} is like $a_0/(3k)!$, except that terms like $3i+1$ are missing from the factorial.
- a_{3k+1} is like $a_1/(3k+1)!$, except that terms like $3i+2$ are missing.
- The radius of convergence is infinite.

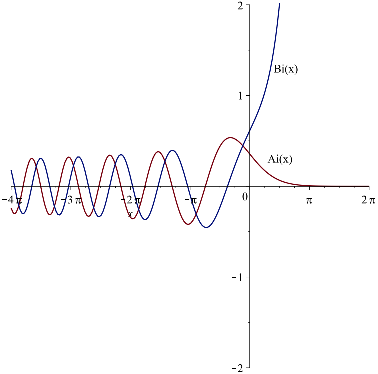
Four solutions for Airy's equation





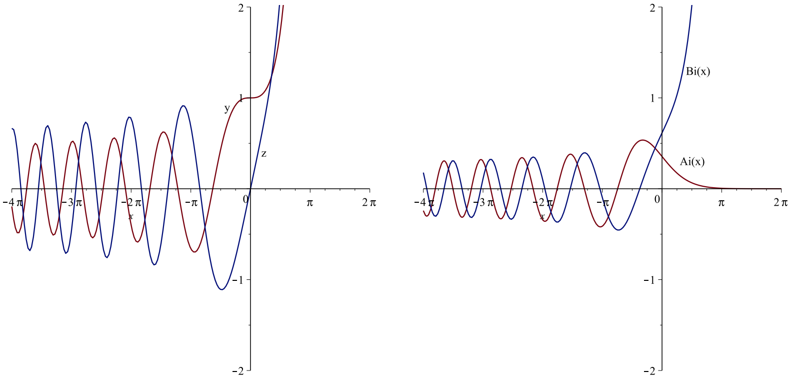
Four solutions for Airy's equation





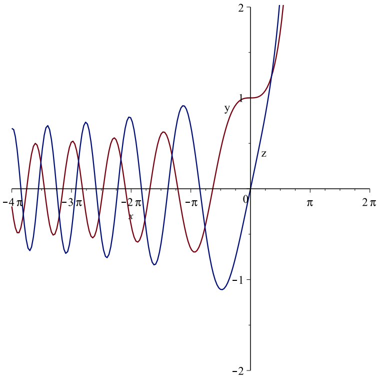
- $y = 1 + O(x^2)$ and $z = x + O(x^2)$

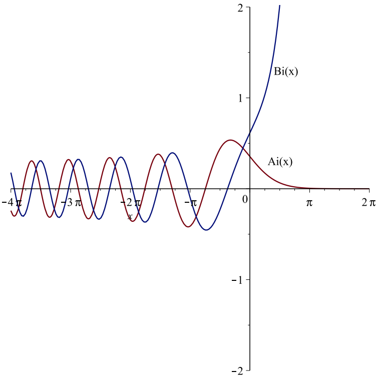
Four solutions for Airy's equation



- $y = 1 + O(x^2)$ and $z = x + O(x^2)$
- It is more traditional to use $Ai(x)$ and $Bi(x)$.

Four solutions for Airy's equation





- $y = 1 + O(x^2)$ and $z = x + O(x^2)$
- It is more traditional to use $Ai(x)$ and $Bi(x)$.
- These are $Ai(x) = \alpha y + \beta z$ and $Bi(x) = \gamma y + \delta z$ for some (complicated) constants $\alpha, \beta, \gamma, \delta$.

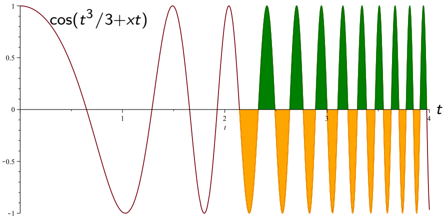
The Airy integral

Another method (relevant for the rainbow): consider the function

$$A(x) = \int_{t=0}^{\infty} \cos(t^3/3 + xt)\, dt$$

The Airy integral

Another method (relevant for the rainbow): consider the function $A(x) = \int_{t=0}^{\infty} \cos(t^3/3 + xt)\, dt$ (which only converges because of cancellation.)



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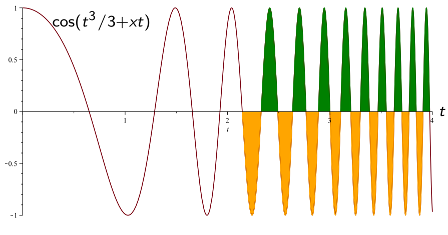

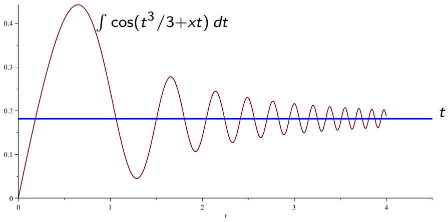

The Airy integral

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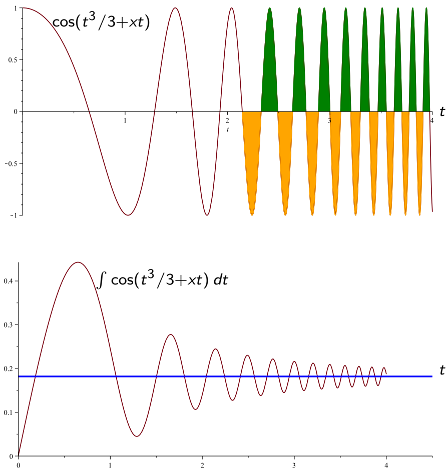
$A(x) = \int_{t=0}^{\infty} \cos(t^3/3 + xt)\,dt$ (which only converges because of cancellation.)



The pictures show $x = 2$; we see that $A(2) \simeq 0.1818604914$.

The Airy integral

Consider the function

$$A(x) = \int_{t=0}^{\infty} \cos(t^3/3 + xt)\, dt$$

The Airy integral

Consider the function

$$\begin{aligned} A(x) &= \int_{t=0}^{\infty} \cos(t^3/3 + xt)\,dt \\ A'(x) &= \int_{t=0}^{\infty} \frac{d}{dx}\cos(t^3/3 + xt)\,dt \end{aligned}$$

The Airy integral

Consider the function

$$A(x) = \int_{t=0}^{\infty} \cos(t^3/3 + xt)\,dt$$
$$A'(x) = \int_{t=0}^{\infty} \frac{d}{dx}\cos(t^3/3 + xt)\,dt = -\int_{t=0}^{\infty} t\,\sin(t^3/3 + xt)\,dt$$

The Airy integral

Consider the function

$$A(x) = \int_{t=0}^{\infty} \cos(t^3/3 + xt)\, dt$$

$$A'(x) = \int_{t=0}^{\infty} \frac{d}{dx} \cos(t^3/3 + xt)\, dt = -\int_{t=0}^{\infty} t\ \sin(t^3/3 + xt)\, dt$$

$$A''(x) = -\int_{t=0}^{\infty} \frac{d}{dx} \left(t\ \sin(t^3/3 + xt)\right)\ dt$$

The Airy integral

Consider the function

$$A(x) = \int_{t=0}^{\infty} \cos(t^3/3 + xt)\,dt$$

$$A'(x) = \int_{t=0}^{\infty} \frac{d}{dx} \cos(t^3/3 + xt)\,dt = -\int_{t=0}^{\infty} t\ \sin(t^3/3 + xt)\,dt$$

$$A''(x) = -\int_{t=0}^{\infty} \frac{d}{dx} \left(t\ \sin(t^3/3 + xt)\right)\,dt = -\int_{t=0}^{\infty} t^2\ \cos(t^3/3 + xt)\,dt$$

The Airy integral

Consider the function

$$A(x) = \int_{t=0}^{\infty} \cos(t^3/3 + xt)\,dt$$

$$A'(x) = \int_{t=0}^{\infty} \frac{d}{dx} \cos(t^3/3 + xt)\,dt = -\int_{t=0}^{\infty} t\,\sin(t^3/3 + xt)\,dt$$

$$A''(x) = -\int_{t=0}^{\infty} \frac{d}{dx} \left(t\,\sin(t^3/3 + xt)\right)\,dt = -\int_{t=0}^{\infty} t^2\,\cos(t^3/3 + xt)\,dt$$

$$A''(x) - x\,A(x) = -\int_{t=0}^{\infty} (t^2 + x)\,\cos(t^3/3 + xt)\,dt.$$

The Airy integral

Consider the function

$$
\begin{aligned}
A(x) &= \int_{t=0}^{\infty} \cos(t^3/3 + xt)\, dt \\
A'(x) &= \int_{t=0}^{\infty} \frac{d}{dx} \cos(t^3/3 + xt)\, dt = -\int_{t=0}^{\infty} t\ \sin(t^3/3 + xt)\, dt \\
A''(x) &= -\int_{t=0}^{\infty} \frac{d}{dx} \left(t\ \sin(t^3/3 + xt)\right)\, dt = -\int_{t=0}^{\infty} t^2\ \cos(t^3/3 + xt)\, dt \\
A''(x) - x\,A(x) &= -\int_{t=0}^{\infty} (t^2 + x)\ \cos(t^3/3 + xt)\, dt.
\end{aligned}
$$

We can integrate this by substituting $u = t^3/3 + xt$, so $(t^2 + x)\, dt = du$.

The Airy integral

Consider the function

$$\begin{aligned}
A(x) &= \int_{t=0}^{\infty} \cos(t^3/3 + xt)\, dt \\
A'(x) &= \int_{t=0}^{\infty} \frac{d}{dx} \cos(t^3/3 + xt)\, dt = -\int_{t=0}^{\infty} t \,\sin(t^3/3 + xt)\, dt \\
A''(x) &= -\int_{t=0}^{\infty} \frac{d}{dx} \left(t \,\sin(t^3/3 + xt)\right)\, dt = -\int_{t=0}^{\infty} t^2 \,\cos(t^3/3 + xt)\, dt \\
A''(x) - x\, A(x) &= -\int_{t=0}^{\infty} (t^2 + x)\, \cos(t^3/3 + xt)\, dt.
\end{aligned}$$

We can integrate this by substituting $u = t^3/3 + xt$, so $(t^2 + x)\, dt = du$. Moreover, we have $u = 0$ when $t = 0$, and $u = \infty$ when $t = \infty$.

The Airy integral

Consider the function

$$\begin{aligned}
A(x) &= \int_{t=0}^{\infty} \cos(t^3/3 + xt)\, dt \\
A'(x) &= \int_{t=0}^{\infty} \frac{d}{dx} \cos(t^3/3 + xt)\, dt = -\int_{t=0}^{\infty} t \sin(t^3/3 + xt)\, dt \\
A''(x) &= -\int_{t=0}^{\infty} \frac{d}{dx} \Big(t \sin(t^3/3 + xt) \Big)\, dt = -\int_{t=0}^{\infty} t^2 \cos(t^3/3 + xt)\, dt \\
A''(x) - x\, A(x) &= -\int_{t=0}^{\infty} (t^2 + x) \cos(t^3/3 + xt)\, dt.
\end{aligned}$$

We can integrate this by substituting $u = t^3/3 + xt$, so $(t^2 + x)\, dt = du$. Moreover, we have $u = 0$ when $t = 0$, and $u = \infty$ when $t = \infty$. This gives

$$A''(x) - x\, A(x) = \int_{u=0}^{\infty} \cos(u)\, du.$$

The Airy integral

Consider the function

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This integral does not really converge, but it is natural to think that the value should be zero, because the graph of $\cos(u)$ is symmetrical about the u-axis. However, some difficult arguments are needed to justify this.

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We will see that many equations of the form $y'' + Py' + Qy = 0$ have similar properties.

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- In any other case, we say that $x = 0$ is an *irregular singular point*.

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Thus $y = x^{\alpha}$ is a solution if and only if $\chi(\alpha) = 0$; in other words, α should be a root of the indicial polynomial.

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In all three cases, every solution is $Ay + Bz$ for some constants A and B.

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We will prove the first theorem but not the other two.

Question: types of points

For each equation, what type of point is $x = 0$?

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We have $P = 4x^{-1}$ and $Q = 2x^{-2}$, so $p_0 = 4$ and $q_0 = 2$, so

$$\chi(t) = t(t-1) + p_0 t + q_0 = t^2 + 3t + 2 = (t+1)(t+2).$$

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- If $\chi(t)$ has a repeated root, or two roots separated by an integer, then the situation is more complicated.

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There is a regular singular point at $x = 0$, with $p_0 = \frac{1}{2}$ and $q_0 = -\frac{1}{2}$, so the indicial polynomial is $\alpha(\alpha - 1) + \frac{1}{2}\alpha - \frac{1}{2} = 0$

Regular singular point example — non-integer gap

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$2x^2y'' + xy' - (x+1)y = 0$ or $y'' + \frac{1}{2}x^{-1}y' + (-\frac{1}{2}x^{-1} - \frac{1}{2}x^{-2})y = 0.$

There is a regular singular point at $x = 0$, with $p_0 = \frac{1}{2}$ and $q_0 = -\frac{1}{2}$,
so the indicial polynomial is $\alpha(\alpha - 1) + \frac{1}{2}\alpha - \frac{1}{2} = 0$ or $\alpha^2 - \frac{1}{2}\alpha - \frac{1}{2} = 0.$

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The roots are $-\frac{1}{2}$ and 1; the difference is not an integer.

Regular singular point example — non-integer gap

Consider the equation $2x^2y'' + xy' - (x+1)y = 0$; indicial roots $\alpha = -\frac{1}{2}, 1$.

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$$a_0 = 1 \qquad a_1 = \frac{1}{5}$$

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The first few terms are

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a_0 = 1 \qquad a_1 = \frac{1}{5} \qquad a_2 = \frac{1}{5 \times 14} = \frac{1}{70}
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so

$$y = x + \frac{1}{5}x^2 + \frac{1}{70}x^3 + \frac{1}{1890}x^4 + \cdots$$

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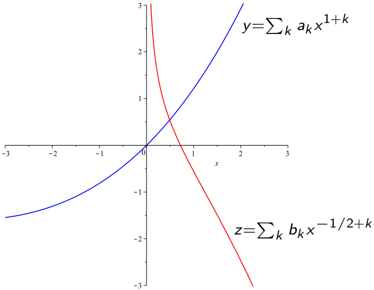

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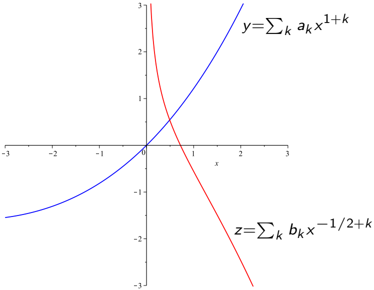

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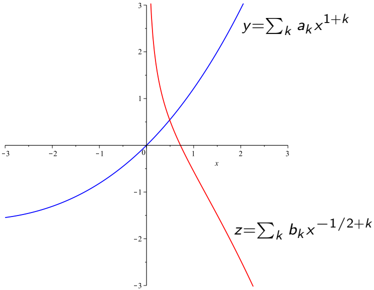

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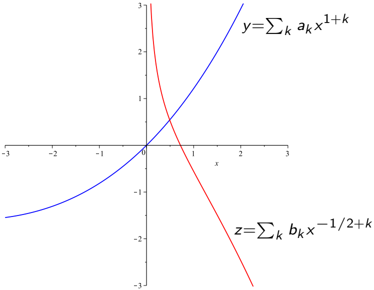

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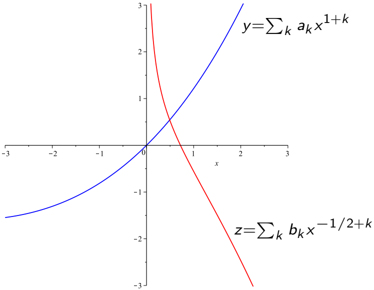

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$$y = \frac{3}{4}(u+v)$$

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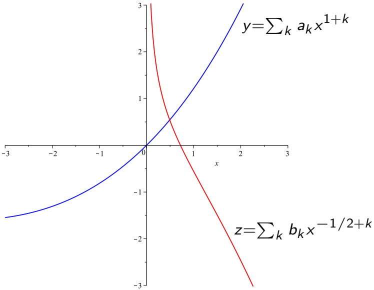

In fact, in this case it is possible to find exact solutions:

$$u = e^{\sqrt{2x}}(1 - 1/\sqrt{2x}) \qquad v = e^{-\sqrt{2x}}(1 + 1/\sqrt{2x})$$

$$\begin{aligned} y &= \frac{3}{4}(u+v) \\ &= \frac{3}{2}\left(\cosh(\sqrt{2x}) - \frac{\sinh(\sqrt{2x})}{\sqrt{2x}}\right) \end{aligned}$$

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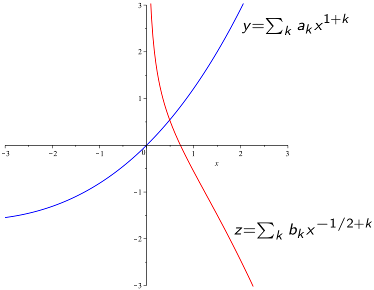

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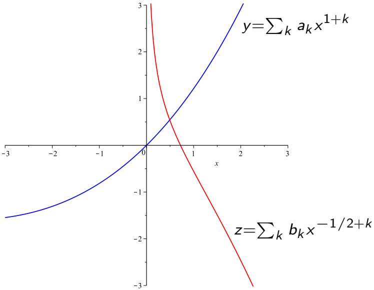

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Put $m = j+2$ to get $a_m = -(m+1)m^{-2}a_{m-1}$ for $m > 0$.

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$$y'' + (x^{-1} + 1)y' + 2x^{-1}y = 0 \qquad y = \sum_{k=0}^{\infty} a_k x^k \qquad a_k = -(k+1)k^{-2}a_{k-1}$$

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$$= -\frac{4}{1.2.3}$$

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$$= -\frac{4}{1.2.3} = -\frac{4}{3!}$$

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In general, $a_k = (-1)^k \frac{k+1}{k!}$

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$$y'' + (x^{-1}+1)y' + 2x^{-1}y = 0 \qquad y = \sum_{k=0}^{\infty} a_k x^k \qquad a_k = -(k+1)k^{-2}a_{k-1}$$

$$a_0 = 1 \qquad a_1 = -\frac{2}{1^2} \qquad a_2 = +\frac{2.3}{1^2 2^2}$$

$$a_3 = -\frac{2.3.4}{1^2 2^2 3^2} \qquad a_4 = +\frac{2.3.4.5}{1^2 2^2 3^2 4^2} \qquad a_5 = -\frac{2.3.4.5.6}{1^2 2^2 3^2 4^2 5^2}$$

$$= -\frac{4}{1.2.3} = -\frac{4}{3!} \qquad = +\frac{5}{1.2.3.4} = +\frac{5}{4!} \qquad = -\frac{6}{1.2.3.4.5} = -\frac{6}{5!}$$

In general, $a_k = (-1)^k \frac{k+1}{k!} = (-1)^k (\frac{1}{k!} + \frac{k}{k!}) = (-1)^k (\frac{1}{k!} + \frac{1}{(k-1)!})$

Regular singular point example — repeated root

$$y'' + (x^{-1}+1)y' + 2x^{-1}y = 0 \qquad y = \sum_{k=0}^{\infty} a_k x^k \qquad a_k = -(k+1)k^{-2}a_{k-1}$$

$$
\begin{aligned}
a_0 &= 1 & a_1 &= -\frac{2}{1^2} & a_2 &= +\frac{2.3}{1^2 2^2} \\
a_3 &= -\frac{2.3.4}{1^2 2^2 3^2} & a_4 &= +\frac{2.3.4.5}{1^2 2^2 3^2 4^2} & a_5 &= -\frac{2.3.4.5.6}{1^2 2^2 3^2 4^2 5^2} \\
&= -\frac{4}{1.2.3} = -\frac{4}{3!} & &= +\frac{5}{1.2.3.4} = +\frac{5}{4!} & &= -\frac{6}{1.2.3.4.5} = -\frac{6}{5!}
\end{aligned}
$$

In general, $a_k = (-1)^k \frac{k+1}{k!} = (-1)^k(\frac{1}{k!} + \frac{k}{k!}) = (-1)^k(\frac{1}{k!} + \frac{1}{(k-1)!})$. Thus

$$y = \sum_k \frac{(-x)^k}{k!} + \sum_k \frac{(-x)^k}{(k-1)!}$$

Regular singular point example — repeated root

$$y''+(x^{-1}+1)y'+2x^{-1}y=0 \qquad y=\sum_{k=0}^{\infty} a_k x^k \qquad a_k=-(k+1)k^{-2}a_{k-1}$$

$$a_0=1 \qquad a_1=-\frac{2}{1^2} \qquad a_2=+\frac{2.3}{1^2 2^2}$$

$$a_3=-\frac{2.3.4}{1^2 2^2 3^2} \qquad a_4=+\frac{2.3.4.5}{1^2 2^2 3^2 4^2} \qquad a_5=-\frac{2.3.4.5.6}{1^2 2^2 3^2 4^2 5^2}$$

$$=-\frac{4}{1.2.3}=-\frac{4}{3!} \qquad =+\frac{5}{1.2.3.4}=+\frac{5}{4!} \qquad =-\frac{6}{1.2.3.4.5}=-\frac{6}{5!}$$

In general, $a_k=(-1)^k\frac{k+1}{k!}=(-1)^k(\frac{1}{k!}+\frac{k}{k!})=(-1)^k(\frac{1}{k!}+\frac{1}{(k-1)!})$. Thus

$$y=\sum_k\frac{(-x)^k}{k!}+\sum_k\frac{(-x)^k}{(k-1)!}=e^{-x}-x\sum_k\frac{(-x)^{k-1}}{(k-1)!}$$

Regular singular point example — repeated root

$$y''+(x^{-1}+1)y'+2x^{-1}y=0 \qquad y=\sum_{k=0}^{\infty}a_kx^k \qquad a_k=-(k+1)k^{-2}a_{k-1}$$

$$\begin{aligned} a_0 &= 1 & a_1 &= -\frac{2}{1^2} & a_2 &= +\frac{2.3}{1^2 2^2} \\ a_3 &= -\frac{2.3.4}{1^2 2^2 3^2} & a_4 &= +\frac{2.3.4.5}{1^2 2^2 3^2 4^2} & a_5 &= -\frac{2.3.4.5.6}{1^2 2^2 3^2 4^2 5^2} \\ &= -\frac{4}{1.2.3} = -\frac{4}{3!} & &= +\frac{5}{1.2.3.4} = +\frac{5}{4!} & &= -\frac{6}{1.2.3.4.5} = -\frac{6}{5!} \end{aligned}$$

In general, $a_k=(-1)^k\frac{k+1}{k!}=(-1)^k(\frac{1}{k!}+\frac{k}{k!})=(-1)^k(\frac{1}{k!}+\frac{1}{(k-1)!})$. Thus

$$y=\sum_k\frac{(-x)^k}{k!}+\sum_k\frac{(-x)^k}{(k-1)!}=e^{-x}-x\sum_k\frac{(-x)^{k-1}}{(k-1)!}=e^{-x}-xe^{-x}=(1-x)e^{-x}.$$

Regular singular point example — repeated root

$y''+(x^{-1}+1)y'+2x^{-1}y = 0$ first solution: $y = (1-x)e^{-x} = \sum \frac{k+1}{k!}(-x)^k$

Regular singular point example — repeated root

$$y''+(x^{-1}+1)y'+2x^{-1}y=0 \qquad \text{first solution: } y=(1-x)e^{-x}=\sum \frac{k+1}{k!}(-x)^k$$

Second solution: $z = y\ln(x) + u$, where $u = \sum_k b_k x^k$ with $b_0 = 1$.

Regular singular point example — repeated root

$$y'' + (x^{-1} + 1)y' + 2x^{-1}y = 0 \qquad \text{first solution: } y = (1-x)e^{-x} = \sum \frac{k+1}{k!}(-x)^k$$

Second solution: $z = y\ln(x) + u$, where $u = \sum_k b_k x^k$ with $b_0 = 1$.

$$z = y\ln(x) + u$$

Regular singular point example — repeated root

$$y''+(x^{-1}+1)y'+2x^{-1}y=0 \qquad \text{first solution: } y=(1-x)e^{-x}=\sum\frac{k+1}{k!}(-x)^k$$

Second solution: $z = y\ln(x) + u$, where $u = \sum_k b_k x^k$ with $b_0 = 1$.

$$\begin{aligned} z &= y\ln(x) + u \\ z' &= y'\ln(x) + yx^{-1} + u' \end{aligned}$$

Regular singular point example — repeated root

$$y''+(x^{-1}+1)y'+2x^{-1}y=0 \qquad \text{first solution: } y=(1-x)e^{-x}=\sum \frac{k+1}{k!}(-x)^k$$

Second solution: $z = y\ln(x) + u$, where $u = \sum_k b_k x^k$ with $b_0 = 1$.

$$\begin{aligned} z &= y\ln(x) + u \\ z' &= y'\ln(x) + yx^{-1} + u' \\ z'' &= y''\ln(x) + y'x^{-1} + y'x^{-1} - yx^{-2} + u'' \end{aligned}$$

Regular singular point example — repeated root

$$y''+(x^{-1}+1)y'+2x^{-1}y=0 \qquad \text{first solution: } y=(1-x)e^{-x}=\sum\frac{k+1}{k!}(-x)^k$$

Second solution: $z = y\ln(x) + u$, where $u = \sum_k b_k x^k$ with $b_0 = 1$.

$$\begin{aligned} z &= y\ln(x) + u \\ z' &= y'\ln(x) + yx^{-1} + u' \\ z'' &= y''\ln(x) + y'x^{-1} + y'x^{-1} - yx^{-2} + u'' \\ &= y''\ln(x) + 2y'x^{-1} - yx^{-2} + u'' \end{aligned}$$

Regular singular point example — repeated root

$$y''+(x^{-1}+1)y'+2x^{-1}y=0 \qquad \text{first solution: } y=(1-x)e^{-x}=\sum\frac{k+1}{k!}(-x)^k$$

Second solution: $z = y\ln(x) + u$, where $u = \sum_k b_k x^k$ with $b_0 = 1$.

$$\begin{aligned}
z &= y\ln(x) + u \\
z' &= y'\ln(x) + yx^{-1} + u' \\
z'' &= y''\ln(x) + y'x^{-1} + y'x^{-1} - yx^{-2} + u'' \\
&= y''\ln(x) + 2y'x^{-1} - yx^{-2} + u''
\end{aligned}$$

$$\begin{aligned}
z'' + (x^{-1}+1)z' + 2x^{-1}z = {} & (y'' + (x^{-1}+1)y' + 2x^{-1}y)\ln(x) + \\
& u'' + (x^{-1}+1)u' + 2x^{-1}u + \\
& 2y'x^{-1} - yx^{-2} + (x^{-1}+1)yx^{-1}
\end{aligned}$$

Regular singular point example — repeated root

$$y'' + (x^{-1} + 1)y' + 2x^{-1}y = 0 \qquad \text{first solution: } y = (1-x)e^{-x} = \sum \frac{k+1}{k!}(-x)^k$$

Second solution: $z = y\ln(x) + u$, where $u = \sum_k b_k x^k$ with $b_0 = 1$.

$$\begin{aligned}
z &= y\ln(x) + u \\
z' &= y'\ln(x) + yx^{-1} + u' \\
z'' &= y''\ln(x) + y'x^{-1} + y'x^{-1} - yx^{-2} + u'' \\
&= y''\ln(x) + 2y'x^{-1} - yx^{-2} + u''
\end{aligned}$$

$$\begin{aligned}
z'' + (x^{-1} + 1)z' + 2x^{-1}z = {} & (y'' + (x^{-1} + 1)y' + 2x^{-1}y)\ln(x) + \\
& u'' + (x^{-1} + 1)u' + 2x^{-1}u + \\
& 2y'x^{-1} - yx^{-2} + (x^{-1} + 1)yx^{-1}
\end{aligned}$$

$$= u'' + (x^{-1} + 1)u' + 2x^{-1}u + x^{-1}(2y' + y).$$

Regular singular point example — repeated root

$$y''+(x^{-1}+1)y'+2x^{-1}y=0 \qquad \text{first solution: } y=(1-x)e^{-x}=\sum \frac{k+1}{k!}(-x)^k$$

Second solution: $z = y\ln(x) + u$, where $u = \sum_k b_k x^k$ with $b_0 = 1$.

$$\begin{aligned}
z &= y\ln(x) + u \\
z' &= y'\ln(x) + yx^{-1} + u' \\
z'' &= y''\ln(x) + y'x^{-1} + y'x^{-1} - yx^{-2} + u'' \\
&= y''\ln(x) + 2y'x^{-1} - yx^{-2} + u''
\end{aligned}$$

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z'' + (x^{-1}+1)z' + 2x^{-1}z = {} & (y'' + (x^{-1}+1)y' + 2x^{-1}y)\ln(x) + \\
& u'' + (x^{-1}+1)u' + 2x^{-1}u + \\
& 2y'x^{-1} - yx^{-2} + (x^{-1}+1)yx^{-1}
\end{aligned}$$

$$= u'' + (x^{-1}+1)u' + 2x^{-1}u + x^{-1}(2y'+y).$$

We need to find u such that this last expression is zero.

Regular singular point example — repeated root

$$y'' + (x^{-1}+1)y' + 2x^{-1}y = 0 \qquad \text{solutions: } y = (1-x)e^{-x}, \quad z = y\ln(x) + u$$

$$u'' + (x^{-1}+1)u' + 2x^{-1}u + x^{-1}(2y'+y) = 0$$

Regular singular point example — repeated root

$$y'' + (x^{-1} + 1)y' + 2x^{-1}y = 0 \qquad \text{solutions: } y = (1 - x)e^{-x}, \quad z = y\ln(x) + u$$

$$u'' + (x^{-1} + 1)u' + 2x^{-1}u + x^{-1}(2y' + y) = 0$$

$$u = \sum_{k=0}^{\infty} b_k x^k, \qquad b_0 = 1$$

Regular singular point example — repeated root

$$y'' + (x^{-1} + 1)y' + 2x^{-1}y = 0 \qquad \text{solutions: } y = (1-x)e^{-x}, \quad z = y\ln(x) + u$$

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$$u = \sum_{k=0}^{\infty} b_k x^k, \qquad b_0 = 1$$

$$u'' + (x^{-1} + 1)u' + 2x^{-1}u = \sum_k ((k+2)^2 b_{k+2} + (k+3)b_{k+1})x^k$$

Regular singular point example — repeated root

$$y'' + (x^{-1}+1)y' + 2x^{-1}y = 0 \qquad \text{solutions: } y = (1-x)e^{-x}, \quad z = y\ln(x) + u$$

$$u'' + (x^{-1}+1)u' + 2x^{-1}u + x^{-1}(2y'+y) = 0$$

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$$y' = -e^{-x} + (1-x)(-e^{-x})$$

Regular singular point example — repeated root

$$y'' + (x^{-1} + 1)y' + 2x^{-1}y = 0 \qquad \text{solutions: } y = (1-x)e^{-x}, \quad z = y\ln(x) + u$$

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$$u'' + (x^{-1} + 1)u' + 2x^{-1}u = \sum_k ((k+2)^2 b_{k+2} + (k+3)b_{k+1})x^k$$

$$y' = -e^{-x} + (1-x)(-e^{-x}) = (x-2)e^{-x}$$

Regular singular point example — repeated root

$$y'' + (x^{-1} + 1)y' + 2x^{-1}y = 0 \qquad \text{solutions: } y = (1-x)e^{-x}, \quad z = y\ln(x) + u$$

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$$u'' + (x^{-1} + 1)u' + 2x^{-1}u = \sum_k ((k+2)^2 b_{k+2} + (k+3)b_{k+1})x^k$$

$$y' = -e^{-x} + (1-x)(-e^{-x}) = (x-2)e^{-x}$$

$$x^{-1}(2y' + y) = x^{-1}(2(x-2) + (1-x))e^{-x}$$

Regular singular point example — repeated root

$$y'' + (x^{-1} + 1)y' + 2x^{-1}y = 0 \qquad \text{solutions: } y = (1 - x)e^{-x}, \quad z = y\ln(x) + u$$

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$$u'' + (x^{-1} + 1)u' + 2x^{-1}u = \sum_k ((k+2)^2 b_{k+2} + (k+3)b_{k+1})x^k$$

$$y' = -e^{-x} + (1 - x)(-e^{-x}) = (x - 2)e^{-x}$$

$$x^{-1}(2y' + y) = x^{-1}(2(x - 2) + (1 - x))e^{-x} = (1 - 3x^{-1})e^{-x}$$

Regular singular point example — repeated root

$$y'' + (x^{-1} + 1)y' + 2x^{-1}y = 0 \qquad \text{solutions: } y = (1-x)e^{-x}, \quad z = y\ln(x) + u$$

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$$x^{-1}(2y' + y) = x^{-1}(2(x-2) + (1-x))e^{-x} = (1 - 3x^{-1})e^{-x}$$

$$= \sum_k \frac{(-1)^k}{k!}x^k - \sum_k 3\frac{(-1)^k}{k!}x^{k-1}$$

Regular singular point example — repeated root

$$y'' + (x^{-1}+1)y' + 2x^{-1}y = 0 \qquad \text{solutions: } y = (1-x)e^{-x}, \quad z = y\ln(x) + u$$

$$u'' + (x^{-1}+1)u' + 2x^{-1}u + x^{-1}(2y'+y) = 0$$

$$u = \sum_{k=0}^{\infty} b_k x^k, \qquad b_0 = 1$$

$$u'' + (x^{-1}+1)u' + 2x^{-1}u = \sum_k ((k+2)^2 b_{k+2} + (k+3)b_{k+1})x^k$$

$$y' = -e^{-x} + (1-x)(-e^{-x}) = (x-2)e^{-x}$$

$$x^{-1}(2y'+y) = x^{-1}(2(x-2)+(1-x))e^{-x} = (1-3x^{-1})e^{-x}$$

$$= \sum_k \frac{(-1)^k}{k!}x^k - \sum_k 3\frac{(-1)^k}{k!}x^{k-1}$$

$$\text{(warning: limits)} = \sum_k (-1)^k \left(\frac{1}{k!} + 3\frac{1}{(k+1)!}\right) x^k$$

Regular singular point example — repeated root

$$y'' + (x^{-1}+1)y' + 2x^{-1}y = 0 \qquad \text{solutions: } y = (1-x)e^{-x}, \quad z = y\ln(x) + u$$

$$u'' + (x^{-1}+1)u' + 2x^{-1}u + x^{-1}(2y'+y) = 0$$

$$u = \sum_{k=0}^{\infty} b_k x^k, \qquad b_0 = 1$$

$$u'' + (x^{-1}+1)u' + 2x^{-1}u = \sum_k ((k+2)^2 b_{k+2} + (k+3)b_{k+1})x^k$$

$$y' = -e^{-x} + (1-x)(-e^{-x}) = (x-2)e^{-x}$$

$$x^{-1}(2y'+y) = x^{-1}(2(x-2)+(1-x))e^{-x} = (1-3x^{-1})e^{-x}$$

$$= \sum_k \frac{(-1)^k}{k!}x^k - \sum_k 3\frac{(-1)^k}{k!}x^{k-1}$$

$$\text{(warning: limits)} = \sum_k (-1)^k \left(\frac{1}{k!} + 3\frac{1}{(k+1)!}\right)x^k = \sum_k (-1)^k \frac{k+4}{(k+1)!}x^k$$

Regular singular point example — repeated root

$$y'' + (x^{-1}+1)y' + 2x^{-1}y = 0 \qquad \text{solutions: } y = (1-x)e^{-x}, \quad z = y\ln(x) + u$$

$$u'' + (x^{-1}+1)u' + 2x^{-1}u + x^{-1}(2y'+y) = 0$$

$$u = \sum_{k=0}^{\infty} b_k x^k, \qquad b_0 = 1$$

$$u'' + (x^{-1}+1)u' + 2x^{-1}u = \sum_k ((k+2)^2 b_{k+2} + (k+3)b_{k+1})x^k$$

$$y' = -e^{-x} + (1-x)(-e^{-x}) = (x-2)e^{-x}$$

$$x^{-1}(2y'+y) = x^{-1}(2(x-2)+(1-x))e^{-x} = (1-3x^{-1})e^{-x}$$

$$= \sum_k \frac{(-1)^k}{k!}x^k - \sum_k 3\frac{(-1)^k}{k!}x^{k-1}$$

$$\text{(warning: limits)} = \sum_k (-1)^k \left(\frac{1}{k!} + 3\frac{1}{(k+1)!}\right)x^k = \sum_k (-1)^k \frac{k+4}{(k+1)!}x^k$$

We therefore need $(k+2)^2 b_{k+2} + (k+3)b_{k+1} + (-1)^k(k+4)/(k+1)! = 0$.

Regular singular point example — repeated root

$$y'' + (x^{-1} + 1)y' + 2x^{-1}y = 0 \qquad \text{solutions: } y = (1-x)e^{-x}, \quad z = y\ln(x) + u$$

$$u = \sum_k b_k x^k \qquad (k+2)^2 b_{k+2} + (k+3)b_{k+1} + (-1)^k (k+4)/(k+1)! = 0$$

Regular singular point example — repeated root

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The above equation for b_{k+1} and b_{k+2} is valid when $k \geq 0$.

Regular singular point example — repeated root

$$y'' + (x^{-1}+1)y' + 2x^{-1}y = 0 \qquad \text{solutions: } y = (1-x)e^{-x}, \quad z = y\ln(x) + u$$

$$u = \sum_k b_k x^k \qquad (k+2)^2 b_{k+2} + (k+3)b_{k+1} + (-1)^k(k+4)/(k+1)! = 0$$

The above equation for b_{k+1} and b_{k+2} is valid when $k \geq 0$. It gives

$$b_m = -\frac{1}{m^2}\left((m+1)b_{m-1} + (-1)^m \frac{m+2}{(m-1)!}\right) \qquad \text{for } m \geq 2.$$

Regular singular point example — repeated root

$$y'' + (x^{-1}+1)y' + 2x^{-1}y = 0 \qquad \text{solutions: } y = (1-x)e^{-x}, \quad z = y\ln(x) + u$$

$$u = \sum_k b_k x^k \qquad (k+2)^2 b_{k+2} + (k+3)b_{k+1} + (-1)^k (k+4)/(k+1)! = 0$$

The above equation for b_{k+1} and b_{k+2} is valid when $k \geq 0$. It gives

$$b_m = -\frac{1}{m^2}\left((m+1)b_{m-1} + (-1)^m \frac{m+2}{(m-1)!}\right) \qquad \text{for } m \geq 2.$$

If we look more carefully at the first few terms, we get

$$u'' + (x^{-1}+1)u' + 2x^{-1}u = (b_1+2)x^{-1} + \text{ terms in } x^0 \text{ and above}$$

Regular singular point example — repeated root

$$y'' + (x^{-1} + 1)y' + 2x^{-1}y = 0 \qquad \text{solutions: } y = (1-x)e^{-x}, \quad z = y\ln(x) + u$$

$$u = \sum_k b_k x^k \qquad (k+2)^2 b_{k+2} + (k+3)b_{k+1} + (-1)^k (k+4)/(k+1)! = 0$$

The above equation for b_{k+1} and b_{k+2} is valid when $k \geq 0$. It gives

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If we look more carefully at the first few terms, we get

$$u'' + (x^{-1} + 1)u' + 2x^{-1}u = (b_1 + 2)x^{-1} + \text{ terms in } x^0 \text{ and above}$$

$$x^{-1}(2y' + y) = -3x^{-1} + \text{ terms in } x^0 \text{ and above.}$$

Regular singular point example — repeated root

$$y'' + (x^{-1}+1)y' + 2x^{-1}y = 0 \qquad \text{solutions: } y = (1-x)e^{-x}, \quad z = y\ln(x) + u$$

$$u = \sum_k b_k x^k \qquad (k+2)^2 b_{k+2} + (k+3)b_{k+1} + (-1)^k(k+4)/(k+1)! = 0$$

The above equation for b_{k+1} and b_{k+2} is valid when $k \geq 0$. It gives

$$b_m = -\frac{1}{m^2}\left((m+1)b_{m-1} + (-1)^m \frac{m+2}{(m-1)!}\right) \qquad \text{for } m \geq 2.$$

If we look more carefully at the first few terms, we get

$$u'' + (x^{-1}+1)u' + 2x^{-1}u = (b_1+2)x^{-1} + \text{ terms in } x^0 \text{ and above}$$

$$x^{-1}(2y'+y) = -3x^{-1} + \text{ terms in } x^0 \text{ and above.}$$

As $u'' + (x^{-1}+1)u' + 2x^{-1}u + x^{-1}(2y'+y) = 0$, we must have
$(b_1+2) + (-3) = 0$

Regular singular point example — repeated root

$$y'' + (x^{-1}+1)y' + 2x^{-1}y = 0 \qquad \text{solutions: } y = (1-x)e^{-x}, \quad z = y\ln(x) + u$$

$$u = \sum_k b_k x^k \qquad (k+2)^2 b_{k+2} + (k+3)b_{k+1} + (-1)^k (k+4)/(k+1)! = 0$$

The above equation for b_{k+1} and b_{k+2} is valid when $k \geq 0$. It gives

$$b_m = -\frac{1}{m^2}\left((m+1)b_{m-1} + (-1)^m \frac{m+2}{(m-1)!}\right) \qquad \text{for } m \geq 2.$$

If we look more carefully at the first few terms, we get

$$u'' + (x^{-1}+1)u' + 2x^{-1}u = (b_1+2)x^{-1} + \text{ terms in } x^0 \text{ and above}$$

$$x^{-1}(2y'+y) = -3x^{-1} + \text{ terms in } x^0 \text{ and above.}$$

As $u'' + (x^{-1}+1)u' + 2x^{-1}u + x^{-1}(2y'+y) = 0$, we must have $(b_1+2) + (-3) = 0$, or in other words $b_1 = 1$.

Regular singular point example — repeated root

$$y'' + (x^{-1}+1)y' + 2x^{-1}y = 0 \qquad \text{solutions: } y = (1-x)e^{-x}, \quad z = y\ln(x) + u$$

$$u = \sum_k b_k x^k \qquad (k+2)^2 b_{k+2} + (k+3)b_{k+1} + (-1)^k (k+4)/(k+1)! = 0$$

The above equation for b_{k+1} and b_{k+2} is valid when $k \geq 0$. It gives

$$b_m = -\frac{1}{m^2}\left((m+1)b_{m-1} + (-1)^m \frac{m+2}{(m-1)!}\right) \qquad \text{for } m \geq 2.$$

If we look more carefully at the first few terms, we get

$$u'' + (x^{-1}+1)u' + 2x^{-1}u = (b_1+2)x^{-1} + \text{ terms in } x^0 \text{ and above}$$

$$x^{-1}(2y'+y) = -3x^{-1} + \text{ terms in } x^0 \text{ and above.}$$

As $u'' + (x^{-1}+1)u' + 2x^{-1}u + x^{-1}(2y'+y) = 0$, we must have $(b_1+2) + (-3) = 0$, or in other words $b_1 = 1$. The recurrence relation now gives

$$b_0 = 1 \qquad\qquad b_1 = 1 \qquad\qquad b_2 = -\frac{7}{4}$$

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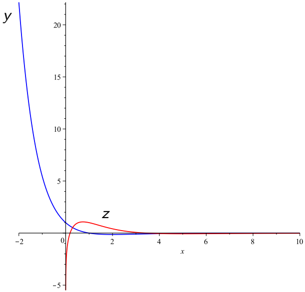
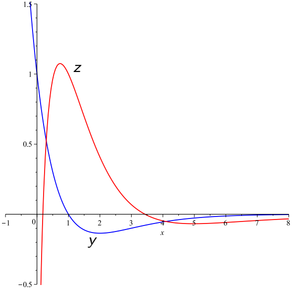
Regular singular point example — repeated root

Solutions for $y'' + (x^{-1} + 1)y' + 2x^{-1}y = 0$ are

$$y = (1 - x)e^{-x}$$

$$z = \ln(x)(1 - x)e^{-x} + 1 + x - \frac{7}{4}x^2 + \frac{19}{18}x^3 - \frac{113}{288}x^4 + \frac{127}{1200}x^5 + \cdots$$





Reduction of order

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This method is called *reduction of order*.

Reduction of order example

Consider the equation $y'' - 2(1 + x^{-1})y' + (1 + 2x^{-1})y = 0$.

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Check: $z' = (x^2 + \frac{1}{3}x^3)e^x$

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\end{aligned}$$

Reduction of order example

Consider the equation $y'' - 2(1 + x^{-1})y' + (1 + 2x^{-1})y = 0$.
One solution is $y = e^x$ (because then $y'' = y' = y$ and everything cancels).
We use reduction of order to find another solution.

$$\begin{aligned} P &= -2(1 + x^{-1}) \\ v &= \int P\,dx = -2(x + \ln(x)) \\ y^{-2}e^{-v} &= e^{-2x}e^{2x+2\ln(x)} = e^{2\ln(x)} = (e^{\ln(x)})^2 = x^2 \\ u &= \int y^{-2}e^{-v}\,dx = \int x^2\,dx = \frac{1}{3}x^3 \\ z &= uy = \frac{1}{3}x^3e^x. \end{aligned}$$

Check:

$$\begin{aligned} z' &= (x^2 + \tfrac{1}{3}x^3)e^x \\ z'' &= (2x + x^2)e^x + (x^2 + \tfrac{1}{3}x^3)e^x = (2x + 2x^2 + \tfrac{1}{3}x^3)e^x \end{aligned}$$

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So $z'' - 2(1+x^{-1})z' + (1+2x^{-1})z = 0$.

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The second solution is

$$z = \sum_{p=0}^{\infty} \frac{(-1/4)^p}{p!^2} x^{2p}(\log(x) - \phi(p)).$$

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eg: $L_{\text{Bessel}}(J_n) = n^2 J_n$

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Reminder about matrices and eigenvectors

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We will only discuss Dirichlet conditions. Other cases are similar.

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Conclusion:
the only real eigenvalues are $\lambda = -n^2\pi^2$; eigenfunctions are $\sin(n\pi x)$.

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Corollary: If $Lf = \lambda f$ and $Lg = \mu g$ with $\lambda \neq \mu$, and $f(a) = g(a) = f(b) = g(b) = 0$, then $\langle f, g \rangle = 0$.

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Corollary: If $Lf = \lambda f$ and $Lg = \mu g$ with $\lambda \neq \mu$, and $f(a) = g(a) = f(b) = g(b) = 0$, then $\langle f, g \rangle = 0$.

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Now $r > 0$ everywhere and $f = g + ih$ is not the zero function so $\int_a^b r(g^2 + h^2)\, dx > 0$. This means that $\nu = 0$, so λ is real.
We also see that $Lg = \lambda g$ and $Lh = \lambda h$, so g and h are real eigenfunctions. $\square$

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Thus, if $z'' + Rz = 0$ then $y'' + Pz' + Qz = 0$. $\square$

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$$y'' + Py' + Qy = 0 \qquad m = \exp(-\tfrac{1}{2}\int P\,dx) \qquad R = Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2$$

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which is the same as before.

Normal form example

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Conclusion: the solution for $x^2y'' - 2\mu xy' + (\mu(\mu+1) + x^2)y = 0$ is

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Normal form for the Bessel equation

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Conclusion: the solutions for $x^2y'' + xy' + (x^2 - n^2)y = 0$ have the form $y = x^{-1/2}z$, where $z'' + \left(1 + \frac{1-4n^2}{4x^2}\right)z = 0$.

Normal form for the Bessel equation

$$y'' + Py' + Qy = 0 \qquad m = \exp(-\tfrac{1}{2}\int P\,dx) \qquad R = Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2$$

$$y = mz \qquad z'' + Rz = 0$$

Consider again the Bessel equation $x^2y'' + xy' + (x^2 - n^2)y = 0$.
We can divide by x^2 to get $y'' + x^{-1}y' + (1 - n^2x^{-2})y = 0$.
This is $y'' + Py' + Qy = 0$, where $P = x^{-1}$ and $Q = 1 - n^2x^{-2}$.
We now have

$$m = \exp(-\tfrac{1}{2}\int P\,dx) = \exp(-\tfrac{1}{2}\ln(x)) = x^{-1/2}$$

$$R = Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2 = 1 - n^2x^{-2} + \tfrac{1}{2}x^{-2} - \tfrac{1}{4}x^{-2} = 1 + \frac{1 - 4n^2}{4x^2}$$

Conclusion: the solutions for $x^2y'' + xy' + (x^2 - n^2)y = 0$ have the form $y = x^{-1/2}z$, where $z'' + \left(1 + \frac{1-4n^2}{4x^2}\right)z = 0$.
For large x, this is approximately $z'' + z = 0$, so z is like $A\cos(x + \phi)$, so y is like $Ax^{-1/2}\cos(x + \phi)$.

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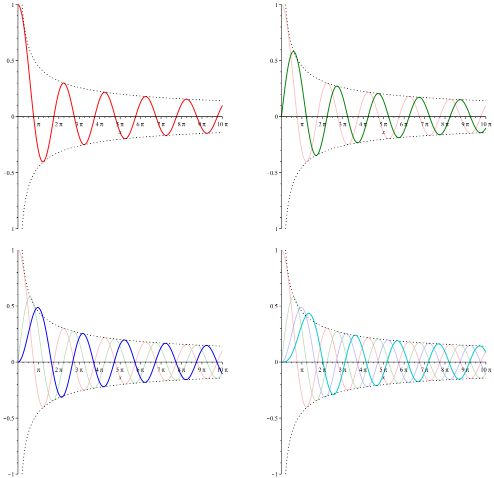

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(c) Using $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}^{-1} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ and (a) and (b) we get

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is equivalent to $\ddot{y} + (e^{2t} - n^2)y = 0$.
Proof:

$$\dot{x} = \frac{dx}{dt} = \frac{d}{dt}e^t = e^t = x$$

Transformations

Suppose that $x = e^t$, and write $u' = du/dx$ and $\dot{u} = du/dt$.
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Proof:

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$$\dot{y} = \frac{dy}{dt}$$

Transformations

Suppose that $x = e^t$, and write $u' = du/dx$ and $\dot{u} = du/dt$.
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$$\dot{x} = \frac{dx}{dt} = \frac{d}{dt}e^t = e^t = x$$

$$\dot{y} = \frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx}$$

Transformations

Suppose that $x = e^t$, and write $u' = du/dx$ and $\dot{u} = du/dt$.
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Transformations

Suppose that $x = e^t$, and write $u' = du/dx$ and $\dot{u} = du/dt$.
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$$\frac{d}{dx}(\dot{y}) = \frac{d}{dx}(xy')$$

Transformations

Suppose that $x = e^t$, and write $u' = du/dx$ and $\dot{u} = du/dt$.
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Proof:

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$$\frac{d}{dx}(\dot{y}) = \frac{d}{dx}(xy') = y' + xy''$$

Transformations

Suppose that $x = e^t$, and write $u' = du/dx$ and $\dot{u} = du/dt$.
Claim: the Bessel equation $x^2y'' + xy' + (x^2 - n^2)y = 0$
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Proof:

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\dot{x} &= \frac{dx}{dt} = \frac{d}{dt}e^t = e^t = x \\
\dot{y} &= \frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx} = xy' \\
\frac{d}{dx}(\dot{y}) &= \frac{d}{dx}(xy') = y' + xy'' \\
\ddot{y} &= \frac{d}{dt}(\dot{y})
\end{aligned}$$

Transformations

Suppose that $x = e^t$, and write $u' = du/dx$ and $\dot{u} = du/dt$.
Claim: the Bessel equation $x^2y'' + xy' + (x^2 - n^2)y = 0$
is equivalent to $\ddot{y} + (e^{2t} - n^2)y = 0$.
Proof:

$$\begin{aligned}
\dot{x} &= \frac{dx}{dt} = \frac{d}{dt}e^t = e^t = x \\
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\ddot{y} &= \frac{d}{dt}(\dot{y}) = \frac{dx}{dt}\frac{d}{dx}(\dot{y})
\end{aligned}$$

Transformations

Suppose that $x = e^t$, and write $u' = du/dx$ and $\dot{u} = du/dt$.
Claim: the Bessel equation $x^2 y'' + xy' + (x^2 - n^2)y = 0$
is equivalent to $\ddot{y} + (e^{2t} - n^2)y = 0$.
Proof:

$$\begin{aligned}
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\dot{y} &= \frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx} = xy' \\
\frac{d}{dx}(\dot{y}) &= \frac{d}{dx}(xy') = y' + xy'' \\
\ddot{y} &= \frac{d}{dt}(\dot{y}) = \frac{dx}{dt}\frac{d}{dx}(\dot{y}) = x(y' + xy'')
\end{aligned}$$

Transformations

Suppose that $x = e^t$, and write $u' = du/dx$ and $\dot{u} = du/dt$.
Claim: the Bessel equation $x^2y'' + xy' + (x^2 - n^2)y = 0$
is equivalent to $\ddot{y} + (e^{2t} - n^2)y = 0$.
Proof:

$$
\begin{aligned}
\dot{x} &= \frac{dx}{dt} = \frac{d}{dt}e^t = e^t = x \\
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\frac{d}{dx}(\dot{y}) &= \frac{d}{dx}(xy') = y' + xy'' \\
\ddot{y} &= \frac{d}{dt}(\dot{y}) = \frac{dx}{dt}\frac{d}{dx}(\dot{y}) = x(y' + xy'') = x^2y'' + xy'
\end{aligned}
$$

Transformations

Suppose that $x = e^t$, and write $u' = du/dx$ and $\dot{u} = du/dt$.
Claim: the Bessel equation $x^2y'' + xy' + (x^2 - n^2)y = 0$
is equivalent to $\ddot{y} + (e^{2t} - n^2)y = 0$.
Proof:

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\end{aligned}$$

so

$$\ddot{y} + (e^{2t} - n^2)y = x^2y'' + xy' + (x^2 - n^2)y$$

Transformations

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Claim: the Bessel equation $x^2y'' + xy' + (x^2 - n^2)y = 0$
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$$\begin{aligned}
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\end{aligned}$$

so

$$\ddot{y} + (e^{2t} - n^2)y = x^2y'' + xy' + (x^2 - n^2)y$$

so solutions to $\ddot{y} + (e^{2t} - n^2)y = 0$ are $y = AJ_n(e^t) + BY_n(e^t)$.

Transformations

Suppose that $x = t^2$ and $z = y/x^3$.

Transformations

Suppose that $x = t^2$ and $z = y/x^3$.
Claim: Legendre equation $(1 - x^2)y'' - 2xy' + 12y = 0$
is equivalent to $(t^6 - t^2)\ddot{z} + (15t^5 - 11t)\dot{z} - 24z = 0$.

Suppose that $x = t^2$ and $z = y/x^3$.
Claim: Legendre equation $(1 - x^2)y'' - 2xy' + 12y = 0$
is equivalent to $(t^6 - t^2)\ddot{z} + (15t^5 - 11t)\dot{z} - 24z = 0$.
Proof:

$$z' = x^{-3}y' - 3x^{-4}y$$

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Proof:

$$z' = x^{-3}y' - 3x^{-4}y = t^{-6}y' - 3t^{-8}y$$

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$$\dot{z} = \frac{dz}{dt} = \frac{dx}{dt}\frac{dz}{dx}$$

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$$\dot{z} = \frac{dz}{dt} = \frac{dx}{dt}\frac{dz}{dx} = 2tz'$$

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$$\ddot{z} = 2z' + 2t\frac{d}{dt}(z')$$

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Proof:

$$
\begin{aligned}
z' &= x^{-3}y' - 3x^{-4}y = t^{-6}y' - 3t^{-8}y \\
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\dot{z} &= \frac{dz}{dt} = \frac{dx}{dt}\frac{dz}{dx} = 2tz' \\
\ddot{z} &= 2z' + 2t\frac{d}{dt}(z') = 2z' + 2t\frac{dx}{dt}\frac{d}{dx}(z') \\
&= 2z' + 2t.2tz''
\end{aligned}
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\ddot{z} &= 2z' + 2t\frac{d}{dt}(z') = 2z' + 2t\frac{dx}{dt}\frac{d}{dx}(z') \\
&= 2z' + 2t.2tz'' = 2z' + 4t^2z''
\end{aligned}
$$

Transformations

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Proof:

$$z' = x^{-3}y' - 3x^{-4}y = t^{-6}y' - 3t^{-8}y$$

$$z'' = x^{-3}y'' - 6x^{-4}y' + 12x^{-5}y = t^{-6}y'' - 6t^{-8}y' + 12t^{-10}y$$

$$\dot{z} = \frac{dz}{dt} = \frac{dx}{dt}\frac{dz}{dx} = 2tz' = 2t^{-5}y' - 6t^{-7}y$$

$$\begin{aligned}\ddot{z} &= 2z' + 2t\frac{d}{dt}(z') = 2z' + 2t\frac{dx}{dt}\frac{d}{dx}(z') \\ &= 2z' + 2t.2tz'' = 2z' + 4t^2z''\end{aligned}$$

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z' &= x^{-3}y' - 3x^{-4}y = t^{-6}y' - 3t^{-8}y \\
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\ddot{z} &= 2z' + 2t\frac{d}{dt}(z') = 2z' + 2t\frac{dx}{dt}\frac{d}{dx}(z') \\
&= 2z' + 2t.2tz'' = 2z' + 4t^2z'' \\
&= 2(t^{-6}y' - 3t^{-8}y) + 4t^2(t^{-6}y'' - 6t^{-8}y' + 12t^{-10}y)
\end{aligned}
$$

Transformations

Suppose that $x = t^2$ and $z = y/x^3$.
Claim: Legendre equation $(1 - x^2)y'' - 2xy' + 12y = 0$
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\begin{aligned}
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\dot{z} &= \frac{dz}{dt} = \frac{dx}{dt}\frac{dz}{dx} = 2tz' = 2t^{-5}y' - 6t^{-7}y \\
\ddot{z} &= 2z' + 2t\frac{d}{dt}(z') = 2z' + 2t\frac{dx}{dt}\frac{d}{dx}(z') \\
&= 2z' + 2t.2tz'' = 2z' + 4t^2z'' \\
&= 2(t^{-6}y' - 3t^{-8}y) + 4t^2(t^{-6}y'' - 6t^{-8}y' + 12t^{-10}y) \\
&= 4t^{-4}y'' - 22t^{-6}y' + 42t^{-8}y
\end{aligned}
$$

Transformations

$$x = t^2 \qquad z = y/x^3 = y/t^6$$
$$\dot{z} = 2t^{-5}y' - 6t^{-7}y \qquad \ddot{z} = 4t^{-4}y'' - 22t^{-6}y' + 42t^{-8}y$$

Transformations

$$x = t^2 \qquad z = y/x^3 = y/t^6$$
$$\dot{z} = 2t^{-5}y' - 6t^{-7}y \qquad \ddot{z} = 4t^{-4}y'' - 22t^{-6}y' + 42t^{-8}y$$

$$\begin{aligned}
&(t^6 - t^2)\ddot{z} + (15t^5 - 11)\dot{z} - 24z \\
= &(t^6 - t^2)(4t^{-4}y'' - 22t^{-6}y' + 42t^{-8}y) + \\
&(15t^5 - 11t)(2t^{-5}y' - 6t^{-7}y) - 24t^{-6}y
\end{aligned}$$

Transformations

$$x = t^2 \qquad z = y/x^3 = y/t^6$$
$$\dot{z} = 2t^{-5}y' - 6t^{-7}y \qquad \ddot{z} = 4t^{-4}y'' - 22t^{-6}y' + 42t^{-8}y$$

$$\begin{aligned}
&(t^6 - t^2)\ddot{z} + (15t^5 - 11)\dot{z} - 24z \\
={}& (t^6 - t^2)(4t^{-4}y'' - 22t^{-6}y' + 42t^{-8}y) + \\
&(15t^5 - 11t)(2t^{-5}y' - 6t^{-7}y) - 24t^{-6}y \\
={}& 4(t^2 - t^{-2})y'' + (-22 + 22t^{-4} + 30 - 22t^{-4})y' + \\
&(42t^{-2} - 42t^{-6} - 90t^{-2} + 66t^{-6} - 24t^{-6})y
\end{aligned}$$

Transformations

$x = t^2 \qquad z = y/x^3 = y/t^6$
$\dot{z} = 2t^{-5}y' - 6t^{-7}y \qquad \ddot{z} = 4t^{-4}y'' - 22t^{-6}y' + 42t^{-8}y$

$$
\begin{aligned}
&(t^6 - t^2)\ddot{z} + (15t^5 - 11)\dot{z} - 24z \\
= &(t^6 - t^2)(4t^{-4}y'' - 22t^{-6}y' + 42t^{-8}y) + \\
&(15t^5 - 11t)(2t^{-5}y' - 6t^{-7}y) - 24t^{-6}y \\
= &4(t^2 - t^{-2})y'' + (-22 + 22t^{-4} + 30 - 22t^{-4})y' + \\
&(42t^{-2} - 42t^{-6} - 90t^{-2} + 66t^{-6} - 24t^{-6})y \\
= &4(t^2 - t^{-2})y'' + 8y' - 48t^{-2}y
\end{aligned}
$$

Transformations

$$x = t^2 \qquad z = y/x^3 = y/t^6$$
$$\dot{z} = 2t^{-5}y' - 6t^{-7}y \qquad \ddot{z} = 4t^{-4}y'' - 22t^{-6}y' + 42t^{-8}y$$

$$\begin{aligned}
&(t^6 - t^2)\ddot{z} + (15t^5 - 11)\dot{z} - 24z \\
=\,&(t^6 - t^2)(4t^{-4}y'' - 22t^{-6}y' + 42t^{-8}y) + \\
&(15t^5 - 11t)(2t^{-5}y' - 6t^{-7}y) - 24t^{-6}y \\
=\,&4(t^2 - t^{-2})y'' + (-22 + 22t^{-4} + 30 - 22t^{-4})y' + \\
&(42t^{-2} - 42t^{-6} - 90t^{-2} + 66t^{-6} - 24t^{-6})y \\
=\,&4(t^2 - t^{-2})y'' + 8y' - 48t^{-2}y \\
=\,&-4t^{-2}((1 - t^4)y'' - 2t^2y' + 12y)
\end{aligned}$$

Transformations

$x = t^2 \qquad z = y/x^3 = y/t^6$
$\dot{z} = 2t^{-5}y' - 6t^{-7}y \qquad \ddot{z} = 4t^{-4}y'' - 22t^{-6}y' + 42t^{-8}y$

$$
\begin{aligned}
&(t^6 - t^2)\ddot{z} + (15t^5 - 11)\dot{z} - 24z \\
&= (t^6 - t^2)(4t^{-4}y'' - 22t^{-6}y' + 42t^{-8}y) + \\
&\quad (15t^5 - 11t)(2t^{-5}y' - 6t^{-7}y) - 24t^{-6}y \\
&= 4(t^2 - t^{-2})y'' + (-22 + 22t^{-4} + 30 - 22t^{-4})y' + \\
&\quad (42t^{-2} - 42t^{-6} - 90t^{-2} + 66t^{-6} - 24t^{-6})y \\
&= 4(t^2 - t^{-2})y'' + 8y' - 48t^{-2}y \\
&= -4t^{-2}((1 - t^4)y'' - 2t^2y' + 12y) = -4x^{-1}((1 - x^2)y'' - 2xy' + 12y)
\end{aligned}
$$

Transformations

$x = t^2 \qquad z = y/x^3 = y/t^6$
$\dot{z} = 2t^{-5}y' - 6t^{-7}y \qquad \ddot{z} = 4t^{-4}y'' - 22t^{-6}y' + 42t^{-8}y$

$$\begin{aligned}
&(t^6 - t^2)\ddot{z} + (15t^5 - 11)\dot{z} - 24z \\
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&(15t^5 - 11t)(2t^{-5}y' - 6t^{-7}y) - 24t^{-6}y \\
=\,&4(t^2 - t^{-2})y'' + (-22 + 22t^{-4} + 30 - 22t^{-4})y' + \\
&(42t^{-2} - 42t^{-6} - 90t^{-2} + 66t^{-6} - 24t^{-6})y \\
=\,&4(t^2 - t^{-2})y'' + 8y' - 48t^{-2}y \\
=\,&-4t^{-2}((1 - t^4)y'' - 2t^2y' + 12y) = -4x^{-1}((1 - x^2)y'' - 2xy' + 12y)
\end{aligned}$$

So $(1 - x^2)y'' - 2xy' + 12y = 0$ is equivalent to
$(t^6 - t^2)\ddot{z} + (15t^5 - 11)\dot{z} - 24z = 0$.

Transformations

$x = t^2 \qquad z = y/x^3 = y/t^6$
$\dot{z} = 2t^{-5}y' - 6t^{-7}y \qquad \ddot{z} = 4t^{-4}y'' - 22t^{-6}y' + 42t^{-8}y$

$$
\begin{aligned}
&(t^6 - t^2)\ddot{z} + (15t^5 - 11)\dot{z} - 24z \\
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&(42t^{-2} - 42t^{-6} - 90t^{-2} + 66t^{-6} - 24t^{-6})y \\
=\,&4(t^2 - t^{-2})y'' + 8y' - 48t^{-2}y \\
=\,&-4t^{-2}((1 - t^4)y'' - 2t^2y' + 12y) = -4x^{-1}((1 - x^2)y'' - 2xy' + 12y)
\end{aligned}
$$

So $(1 - x^2)y'' - 2xy' + 12y = 0$ is equivalent to $(t^6 - t^2)\ddot{z} + (15t^5 - 11)\dot{z} - 24z = 0$. So solutions to $(t^6 - t^2)\ddot{z} + (15t^5 - 11)\dot{z} - 24z = 0$ are $(AP_n(t^2) + BQ_n(t^2))/t^6$.