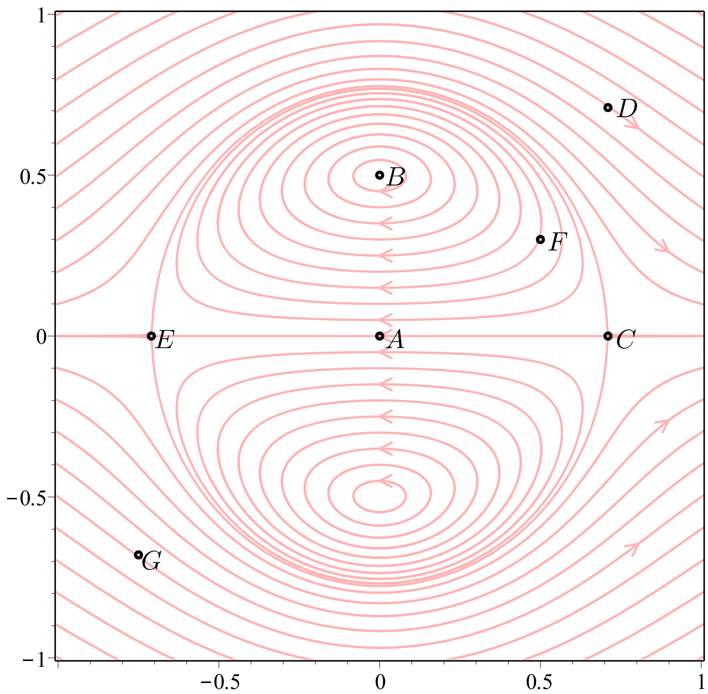
MAS290 PROBLEMS

Exercise 1. Consider the following phase portrait:



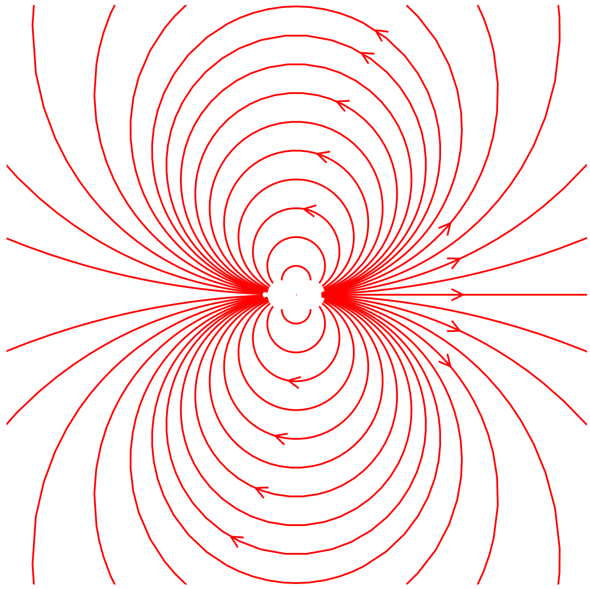
Which of the following statements are true?

(a) Point A is an equilibrium point.
(b) Point B is a saddle.
(c) If a solution has $(x, y) = C$ at $t = 0$, then $(x, y) = E$ at some time $t > 0$.
(d) At point D we have $\dot{x} > 0$ and $\dot{y} > 0$.
(e) Point E lies on the x-nullcline.
(f) If a solution has $(x, y) = F$ at $t = 0$, then it also has $(x, y) = F$ at some time $t > 0$.
(f) If a solution has $(x, y) = G$ at $t = 0$, then it also has $(x, y) = G$ at some time $t > 0$.

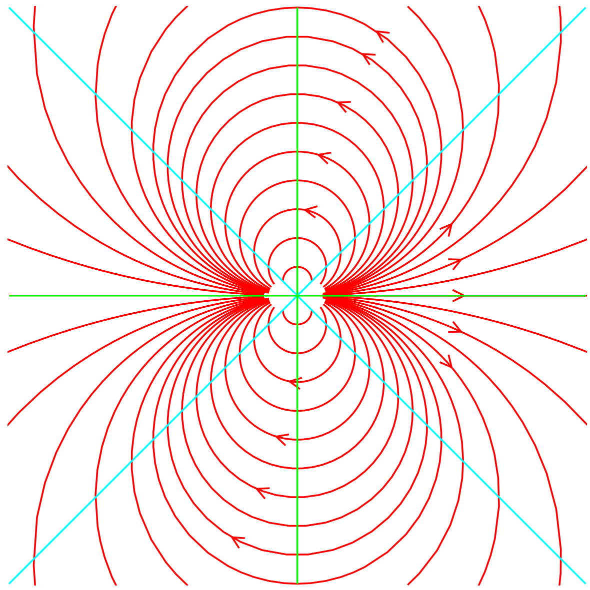
Solution:

(a) This is false. A solution starting at A will move immediately to the left.
(b) This is false. The flow lines near B circulate around B, so it is a centre, not a saddle.
(c) This is false. Point C is the limit of two different flow lines, which means it must be an equilibrium point (and in fact a saddle). This means that a flow line starting at C just stays there, it does not move towards E.
(d) This is false. The flow line through D points to the right (so $\dot{x} > 0$) and down (so $\dot{y} < 0$).
(e) This is true. In fact E is an equilibrium point (just like C), so it lies on both the x-nullcline and the y-nullcline.
(f) This is true. The flow line through F is a closed curve, so after a finite time the solution starting at F will travel all the way around the curve and return to F.
(g) This is false (assuming that the flow lines behave in the obvious way outside the box that we have shown). The solution starting at G will move continually to the right and will never return to G.

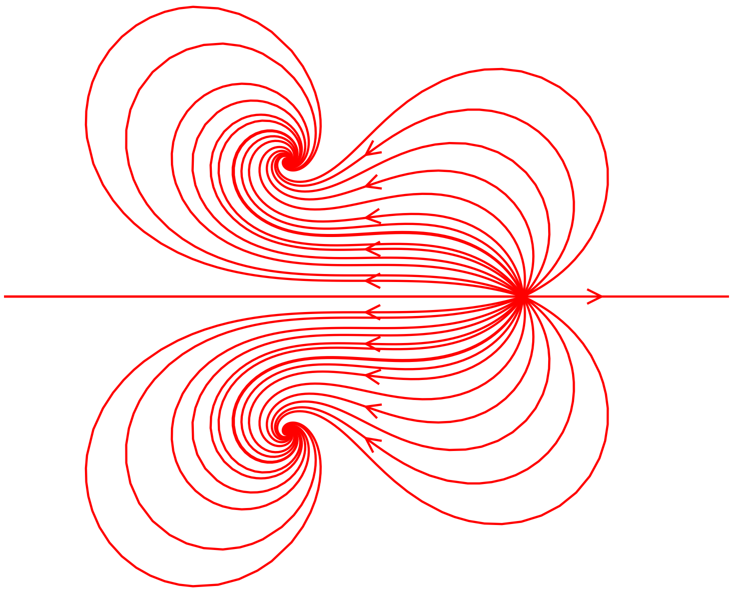
Exercise 2. Draw the nullclines and equilibrium points for the following phase portrait.



Solution: Each circular flow line has two points (one on the left and one on the right) where the line is vertical and so $\dot{x} = 0$. Joining up these points gives the x-nullcline, which consists of two diagonal lines. Similarly, at the top and bottom of each circular flow line the line is horizontal, so $\dot{y} = 0$. Joining up these points gives the y-axis, which is part of the y-nullcline. Moreover, the whole of the x-axis is another flow line which is horizontal and so is part of the y-nullcline. Thus, the y-nullcline is the union of the two axes.

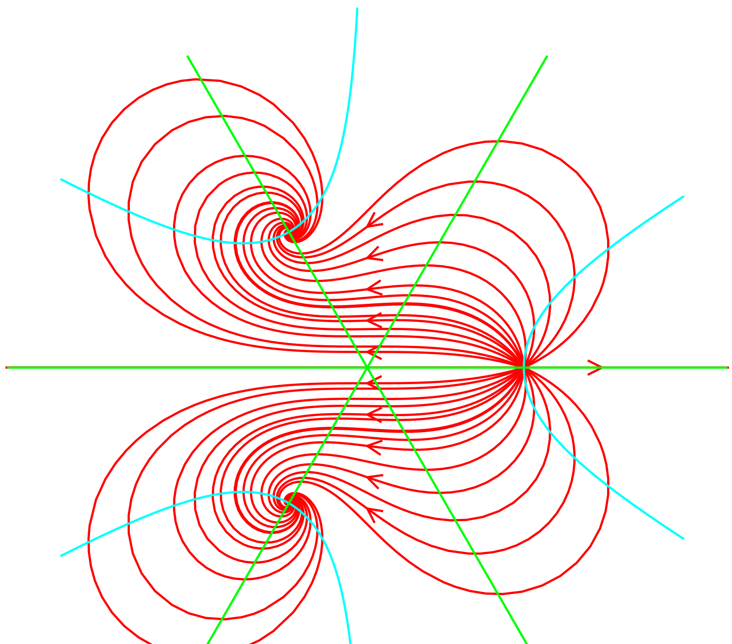


Exercise 3. Draw the nullclines and equilibrium points for the following phase portrait.



Hint: the y-nullcline consists of three straight lines, and the x-nullcline consists of three curves. The whole picture should be very symmetrical.

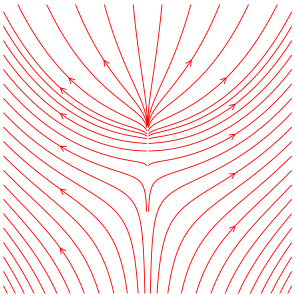
Solution: The x-axis is a flow line which is horizontal and so is part of the y-nullcline. On each of the other flow lines, there is a point where the line is horizontal. These points are easy to find on the flow lines that are highly curved, and the hint tells us that we can just connect them together with straight lines. This gives the y-nullcline. Similarly, to find the x-nullcline, we look for points where the flow lines are vertical, and join them together with a smooth curve.

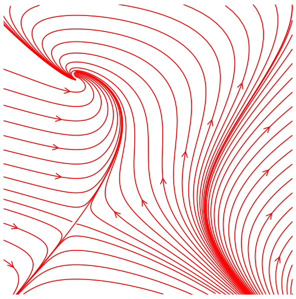


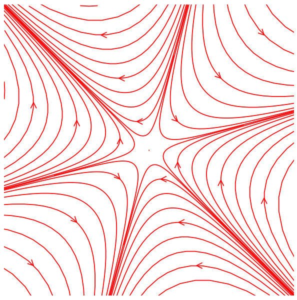
Exercise 4. Consider the following four systems:

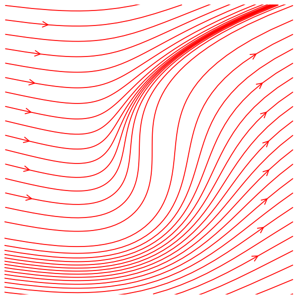
(a) $\dot{x} = x^2 - y^2$, $\dot{y} = x + 1$
(b) $\dot{x} = x^2 + y^2$, $\dot{y} = x + 1$
(c) $\dot{x} = x$, $\dot{y} = x^2 + y^2$
(d) $\dot{x} = x^2 + 2xy - y^2$, $\dot{y} = x^2 - 2xy - y^2$

The phase portraits are shown below, in a different order:


(p)


(q)


(r)


(s)

By considering equilibrium points, nullclines and so on, work out which phase portrait belongs to which system.

Solution: The nullclines and equilibrium points are as follows.

(a) The x-nullcline is given by $x^2 - y^2 = 0$, which means that $y = \pm x$, so we have two diagonal lines through the origin. The y-nullcline is given by $x = -1$. The equilibrium points are where $x = -1$ and $y = \pm x$, so they are $(-1, -1)$ and $(-1, 1)$.

(b) Here the x-nullcline is given by $x^2 + y^2 = 0$, which is only satisfied when $x = y = 0$. Thus, the x-nullcline is a single point at the origin. At all other points we have $\dot{x} \geq 0$, so the flow lines move to the right. The y-nullcline is a vertical line given by $x = -1$. The x-nullcline and the y-nullcline do not intersect, so there are no equilibrium points.

(c) Here the y-nullcline is just a single point at the origin, and at all other points the flow lines move upwards. The x-nullcline is given by $x = 0$. To the left of this line we have $\dot{x} < 0$ and so the flow lines move further to the left. Similarly, to the right of $x = 0$ we have $\dot{x} > 0$ and so the lines move further to the right.

(d) The x-nullcline is given by $x^2 + 2xy - y^2 = 0$. This is a quadratic equation for y, which we can solve to get $y = (-2x \pm \sqrt{4x^2 + 4x^2})/(-2)$, or $y = (1 \pm \sqrt{2})x$. Thus, the x-nullcline consists of two straight lines through the origin, with slopes $1 - \sqrt{2} \simeq -0.41$ and $1 + \sqrt{2} \simeq 2.41$. Similarly, the y-nullcline is given by $x^2 - 2xy - y^2 = 0$ or $y = (-1 \pm \sqrt{2})x$, so it consists of two more straight lines through th origin, of slope approximately 0.41 and -2.41

Now consider the pictures. In picture (p) the lines all move upwards, and they move left when $x < 0$ and right when $x > 0$; this matches with system (c). In picture (q) there are two equilibrium points: a saddle near the bottom left, and a focus directly above the saddle near the top left. This matches with the behaviour of system (a). In picture (r) we can see that the only equilibrium point is at the origin. We can sketch the x-nullcline by looking for points where the flow lines are vertical. We find that these points lie on two straight lines through the origin. Similarly, we can sketch that y-nullcline by by looking for points where the flow lines are horizontal. We find that these points again lie on two straight lines through the origin. This matches with system (d). Finally, picture (s) shows no equilibrium points at all, so this must be system (b).

Exercise 5. Find the trace, determinant and eigenvalues of the following matrices.

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \qquad B = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \qquad C = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \qquad D = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}.$$

Solution:

(a) Matrix A has trace $\tau = a + c$ and determinant $\delta = ac$, so the characteristic polynomial is

$$t^2 - \tau t + \delta = t^2 - (a + c)t + ac = (t - a)(t - c),$$

so the eigenvalues are a and c.

(b) Matrix B has trace $\tau = 2\cos(\theta) = e^{i\theta} + e^{-i\theta}$ and determinant $\delta = \cos(\theta)^2 + \sin(\theta)^2 = 1$. It follows that the characteristic polynomial is

$$t^2 - \tau t + \delta = t^2 - (e^{i\theta} + e^{-i\theta})t + 1 = (t - e^{i\theta})(t - e^{-i\theta}),$$

and the eigenvalues are $e^{i\theta}$ and $e^{-i\theta}$.

(c) Matrix C has trace $\tau = 2a$ and determinant $\delta = a^2 - b^2$. It follows that $\tau^2 - 4\delta = 4a^2 - 4a^2 + 4b^2 = 4b^2$, so the eigenvalues are

$$\tfrac{1}{2}(\tau \pm \sqrt{\tau^2 - 4\delta}) = \tfrac{1}{2}(2a \pm 2b) = a + b,\ a - b.$$

(d) Matrix D has trace $\tau = 0$ and determinant $\delta = -ab$, so the characteristic polynomial is $t^2 - ab$, and the eigenvalues are $\pm\sqrt{ab}$.

Exercise 6. Solve the system $\dot{u} = 13v$, $\dot{v} = -13u - 10v$ with $u = a$ and $v = b$ at $t = 0$.

Solution: The problem can be written as $\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix}$, where $A = \begin{bmatrix} 0 & 13 \\ -13 & -10 \end{bmatrix}$. The trace is $\tau = -10$ and the determinant is $\delta = 13^2 = 169$, so $\tau^2 - 4\delta = -576 < 0$. Thus, the eigenvalues are $(-10 \pm \sqrt{576}i)/2 = -5 \pm 12i$. We therefore have a stable focus with $\lambda = -5$ and $\omega = 12$. The corresponding matrix P is

$$P = e^{\lambda t}(\cos(\omega t)I + \omega^{-1}\sin(\omega t)(A - \lambda I)) = e^{-5t}(\cos(12t)I + \frac{1}{12}\sin(12t)(A + 5I)).$$

Here

$$\frac{A + 5I}{12} = \frac{1}{12}\begin{bmatrix} 5 & 13 \\ -13 & -5 \end{bmatrix} = \begin{bmatrix} 5/12 & 13/12 \\ -13/12 & -5/12 \end{bmatrix},$$

so

$$P = e^{-5t}\begin{bmatrix} \cos(12t) + \frac{5}{12}\sin(12t) & \frac{13}{12}\sin(12t) \\ -\frac{13}{12}\sin(12t) & \cos(12t) - \frac{5}{12}\sin(12t) \end{bmatrix}.$$

Thus, the solution starting at $\begin{bmatrix} a \\ b \end{bmatrix}$ is

$$\begin{bmatrix} u \\ v \end{bmatrix} = P\begin{bmatrix} a \\ b \end{bmatrix} = e^{-5t}\begin{bmatrix} a\cos(12t) + \frac{5a+13b}{12}\sin(12t) \\ b\cos(12t) - \frac{13a+5b}{12}\sin(12t) \end{bmatrix}.$$

Exercise 7. For each of the following linear systems, classify the equilibrium point at the origin.

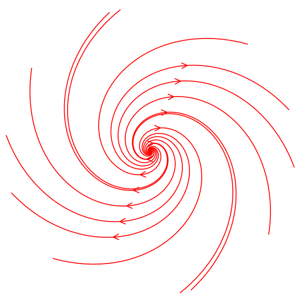
(a) $\dot{x} = 2x + y$, $\dot{y} = x + 2y$
(b) $\dot{x} = 2x + y$, $\dot{y} = x - 3y$
(c) $\dot{x} = x - 4y$, $\dot{y} = 2x - y$
(d) $\dot{x} = 2y$, $\dot{y} = -3x - y$
(e) $\dot{x} = 8y - x$, $\dot{y} = 7y - 2x$
(f) $\dot{x} = 3y - 7x$, $\dot{y} = 3y - 8x$.

Solution:

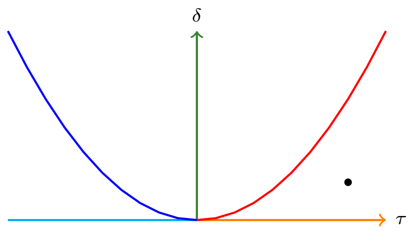
system	A	τ	δ	$\tau^2-4\delta$	λ_1,λ_2	type
(a)	$\begin{bmatrix}2&1\\1&2\end{bmatrix}$	4	3	4	$1,3$	unstable node
(b)	$\begin{bmatrix}2&1\\1&-3\end{bmatrix}$	-1	-7	29	$\frac{-1\pm\sqrt{29}}{2}$	saddle
(c)	$\begin{bmatrix}1&-4\\2&-1\end{bmatrix}$	0	7	-28	$\pm\sqrt{7}i$	centre
(d)	$\begin{bmatrix}0&2\\-3&-1\end{bmatrix}$	-1	6	-23	$\frac{-1\pm\sqrt{23}i}{2}$	stable focus
(e)	$\begin{bmatrix}-1&8\\-2&7\end{bmatrix}$	6	9	0	$3,3$	unstable node
(f)	$\begin{bmatrix}-7&3\\-8&3\end{bmatrix}$	-4	3	4	$-3,-1$	stable node

Exercise 8. Suppose we have five different linear differential equations with properties described below. In each case, find the type of equilibrium point at the origin.

- The matrix for system A has eigenvalues -2 and 3.
- The matrix for system B has $\tau = 0$ and $\delta = 16$.
- One of the solutions for system C is $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3e^{-2t}+2e^{-3t}\\3e^{-3t}+2e^{-2t}\end{bmatrix}$.
- System D has the following phase portrait:



- System E corresponds to the following point in the (τ,δ) plane:



Solution:

A This system has one negative eigenvalue and one positive eigenvalue, so it must be a saddle.

B This system has $\tau = 0$ and $\delta > 0$ so it is a centre. The eigenvalues are $\pm\sqrt{-4\times 16} = \pm 8i$, so the angular frequency is $\omega = 8$. We do not have enough information to decide whether the rotation is clockwise or anticlockwise.

C As e^{-2t} and e^{-3t} appear in the solution, the eigenvalues must be -2 and -3. As these are both negative real numbers, we have a stable node.

D As the flow lines spiral outwards, this is an unstable focus.

E The given point is in the top right quadrant where $\tau > 0$ and $\delta > 0$. It lies below the parabola with equation $\delta = \tau^2/4$, so $\delta < \tau^2/4$, so $\tau^2 - 4\delta > 0$. This means that the eigenvalues $(\tau \pm \sqrt{\tau^2 - 4\delta})/2$ are both positive, so we have an unstable node.

Exercise 9. Give examples of linear differential equations of the following types. Try to make your examples as simple as possible.

(a) A stable node.
(b) A saddle.
(c) A clockwise centre.
(d) An anticlockwise unstable focus.

Solution:

(a) We need a system with two distinct, negative eigenvalues. The simplest example is the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, or the corresponding system $\dot{x} = -x$, $\dot{y} = -2y$.

(b) We need a system with one positive eigenvalue and one negative eigenvalue. The simplest example is the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, or the corresponding system $\dot{x} = x$, $\dot{y} = -y$.

(c) We need a system with $\tau = 0$ and $\delta > 0$. The two simplest examples are $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The first of these gives an anticlockwise rotation, so we use the second one. The corresponding equations are $\dot{x} = y$ and $\dot{y} = -x$.

(d) We need a system with $\tau^2 - 4\delta < 0$ (for a focus) and $\tau > 0$ (to make it unstable). The bottom left entry in the matrix should also be positive, to ensure that the rotation is anticlockwise. A simple example is $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$, which has $\tau = \delta = 1$. The corresponding equations are $\dot{x} = -y$ and $\dot{y} = x + y$.

Exercise 10. Consider the matrix $A = \begin{bmatrix} -4 & 3 \\ -10 & 7 \end{bmatrix}$.

(a) Find the trace, determinant, eigenvalues and eigenvectors of A.
(b) Hence find a diagonal matrix D and an invertible matrix V such that $A = VDV^{-1}$.
(c) Find a matrix P depending on t such that $P = I$ when $t = 0$, and $\dot{P} = AP$.
(d) Find a vector u depending on t such that $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ at $t = 0$, and $\dot{u} = Au$.

Solution:

(a) The trace is $\tau = -4 + 7 = 3$, and the determinant is $\delta = (-4) \times 7 - 3 \times (-10) = 2$. This means that $\tau^2 - 4\delta = 9 - 8 = 1$. The eigenvalues are $(\tau \pm \sqrt{\tau^2 - 4\delta})/2$, which gives $\lambda_1 = 1$ and $\lambda_2 = 2$. An eigenvector v_1 of eigenvalue 1 must satisfy $(A - I)v_1 = 0$, but $A - I = \begin{bmatrix} -5 & 3 \\ -10 & 6 \end{bmatrix}$, so we can take $v_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Similarly, an eigenvector v_2 of eigenvalue 2 must satisfy $(A - 2I)v_2 = 0$, but $A - 2I = \begin{bmatrix} -6 & 3 \\ -10 & 5 \end{bmatrix}$, so we can take $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(b) The general method here is to take

$$V = \left[\begin{array}{c|c} v_1 & v_2 \end{array}\right] = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$
$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Note here that $\det(V) = 3 \times 2 - 1 \times 5 = 1$ and so

$$V^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}.$$

(c) In the lectures we gave two different methods for this. The first method says that $P = VEV^{-1}$, where

$$E = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}.$$

This works out as

$$P = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 2e^t & -e^t \\ -5e^{2t} & 3e^{2t} \end{bmatrix} = \begin{bmatrix} 6e^t - 5e^{2t} & -3e^t + 3e^{2t} \\ 10e^t - 10e^{2t} & -5e^t + 6e^{2t} \end{bmatrix}.$$

The second approach uses the formula

$$P = (\lambda_2 - \lambda_1)^{-1}((\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})I + (e^{\lambda_2 t} - e^{\lambda_1 t})A).$$

In our case this becomes

$$\begin{aligned} P &= (2e^t - e^{2t})I + (e^{2t} - e^t)A \\ &= \begin{bmatrix} 2e^t - e^{2t} & 0 \\ 0 & 2e^t - e^{2t} \end{bmatrix} + (e^{2t} - e^t) \begin{bmatrix} -4 & 3 \\ -10 & 7 \end{bmatrix} = \begin{bmatrix} 6e^t - 5e^{2t} & -3e^t + 3e^{2t} \\ 10e^t - 10e^{2t} & -5e^t + 6e^{2t} \end{bmatrix}. \end{aligned}$$

(d) The relevant vector is $u = P \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3e^t - 2e^{2t} \\ 5e^t - 4e^{2t} \end{bmatrix}$

Exercise 11. For each of the following matrices A_k, find a matrix P_k (depending on t) such that $\dot{P}_k = AP_k$, and $P_k = I$ when $t = 0$.

$$A_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Solution: All three matrices A_k have trace $\tau = 2$, and the determinants are $0, 1$ and 2. The corresponding values of $\tau^2 - 4\delta$ are 4, 0 and -4.

(a) For A_0, the eigenvalues are $(2 \pm \sqrt{4})/2$, which gives $\lambda_1 = 0$ and $\lambda_2 = 2$ (both real). The standard formula in this context is

$$P = \frac{1}{\lambda_2 - \lambda_1} \left((\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})I + (e^{\lambda_2 t} - e^{\lambda_1 t})A \right).$$

In the present case, this becomes

$$\begin{aligned} P_0 &= \frac{1}{2} \left((2e^0 - 0e^{2t})I + (e^{2t} - e^0)A_0 \right) = \frac{1}{2} \left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + (e^{2t} - 1) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) \\ &= \frac{1}{2} \begin{bmatrix} e^{2t} + 1 & e^{2t} - 1 \\ e^{2t} - 1 & e^{2t} + 1 \end{bmatrix}. \end{aligned}$$

(b) For A_1, the eigenvalues are $(2 \pm \sqrt{0})/2$, so $\lambda = 1$ is a repeated eigenvalue. The standard formula in this context is

$$P = e^{\lambda t}(I + t(A - \lambda I)).$$

In the present case, this becomes

$$P_1 = e^t \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} e^t & t\,e^t \\ 0 & e^t \end{bmatrix}.$$

(c) For A_2, the eigenvalues are $(2 \pm \sqrt{-4})/2$, which gives $1 \pm i$. The standard formula in this context is

$$P = e^{\lambda t} \left(\cos(\omega t)I + \omega^{-1} \sin(\omega t)(A - \lambda I) \right).$$

In the present case we have $\lambda = \omega = 1$, giving

$$P = e^t \left(\cos(t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin(t) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) = e^t \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}.$$

Exercise 12. Consider the matrix

$$P = e^{2t} \begin{bmatrix} 7\cos(3t) + \sin(3t) & 4\cos(3t) - 3\sin(3t) \\ \cos(3t) + \sin(3t) & \cos(3t) \end{bmatrix}.$$

Find a matrix A such that $\dot{P} = AP$. Is P the fundamental solution for A?

Solution: Write $s = \sin(3t)$ and $c = \cos(3t)$ and $A = \begin{bmatrix} m & n \\ p & q \end{bmatrix}$. We then have

$$\begin{aligned}
\dot{P} &= 2e^{2t}\begin{bmatrix} 7c+s & 4c-3s \\ c+s & c \end{bmatrix} + e^{2t}\begin{bmatrix} -21s+3c & -12s-9c \\ -3s+3c & -3s \end{bmatrix} \\
&= e^{2t}\begin{bmatrix} 17c-19s & -c-18s \\ 5c-s & 2c-3s \end{bmatrix} \\
AP &= e^{2t}\begin{bmatrix} m & n \\ p & q \end{bmatrix}\begin{bmatrix} 7c+s & 4c-3s \\ c+s & c \end{bmatrix} \\
&= e^{2t}\begin{bmatrix} (7m+n)c+(m+n)s & (4m+n)c-3ms \\ (7p+q)c+(p+q)s & (4p+q)c-3ps \end{bmatrix}
\end{aligned}$$

We therefore want

$$\begin{aligned}
7m+n &= 17 & m+n &= -19 \\
4m+n &= -1 & -3m &= -18 \\
7p+q &= 5 & p+q &= -1 \\
4p+q &= 2 & -3p &= -3.
\end{aligned}$$

These are easily solved to give $m = 6$ and $n = -25$ and $p = 1$ and $q = -2$, so $A = \begin{bmatrix} 6 & -25 \\ 1 & -2 \end{bmatrix}$. If P was the fundamental solution for A then we would not only have $\dot{P} = AP$, but also $P = I$ when $t = 0$. In fact we have $P = \begin{bmatrix} 7 & 4 \\ 1 & 1 \end{bmatrix}$ when $t = 0$, so P is not the fundamental solution.

Exercise 13. Give examples as follows. The numbers in every matrix should be real numbers.

(a) Give an example of a linear system with an anticlockwise centre at the origin.
(b) Give an example of a matrix B where $\tau = 3$ and $\delta = 0$.
(c) Give an example of a matrix C where the eigenvalues are $1+i$ and $1-i$.
(d) Give an example of a linear system for which the function $U = xy$ is a conserved quantity.
(e) Give an example of a linear system with solution $(x, y) = (e^t, te^t)$.

Solution:

(a) We need $\dot{x} = ax + by$ and $\dot{y} = cx + dy$ with $\tau = a + d = 0$ and $\delta = ad - bc > 0$ and $c > 0$. The simplest way to do this is with $a = d = 0$ and $b = -1$ and $c = 1$, giving $\dot{x} = -y$ and $\dot{y} = x$.
(b) We need $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a + d = 3$ and $ad - bc = 0$. The simplest way to do this is with $a = 3$ and $b = c = d = 0$ giving $B = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$. Another possibility is $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.
(c) We need

$$\begin{aligned}
\tau &= \lambda_1 + \lambda_2 = (1-i) + (1+i) = 2 \\
\delta &= \lambda_1\lambda_2 = (1-i)(1+i) = 1 - i^2 = 2.
\end{aligned}$$

The simplest way to do this is with $D = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.
(d) The simplest way to do this is with $\dot{x} = x$ and $\dot{y} = -y$. (This gives $\dot{U} = \dot{x}y + x\dot{y} = xy - xy = 0$. Alternatively, the solutions have the form $(x, y) = (x_0e^t, y_0e^{-t})$, giving $U = x_0y_0$ for all t.)
(e) These functions satisfy $\dot{x} = x$ and $\dot{y} = x + y$. (This is easy to see, because $\dot{x} = e^t = x$ and $\dot{y} = e^t + te^t = x + y$.)

Exercise 14. For each of the following (essentially linear) systems, find the equilibrium points.

(a) $\dot{x} = 10x - 100$, $\dot{y} = y - 11x$
(b) $\dot{x} = 10x - 100y$, $\dot{y} = y - 11x$
(c) $\dot{u} = 100u + v - 506$, $\dot{v} = u + 10v - 65$
(d) $\dot{p} = p + q + 1$, $\dot{q} = p + q + 1$
(e) $\dot{x} = x - y + 1$, $\dot{y} = y - x + 1$.

Solution:

(a) The equilibrium points are the points where $\dot{x} = 0$ and $\dot{y} = 0$, so $10x - 100 = 0$ and $y - 11x = 0$. The first equation gives $x = 10$, and we can put $x = 10$ in the second equation to get $y = 110$. Thus, the only equilibrium point is $(10, 110)$.

(b) Here we need to solve $10x - 100y = 0$ and $y - 11x = 0$. The first equation gives $x = 10y$, and the second gives $y = 11x$, so $x = 110x$, so $x = 0$, so $y = 0$. Thus, the only equilibrium point is $(0, 0)$.

(c) Here we need to solve $100u + v = 506$ and $u + 10v = 65$. You can see by inspection that $u = 5$ and $v = 6$. Alternatively, you can multiply the first equation by 10 and subtract the second equation to get $999u = 5060 - 65 = 4995$, so $u = 4995/999 = 5$. The second equation then gives $v = (65 - u)/10 = 6$. Thus, the only equilibrium point is $(5, 6)$.

(d) Here both of the equations $\dot{p} = 0$ and $\dot{q} = 0$ give $p + q + 1 = 0$, so $q = -1 - p$ with p arbitrary. This means that all points of the form $(p, -1 - p)$ are equilibrium points.

(e) Here we must solve $x - y + 1 = 0$ and $y - x + 1 = 0$. Adding these equations gives $2 = 0$, which is impossible. Thus, this system does not have any equilibrium points.

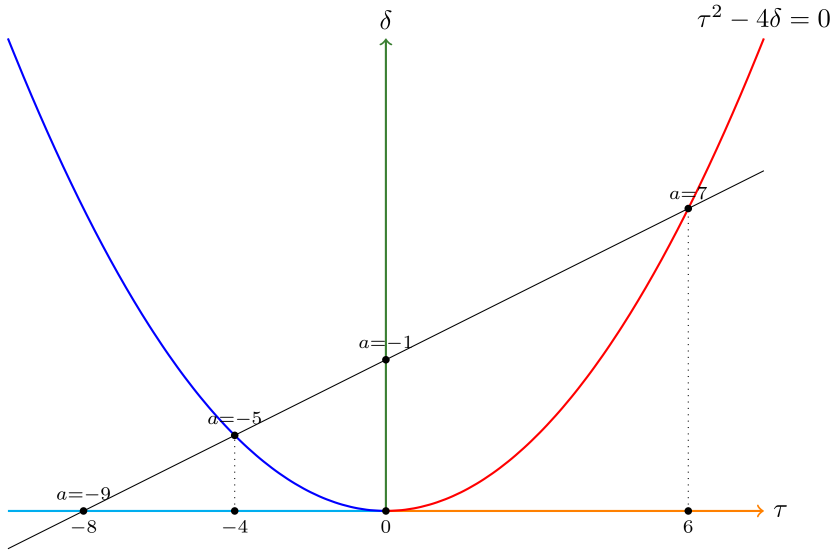
Exercise 15. Consider the matrix $A = \begin{bmatrix} 1 & -3 \\ 3 & a \end{bmatrix}$. Analyse how the equilibrium type of A depends on the parameter a.

Solution: We have $\tau = 1 + a$ and $\delta = a + 9 = \tau + 8$ and

$$\Delta = \tau^2 - 4\delta = (a + 1)^2 - 4a - 36 = (a - 1)^2 - 36.$$

In particular, we have $\tau < 0$ iff $a < -1$ and $\delta < 0$ iff $a < -9$ and $\Delta < 0$ iff $(a - 1)^2 < 36$ iff $-5 < a < 7$. Thus:

- For $a < -9$ we have $\delta < 0$, so the equilibrium point is a saddle.
- For $-9 < a < -5$ we have $\delta > 0$ and $\Delta > 0$ and $\tau < 0$, so the equilibrium point is a stable node.
- For $-5 < a < -1$ we have $\delta > 0$ and $\Delta < 0$ and $\tau < 0$, so the equilibrium point is a stable focus.
- For $-1 < a < 7$ we have $\delta > 0$ and $\Delta < 0$ and $\tau > 0$, so the equilibrium point is an unstable focus.
- For $7 < a$ we have $\delta > 0$ and $\Delta > 0$ and $\tau > 0$, so the equilibrium point is an unstable node.



Exercise 16. Consider the equations

$$\dot{x} = (a + b)x + 2by \qquad \dot{y} = -bx + (a - b)y,$$

where a and b are nonzero real constants. Show that the system always has a focus at $(0, 0)$. Give examples to show that the focus can be stable or unstable, and clockwise or anticlockwise, depending on the values of a and b.

Solution: The corresponding matrix is $A = \begin{bmatrix} a + b & 2b \\ -b & a - b \end{bmatrix}$, with trace $\tau = 2a$ and determinant

$$\delta = (a + b)(a - b) - 2b.(-b) = a^2 - b^2 + 2b^2 = a^2 + b^2.$$

This gives

$$\tau^2 - 4\delta = 4a^2 - 4(a^2 + b^2) = -4b^2 < 0,$$

so we have a focus or centre . However, we have $\tau = 2a$ and $a \neq 0$ by assumption so we cannot have a centre, and we must instead have a focus . If $a < 0$ then $\tau < 0$ so the focus is stable; similarly, if $a > 0$ then the focus is unstable . The direction of rotation is controlled by the bottom left entry in A, which is $-b$. If $b < 0$ then $-b > 0$ so the rotation is anticlockwise, but if $b > 0$ then the rotation is clockwise.

Exercise 17. Which of the following matrices corresponds to a system with a clockwise centre at the origin?

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \qquad \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

Solution: Consider a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\tau = a + b$ and $\delta = ad - bc$. This corresponds to a centre if $\tau = 0$ and $\delta > 0$; if so, then the rotation is clockwise if $c < 0 < b$ and anticlockwise if $b < 0 < c$. Only the 4th and 5th matrices in the list have $c < 0$. The 4th one has $\delta = 3$, and the 5th has $\delta = -3$, and both have $\tau = 0$. It follows that the 4th matrix $\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$ is the only one that gives an clockwise centre.

Exercise 18. For each of the following nonlinear systems, find the equilibrium points:

(a) $\dot{x} = y^2 - 5y + 6,\ \dot{y} = x^2 - 9x + 20$
(b) $\dot{x} = x^2 + y^2 + 1,\ \dot{y} = x^2 - y^2 + 1$
(c) $\dot{x} = x^2 + y^2 - 1,\ \dot{y} = x^2 + y^2 - 1$
(d) $\dot{x} = x^2 + y^2 - 1,\ \dot{y} = x^2 - y^2$

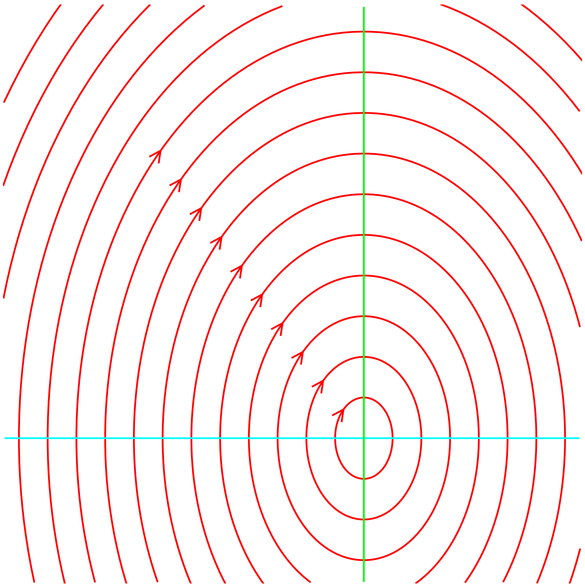
Solution:

(a) At an equilibrium point we must have $\dot{x} = y^2 - 5y + 6 = 0$, but this factorises as $(y-2)(y-3) = 0$, so $y = 2$ or $y = 3$. We must also have $\dot{y} = x^2 - 9x + 20 = 0$, but this factorises as $(x-4)(x-5) = 0$, so $x = 4$ or $x = 5$. It follows that there are four equilibrium points: $(4, 2)$, $(4, 3)$, $(5, 2)$ and $(5, 3)$.
(b) At an equilibrium point we must have $\dot{x} = x^2 + y^2 + 1 = 0$, but this is impossible because $x^2 \geq 0$ and $y^2 \geq 0$ so $x^2 + y^2 + 1 \geq 1$. Thus, there are no equilibrium points.
(c) At an equilibrium point we must have $\dot{x} = x^2 + y^2 - 1 = 0$, which means that (x, y) lies on the unit circle, so it can be written in the form $(x, t) = (\cos(\theta), \sin(\theta))$ for some angle θ. Moreover, $\dot{y}$ is the same as $\dot{x}$, so the equation $\dot{y} = 0$ does not give anything new. Thus, all the points on the unit circle are equilibrium points, and there are no other equilibrium points.
(d) Here again the equation $\dot{x} = x^2 + y^2 - 1 = 0$ means that all equilibrium points lie on the unit circle. However, we now have $\dot{y} = x^2 - y^2 = 0$, so $x^2 = y^2$, so $x = \pm y$. Putting this in the equation $x^2 + y^2 - 1 = 0$ gives $2x^2 = 1$, so $x = \pm\sqrt{2}/2$. It follows that there are four equilibrium points: $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$, $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$.

Exercise 19. Find and classify the equilibrium points for the system

$$\begin{aligned} \dot{x} &= f(x, y) = 1 + y \\ \dot{y} &= g(x, y) = 1 - 2x. \end{aligned}$$

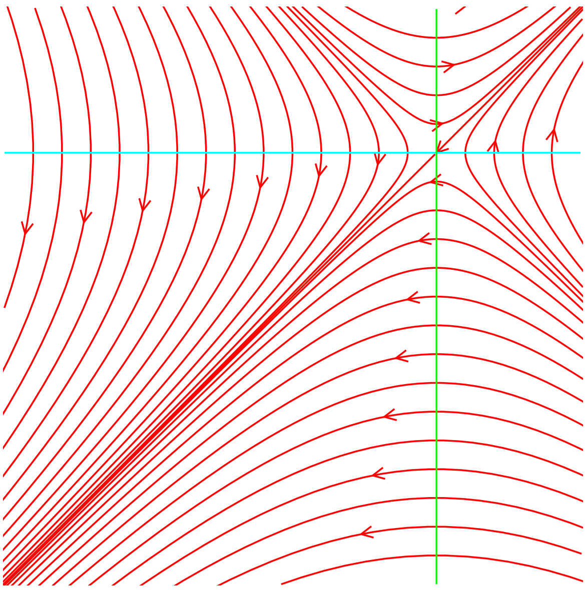
Solution: For an equilibrium point, we need $1 + y = 0$ and $1 - 2x = 0$, so $x = 1/2$ and $y = -1$. Thus, there is a unique equilibrium point at $(1/2, -1)$. The Jacobian is $\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$, which has trace $\tau = 0$ and determinant $\delta = 2$. As $\tau = 0$ and $\delta > 0$, this is a centre. The bottom left entry in J is $-2 < 0$, so the rotation is clockwise. In this case the functions f and g are just linear + constant, so there is no error in linearization, and the whole phase diagram is exactly the same as the usual phase diagram for a centre, except that it has been shifted away from the origin. The picture is as follows:



Exercise 20. Find and classify the equilibrium points for the system

$$\dot{x} = f(x, y) = y - 1$$
$$\dot{y} = g(x, y) = x - 1.$$

Solution: For an equilibrium point, we need $y - 1 = x - 1 = 0$. Thus, there is a unique equilibrium point at $(1, 1)$. The Jacobian is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which has trace $\tau = 0$ and determinant $\delta = -1$. As $\delta < 0$, this is a saddle. In this case the functions f and g are just linear + constant, so there is no error in linearization, and the whole phase diagram is exactly the same as the usual phase diagram for a saddle, except that it has been shifted away from the origin. The picture is as follows:



Exercise 21. Find and classify the equilibrium points for the system

$$\dot{x} = f(x, y) = 25 - 16x^2 - 9y^2$$
$$\dot{y} = g(x, y) = 9x^2 + 16y^2 - 25.$$

Sketch the nullclines.

Solution: If we divide the equation $\dot{x} = 0$ by 16 and divide the equation $\dot{y} = 0$ by 9 and add them together we get

$$\frac{25}{16} - x^2 - \frac{9}{16}y^2 + x^2 + \frac{16}{9}y^2 - \frac{25}{9} = 0.$$

This simplifies to $(\frac{16}{9} - \frac{9}{16})y^2 = \frac{25}{9} - \frac{25}{16}$, but $\frac{16}{9} - \frac{9}{16} = \frac{175}{144} = \frac{25}{9} - \frac{25}{16}$ so we get $y^2 = 1$. We can substitute $y^2 = 1$ in the equation $\dot{x} = 25 - 16x^2 - 9y^2 = 0$ to get $16 - 16x^2 = 0$, so $x^2 = 1$ as well. We now see that $x = \pm 1$ and $y = \pm 1$, so there are four equilibrium points:

$$a_1 = (1,1) \qquad a_2 = (-1,-1) \qquad a_3 = (1,-1) \qquad a_4 = (-1,1).$$

To classify these, we use the Jacobian:

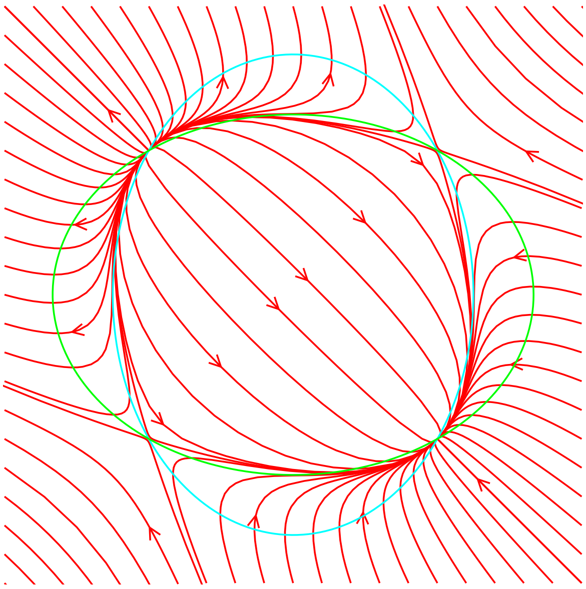
$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} -32x & -18y \\ 18x & 32y \end{bmatrix}.$$

so the trace is $\tau = 32(y - x)$ and the determinant is $\delta = (-32^2 + 18^2)xy = -700xy$. We therefore have the following table:

	a_1	a_2	a_3	a_4
τ	0	0	-64	64
δ	-700	-700	700	700
$\tau^2 - 4\delta$	2800	2800	1296	1296

At a_1 and a_2 we have $\delta < 0$, so these points are saddles. At a_2 and a_3 we have $\tau^2 - 4\delta > 0$, so these points are nodes. At a_2 we have $\tau < 0$, so this is a stable node. At a_3 we have $\tau > 0$, so this is an unstable node.

The x-nullcline has equation $16x^2 + 9y^2 = 25$, which describes an ellipse. In more detail, we have $\frac{16}{25}x^2 + \frac{9}{25}y^2 = 1$, or in other words $(\frac{4}{5}x)^2 + (\frac{3}{5}y)^2 = 1$. This means that the point $(\frac{4}{5}x, \frac{3}{5}y)$ lies on the unit circle, so it is $(\cos(\theta), \sin(\theta))$ for some θ. This gives $x = \frac{5}{4}\cos(\theta)$ and $y = \frac{5}{3}\sin(\theta)$, which is an ellipse. As $\frac{5}{3} > \frac{5}{4}$, the height of this ellipse is larger than the width. Similarly, the y-nullcline is given by $9x^2 + 16y^2 = 25$, or $(x, y) = (\frac{5}{3}\cos(\theta), \frac{5}{4}\sin(\theta))$. This is another ellipse, but in this case the width is larger than the height. The phase portrait is as follows:



Exercise 22. Consider the system

$$\dot{x} = y^3 - y = y(y+1)(y-1)$$
$$\dot{y} = x - x^3 = x(1+x)(1-x).$$

(a) Show that the function $U = (x^2 - 1)^2/4 + (y^2 - 1)^2/4$ is a conserved quantity.

(b) Find and classify the equilibrium points.

Solution:

(a) We have

$$\begin{aligned}
U_x &= \frac{\partial U}{\partial x} = 2(x^2-1)\times 2x/4 = (x^2-1)x = x^3-x \\
U_y &= \frac{\partial U}{\partial y} = 2(y^2-1)\times 2y/4 = (y^2-1)y = y^3-y \\
\dot{U} &= U_x f + U_y g = (x^3-x)(y^3-y)+(y^3-y)(x-x^3) = 0.
\end{aligned}$$

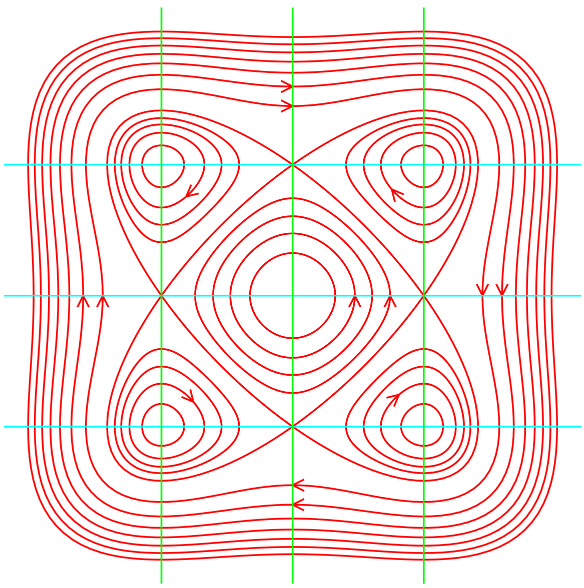
Thus, U is conserved.

(b) The x-nullcline consists of three horizontal lines, with equations $y=0$, $y=1$ and $y=-1$. Similarly, the y-nullcline consists of three vertical lines, with equations $x=0$, $x=1$ and $x=-1$. This means that there are nine equilibrium points (n,m) with $n,m\in\{-1,0,1\}$. The Jacobian is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 0 & 3y^2-1 \\ 1-3x^2 & 0 \end{bmatrix},$$

so the trace is $\tau=0$ and the determinant is $\delta=(1-3x^2)(1-3y^2)$. As $\tau=0$ we see that the equilibrium points are centres if $\delta>0$, and saddles if $\delta<0$. If x and y are both zero then $\delta=1$, corresponding to a centre. We have seen other examples where the linearised system has a centre but the original nonlinear system has a slow spiral. However, that cannot happen here because we have a conserved quantity, which forces the flow lines to close up as circles. The bottom left entry in J is $1-3x^2=1>0$, so the rotation is anticlockwise. If $x=0$ and $y=\pm1$ then $\delta=-2<0$, corresponding to a saddle. The same applies if $x=\pm1$ and $y=0$. Finally, if both x and y are ±1 then $\delta>0$, which means we have another centre. The bottom left entry is $-2<0$, so the rotation is clockwise.

The phase portrait for this system was shown in lectures:



Exercise 23. Consider the system

$$\begin{aligned}
\dot{x} &= f(x,y) = \sin(\pi y) \\
\dot{y} &= g(x,y) = \sin(\pi x).
\end{aligned}$$

(a) Show that the function $U=\cos(\pi x)-\cos(\pi y)$ is a conserved quantity.
(b) Find and classify the equilibrium points.
(c) Sketch the phase portrait.

Solution:

(a) We have

$$\dot{U} = U_x f + U_y g = -\pi \sin(\pi x).\sin(\pi y) + \pi \sin(\pi y)\sin(\pi x) = 0,$$

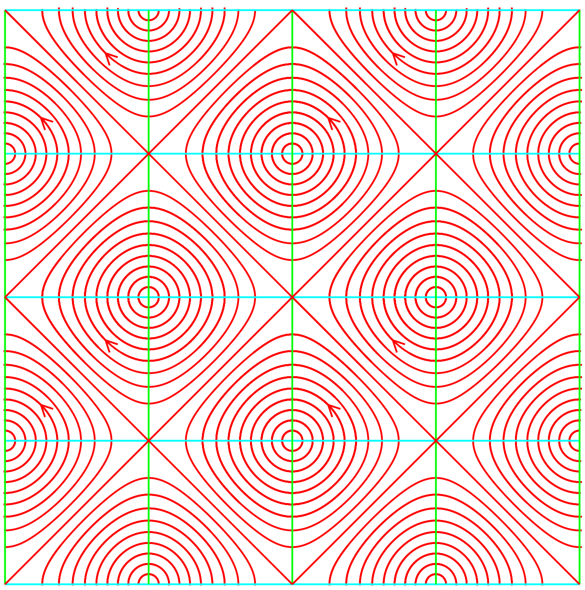
so U is conserved.

(b) The x-nullcline is given by $\sin(\pi y) = 0$, which means that y must be an integer. Similarly, the y-nullcline is given by $\sin(\pi x) = 0$, which means that x must be an integer. Thus, the equilibrium points are of the form (n, m), where n and m are both integers. The Jacobian is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 0 & \pi\cos(\pi y) \\ \pi\cos(\pi x) & 0 \end{bmatrix},$$

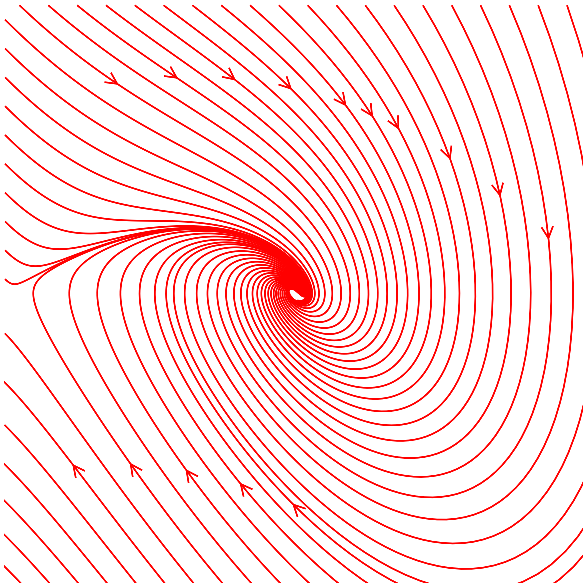
so the trace is $\tau = 0$ and the determinant is $\delta = -\pi^2 \cos(\pi x)\cos(\pi y)$. At an equilibrium point (n, m) we have $\cos(\pi x) = (-1)^n$ and $\cos(\pi y) = (-1)^m$, so $\delta = (-1)^{n+m+1}\pi^2$. If n and m are both odd, or both even, then we have $\delta = -\pi^2 < 0$, so (n, m) is a saddle. If n is odd and m is even then $\delta = \pi^2 > 0$ and $\tau = 0$ so we have a centre, at least for the linearised system. We have seen other examples where the linearised system has a centre but the original nonlinear system has a slow spiral. However, that cannot happen here because we have a conserved quantity, which forces the flow lines to close up as circles. The bottom left entry in J is $\cos(\pi x) = (-1)^n = -1$, so the rotation is clockwise. Similarly, if n is even and m is odd then we again have a centre, but in this case the rotation is anticlockwise.

(c) The phase portrait is as follows:



Exercise 24. Sketch the phase portrait of the system $\dot{x} = y$, $\dot{y} = -x - x^2 - y$ in a neighbourhood of the origin.

Solution:



Exercise 25. Let a, b, c, d, p and q be constants with $a \neq 0$ and $d \neq 0$ and $ad - bc \neq 0$. Consider the system

$$\begin{aligned}\dot{x} &= f(x, y) = x(ax + by - p)\\ \dot{y} &= g(x, y) = y(cx + dy - q).\end{aligned}$$

Find the four equilibrium points, and give a formula for the Jacobian matrix at each of these points.

Solution: For an equilibrium point, we must have either $x = 0$ or $ax + by = p$, and also $y = 0$ or $cx + dy = q$. If $x = 0$ then the equation $cx + dy = q$ becomes $y = q/d$, and if $y = 0$ then the equation $ax + by = p$ becomes $x = p/a$. If x and y are both nonzero then we must have $ax + by = p$ and $cx + dy = q$. If we multiply the first equation by d and the second equation by b and subtract, we get $(ad - bc)x = dp - bq$, so $x = \frac{dp-bq}{ad-bc}$. Similarly, we can multiply the first equation by c and the second equation by a and subtract to get $(bc - ad)y = cp - aq$, so $y = \frac{aq-cp}{ad-bc}$. We thus have four equilibrium points:

$$a_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 0 \\ q/d \end{bmatrix} \qquad a_3 = \begin{bmatrix} p/a \\ 0 \end{bmatrix} \qquad a_4 = \frac{1}{ad - bc} \begin{bmatrix} dp - bq \\ aq - cp \end{bmatrix}.$$

The Jacobian matrix is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 2ax + by - p & bx \\ cy & cx + 2dy - q \end{bmatrix}.$$

Evaluating this at the equilibrium points $a_1, \dots, a_4$ gives

$$J_1 = \begin{bmatrix} -p & 0 \\ 0 & -q \end{bmatrix} \qquad J_2 = \begin{bmatrix} (bq - dp)/d & 0 \\ cq/d & q \end{bmatrix} \qquad J_3 = \begin{bmatrix} p & bp/a \\ 0 & (cp - aq)/a \end{bmatrix}$$

and

$$J_4 = \frac{1}{ad - bc} \begin{bmatrix} a(dp - bq) & b(dp - bq) \\ c(aq - cp) & d(aq - cp) \end{bmatrix}.$$

Exercise 26. In lectures we considered the system

$$\dot{x} = x - x^3 \qquad\qquad \dot{y} = y - y^3.$$

Suppose that $x_0, y_0 > 0$. Show that the formulae

$$x = (1 + (x_0^{-2} - 1)e^{-2t})^{-1/2} \qquad\qquad y = (1 + (y_0^{-2} - 1)e^{-2t})^{-1/2}$$

give an explicit solution for this system, with $(x, y) = (x_0, y_0)$ at $t = 0$.

Solution: It will be convenient to put $C = x_0^{-2} - 1$, so $x = (1 + Ce^{-2t})^{-1/2}$. This gives

$$\dot{x} = -\tfrac{1}{2}(1 + Ce^{-2t})^{-3/2} \times (-2) \times Ce^{-2t} = (1 + Ce^{-2t})^{-3/2} Ce^{-2t}$$
$$x - x^3 = x^3(x^{-2} - 1) = (1 + Ce^{-2t})^{-3/2}(1 + Ce^{-2t} - 1) = (1 + Ce^{-2t})^{-3/2} Ce^{-2t} = \dot{x},$$

as required. Also, at $t = 0$ we have $e^{-2t} = 1$ so $x = (1 + C)^{-1/2} = (x_0^{-2})^{-1/2} = x_0$. Essentially the same argument gives $\dot{y} = y - y^3$ and $y = y_0$ at $t = 0$.

Exercise 27. Consider the system $\dot{x} = \dot{y} = xy$.

(a) Find a conserved quantity. (There is a very simple one.)
(b) Show that for any $x_0, y_0 > 0$ we have a solution

$$x = \frac{x_0(x_0 - y_0)}{x_0 - y_0 e^{t(x_0 - y_0)}} \qquad y = \frac{y_0(y_0 - x_0)}{y_0 - x_0 e^{t(y_0 - x_0)}}.$$

(c) Check that the solution in (b) becomes infinite when $t = (\ln(x_0) - \ln(y_0))/(x_0 - y_0)$.
(d) What is the value of the conserved quantity on the solution in (b)?

Solution:

(a) As $\dot{x} = \dot{y}$ we see that the function $U = x - y$ has $\dot{U} = 0$, so U is a conserved quantity.
(b) It will be convenient to put $a = x_0 - y_0$ and $u = e^{at}$. In terms of these, we have

$$x = \frac{x_0 a}{x_0 - y_0 u} \qquad y = \frac{-y_0 a}{y_0 - x_0/u} = \frac{y_0 a u}{x_0 - y_0 u}.$$

Note also that a is constant and $\dot{u} = ae^{at} = au$. This gives

$$\dot{x} = -\frac{x_0 a}{(x_0 - y_0 u)^2}(-y_0 \dot{u}) = -\frac{x_0 a}{(x_0 - y_0 u)^2}(-y_0 a u) = \frac{x_0 y_0 a^2 u}{(x_0 - y_0 u)^2} = xy.$$

A similar argument also gives $\dot{y} = xy$.
(c) Put $t^* = (\ln(x_0) - \ln(y_0))/(x_0 - y_0)$. We then have $at^* = \ln(x_0) - \ln(y_0) = \ln(x_0/y_0)$, so when $t = t^*$ we have $u = e^{at^*} = x_0/y_0$. This means that the denominator $x_0 - y_0 u$ is zero, so the solution goes to infinity.
(d) At $t = 0$ we have $u = 1$ and so $x = x_0$ and $y = y_0$ and $U = x - y = x_0 - y_0$. As U is conserved, we must have $U = x_0 - y_0$ for all t. This can also be seen explicitly:

$$U = x - y = \frac{x_0 a}{x_0 - y_0 u} - \frac{y_0 a u}{x_0 - y_0 u} = \frac{(x_0 - y_0 u)a}{x_0 - y_0 u} = a = x_0 - y_0.$$

Exercise 28. Consider the system

$$\dot{x} = x^2 - y^2 \qquad \dot{y} = 2xy.$$

(a) Show that the origin is the only equilibrium point, and that the Jacobian is zero at the origin.
(b) Show that the formulae

$$\begin{aligned} A &= x_0^2 + y_0^2 & & \\ u &= x_0 - tA & v &= 1 - 2tx_0 + t^2 A \\ x &= u/v & y &= y_0/v \end{aligned}$$

give an explicit solution with $(x, y) = (x_0, y_0)$ when $t = 0$.
(c) Let R be a large positive number. Show that the solution starting at $(R^3/(R^4+1),\ R/(R^4+1))$ (very close to the origin) reaches the point $(0, R)$ (very far from the origin) at $t = R$. This shows that the origin is unstable.

Solution:

(a) At an equilibrium point we must have $x^2 - y^2 = 0$ (so $x = \pm y$) and $2xy = 0$ (so either x or y is zero). The only way to satisfy both conditions is if $x = y = 0$, so the origin is the only equilibrium point. The Jacobian is

$$J = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix},$$

which becomes zero when $x = y = 0$.

(b) First note that when $t = 0$ we have $u = x_0$ and $v = 1$ so $x = u/v = x_0$ and $y = y_0/v = y_0$ as expected. We also have

$$\begin{aligned}
\dot{u} &= -A \\
\dot{v} &= -2x_0 + 2tA \\
\dot{x} &= \frac{\dot{u}v - u\dot{v}}{v^2} = (-A(1 - 2tx_0 + t^2A^2) - (x_0 - tA)(-2x_0 + 2tA))/v^2 \\
&= (-A + 2tAx_0 - t^2A^3 + x_0^2 - 2tAx_0 - 2tAx_0 + 2t^2A^2)/v^2 \\
&= (-A + 2x_0^2 - 2tAx_0 + t^2A^2)/v^2 \\
&= (x_0^2 - y_0^2 - 2tAx_0 + t^2A^2)/v^2 \\
\dot{y} &= -\frac{y_0\dot{v}}{v^2} = 2y_0(x_0 - tA)/v^2 \\
x^2 - y^2 &= \frac{u^2 - y_0^2}{v^2} \\
&= (x_0^2 - 2tAx_0 + t^2A^2 - y_0^2)/v^2 \\
2xy &= 2uy_0/v^2 = 2(x_0 - tA)y_0/v^2.
\end{aligned}$$

From this it is clear that $\dot{x} = x^2 - y^2$ and $\dot{y} = 2xy$, as required.

(c) Now take $x_0 = R^3/(R^4 + 1)$ and $y_0 = R/(R^4 + 1)$ and $t = R$. This gives

$$\begin{aligned}
A &= x_0^2 + y_0^2 = \frac{R^6 + R^2}{(R^4 + 1)^2} = \frac{R^2(R^4 + 1)}{(R^4 + 1)^2} = \frac{R^2}{R^4 + 1} \\
u &= x_0 - tA = \frac{R^3}{R^4 + 1} - R\frac{R^2}{R^4 + 1} = 0 \\
v &= 1 - 2tx_0 + t^2A = 1 - 2R\frac{R^3}{R^4 + 1} + R^2\frac{R^2}{R^4 + 1} = 1 - \frac{R^4}{R^4 + 1} = \frac{1}{R^4 + 1} \\
x &= u/v = 0 \\
y &= y_0/v = \frac{R}{R^4 + 1} \Big/ \frac{1}{R^4 + 1} = R.
\end{aligned}$$

Exercise 29. Consider the system

$$\begin{aligned}
\dot{x} &= f(x, y) = x^2 - y^2 - 1/4 \\
\dot{y} &= g(x, y) = 2xy.
\end{aligned}$$

(a) Show that the x-nullcline consists of all points of the form $(\pm\cosh(s)/2, \sinh(s)/2)$.
(b) Find the y-nullcline.
(c) Find and classify the equilibrium points.
(d) Sketch the phase portrait.

Solution:

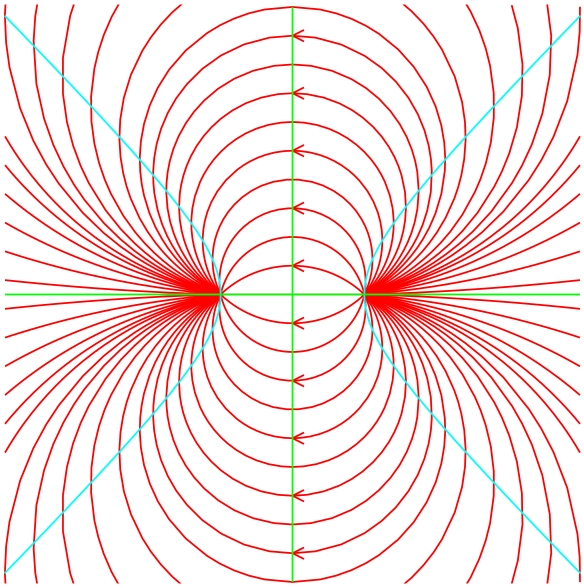
(a) If (x, y) lies on the x-nullcline we must have $x^2 - y^2 - 1/4 = 0$, which can be rearranged as $(2x - 2y)(2x + 2y) = 1$. Put $u = 2x + 2y$ so $u^{-1} = 2x - 2y$. Adding these equations gives $x = (u + u^{-1})/4$, and subtracting them gives $y = (u - u^{-1})/4$. If $u > 0$ we can put $s = \ln(u)$ and we find that $x = (e^s + e^{-s})/4 = \cosh(s)/2$ and $y = (e^s - e^{-s})/4 = \sinh(s)/2$. On the other hand, if $u < 0$ we can put $s = -\ln(-u)$ and we find that $x = (-e^{-s} - e^s)/4 = -\cosh(s)/2$ and $y = (-e^{-s} + e^s)/4 = \sinh(s)/2$.

(b) The y-nullcline is given by $2xy = 0$, so $x = 0$ or $y = 0$.

(c) At an equilibrium point, we must have $x^2 - y^2 - 1/4 = 0$ and also $xy = 0$, so either $x = 0$ or $y = 0$. If $x = 0$ then the equation $x^2 - y^2 - 1/4 = 0$ becomes $y^2 + 1/4 = 0$, which is impossible because $y^2 \geq 0$. If $y = 0$ then the equation $x^2 - y^2 - 1/4 = 0$ becomes $x^2 - 1/4 = 0$, which gives $x = \pm\frac{1}{2}$. Thus, the equilibrium points are $a_1 = (-1/2, 0)$ and $a_2 = (1/2, 0)$.

The Jacobian is $J = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$, which is $-I$ at a_1 and $+I$ at a_2. This means that a_1 is an improper stable node and a_2 is an improper unstable node.

(d) The phase portrait is as follows:



Exercise 30. The system

$$\dot{x} = x^2 - y^2 + 2xy \qquad\qquad \dot{y} = x^2 - y^2 - 2xy$$

has some solutions of the form $(x, y) = (a/t, b/t)$. Find the relevant values of a and b.

Solution: If $x = a/t$ and $y = b/t$ then

$$\begin{aligned} \dot{x} &= -a/t^2 \\ \dot{y} &= -b/t^2 \\ x^2 - y^2 + 2xy &= (a^2 - b^2 + 2ab)/t^2 \\ x^2 - y^2 - 2xy &= (a^2 - b^2 - 2ab)/t^2. \end{aligned}$$

Thus, we have a solution to the given system if

(A) $$-a = a^2 - b^2 + 2ab$$

(B) $$-b = a^2 - b^2 - 2ab.$$

If we add and subtract these equations, we get

(C) $$-a - b = 2(a^2 - b^2) = 2(a + b)(a - b)$$

(D) $$b - a = 4ab.$$

Equation (C) can be rearranged as $(a + b)(2a - 2b + 1) = 0$, so either $b = -a$ or $b = a + \frac{1}{2}$.

(a) If $b = -a$ then equation (D) becomes $-2a = -4a^2$, so $2a^2 = a$, so $a(1 - 2a) = 0$, so $a = 0$ or $a = \frac{1}{2}$. If $a = 0$ then $b = 0$, and if $a = \frac{1}{2}$ then $b = -\frac{1}{2}$. We thus have two solutions to the original system: the constant solution $(x, y) = (0, 0)$, and the solution $(x, y) = (1/(2t), -1/(2t))$.

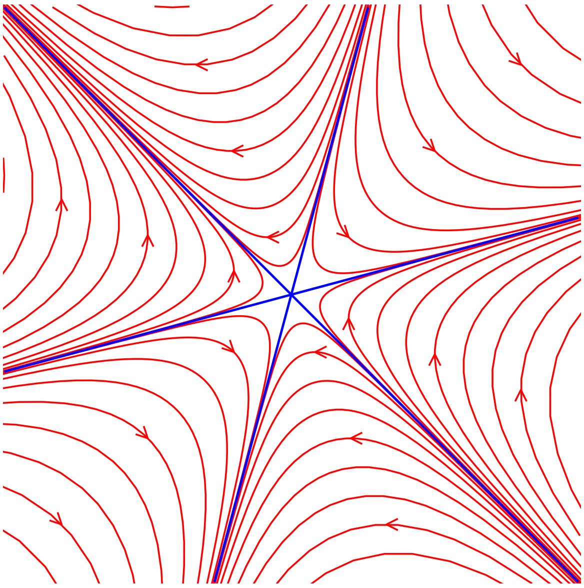
(b) Suppose instead that $b = a + \frac{1}{2}$. Then equation (D) becomes $\frac{1}{2} = 4a(a + \frac{1}{2})$, and this can be rearranged as $4a^2 + 2a - \frac{1}{2} = 0$, or $a^2 + \frac{1}{2}a - \frac{1}{8} = 0$. This has solutions

$$a = (-\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} + \tfrac{1}{2}})/2 = (-1 \pm \sqrt{3})/4.$$

The corresponding values of $b = a + \frac{1}{2}$ are $(+1 \pm \sqrt{3})/4$. We therefore have two solutions to the original system:

$$(x, y) = \left(\frac{-1 + \sqrt{3}}{4t}, \frac{1 + \sqrt{3}}{4t} \right) \qquad \text{or} \qquad (x, y) = \left(\frac{-1 - \sqrt{3}}{4t}, \frac{1 - \sqrt{3}}{4t} \right).$$

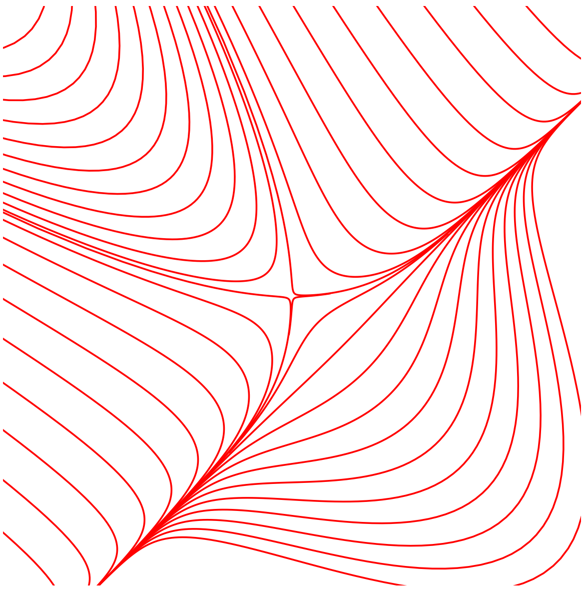
In the following picture, the above solutions are the straight blue lines. Some other flow lines are shown in red.



Exercise 31. The system

$$\dot{x} = -x + (x^2 - y^2)/4$$
$$\dot{y} = y - (x^2 - y^2)/4.$$

has the following phase portrait:



(a) Find and classify the equilibrium points.
(b) Add some arrows to the diagram to indicate the direction of flow.
(c) There are nonzero constants a, b, c such that $(x, y) = (at, bt + c)$ is a solution. Find these constants.
(d) Where does your solution to (c) appear in the phase portrait?

Solution:

(a) At an equilibrium point we must have $\dot{x} = -x + (x^2 - y^2)/4 = 0$ and $\dot{y} = y - (x^2 - y^2)/4 = 0$. Adding these equations gives $y - x = 0$, so $y = x$. Substituting $y = x$ in the relation $-x + (x^2 - y^2)/4 = 0$ gives $x = 0$. Thus, the only equilibrium point is at the origin. The

Jacobian is

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} -1 + x/2 & -y/2 \\ -x/2 & 1 + y/2 \end{bmatrix}.$$

At the origin this becomes $J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. As this is a diagonal matrix, the eigenvalues are just the diagonal entries, which are 1 and -1. As one eigenvalue is positive and the other is negative, we have a saddle.

(b)
- At a point $(x, y) = (u, u)$ with $u > 0$ we have $\dot{x} = -u + (u^2 - u^2)/4 = -u < 0$ and $\dot{y} = u > 0$ so the flow line points up and to the left.
- At a point $(x, y) = (u, -u)$ with $u > 0$ we have $\dot{x} = -u = -u < 0$ and $\dot{y} = -u < 0$ so the flow line points down and to the left.
- At a point $(x, y) = (-u, u)$ with $u > 0$ we have $\dot{x} = u > 0$ and $\dot{y} = u > 0$ so the flow line points up and to the right.
- At a point $(x, y) = (-u, -u)$ with $u > 0$ we have $\dot{x} = u > 0$ and $\dot{y} = -u < 0$ so the flow line points down and to the right.

For the actual picture, see part (d).

(c) If $x = at$ and $y = bt + c$ then

$$\begin{aligned} \dot{x} &= a \\ \dot{y} &= b \\ -x + (x^2 - y^2)/4 &= -at + (a^2t^2 - b^2t^2 - 2bct - c^2)/4 \\ &= \tfrac{1}{4}(a^2 - b^2)t^2 - (\tfrac{1}{2}bc + a)t - \tfrac{1}{4}c^2 \\ y - (x^2 - y^2)/4 &= bt + c - (a^2t^2 - b^2t^2 - 2bct - c^2)/4 \\ &= \tfrac{1}{4}(b^2 - a^2)t^2 + (\tfrac{1}{2}bc + b)t + (c + \tfrac{1}{4}c^2). \end{aligned}$$

Thus, we have a solution if the following equations hold for all t:

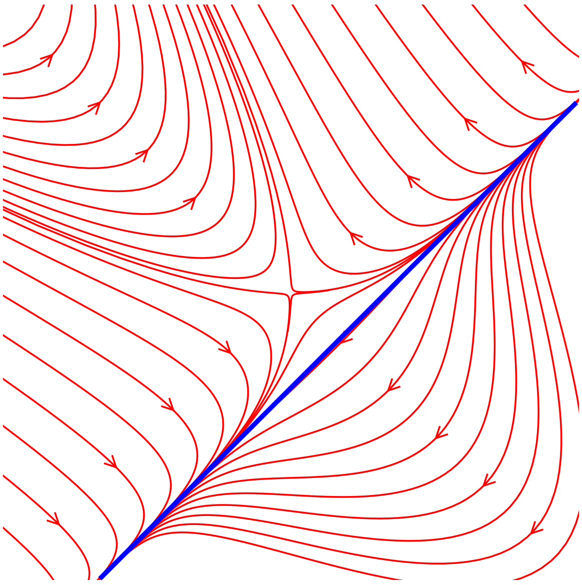
$$\begin{aligned} a &= \tfrac{1}{4}(a^2 - b^2)t^2 - (\tfrac{1}{2}bc + a)t - \tfrac{1}{4}c^2 \\ b &= \tfrac{1}{4}(b^2 - a^2)t^2 + (\tfrac{1}{2}bc + b)t + (c - \tfrac{1}{4}c^2). \end{aligned}$$

We can equate the coefficients of 1, t and t^2 to get

$$\begin{aligned} a &= -\tfrac{1}{4}c^2 & \tfrac{1}{2}bc + a &= 0 & a^2 - b^2 &= 0 \\ b &= c - \tfrac{1}{4}c^2 & \tfrac{1}{2}bc + b &= 0 & b^2 - a^2 &= 0. \end{aligned}$$

As $\frac{1}{2}bc + a$ and $\frac{1}{2}bc + b$ are both zero, we must have $b = a$. We can substitute this in $\frac{1}{2}bc + a = 0$ to get $a(c + 2) = 0$ but $a \neq 0$ by assumption, so $c = -2$. We now have $a = -\frac{1}{4}c^2 = -1$ and $b = a = -1$. Thus, the relevant solution is $(x, y) = (-t, -t - 2)$. This covers the line where $x - y = 2$.

(d) This picture shows the direction arrows from (a) as well as the solution from (c) (in blue).



Exercise 32. Consider the system

$$\dot{x} = f(x, y) = x - y - x^3 - xy^2$$
$$\dot{y} = g(x, y) = x + y - x^2y - y^3.$$

Put $U = x^2 + y^2$ and $V = 1/U - 1$ and $W = \arctan(y/x)$. Show that $\dot{U} = 2(U - U^2)$ and $\dot{V} = -2V$ and $\dot{W} = 1$. Using this, solve the equations, and show that there is a limit cycle.

Solution: First, we have

$$\begin{aligned}\dot{U} &= U_x f + U_y g = 2x(x - y - x^3 - xy^2) + 2y(x + y - x^2y - y^3) \\ &= 2x^2 - 2xy - 2x^4 - 2x^2y^2 + 2xy + 2y^2 - 2x^2y^2 - 2y^4 \\ &= 2x^2 + 2y^2 - 2x^4 - 4x^2y^2 - 2y^4 = 2U - 2U^2.\end{aligned}$$

This gives

$$\dot{V} = -U^{-2}\dot{U} = -2U^{-2}(U - U^2) = -2(U^{-1} - 1) = -2V.$$

It follows that $V = V_0e^{-2t}$ for some constant V_0, and so $U = 1/(1 + V) = 1/(1 + V_0e^{-2t})$.

Next, remember that $\arctan'(t) = 1/(1 + t^2)$. This gives

$$\begin{aligned}\dot{W} &= \frac{d(y/x)/dt}{1 + (y/x)^2} = \frac{(\dot{y}x - y\,\dot{x})/x^2}{1 + (y/x)^2} = \frac{gx - yf}{x^2 + y^2} \\ gx - yf &= (x + y - x^2y - y^3)x - y(x - y - x^3 - xy^2) \\ &= x^2 + xy - x^3y - xy^3 - xy + y^2 + x^3y + xy^3 = x^2 + y^2 \\ \dot{W} &= 1.\end{aligned}$$

This means that $W = W_0 + t$ for some constant t. Now, the polar coordinates of (x, y) are $r = \sqrt{U} = (1 + V_0e^{-2t})^{-1/2}$ and $\theta = W = W_0 + t$, so we have

$$x = (1 + V_0e^{-2t})^{-1/2}\cos(W_0 + t)$$
$$y = (1 + V_0e^{-2t})^{-1/2}\sin(W_0 + t).$$

As $t \to \infty$ we have $e^{-2t} \to 0$ and so $U \to 1$, which means that

$$\begin{bmatrix} x \\ y \end{bmatrix} \simeq \begin{bmatrix} \cos(W_0 + t) \\ \sin(W_0 + t) \end{bmatrix},$$

so the solution curve approaches the unit circle, which is a limit cycle for the system.

Exercise 33. Find and classify the equilibrium points for the system

$$\dot{x} = f(x,y) = x^2 + 2y^2 - y$$
$$\dot{y} = g(x,y) = 2x + 2y.$$

Solution: For an equilibrium point we must have $x^2+2y^2-y=0$ and $2x+2y=0$. The second equation gives $y=-x$, and substituting this into the first equation gives $3x^2+x=0$ or $x(3x+1)=0$. We thus have $x=0$ or $x=-1/3$, so the two equilibria are $a_1=(0,0)$ and $a_2=(-1/3,1/3)$. The Jacobian is

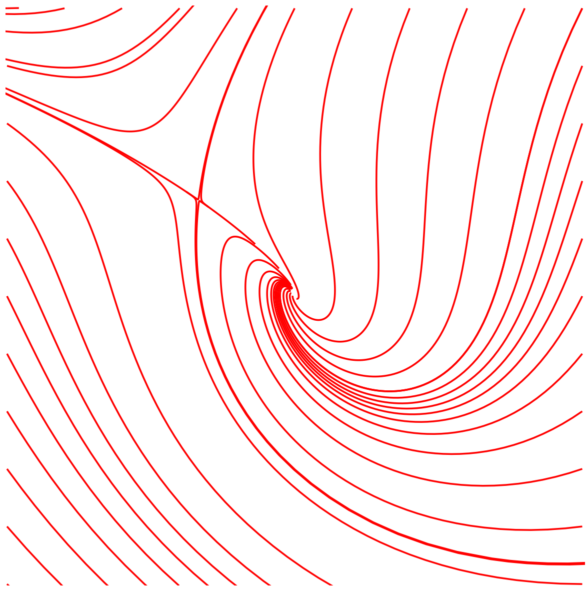
$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 2x & 4y-1 \\ 2 & 2 \end{bmatrix}.$$

Evaluating this at the equilibrium points gives

$$J_1 = \begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix} \qquad\qquad J_2 = \begin{bmatrix} -2/3 & 1/3 \\ 2 & 2 \end{bmatrix}.$$

For J_1 we have $\tau=2$ and $\delta=2$ so $\tau^2-4\delta=-4<0$. As $\tau>0$ and $\tau^2-4\delta<0$ we see that the point a_1 is an unstable focus. As the bottom left entry in J is positive, it is an anticlockwise focus.

For J_2 we have $\tau=4/3$ and $\delta=-2$. As $\delta<0$, this is a saddle.



Exercise 34. Analyse the following system (which describes the populations of two species that compete for food):

$$\dot{x} = 2x(1-x-y) \qquad\qquad \dot{y} = y(1-2y-2x).$$

Solution: On the x-nullcline we have $2x(1-x-y)=0$, so either $x=0$ or $y=1-x$. On the y-nullcline we have $y(1-2y-2x)=0$, so either $y=0$ or $y=\frac{1}{2}-x$. In principle there are now four possibilities for (x,y), but the equations $y=1-x$ and $y=\frac{1}{2}-x$ are incompatible, so in fact there are only three:

- The origin is an equilibrium point $a_1=(0,0)$.
- There is another equilibrium point with $x=0$ and $y=\frac{1}{2}-x$, so $y=\frac{1}{2}$, so the point is $a_2=(0,\frac{1}{2})$.
- There is another equilibrium point with $y=1-x$ and $y=0$, so $x=1$, so the point is $a_3=(1,0)$.

Recall that

$$f = 2x(1-x-y) = 2x - 2x^2 - 2xy$$
$$g = y(1-2y-2x) = y - 2y^2 - 2xy.$$

The Jacobian is

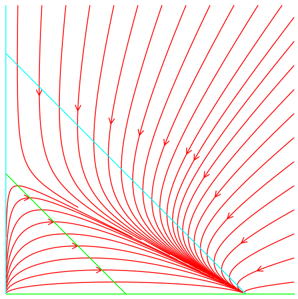
$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 2-4x-2y & -2x \\ -2y & 1-4y-2x \end{bmatrix}.$$

Evaluating this at the equilibrium points gives

$$J_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$
$$J_2 = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$
$$J_3 = \begin{bmatrix} -2 & -2 \\ 0 & -1 \end{bmatrix}.$$

Recall that for a matrix of the form $\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ or $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, the eigenvalues are just the same as the diagonal entries a and d. Thus, the eigenvalues for J_1 are 1 and 2 (so a_1 is an unstable node), the eigenvalues for J_2 are 1 and -1 (so a_2 is a saddle), and the eigenvalues for J_3 are -2 and -1 (so a_3 is a stable node).

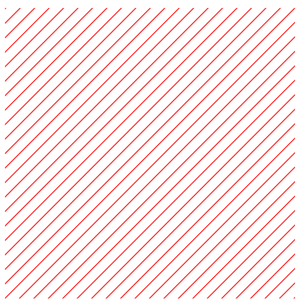
The phase portrait is as follows:



The variables x and y represent populations of competing species, so they cannot be negative, so we have shown only the region where $x \geq 0$ and $y \geq 0$. All the flow lines converge to a_3, where $x = 1$ and $y = 0$. This means that the y species will become extinct and will be replaced by the x species.

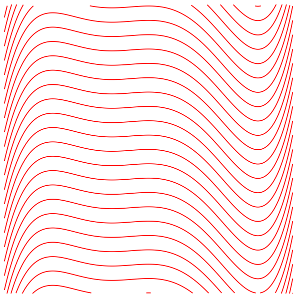
Exercise 35. Consider a system of the form $\dot{x} = \dot{y} = f(x, y)$. What can you say about the phase portrait?

Solution: As $\dot{y} = \dot{x}$, the difference $y - x$ is constant, so each flow line is (part of) a straight line $y = x + c$. This means that the exact form of the function f is mostly irrelevant, the phase portrait just looks like this:



Exercise 36. Consider a system of the form $\dot{x} = 1$, $\dot{y} = f'(x)$, for some given function f with $f(0) = 0$. What can you say about the solutions and the phase portrait?

Solution: Consider a solution that starts at a point $(0, y_0)$ on the y-axis. We have $\dot{x} = 1$ so $x = t + c$ for some constant c, but $x = 0$ when $t = 0$, so $x = t$. We also have $\dot{y} = f'(x) = f'(t)$, so $y - f(t)$ is constant, say $y = f(t) + d$. At $t = 0$ we have $y = y_0$ and $f(0) = 0$ so $d = y_0$. This means that the solution is just $y = f(t) + y_0$. In particular, all the solutions are the same except for the vertical shift by y_0. Thus, the phase portrait might look something like this:



Exercise 37. For each of the following quadratic functions, decide whether it is positive definite, negative definite or indefinite.

$$Q_1 = xy \qquad Q_2 = x^2 - y^2 \qquad Q_3 = (x+y)^2$$
$$Q_4 = x^2 + xy + y^2 \qquad Q_5 = 3x^2 - 4xy + 5y^2 \qquad Q_6 = 8xy - 3x^2 - 5y^2.$$

Solution:

- Q_1 is indefinite because $Q_1 = 0$ at $(x, y) = (0, 1)$.
- Q_2 is indefinite because $Q_2 = 0$ at $(x, y) = (1, 1)$.
- Q_3 is indefinite because $Q_3 = 0$ at $(x, y) = (1, -1)$.
- Q_4 is positive definite. To see this, note that $Q_4 = ax^2 + 2bxy + cy^2$ with $a = c = 1$ and $b = \frac{1}{2}$. Here $ac - b^2 = \frac{3}{4} > 0$ and $a, c > 0$, so the standard method tells us that Q_4 is positive definite. Alternatively, one can check that
$$Q_4 = \frac{3}{4}(x+y)^2 + \frac{1}{4}(x-y)^2.$$
From this it is clear that $Q_4 \geq 0$, and that Q_4 can only be equal to 0 if $x + y = 0$ and $x - y = 0$, which means that $x = y = 0$.
- Q_5 is again positive definite. Indeed, here we have $a = 3$ and $b = -2$ and $c = 5$ so $a, c > 0$ and $ac - b^2 = 11 > 0$.
- Q_6 is indefinite. Indeed, here we have $a = -3$ and $b = 4$ and $c = -5$ so $ac - b^2 = -1 < 0$. Alternatively, we can just note that $Q_6 = 0$ when $(x, y) = (1, 1)$.

Exercise 38. For each of the following functions, discuss whether it is positive definite, negative definite or indefinite about the origin, either on the whole plane or on some smaller domain.

$$V_1 = e^{-x^2-y^2} \qquad V_2 = e^{-x^2-y^2} - 1 \qquad V_3 = 2 - \cos(x) - \cos(y)$$
$$V_4 = x^3 + x^2y + xy^2 + y^3 \qquad V_5 = \frac{x^2+y^2}{1+x^2+y^2} \qquad V_6 = \frac{x^2+y^2}{1-x^2-y^2}$$

Solution:

- As $e^t > 0$ for all t, we have $V_1 > 0$ at all points (x, y). In particular, $V_1 = 1$ at $(0, 0)$, but we only say that a function is positive definite around the origin if it takes the value 0 at the origin. Thus, V_1 is not positive definite.
- At $(x, y) = (0, 0)$ we have $x^2 + y^2 = 0$ so $V_2 = e^0 - 1 = 0$. At all other points, we have $x^2 + y^2 > 0$ so $e^{-x^2-y^2} < 1$ so $V_2 < 0$. Thus, V_2 is negative definite.
- For all x and y we have $-1 \leq \cos(x), \cos(y) \leq 1$ and so $V_3 = (1 - \cos(x)) + (1 - \cos(y)) \geq 0$. From this it is also clear that V_3 can only be zero if $\cos(x) = 1$ and $\cos(y) = 1$, which means that x and y are both multiples of 2π. In particular, V_3 is zero at the point $(2\pi, 0)$, which means that V_3 is not positive definite on the whole plane. However, we can instead put
$$R = \{(x, y) \mid -2\pi < x, y < 2\pi\}.$$
We find that the only point in R where $V_3 = 0$ is $(0, 0)$, so V_3 is positive definite about the origin on R.
- The function V_4 is zero at $(x, y) = (1, -1)$, so V_4 is neither positive definite nor negative definite on the whole plane. In fact, for any small $\epsilon > 0$ we have a point $(\epsilon, -\epsilon)$ (very close to the origin)

where $V_4 = 0$. This means that V_4 is not positive or negative definite on any neighbourhood of the origin.

- As $x^2 + y^2 \geq 0$ and $1 + x^2 + y^2 > 0$ for all x, y we find that $V_5 \geq 0$ everywhere. Moreover, V_5 can only be zero if $x^2 + y^2 = 0$ which means that $x = y = 0$. Thus, V_5 is positive definite on the whole plane.
- The function V_6 is undefined if $x^2 + y^2 = 1$, and is negative if $x^2 + y^2 > 1$. Thus, we should consider only the domain
$$R = \{(x, y) \mid x^2 + y^2 < 1\},$$
which is an open disc centred at the origin. In this region we have $x^2+y^2 \geq 0$ and $1-x^2-y^2 > 0$ so $V_6 \geq 0$. Moreover, V_6 can only be zero if $x^2 + y^2 = 0$ which means that $x = y = 0$. Thus, V_6 is positive definite on the region R.

Exercise 39. Consider a function of the form $V = ax^5 + bx^4y + cxy^4 + dy^5$, where a, b, c and d are constants. Show that V is neither positive definite nor negative definite around the origin.

Solution: At $(1, 0)$ we have $V = a$, and at $(-1, 0)$ we have $V = -a$. These cannot both be strictly positive, so V is not positive definite. Similarly, a and $-a$ cannot both be strictly negative, so V is not negative definite.

Exercise 40. Suppose we have a linear system $\dot{u} = Au$ with a stable node or a stable focus at the origin, and that U is a conserved quantity that is defined and continuous over the whole plane. Show that U is just a constant.

Solution: Put $C = U(0, 0)$ (the value of the conserved quantity at the origin). Consider an arbitrary point (x_0, y_0). Let (x, y) be the solution that starts at (x_0, y_0) at time $t = 0$. As the system is a stable node or focus, we have $(x, y) \to (0, 0)$ as $t \to \infty$. As U is continuous, we have $U(x, y) \to U(0, 0) = C$ as $t \to \infty$. However, U is a conserved quantity, so $U(x, y)$ is independent of t. The only way it can be independent of t and converge to C is if it is equal to C all the time. We thus have $U(x, y) = C$ for all t, so in particular this holds when $t = 0$, which means that $U(x_0, y_0) = C$. This holds for all x_0 and y_0, so U is constant.

Exercise 41. Consider the system
$$\dot{x} = -2y^3 \qquad\qquad \dot{y} = x - 3y^3.$$
Find a Lyapunov function V of the form $V = ax^2 + by^4$ (for some constants a and b). Deduce that the origin is a stable equilibrium point.

Solution: Consider a function $V = ax^2 + by^4$. We want V to be positive definite, so we must have $a, b > 0$. Note that
$$\dot{V} = V_x\dot{x} + V_y\dot{y} = 2ax(-2y^3) + 4by^3(x - 3y^3) = (4b - 4a)xy^3 - 12by^6.$$
The term xy^3 can be positive or negative depending on the signs of x and y. It is simplest to make this term go away by choosing $a = b = 1$. Out function is then $V = x^2 + y^4$, with $\dot{V} = -12y^6$. This means we always have $\dot{V} \leq 0$, but $\dot{V} = 0$ everywhere on the x-axis, so $\dot{V}$ is negative semidefinite but not negative definite. Thus, V is a weak Lyapunov function but not a strong Lyapunov function, and the origin is stable but not necessarily asymptotically stable. (In fact the origin *is* asymptotically stable, but we need a different argument to prove it.)

Exercise 42. Consider the system
$$\dot{x} = -3x^3 - y \qquad\qquad \dot{y} = x^5 - 2y^3.$$
Find a Lyapunov function of the form $V = \alpha x^{2n} + \beta y^{2m}$ (for certain constants α, β, n, m), and deduce that the origin is a stable equilibrium point. Can we conclude that it is asymptotically stable?

Solution: Consider a function $V = \alpha x^{2n} + \beta y^{2m}$. If we take n and m to be positive integers and $\alpha, \beta > 0$ then V will be positive definite. We also have
$$\begin{aligned}\dot{V} = V_x\dot{x} + V_y\dot{y} &= 2n\alpha x^{2n-1}(-3x^3 - y) + 2m\beta y^{2m-1}(x^5 - 2y^3)\\ &= -6n\alpha x^{2n+2} - 2n\alpha x^{2n-1}y + 2m\beta x^5y^{2m-1} - 4m\beta y^{2m+2}.\end{aligned}$$
The middle two terms have odd exponents, so they can be either positive or negative. It is best to make them cancel out. For that, we need $2n\alpha = 2m\beta$ and $2n - 1 = 5$ and $2m - 1 = 1$. This gives $m = 1$ and

$n = 3$ and $6\alpha = 2\beta$. We can choose $\alpha = 1$ and then $\beta = 3$. With these choices, the function becomes $V = x^6 + 3y^2$, and

$$\dot{V} = -6n\alpha x^{2n+2} - 4m\beta y^{2m+2} = -18x^8 - 12y^4.$$

It is clear from this that $\dot{V}$ is negative definite, so V is a strong Lyapunov function, and the origin is an asymptotically stable equilibrium.

Exercise 43. Consider the system

$$\dot{x} = y \qquad\qquad \dot{y} = -x + y(x^2 + y^2 - 1).$$

Find a Lyapunov function of the form $V = \alpha x^{2n} + \beta y^{2m}$ (for certain constants α, β, n, m), and deduce that the origin is a stable equilibrium point. Can we conclude that it is asymptotically stable?

Solution: Consider a function $V = \alpha x^{2n} + \beta y^{2m}$. If we take n and m to be positive integers and $\alpha, \beta > 0$ then V will be positive definite. We also have

$$\begin{aligned}\dot{V} = V_x\dot{x} + V_y\dot{y} &= 2n\alpha x^{2n-1}y + 2m\beta y^{2m-1}(-x + y(x^2 + y^2 - 1)) \\ &= 2n\alpha x^{2n-1}y - 2m\beta x y^{2m-1} - 2m\beta y^{2m}(1 - x^2 - y^2).\end{aligned}$$

The first two terms have odd exponents, so they can be either positive or negative. It is best to make them cancel out. For that, we need $2n - 1 = 1$ and $2m - 1 = 1$ and $2n\alpha = 2m\beta$, so $n = m = 1$ and $\alpha = \beta$. We choose $\alpha = 1$, so $V = x^2 + y^2$ and

$$\dot{V} = -2m\beta y^2(1 - x^2 - y^2) = -2y^2(1 - x^2 - y^2).$$

If we restrict attention to the unit disc

$$R = \{(x, y) \mid x^2 + y^2 < 1\},$$

then we have $\dot{V} \leq 0$. This means that V is a weak Lyapunov function, so the origin is a stable equilibrium point. However, we have $\dot{V} = 0$ at all points where $y = 0$, so V is not a strong Lyapunov function, and we cannot immediately conclude that the origin is asymptotically stable.

Exercise 44. Consider a linear system $\dot{x} = py - qx,\ \ \dot{y} = rx - sy$ where $p, q, r, s > 0$ and $4qs > (p+r)^2$.

(a) Show that the function $x^2 + y^2$ is a strong Lyapunov function.
(b) Show that $qs - pr > 0$.
(c) Classify the equilibrium point at the origin.

Solution:

(a) It is clear that V is positive definite. First, we have

$$\dot{V} = 2x\dot{x} + 2y\dot{y} = 2pxy - 2qx^2 + 2rxy - 2sy^2 = -2qx^2 + 2(p+r)xy - 2sy^2.$$

In other words, we have $\dot{V} = ax^2 + 2bxy + cy^2$, where $a = -2q$ and $b = p + r$ and $c = -2s$. The standard criterion says that $\dot{V}$ is negative definite if $a, c < 0$ and $ac > b^2$, or equivalently $q, s > 0$ and $4qs > (p+r)^2$. These are precisely the assumptions in the question, so $\dot{V}$ is negative definite. This means that V is a strong Lyapunov function, and that the origin is an asymptotically stable equilibrium point.

(b) We are given that $4qs > (p + r)^2$, so

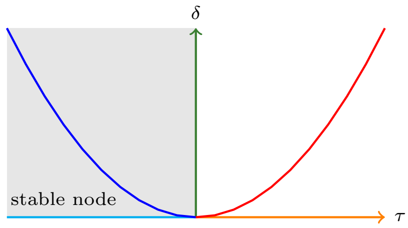
$$4qs - 4pr > (p + r)^2 - 4pr = p^2 + 2pr + r^2 - 4pr = p^2 - 2pr + r^2 = (p - r)^2 \geq 0.$$

This shows that $4qs - 4pr > 0$, so $qs - pr > 0$.

(c) The system can be written as $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = A\begin{bmatrix}x\\ y\end{bmatrix}$, where $A = \begin{bmatrix}-q & p\\ r & -s\end{bmatrix}$. This has trace $\tau = -(q+s) < 0$ and determinant $\delta = qs - pr > 0$ (by part (b)). We also have

$$\begin{aligned}\tau^2 - 4\delta &= (q + s)^2 - 4qs + 4pr = q^2 + 2qs + s^2 - 4qs + 4pr \\ &= q^2 - 2qs + s^2 + 4pr = (q - s)^2 + 4pr.\end{aligned}$$

Now $p, r > 0$ by assumption, so $4pr > 0$, and $(q - s)^2 \geq 0$ automatically, so $\tau^2 - 4\delta > 0$. As $\delta > 0$ and $\tau < 0$ and $\tau^2 - 4\delta > 0$, we have a stable node at the origin.



Exercise 45. Let $\alpha, \beta, \gamma, \delta$ be positive constants, and let m, n, p, q be nonnegative integers. Put

$$f(x,y) = -\alpha x^{2p+1} - (m+1)\beta y^{2m+1}$$
$$g(x,y) = (n+1)\gamma x^{2n+1} - \delta y^{2q+1}$$
$$V(x,y) = \gamma x^{2n+2} + \beta y^{2m+2}.$$

Show that if $\dot{x} = f$ and $\dot{y} = g$ then

$$\dot{V} = -2(n+1)\alpha\gamma x^{2(n+p+1)} - 2(m+1)\beta\delta y^{2(m+q+1)},$$

and so V is a strong Lyapunov function for this system.

Solution: Write this.

Exercise 46. Find a Lyapunov function of the form $ax^2 + by^2$ for the system $\dot{x} = xy^2 - x^3$, $\dot{y} = -(2x^2y + y^3)$. What can we conclude about the nature of the equilibrium at the origin?

Solution: If $V = ax^2 + by^2$ then

$\dot{V} = 2ax(xy^2 - x^3) + 2by(-2x^2y - y^3) = 2ax^2y^2 - 2ax^4 - 4bx^2y^2 - 2by^4 = -2ax^4 + (2a - 4b)x^2y^2 - 2by^4.$

The simplest thing to do is to choose $a = 2$ and $b = 1$, so $V = 2x^2 + y^2$, which is positive definite. The equation above then becomes $\dot{V} = -4x^4 - 2y^4$, which is negative definite. This means that V is a strong Lyapunov function, and the origin is asymptotically stable.

Exercise 47. Find a Lyapunov function of the form $ax^2 + by^2$ for the system $\dot{x} = -\frac{1}{2}x^3 + 2xy^2$, $\dot{y} = -y^3$. What can we conclude about the nature of the equilibrium at the origin?

Solution: It will work out that $ax^2 + by^2$ is a strong Lyapunov function provided that $a, b > 0$ and $b > 2a$. To be definite, we will take $a = 1$ and $b = 3$, so $V = x^2 + 3y^2$. This is clearly positive definite. We also have

$$\dot{V} = 2x(-\tfrac{1}{2}x^3 + 2xy^2) + 6y(-y^3) = -x^4 + 4x^2y^2 - 6y^4 = -(x^2 - 2y^2)^2 - 2y^4.$$

From this representation it is clear that $\dot{V} \leq 0$, and that $\dot{V}$ can only equal zero if $x^2 - 2y^2 = 0$ and $y = 0$, which gives $(x, y) = (0, 0)$. Thus, $\dot{V}$ is negative definite, so V is a strong Lyapunov function, so the origin is asymptotically stable.

Exercise 48. Use quadratic Lyapunov functions to determine the stability of the origin for the following systems:

(a) $\dot{x} = -x - 2y^2, \;\; \dot{y} = xy - y^3$.
(b) $\dot{x} = 2y^2 - x^3, \;\; \dot{y} = -4xy$.

Solution:

(a) Consider the function $V = x^2 + 2y^2$, which is positive definite. We then have

$$\dot{V} = 2x(-x - 2y^2) + 2y(xy - y^3) = -2x^2 - 4xy^2 + 4xy^2 - 2y^4 = -2(x^2 + y^4),$$

which is negative definite. This means that V is a strong Lyapunov function, so the origin is asymptotically stable.

(b) Consider the function $V = 2x^2 + y^2$, which is positive definite. We then have

$$\dot{V} = 4x(2y^2 - x^3) + 2y(-4xy) = 8xy^2 - 4x^4 - 8xy^2 = -4x^4.$$

This is negative semidefinite, but not negative definite (because $\dot{V} = 0$ at all points of the form $(0, y)$). It follows that the origin is stable, but not necessarily asymptotically stable.

Exercise 49. Consider a system of the form $\dot{x} = px - qy, \;\; \dot{y} = rx - py$. Show that the function $U = rx^2 - 2pxy + qy^2$ is a conserved quantity. When is it positive definite?

Solution: First, we have

$$\begin{aligned}\dot{U} &= U_x\dot{x} + U_y\dot{y} = (2rx - 2py)(px - qy) + (-2px + 2qy)(rx - py) \\ &= 2(rx - py)(px - qy) - 2(px - qy)(rx - py) = 0.\end{aligned}$$

This shows that U is conserved. Moreover, U is a quadratic function so we can use the standard test for definiteness. This says that U is positive definite if $q, r > 0$ and $qr > p^2$.

Exercise 50. Consider the system $\dot{x} = x(1+y^2)$, $\dot{y} = -y(1+x^2)$. Use Lyapunov's method to show that the origin is an unstable equilibrium.

Solution: Put $V = x^2 - y^2$. Then

$$\dot{V} = V_x\dot{x} + V_y\dot{y} = 2x\dot{x} - 2y\dot{y} = 2x^2(1+y^2) + 2y^2(1+x^2) = 2x^2 + 2y^2 + 4x^2y^2.$$

This is clearly positive definite. Moreover, for any small $\epsilon > 0$ we have a point $(\epsilon, 0)$ (very close to the origin) where $V = \epsilon^2 > 0$. Thus, V has the conditions required for Lyapunov's method, and we conclude that the origin is unstable.

Exercise 51. Consider the linear system where $\dot{x} = -x - y$ and $\dot{y} = x - y$, and the function

$$U = \arctan(y/x) + \tfrac{1}{2}\ln(x^2 + y^2).$$

(a) Show that there is a stable focus at the origin.
(b) Find the solution starting at $(r, 0)$ at $t = 0$.
(c) Use the rule $\dot{U} = U_x f + U_y g$ to show that U is conserved.
(d) Use the solution from (b) to give another proof that U is conserved.
(e) In the lectures we explained that a focus cannot have a conserved quantity. How can this be correct?

Solution:

(a) The system has matrix $A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$, with $\tau = -2$ and $\delta = 2$ so $\tau^2 - 4\delta = -4 < 0$. This means that the eigenvalues are $\lambda \pm i\omega$ where $\lambda = -1$ and $\omega = 1$. As the eigenvalues are complex with negative real part, we must have a stable focus.
(b) We use the formula

$$\begin{aligned}P &= e^{\lambda t}\left(\cos(\omega t)I + \omega^{-1}\sin(\omega t)(A - \lambda I)\right) \\ &= e^{-t}\left(\cos(t)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin(t)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = e^{-t}\begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= P\begin{bmatrix} r \\ 0 \end{bmatrix} = \begin{bmatrix} re^{-t}\cos(t) \\ re^{-t}\sin(t) \end{bmatrix}.\end{aligned}$$

(c) Put $V = \arctan(y/x)$ and $W = \frac{1}{2}\ln(x^2+y^2)$ so $U = V + W$. Recall that $\arctan'(z) = 1/(1+z^2)$. Using this, we get

$$\begin{aligned}\dot{U} &= \arctan'(y/x)\frac{d}{dt}(y/x) = \frac{1}{1 + y^2/x^2}\,\frac{\dot{y}x - y\dot{x}}{x^2} \\ &= \frac{(x-y)x - y(-x-y)}{x^2+y^2} = \frac{x^2+y^2}{x^2+y^2} = 1 \\ \dot{V} &= \frac{1}{2}\frac{1}{x^2+y^2}\frac{d}{dt}(x^2+y^2) = \frac{2x\dot{x} + 2y\dot{y}}{2(x^2+y^2)} \\ &= \frac{x(-x-y) + y(x-y)}{x^2+y^2} = \frac{-x^2-y^2}{x^2+y^2} = -1 \\ \dot{U} &= \dot{V} + \dot{W} = 1 - 1 = 0.\end{aligned}$$

(d) If $x = re^{-t}\cos(t)$ and $y = re^{-t}\sin(t)$ as in (b), then we have

$$\begin{aligned}
y/x &= \frac{re^{-t}\sin(t)}{re^{-t}\cos(t)} = \frac{\sin(t)}{\cos(t)} = \tan(t) \\
\arctan(y/x) &= \arctan(\tan(t)) = t \\
x^2 + y^2 &= r^2 e^{-2t}(\cos^2(t) + \sin^2(t)) = r^2 e^{-2t} \\
\tfrac{1}{2}\ln(x^2 + y^2) &= \ln(r) - t \\
U &= \arctan(y/x) + \tfrac{1}{2}\ln(x^2 + y^2) = t + (\ln(r) - t) = \ln(r).
\end{aligned}$$

As expected, this does not depend on t.

(e) The point is that U is not really well-defined (because you can add multiples of π to the arctan term). We can make try to make it well-defined by always taking the value of arctan that lies in $(-\pi/2, \pi/2]$. However, with this convention, U is discontinuous when $y = 0$. Also, U will always be discontinuous at the point $(0, 0)$, whatever convention we make. The theorem in the lectures only covers the case where U is well-defined and continuous, so there is no contradiction.

Exercise 52. Find the general solution for the equation $y'' + 3py' + 2p^2 y = 0$ (where p is constant).

Solution: Here the coefficients are constant so we do not need power series methods. The auxiliary polynomial is $\lambda^2 + 3p\lambda + 2p^2 = (\lambda + p)(\lambda + 2p)$, with roots $-p$ and $-2p$. If $p \neq 0$ then these roots are distinct and so the general solution is $y = Ae^{-px} + Be^{-2px}$. If $p = 0$ then the equation is just $y'' = 0$ and the solutions are $y = Ax + B$.

Exercise 53. Find the general solution for the equation $y'' + 2py' + p^2 y = 0$ (where p is constant).

Solution: Here the coefficients are constant so we do not need power series methods. The auxiliary polynomial is $\lambda^2 + 2p\lambda + p^2 = (\lambda + p)^2$, so $-p$ is the only root. The general solution is $y = (A + Bx)e^{-px}$.

Exercise 54. Solve the following boundary value problems (or prove that there is no solution).

(a) $y'' - y = 0$ with $y(0) = 0$ and $y(\ln(2)) = 1$.
(b) $y'' + y = 0$ with $y(0) = 1$ and $y(\pi/2) = 2$.
(c) $y'' + y = 0$ with $y(0) = 1$ and $y(2\pi) = 1$.
(d) $y'' + y = 0$ with $y(0) = 1$ and $y(\pi) = 1$.
(e) $y'' + \pi^2 y = 0$ with $y(0) + y(1) = 0$ and $y'(0) + y'(1) = 0$.

Solution:

(a) The equation $y'' - y = 0$ has auxiliary polynomial $t^2 - 1 = 0$, with roots ± 1, so the solutions are $y = Ae^x + Be^{-x}$ with A and B constant. The boundary condition $y(0) = 0$ gives $A + B = 0$. The boundary condition $y(\ln(2)) = 1$ gives $Ae^{\ln(2)} + Be^{-\ln(2)} = 1$, but $e^{\ln(2)} = 2$ and $e^{-\ln(2)} = \frac{1}{2}$ so $2A + \frac{1}{2}B = 1$. The equations $A + B = 0$ and $2A + \frac{1}{2}B = 1$ can be solved to give $A = \frac{2}{3}$ and $B = -\frac{2}{3}$. Thus, the solution is $y = \frac{2}{3}(e^x - e^{-x})$.

(b) The equation $y'' + y = 0$ has solutions $y = A\cos(x) + B\sin(x)$ with A and B constant. When $x = 0$ we have $y = A$, and when $x = \pi/2$ we have $y = B$. Thus, the boundary condition $y(0) = 1$ and $y(\pi/2) = 2$ give $A = 1$ and $B = 2$. It follows that the solution is $y = \cos(x) + 2\sin(x)$.

(c) We again have $y = A\cos(x) + B\sin(x)$. When $x = 0$ we have $y = A$, and when $x = 2\pi$ we again have $y = A$. The boundary conditions $y(0) = y(2\pi) = 1$ therefore just give $A = 1$, and B is arbitrary. Thus, the boundary value problem has solutions $y = \cos(x) + B\sin(x)$, where B can be any constant.

(d) We again have $y = A\cos(x) + B\sin(x)$. As $\sin(0) = \sin(\pi) = 0$ and $\cos(0) = 1$ and $\cos(\pi) = -1$ we see that $y(0) = A$ and $y(\pi) = -A$. Thus, it is impossible to satisfy the given boundary conditions $y(0) = y(\pi) = 1$; there are no solutions.

(e) The equation $y'' + \pi^2 y = 0$ has solutions $y = A\cos(\pi x) + B\sin(\pi x)$. This means that

$$\begin{aligned}
y' &= -\pi A\sin(\pi x) + \pi B\cos(\pi x) \\
y(0) + y(1) &= A\cos(0) + B\sin(0) + A\cos(\pi) + B\sin(\pi) \\
&= A + 0 - A + 0 = 0 \\
y'(0) + y'(1) &= -\pi A\sin(0) + \pi B\cos(0) - \pi A\sin(\pi) + \pi B\cos(\pi) \\
&= 0 + \pi B - 0 - \pi B = 0.
\end{aligned}$$

Thus, the boundary conditions are satisfied automatically, so every function of the form $A\cos(\pi x)+B\sin(\pi x)$ is a solution for the boundary value problem.

Exercise 55. For each of the following problems, find the values of λ for which a nonzero solution exists, and then find y.

(a) $y''+\lambda y=0$ with $y(0)=y'(\pi)=0$.
(b) $y''-2y'+\lambda y=0$ with $y(0)=y(1)=0$.
(c) $(xy')'+\lambda x^{-1}y=0$ with $y(1)=y(e^{\pi})=0$.

Hint for (c): first look for solutions of the form $y=x^p$, remembering that $x^{iu}=e^{iu\ln(x)}=\cos(u\ln(x))+i\sin(u\ln(x))$.

Solution:

(a) First consider the case where $\lambda=0$, so $y''=0$. The solutions have the form $y=A+Bx$, where A and B are constant. The condition $y(0)=0$ gives $A=0$, so $y=Bx$, so $y'=B$. The condition $y'(\pi)=0$ therefore gives $B=0$ and so $y=0$. Thus, there are no nontrivial solutions for y when $\lambda=0$.

Now consider the case where $\lambda<0$, say $\lambda=-\mu^2$ with $\mu>0$. The solutions for $y''+\lambda y=y''-\mu^2y=0$ have the form $Ae^{\mu t}+Be^{-\mu t}$ with A and B constant. The condition $y(0)=0$ gives $A+B=0$, so $y=A(e^{\mu t}-e^{-\mu t})$. It follows that $y'=A\mu(e^{\mu t}+e^{-\mu t})$, and so $y'(\pi)=A\mu(e^{\mu\pi}+e^{-\mu\pi})$. Now μ and $e^{\mu\pi}+e^{-\mu\pi}$ are both strictly positive, but $y'(\pi)$ is required to be zero, so $A=0$, so $y=0$. Thus, there are again no nontrivial solutions.

Finally, consider the case where $\lambda>0$, say $\lambda=\mu^2$ with $\mu>0$. The solutions for $y''+\lambda y=y''+\mu^2y=0$ are $y=A\cos(\mu x)+B\sin(\mu x)$. The condition $y(0)=0$ gives $A=0$, so $y=B\sin(\mu x)$ and $y'=B\mu\cos(\mu x)$. We are asked to find nonzero solutions, so we must have $B\neq 0$. We also have $y'(\pi)=0$, so $\cos(\mu\pi)=0$, which means that $\mu=m+\frac{1}{2}$ for some integer m.

In conclusion, If $\lambda=(m+\frac{1}{2})^2$ for some integer m then we have solutions $y=B\sin((m+\frac{1}{2})x)$, but for all other λ the only solution is $y=0$.

(b) The equation $y''-2y'+\lambda y=0$ has auxiliary polynomial $t^2-2t+\lambda$, with roots $(2\pm\sqrt{4-4\lambda})/2=1\pm\sqrt{1-\lambda}$. If $\lambda=1$ then the roots are both 1, so the solutions have the form $(A+Bx)e^x$. The boundary conditions $y(0)=y(1)=0$ give $A=(A+B)e=0$, so $A=B=0$, so $y=0$, so there are no nontrivial solutions.

Consider instead the case where $\lambda\neq 1$, and write $\alpha=1-\sqrt{1-\lambda}$ and $\beta=1+\sqrt{1-\lambda}$ (which will be complex if $\lambda>1$). Then the solutions must have the form $y=Ae^{\alpha t}+Be^{\beta t}$. The first boundary condition gives $A+B=0$, so

$$y=A(e^{\alpha t}-e^{\beta t})=Ae^t(e^{\sqrt{1-\lambda}t}-e^{-\sqrt{1-\lambda}t}).$$

The second boundary condition therefore gives

$$Ae(e^{\sqrt{1-\lambda}}-e^{-\sqrt{1-\lambda}})=0.$$

For a nontrivial solution we must have $A\neq 0$ and so $e^{\sqrt{1-\lambda}}=e^{-\sqrt{1-\lambda}}$, which means that $e^{2\sqrt{1-\lambda}}=1$, so $2\sqrt{1-\lambda}=2n\pi i$ for some integer n. After squaring both sides and rearranging we get $\lambda=1+n^2\pi^2$. To get a real solution, A must be imaginary, say $A=C/(2i)$ for some real number C. This gives $y=Ce^x\sin(n\pi x)$.

(c) Now consider the operator $Ly=(xy')'+\lambda x^{-1}y$. If $y=x^p$ then $xy'=px^p$ so $(xy')'=p^2x^{p-1}$ and $Ly=(p^2+\lambda)x^{p-1}$.

Suppose that $\lambda<0$, so $\lambda=-\mu^2$ for some $\mu>0$. We then find that the functions $L(x^{\mu})=L(x^{-\mu})=0$. This gives two linearly independent solutions, so all functions with $Ly=0$ have $y=Ax^{\mu}+Bx^{-\mu}$ for some constants A and B. The boundary condition $y(1)=0$ gives $B=-A$, so $y=A(x^{\mu}-x^{-\mu})$. The boundary condition $y(e^{\pi})=0$ then gives $A(e^{\mu\pi}-e^{-\mu\pi})=0$ but $e^{\mu\pi}-e^{-\mu\pi}>0$ so $A=0$ so $y=0$.

Consider instead the case where $\lambda>0$, so $\lambda=\mu^2$ for some $\mu>0$. In the same way, we find that $y=A(x^{i\mu}-x^{-i\mu})$ for some constant A, and we must have

$$A(e^{\mu\pi}-e^{-\mu\pi})=0.$$

For a nontrivial solution we must have $A\neq 0$ so $e^{\mu\pi i}=e^{-\mu\pi i}$ so $e^{2\mu\pi i}=1$, which means that μ must be an integer. Note also that

$$x^{i\mu}-x^{-i\mu}=\cos(\mu\ln(x))+i\sin(\mu\ln(x))-(\cos(\mu\ln(x))-i\sin(\mu\ln(x)))=2i\sin(\mu\ln(x)).$$

We conclude that $y = C\sin(\mu\ln(x))$, where $C = A/(2i)$ is a real constant.

Finally, consider the case $\lambda = 0$. Here the equation is $(xy')' = 0$, so xy' is constant, say $xy' = A$. This gives $y' = A/x$ so $y = \int A/x\,dx = A\ln(x) + B$ for some constant B. The boundary condition $y(1) = 0$ gives $B = 0$, and the condition $y(e^\pi) = 0$ gives $A\pi + B = 0$, so $A = B = 0$. Thus, we only have the trivial solution in this case.

Exercise 56. Consider the differential equation

$$(6x^2 - 5x + 1)y'' + (24x - 10)y' + 12y = 0.$$

Suppose we have a function $y = \sum_k a_k x^k$ (with $a_k = 0$ for $k < 0$). Show that the above equation is satisfied if and only if

$$a_{k+2} - 5a_{k+1} + 6a_k = 0$$

for all $k \geq 0$. Show that this holds if $a_k = 2^k$ or if $a_k = 3^k$. Using this, find the general solution for the differential equation.

Solution:

$$\begin{aligned}
(6x^2 - 5x + 1)y'' &= \sum_k 6k(k-1)a_k x^k - \sum_k 5k(k-1)a_k x^{k-1} + \sum_k k(k-1)a_k x^{k-2} \\
&= \sum_j 6j(j-1)a_j x^j - \sum_j 5(j+1)j a_{j+1} x^j + \sum_j (j+2)(j+1)a_{j+2} x^j \\
(24x - 10)y' &= \sum_k 24k a_k x^k - \sum_k 10k a_k x^{k-1} \\
&= \sum_j 24j a_j x^j - \sum_j 10(j+1)a_{j+1} x^j \\
12y &= \sum_j 12a_j x^j.
\end{aligned}$$

It follows that the coefficient of x^j in $(6x^2 - 5x + 1)y'' + (24x - 10)y' + 12y$ is

$$\begin{aligned}
&6j(j-1)a_j - 5(j+1)j a_{j+1} + (j+2)(j+1)a_{j+2} + 24j a_j - 10(j+1)a_{j+1} + 12a_j \\
=&(6j^2 - 6j + 24j + 12)a_j + (-5j^2 - 5j - 10j - 10)a_{j+1} + (j^2 + 3j + 2)a_{j+2} \\
=&(j^2 + 3j + 2)(a_{j+2} - 5a_{j+1} + 6a_j).
\end{aligned}$$

For $j \geq 0$ we have $j^2 + 3j + 2 > 0$ so the above coefficient can only vanish if $a_{j+2} - 5a_{j+1} + 6a_j = 0$, as claimed.

Now take $a_k = 2^k$. We then have

$$a_{k+2} - 5a_{k+1} + 6a_k = 2^{k+2} - 5.2^{k+1} + 6.2^k = 2^k(4 - 10 + 6) = 0.$$

It follows that the function $y = \sum_k 2^k x^k$ is a solution to the original differential equation. This is just a geometric progression, so the sum is $y = 1/(1 - 2x)$.

Similarly, if $a_k = 3^k$ then

$$a_{k+2} - 5a_{k+1} + 6a_k = 3^{k+2} - 5.3^{k+1} + 6.3^k = 3^k(9 - 15 + 6) = 0,$$

so the function $y = \sum_k 3^k x^k = 1/(1 - 3x)$ is another solution. This gives two linearly independent solutions, so any other solution has the form

$$y = \frac{A}{1 - 2x} + \frac{B}{1 - 3x}$$

for some constants A and B.

Exercise 57. Find a power series solutions for the equation $y'' - x^9 y' - 9x^8 y = 0$ with $y = 1$ and $y' = 0$ at $x = 0$.

Solution: We take $y = \sum_k a_k x^k$ with $a_k = 0$ for $k < 0$. We are given that $y = 1$ and $y' = 0$ when $x = 0$, which means that $a_0 = 1$ and $a_1 = 0$. We also have

$$\begin{aligned}
y'' &= \sum_k k(k-1)a_k x^{k-2} && = \sum_j (j+2)(j+1)a_{j+2}x^j \\
-x^9 y' &= -x^9 \sum_k k a_k x^{k-1} = \sum_k -k a_k x^{k+8} && = \sum_j -(j-8)a_{j-8}x^j \\
-9x^8 y &= \sum_k -9 a_k x^{k+8} && = \sum_j -9a_{j-8}x^j,
\end{aligned}$$

so we need $(j+2)(j+1)a_{j+2} - (j-8)a_{j-8} - 9a_{j-8} = 0$. This simplifies to $(j+2)(j+1)a_{j+2} = (j+1)a_{j-8}$. If we put $m = j+2$, this becomes $m(m-1)a_m = (m-1)a_{m-10}$. If $m > 1$ we can divide both sides by $m(m-1)$ to get $a_m = a_{m-10}/m$. In particular, if $a_{m-10} = 0$ then $a_m = 0$. For example, $a_{-8} = 0$ (because $a_k = 0$ whenever $k < 0$), so $a_2 = 0$, so $a_{12} = 0$, so $a_{22} = 0$ and so on; in general $a_{10i+2} = 0$ for all i. Similarly $a_{-7} = 0$, so $a_3 = 0$, so $a_{13} = 0$ and so on. By this argument we see that $a_{10k+2}, a_{10k+3}, \dots, a_{10k+9}$ are all zero. Moreover, we are given that $a_1 = 0$, so $a_{10k+1} = 0$ for all k. Thus, only the coefficients a_{10k} can be nonzero. Using the relation $a_m = a_{m-10}/m$ (for $m > 1$) we get

$$\begin{aligned}
a_0 &= 1 \\
a_{10} &= \frac{1}{10} \\
a_{20} &= \frac{1}{10.20} = \frac{1}{10^2 \times 2} \\
a_{30} &= \frac{1}{10.20.30} = \frac{1}{10^3 \times 2.3} = \frac{1}{10^3 \times 3!} \\
a_{40} &= \frac{1}{10.20.30.40} = \frac{1}{10^4 \times 4!}
\end{aligned}$$

and in general $a_{10k} = 10^{-k}a_0/k!$. This means that

$$y = \sum_k a_{10k}x^{10k} = \sum_k \frac{x^{10k}}{10^k k!} = \sum_k \frac{1}{k!}\left(\frac{x^{10}}{10}\right)^k = \exp\left(\frac{x^{10}}{10}\right).$$

Exercise 58. Find a power series solution about $x = 0$ for the equation $y'' - x^2 y' + xy = 0$.

Solution: Suppose that $y = \sum_{k=0}^{\infty} a_k x^k$ (and we take $a_k = 0$ for $k < 0$). We then have

$$\begin{aligned}
y'' &= \sum_k k(k-1)a_k x^{k-2} && = \sum_j (j+2)(j+1)a_{j+2}x^j \\
-x^2 y' &= -x^2 \sum_k k a_k x^{k-1} = \sum_k -k a_k x^{k+1} && = \sum_j -(j-1)a_{j-1}x^j \\
xy &= \sum_k a_k x^{k+1} && = \sum_j a_{j-1}x^j,
\end{aligned}$$

so we must have $(j+2)(j+1)a_{j+2} - (j-1)a_{j-1} + a_{j-1} = 0$. This simplifies to $(j+2)(j+1)a_{j+2} = (j-2)a_{j-1}$. If we put $m = j+2$, this becomes $m(m-1)a_m = (m-4)a_{m-3}$. For $m > 1$ this can be rearranged as $a_m = \frac{m-4}{m(m-1)}a_{m-3}$. As $a_{-1} = 0$, it follows that $a_2 = 0$, then $a_5 = 0$, then $a_8 = 0$, and in general $a_{3k+2} = 0$ for all k. The coefficient a_1 can be nonzero, but then $a_4 = \frac{4-4}{4\times 3}a_1 = 0$, and then $a_7 = 0$, then $a_{10} = 0$ and so on; so $a_{3k+1} = 0$ whenever $k > 0$. Next, we have

$$\begin{aligned}
a_3 &= -\frac{1}{6}a_0 \\
a_6 &= \frac{2}{6\times 5}a_3 = \frac{1}{15} \times \frac{-1}{6}a_0 = -\frac{1}{90}a_0 \\
a_9 &= \frac{5}{9\times 8}a_6 = -\frac{1}{1296}a_0
\end{aligned}$$

and so on. The solution is therefore

$$y = a_1 x + a_0\left(1 - \frac{1}{6}x^3 - \frac{1}{90}x^6 - \frac{1}{1296}x^9 - \cdots\right).$$

In fact, it works out that $a_{3k} = -\frac{1}{3^k k!(3k-1)}a_0$; this can easily be checked by induction.

Exercise 59. Let m be a natural number, and consider the Hermite equation $y'' - 2xy' + 2my = 0$.

(a) If m is even, show that there is a solution $y = h_m(x)$ where h_m is a polynomial (not an infinite power series) satisfying $h(0) = 1$ and $h'(0) = 0$. For example, we have $h_0(x) = 1$ and $h_2(x) = 1 - 2x^2$.

(b) If m is odd, show that there is a solution $y = h_m(x)$ where h_m is a polynomial (not an infinite power series) satisfying $h(0) = 0$ and $h'(0) = 1$.

Solution: Put $y = h_m(x) = \sum_k a_k x^k$ (with $a_k = 0$ for $k < 0$). We then have

$$\begin{aligned} y'' &= \sum_k k(k-1)a_k x^{k-2} && = \sum_j (j+2)(j+1)a_{j+2}x^j \\ -2xy' &= -2x\sum_k k a_k x^{k-1} && = \sum_j -2ja_j x^j \\ 2my & && = \sum_j 2ma_j x^j, \end{aligned}$$

so we need $(j+2)(j+1)a_{j+2} + 2(m-j)a_j = 0$. If we put $k = j+2$, this becomes $k(k-1)a_k = 2(k-2-m)a_{k-2}$. For $k > 1$ we can rewrite this as $a_k = 2(k-2-m)/(k(k-1))a_{k-2}$. In particular, this gives $a_{m+2} = 0$, and then $a_{m+4} = 0$ and $a_{m+6} = 0$ and so on.

Now suppose that m is even. We look for the solution where $h_m(0) = 1$ and $h'_m(0) = 0$, or equivalently $a_0 = 1$ and $a_1 = 0$. As $a_1 = 0$ we have $a_3 = 0$, then $a_5 = 0$ and so on, so $a_k = 0$ whenever k is odd. We also have $a_{m+2} = a_{m+4} = \cdots = 0$, so $a_k = 0$ whenever k is even and greater than m. This means that only finitely many of the coefficients are nonzero, so $h_m(x)$ is a polynomial.

The case where m is odd is similar. Here we look for the solution where $a_0 = 0$ and $a_1 = 1$. As $a_0 = 0$ we have $a_2 = 0$, then $a_4 = 0$ and so on, so $a_k = 0$ for all even k. We also have $a_{m+2} = a_{m+4} = \cdots = 0$, so $a_k = 0$ for all odd $k > m$. It follows again that only finitely many coefficients are nonzero, so $h_m(x)$ is polynomial.

Exercise 60. Suppose that y and z are nonzero functions with $y'' + \lambda^2 y = 0$ and $z'' + \mu^2 z = 0$, where $\lambda, \mu > 0$ and $\lambda \neq \mu$. Suppose also that $y(0) = y(\pi/2) = z(0) = z(\pi/2) = 0$.

(a) Integrate $\int_0^{\pi/2} y'z'\,dx$ by parts in two different ways, and use this to show that $\int_0^{\pi/2} yz\,dx = 0$.

(b) Solve the equations explicitly to find λ, μ, y and z. Use this to prove again that $\int_0^{\pi/2} yz\,dx = 0$.

Solution:

(a) First, we have

$$\int_0^{\pi/2} y'z'\,dx = [yz']_0^{\pi/2} - \int_0^{\pi/2} yz''\,dx = y(\pi/2)z'(\pi/2) - y(0)z(0) + \mu^2\int_0^{\pi/2} yz\,dx = \mu^2\int_0^{\pi/2} yz\,dx.$$

(We have used the fact that $y(\pi/2) = y(0) = 0$ and that $z'' = -\mu^2 z$.) Similarly, we have

$$\int_0^{\pi/2} y'z'\,dx = [y'z]_0^{\pi/2} - \int_0^{\pi/2} y''z\,dx = y'(\pi/2)z(\pi/2) - y'(0)z(0) + \lambda^2\int_0^{\pi/2} yz\,dx = \lambda^2\int_0^{\pi/2} yz\,dx.$$

These two answers must be the same, so $(\mu^2 - \lambda^2)\int_0^{\pi/2} yz\,dx = 0$. Moreover, as $\lambda, \mu > 0$ with $\lambda \neq \mu$, we have $\mu^2 - \lambda^2 \neq 0$, so $\int_0^{\pi/2} yz\,dx = 0$.

(b) As $y'' + \lambda^2 y = 0$ we have $y = A\cos(\lambda x) + B\sin(\lambda x)$ for some constants A and B. As $y(0) = 0$ we have $A = 0$, so $y = B\sin(\lambda x)$, and $B \neq 0$ because y is assumed to be nonzero. As $y(\pi/2) = 0$ we also have $B\sin(\lambda\pi/2) = 0$, so $\sin(\lambda\pi/2) = 0$, so $\lambda = 2n$ for some integer $n > 0$. In conclusion, $y = B\sin(2nx)$ for some integer $n > 0$ and some constant $B \neq 0$. Similarly $z = C\sin(2mx)$ for some integer $m > 0$ and some constant $C \neq 0$. This gives

$$\begin{aligned} \int_0^{\pi/2} yz\,dx &= BC\int_0^{\pi/2} \sin(2nx)\sin(2mx)\,dx = \tfrac{1}{2}BC\int_0^{\pi/2} \cos(2(n-m)x) - \cos(2(n+m)x)\,dx \\ &= \tfrac{1}{2}BC\left[\frac{\sin(2(n-m)x)}{2(n-m)} - \frac{\sin(2(n+m)x)}{2(n+m)}\right]_0^{\pi/2}. \end{aligned}$$

Here the functions $\sin(2(n-m)x)$ and $\sin(2(n+m)x)$ are both zero when $x = 0$ or $x = \pi/2$, so we conclude that $\int_0^{\pi/2} yz\,dx = 0$ as expected.

Exercise 61. Consider the equation

$$3xy'' + (1-2x)y' - 2y = 0.$$

(a) Show that the roots of the indicial polynomial are 0 and 2/3.
(b) Find two linearly independent solutions. In each case, you should give the first four nonzero terms of the series.
(c) Find the solution where $y = 7$ and y' is finite when $x = 0$.
(d) For which values of x will your series in (c) converge?

Solution:

(a) First, the equation is equivalent to $y'' + (\frac{1}{3}x^{-1} - \frac{2}{3})y' - \frac{2}{3}x^{-1}y = 0$, so $P = \frac{1}{3}x^{-1} - \frac{2}{3}$ and $Q = -\frac{2}{3}x^{-1}$. Recall that p_0 is the coefficient of x^{-1} in P, which is $\frac{1}{3}$, and q_0 is the coefficient of x^{-2} in Q, which is 0. The indicial polynomial is

$$\alpha(\alpha-1) + p_0\alpha + q_0 = \alpha^2 - \alpha + \tfrac{1}{3}\alpha = \alpha(\alpha - \tfrac{2}{3}),$$

so the roots are $\alpha = 0$ and $\alpha = 2/3$, as required.

(b) We first consider the root $\alpha = 0$ and so look for a solution of the form $y = \sum_{k=0}^{\infty} a_k x^k$ with $a_0 = 1$ (and take $a_k = 0$ for $k < 0$ as usual). We have

$$\begin{aligned}
3xy'' &= \sum_k 3k(k-1)a_k x^{k-1} && = \sum_j 3(j+1)ja_{j+1}x^j \\
y' &= \sum_k ka_k x^{k-1} && = \sum_j (j+1)a_{j+1}x^j \\
-2xy' &= -\sum_k 2ka_k x^k && = -\sum_j 2ja_j a^j \\
-2y & && = -\sum_j 2a_j x^j.
\end{aligned}$$

Thus, the coefficient of x^j in the relation $3xy'' + (1-2x)y' - 2y = 0$ gives

$$3(j+1)ja_{j+1} + (j+1)a_{j+1} = 2ja_j + 2a_j,$$

or equivalently $(3j+1)(j+1)a_{j+1} = 2(j+1)a_j$. If we put $m = j+1$ and assume that $m > 0$ we get $a_m = 2a_{m-1}/(3m-2)$. By assumption we have $a_0 = 1$, so we get

$$\begin{aligned}
a_1 &= 2a_0/1 = 2 \\
a_2 &= 2a_1/4 = 1 \\
a_3 &= 2a_2/7 = 2/7,
\end{aligned}$$

giving

$$y = 1 + 2x + x^2 + \tfrac{2}{7}x^3 + \cdots.$$

We now look for a second solution of the form $z = \sum_k b_k z^{2/3+k}$ with $b_0 = 1$ and $b_k = 0$ for $k < 0$. We have

$$\begin{aligned}
3xz'' &= \sum_k 3(k+\tfrac{2}{3})(k-\tfrac{1}{3})b_k x^{-1/3+k} && = \sum_j 3(j+\tfrac{5}{3})(j+\tfrac{2}{3})b_{j+1}x^{2/3+k} \\
z' &= \sum_k (k+\tfrac{2}{3})b_k x^{-1/3+k} && = \sum_j (j+\tfrac{5}{3})b_{j+1}x^{2/3+j} \\
-2xz' &= -\sum_k 2(k+\tfrac{2}{3})b_k x^{2/3+k} && = -\sum_j 2(j+\tfrac{2}{3})b_j x^{2/3+j} \\
-2z & && = -\sum_j 2b_j x^{2/3+j}.
\end{aligned}$$

Thus, the coefficient of $z^{2/3+j}$ in the relation $3xz'' + (1-2x)z' - 2z = 0$ gives

$$3(j+\tfrac{5}{3})(j+\tfrac{2}{3})b_{j+1} + (j+\tfrac{5}{3})b_{j+1} = 2(j+\tfrac{2}{3})b_j + 2b_j,$$

or equivalently

$$(3j+3)(j+\tfrac{5}{3})b_{j+1} = 2(j+\tfrac{5}{3})b_j.$$

Here j is an integer so $j + \frac{5}{3}$ is never zero so we can divide by it to get $3(j+1)b_{j+1} = 2b_j$, or equivalently $3mb_m = 2b_{m-1}$. If $m > 0$ then we can divide by $3m$ to get $b_m = \frac{2}{3}b_{m-1}/m$. This gives

$$\begin{aligned}
b_1 &= \tfrac{2}{3}b_0/1 = \tfrac{2}{3} \\
b_2 &= \tfrac{2}{3}b_1/2 = \left(\tfrac{2}{3}\right)^2 \times \frac{1}{2} = \frac{2}{9} \\
b_3 &= \tfrac{2}{3}b_2/3 = \left(\tfrac{2}{3}\right)^3 \times \frac{1}{2 \times 3} = \frac{4}{81},
\end{aligned}$$

giving

$$z = x^{2/3} + \frac{2}{3}x^{5/3} + \frac{2}{9}x^{8/3} + \frac{4}{81}x^{11/3} + \cdots .$$

In fact, in this case it is easy to see that $b_k = (2/3)^k/k!$ and so we have

$$z = x^{2/3}\sum_k b_k x^k = x^{2/3}\sum \left(\frac{2x}{3}\right)^k \times \frac{1}{k!} = x^{2/3}e^{2x/3}.$$

(c) We now see that all solutions have the form $u = Ay + Bz$, where A and B are constant, and y and z are as in part (b). This gives $u' = Ay' + Bz'$, where

$$\begin{aligned}
y' &= 2 + 2x + \tfrac{6}{7}x^2 + \cdots \\
z' &= \tfrac{2}{3}x^{-2/3} + \tfrac{10}{9}x^{2/3} + \frac{16}{27}x^{5/3} + \cdots .
\end{aligned}$$

Now y' stays finite as $x \to 0$ but $z' \to \infty$. Thus, for u' to stay finite we must have $B = 0$ and so $u = Ay$. At $x = 0$ we have $y = 1$ but we are asked to find a solution with $u = 7$ so we must take $A = 7$ and so

$$u = 7y = 7 + 14x + 7x^2 + 2x^3 + \cdots .$$

(d) For the coefficients a_m in y, we saw in (b) that $a_m = 2a_{m-1}/(3m-2)$, so $a_{m-1}/a_m = (3m-2)/2$, which tends to infinity. Thus, the series has infinite radius of convergence, so it converges for all x. Similarly, in the series for z we have $b_{m-1}/b_m = 3m/2$, which again tends to infinity. Thus, this series also converges for all x.

Exercise 62. Consider the equation

$$x^2y'' + xy' + (x^2 - \tfrac{1}{4})y = 0.$$

(a) Show that the function $y = x^{-1/2}\sin(x)$ is one solution.
(b) Use reduction of order to find another solution.
(Hint: $\int \frac{dx}{\sin(x)^2} = -\cot(x) + c$.)

Solution:

(a) If $y = x^{-1/2}\sin(x)$ then

$$\begin{aligned}
y' &= -\tfrac{1}{2}x^{-3/2}\sin(x) + x^{-1/2}\cos(x) = x^{-3/2}(-\tfrac{1}{2}\sin(x) + x\cos(x)) \\
y'' &= \tfrac{3}{4}x^{-5/2}\sin(x) + 2 \times (-\tfrac{1}{2}x^{-3/2}\cos(x)) - x^{-1/2}\sin(x) \\
&= x^{-5/2}((\tfrac{3}{4} - x^2)\sin(x) - x\cos(x))
\end{aligned}$$

so

$$\begin{aligned}
&x^2y'' + xy' + (x^2 - \tfrac{1}{4})y \\
=&x^{-1/2}((\tfrac{3}{4} - x^2)\sin(x) - x\cos(x) + (-\tfrac{1}{2}\sin(x) + x\cos(x)) + (x^2 - \tfrac{1}{4})\sin(x)) \\
=&x^{-1/2}((\tfrac{3}{4} - x^2 - \tfrac{1}{2} + x^2 - \tfrac{1}{4})\sin(x) + (-x + x)\cos(x)) = 0.
\end{aligned}$$

(b) Our equation can be written as

$$y'' + x^{-1}y' + (1 - \tfrac{1}{4}x^{-2})y = 0,$$

so $P = x^{-1}$ and $Q = 1 - \frac{1}{4}x^{-2}$. Our first solution is $y = x^{-1/2}\sin(x)$. In the reduction of order method we therefore have $v = \int P\,dx = \ln(x)$, so

$$y^{-2}e^{-v} = x\sin(x)^{-2}x^{-1} = \sin(x)^{-2}.$$

We thus have

$$u = \int y^{-2}e^{-v}\,dx = \int \sin(x)^{-2}\,dx = -\cot(x) = -\frac{\cos(x)}{\sin(x)},$$

and

$$z = uy = -\frac{\cos(x)}{\sin(x)}x^{-1/2}\sin(x) = -x^{-1/2}\cos(x).$$

Exercise 63. Show that the function

$$y = \cosh(\sqrt{2x}) - \sinh(\sqrt{2x})/\sqrt{2x}$$

satisfies $2x^2y'' + xy' - (x+1)y = 0$.

Solution: Put $t = \sqrt{2x}$, so $x = \frac{1}{2}t^2$, so $dx/dt = t$, so $dt/dx = t^{-1}$. Write $\dot{u} = du/dt$ and $u' = du/dx$, so

$$u' = \frac{du}{dx} = \frac{dt}{dx}\frac{du}{dt} = t^{-1}\dot{u}.$$

Thus

$$\begin{aligned}
u'' &= (u')' = (t^{-1}\dot{u})' = t^{-1}(t^{-1}\dot{u})^{\cdot} \\
&= t^{-1}(-t^{-2}\dot{u} + t^{-1}\ddot{u}) = t^{-2}\ddot{u} - t^{-3}\dot{u}.
\end{aligned}$$

This gives

$$\begin{aligned}
&2x^2y'' + xy' - (x+1)y \\
=&2\times(\tfrac{1}{2}t^2)^2 \times (t^{-2}\ddot{y} - t^{-3}\dot{y}) + \tfrac{1}{2}t^2 \times (t^{-1}\dot{y}) - (\tfrac{1}{2}t^2 + 1)y \\
=&\tfrac{1}{2}t^2\ddot{y} - \tfrac{1}{2}t\dot{y} + \tfrac{1}{2}t\dot{y} - (\tfrac{1}{2}t^2 + 1)y \\
=&\tfrac{1}{2}t^2\ddot{y} - (\tfrac{1}{2}t^2 + 1)y.
\end{aligned}$$

Now

$$\begin{aligned}
y &= \cosh(t) - \sinh(t)t^{-1} \\
\dot{y} &= \sinh(t) - \cosh(t)t^{-1} + \sinh(t)t^{-2} \\
&= \sinh(t)(1 + t^{-2}) - \cosh(t)t^{-1} \\
\ddot{y} &= \cosh(t)(1 + t^{-2}) + \sinh(t)(-2t^{-3}) - \sinh(t)t^{-1} + \cosh(t)t^{-2} \\
&= \cosh(t)(1 + 2t^{-2}) + \sinh(t)(-t^{-1} - 2t^{-3}) \\
&= (\cosh(t) - \sinh(t)t^{-1})(1 + 2t^{-2}) = y(1 + 2t^{-2}).
\end{aligned}$$

This gives

$$\tfrac{1}{2}t^2\ddot{y} - (\tfrac{1}{2}t^2 + 1)y = \tfrac{1}{2}t^2(1 + 2t^{-2})y - (\tfrac{1}{2}t^2 + 1)y = 0$$

as required.

Exercise 64. Show that the function $y = (1-x)e^{-x}$ satisfies $y'' + (x^{-1} + 1)y' + 2x^{-1}y = 0$.

Solution:

$$\begin{aligned}
y &= (1-x)e^{-x} \\
y' &= -e^{-x} + (1-x)(-e^{-x}) = (x-2)e^{-x} \\
y'' &= e^{-x} + (x-2)(-e^{-x}) = (3-x)e^{-x}
\end{aligned}$$

so

$$\begin{aligned}
y'' + (x^{-1} + 1)y' + 2x^{-1}y &= e^{-x}((3-x) + (x^{-1} + 1)(x-2) + 2x^{-1}(1-x)) \\
&= e^{-x}(3 - x + 1 - 2x^{-1} + x - 2 + 2x^{-1} - 2) = 0.
\end{aligned}$$

Exercise 65. Solve $Ly = 0$, where $Lu = x^2(1-x)u'' + x(1-3x)u' - u$.

Solution: This is equivalent to $y'' + Py' + Qy = 0$, where

$$P = \frac{x(1-3x)}{x^2(1-x)} = x^{-1}\frac{1-3x}{1-x} = x^{-1}(1-3x)(1+x+x^2+\cdots) = x^{-1} + O(1)$$
$$Q = \frac{-1}{x^2(1-x)} = -x^{-2}(1+x+x^2+\cdots) = -x^{-2} + O(x^{-1}).$$

We therefore have a regular singular point with $p_0 = 1$ and $q_0 = -1$. The indicial polynomial is

$$\chi(t) = t(t-1) + p_0 t + q_0 = t^2 - t + t - 1 = t^2 - 1 = (t-1)(t+1),$$

so the roots are $\alpha = 1$ and $\beta = -1$. There is a solution $y = \sum_k a_k x^{\alpha+k} = \sum_k a_k x^{k+1}$ with $a_0 = 1$. This has

$$\begin{aligned}
x^2y'' &= \sum (1+k)k\, a_k x^{1+k} \\
-x^3y'' &= \sum -(1+k)k\, a_k x^{2+k} = \sum -j(j-1)\, a_{j-1} x^{1+j} \\
xy' &= \sum (1+k) a_k x^{1+k} \\
-3x^2y' &= \sum -3(1+k) a_k x^{2+k} = \sum -3j\, a_{j-1} x^{1+j} \\
-y &= \sum -a_k x^{1+k}.
\end{aligned}$$

It follows that the coefficient of x^{1+j} in Ly is

$$\begin{aligned}
&(1+j)j\, a_j - j(j-1)a_{j-1} + (1+j)a_j - 3j\, a_{j-1} - a_j \\
=&(j + j^2 + 1 + j - 1)a_j - (j^2 - j + 3j)a_{j-1} = (j^2+2j)a_j - (j^2+2j)a_{j-1} \\
=&j(j+2)(a_j - a_{j-1}).
\end{aligned}$$

For $Ly = 0$ we must have $j(j+2)(a_j - a_{j-1}) = 0$, so for $j > 0$ we have $a_j = a_{j-1}$. As $a_0 = 1$ we have $a_j = 1$ for all $j \geq 0$, giving

$$y = \sum_{j\geq 0} x^{j+1} = \frac{x}{1-x}.$$

To find a second solution, we use reduction of order. We have $P = (3x-1)/(x(x-1))$. The method of partial fractions says that $P = A/x + B/(x-1)$ for some constants A and B. This gives

$$\frac{3x-1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1)+Bx}{x(x-1)} = \frac{(A+B)x - A}{x(x-1)},$$

so we need $A + B = 3$ and $A = 1$, so $B = 2$. This means that $P = 1/x + 2/(x-1)$. Next, we have

$$v = \int P\, dx = \int \frac{1}{x} + \frac{2}{x-1}\, dx = \ln(x) + 2\ln(x-1) = \ln(x\,(x-1)^2),$$

so $e^{-v} = x^{-1}(x-1)^{-2}$. Recall also that $y = x/(1-x)$, so

$$u = \int y^{-2}e^{-v}\, dx = \int \frac{(1-x)^2}{x^2} x^{-1}(x-1)^{-2}\, dx = \int x^{-3}\, dx = -\tfrac{1}{2}x^{-2}.$$

Finally, our second solution is

$$z = uy = -\tfrac{1}{2}x^{-2}\frac{x}{1-x} = \frac{1}{2x(x-1)}.$$

Exercise 66. Solve $Ly = 0$, where $Lu = x^2(x^2-2)u'' - (x^2+2)xu' + (x^2+2)u = 0$.

Solution: This is equivalent to $y'' + Py' + Qy = 0$, where

$$P = -\frac{(x^2+2)x}{x^2(x^2-2)} = x^{-1}\frac{1+x^2/2}{1-x^2/2} = x^{-1} + O(1)$$
$$Q = \frac{x^2+2}{x^2(x^2-2)} = -x^{-2}\frac{1+x^2/2}{1-x^2/2} = -x^{-2} + O(x^{-1}).$$

We therefore have a regular singular point at $x = 0$, with $p_0 = 1$ and $q_0 = -1$. The indicial polynomial is

$$\chi(t) = t(t-1) + p_0 t + q_0 = t^2 - t + t - 1 = t^2 - 1 = (t-1)(t+1),$$

so the roots are $\alpha = 1$ and $\beta = -1$. There is a solution $y = \sum_k a_k x^{\alpha+k} = \sum_k a_k x^{k+1}$ with $a_0 = 1$. This has

$$\begin{aligned}
x^4 y'' &= \sum_k (1+k)k\, a_k x^{3+k} = \sum_j (j-1)(j-2) a_{j-2} x^{1+j} \\
-2x^2 y'' &= \sum_k -2(1+k)k\, a_k x^{1+k} \\
-x^3 y' &= \sum_k -(1+k) a_k x^{3+k} = \sum_j -(j-1) a_{j-2} x^{1+j} \\
-2xy' &= \sum_k -2(1+k) a_k x^{1+k} \\
x^2 y &= \sum_k a_k x^{3+k} = \sum_j a_{j-2} x^{1+j} \\
2y &= \sum_k 2a_k x^{1+k}.
\end{aligned}$$

Thus, the coefficient of x^{1+j} in our differential equation is

$$\begin{aligned}
&(j-1)(j-2)a_{j-2} - 2(1+j)ja_j - (j-1)a_{j-2} - 2(1+j)a_j + a_{j-2} + 2a_j \\
=&(j^2 - 3j + 2 - j + 1 + 1)a_{j-2} + (-2j - 2j^2 - 2 - 2j + 2)a_j \\
=&(j^2 - 4j + 4)a_{j-2} - (2j^2 + 4j)a_j \\
=&(j-2)^2 a_{j-2} - 2j(j+2)a_j.
\end{aligned}$$

For $j > 0$ we therefore have

$$a_j = \frac{(j-2)^2}{2j(j+2)} a_{j-2}.$$

As $a_{-1} = 0$ we see that $a_1 = 0$ then $a_3 = 0$ and so on, so $a_j = 0$ whenever j is odd. We can also put $j = 2$ in the above formula to get $a_2 = 0$, and then $a_4 = 0$ and $a_6 = 0$ and so on. It follows that all the coefficients a_j are zero except for $a_0 = 1$, so we just have $y = x$.

We could find a second solution by reduction of order, but we will use a different method instead. The main theorem says that there is a second solution of the form

$$z = c\, y\, \ln(x) + \sum_k b_k x^{-1-k}$$

with $b_0 = 1$ and $b_k = 0$ for $k < 0$. Here $y = x$. Put $u = x\,\ln(x)$ and $v = \sum b_k x^{k-1}$ so that $z = cu + v$. Now $u' = \ln(x) + 1$ and $u'' = x^{-1}$ so

$$\begin{aligned}
Lu &= x^2(x^2-2)u'' - (x^2+2)xu' + (x^2+2)u \\
&= (x^4 - 2x^2)x^{-1} - (x^3 + 2x)(\ln(x) + 1) + (x^2 + 2)(x\ln(x)) \\
&= x^3 - 2x - x^3 - 2x + (-x^3 - 2x + x^3 + 2x)\ln(x) \\
&= -4x,
\end{aligned}$$

so $Ly = cLu + Lv = (Lv) - 4cx$. We therefore want to have $Lv = 4cx$. On the other hand, we have

$$\begin{aligned}
x^4 v'' &= \sum_k (k-1)(k-2)\, b_k x^{1+k} = \sum_j (j-2)(j-3) b_{j-1} x^j \\
-2x^2 v'' &= \sum_k -2(k-1)(k-2)\, b_k x^{k-1} = \sum_j -2j(j-1) b_{j+1} x^j \\
-x^3 v' &= \sum_k -(k-1) b_k x^{k+1} = \sum_j -(j-2) b_{j-1} x^j \\
-2xv' &= \sum_k -2(k-1) b_k x^{k-1} = \sum_j -2j b_{j+1} x^j \\
x^2 v &= \sum_k b_k x^{k+1} = \sum_j b_{j-1} x^j \\
2v &= \sum_k 2b_k x^{k-1} = \sum_j 2b_{j+1} x^j.
\end{aligned}$$

It follows that the coefficient of x^j in Lv is

$$\begin{aligned}&((j-2)(j-3)-(j-2)+1)b_{j-1}+(-2j(j-1)-2j+2)b_{j+1}\\ =&(j^2-6j+9)b_{j-1}+(2-2j^2)b_j\\ =&(j-3)^2b_{j-1}+2(1-j^2)b_{j+1}.\end{aligned}$$

We want $Lv = 4cx$, so the above coefficient should be zero for $j \neq 1$. Taking $j = 0$ gives $b_1 = 0$, then taking $j = 2$ gives $b_3 = 0$ and so on, so $b_j = 0$ whenever j is odd. For $j = 0$ we instead get $4b_0 = 4c$, but $b_0 = 1$ so $c = 1$. For $j = 3$ we get $b_4 = 0$, and then it follows that $b_6 = 0$, then $b_8 = 0$ and so on. We thus have $v = x^{-1} + b_2 x$. We can choose b_2 arbitrarily. It is simplest to take $b_2 = 0$, giving $v = x^{-1}$ and $z = x\ln(x) + x^{-1}$.

Exercise 67. Consider the equation $y'' + y = 0$. One solution is $y = \cos(x)$. Use reduction of order to find another solution. (Of course you know already what is the second solution, but it is nice to see how reduction of order works in this simple case.)

Solution: Our equation is $y'' + Py' + Qy = 0$, where $P = 0$ and $Q = 1$. We put $v = \int P\,dx = 0$, so

$$y^{-2}e^{-v} = \cos(x)^{-2} = \sec(x)^2 = \frac{d}{dx}\tan(x),$$

so

$$u = \int y^{-2}e^{-v}\,dx = \int \sec(x)^2\,dx = \tan(x),$$

so

$$z = uy = \tan(x)\cos(x) = \frac{\sin(x)}{\cos(x)}\cos(x) = \sin(x).$$

Thus, the second solution is $\sin(x)$, as expected.

Exercise 68. Put

$$Lu = x^2u'' - (x^2 + 20x)u' + (110 + 10x)u.$$

The equation $Ly = 0$ has a solution of the form $y = x^\alpha$. Find α, then use reduction of order to find another solution.

Solution: If $y = x^\alpha$ then

$$\begin{aligned}x^2y'' &= \alpha(\alpha-1)x^\alpha\\ -x^2y' &= -\alpha x^{\alpha+1}\\ -20xy' &= -20\alpha x^\alpha\\ 110y &= 110x^\alpha\\ 10xy &= 10x^{\alpha+1},\end{aligned}$$

so

$$\begin{aligned}Ly &= (\alpha^2-\alpha-20\alpha+110)x^\alpha + (-\alpha+10)x^{\alpha+1}\\ &= (\alpha^2-21\alpha+110)x^\alpha - (\alpha-10)x^{\alpha+1}\\ &= (\alpha-10)(\alpha-11)x^\alpha - (\alpha-10)x^{\alpha+1}.\end{aligned}$$

We can make this zero by taking $\alpha = 10$, so $y = x^{10}$. Next, our equation can be written as $y''+Py'+Qy = 0$, where $P = -1 - 20x^{-1}$ and $Q = 110x^{-2} + 10x^{-1}$. We put

$$\begin{aligned}v &= \int P\,dx = -x - 20\ln(x)\\ y^{-2}e^{-v} &= x^{-20}exp(x + 20\ln(x)) = x^{-20}e^x x^{20} = e^x\\ u &= \int y^{-2}e^{-v}\,dx = \int e^x\,dx = e^x\\ z &= uy = x^{10}e^x.\end{aligned}$$

Exercise 69. Find the normal form of the equation $x^2y'' + (1 + 2\alpha)xy' + \alpha^2 y = 0$. You should see that the normal form is the same as an example that was discussed in the lectures. Use this to find the general solution for the original equations.

Solution: We first divide by x^2 to get $y'' + (1+2\alpha)x^{-1}y' + \alpha^2 x^{-2}y = 0$. This is $y'' + Py' + Qy = 0$, where $P = (1+2\alpha)x^{-1}$ and $Q = \alpha^2 x^{-2}$. To normalise it, we put

$$\begin{aligned}
m &= \exp(-\tfrac{1}{2}\int P) = \exp(-\tfrac{1}{2}(1+2\alpha)\ln(x)) = x^{-\alpha-\frac{1}{2}} \\
R &= Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2 = \alpha^2 x^{-2} - \tfrac{1}{2}(1+2\alpha)(-x^{-2}) - \tfrac{1}{4}(1+2\alpha)^2 x^{-2}) \\
&= \alpha^2 x^{-2} + \tfrac{1}{2}x^{-2} + \alpha x^{-2} - \tfrac{1}{4}x^{-2} - \alpha x^{-2} - \alpha^2 x^{-2} \\
&= \tfrac{1}{4}x^{-2}.
\end{aligned}$$

We conclude that the solutions are of the form $y = x^{-\alpha-\frac{1}{2}}z$, where z satisfies $z'' + \frac{1}{4}x^{-2}z = 0$. This is the first example that we discussed of an equation with a regular singular point at the origin. The solution was given there but we will do it again for completeness. We first look for solutions of the form $z = x^\mu$. This gives

$$z'' + \tfrac{1}{4}x^{-2}z = \mu(\mu-1)x^{\mu-2} + \tfrac{1}{4}x^{\mu-2} = (\mu^2 - \mu + \tfrac{1}{4})x^{\mu-2} = (\mu - \tfrac{1}{2})^2 x^{\mu-2}.$$

We can thus take $\mu = \frac{1}{2}$, and we see that $z = x^{1/2}$ is one solution for the normalised equation. We can find another solution by the reduction of order method, which means we look for a solution of the form $z = ux^{1/2}$. We then have

$$\begin{aligned}
z'' + \tfrac{1}{4x^2}z &= u''x^{1/2} + 2u' \times (\tfrac{1}{2}x^{-1/2}) + u \times (-\tfrac{1}{4}x^{-3/2}) + \tfrac{1}{4}x^{-3/2}u \\
&= (u'' + x^{-1}u')x^{1/2}.
\end{aligned}$$

We therefore need $u'' + x^{-1}u' = 0$. One solution is $u' = x^{-1}$ and $u = \ln(x)$. Thus, we have a second solution $z = \ln(x)x^{1/2}$ for the equation $z'' + \frac{1}{4x^2}z = 0$, giving the general solution $z = (A\ln(x) + B)x^{1/2}$. We saw above that the function

$$y = mz = x^{-\alpha-\frac{1}{2}}(A\ln(x) + B)x^{1/2} = (A\ln(x) + B)x^{-\alpha}$$

is the general solution for the original equation $x^2y'' + (1+2\alpha)xy' + \alpha^2 y = 0$.

Exercise 70. Recall the Bessel equation $x^2y'' + xy' + (x^2 - n^2)y = 0$. The normal form of this equation was given in lectures. Use it to solve the Bessel equation when $n = 1/2$.

Solution: We saw in lectures that the normal form is $z'' + \left(1 + \frac{1-4n^2}{4x^2}\right)z = 0$, and that $y = x^{-1/2}z$. When $n = 1/2$ the differential equation is just $z'' + z = 0$, so $z = A\cos(x) + B\sin(x)$ for some constants A and B. It follows that $y = (A\cos(x) + B\sin(x))x^{-1/2}$.

Exercise 71. Find the normal form of the equation $x^2y'' - 2nxy' + (n^2 + n + x^2)y = 0$. Hence find the general solution.

Solution: The equation is equivalent to $y'' - 2nx^{-1}y' + ((n^2+n)x^{-2} + 1)y = 0$, so $P = -2nx^{-1}$ and $Q = 1 + (n^2+n)x^{-2}$. This gives $-\frac{1}{2}\int P\,dx = n\ln(x)$, so $m = \exp(-\frac{1}{2}\int P\,dx) = x^n$. Next, we have

$$\begin{aligned}
P' &= 2nx^{-2} \\
R &= Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2 = 1 + (n^2+n)x^{-2} - nx^{-2} - n^2x^{-2} = 1.
\end{aligned}$$

This means that the normal form is $z'' + z = 0$, with solutions $z = A\cos(x) + B\sin(x)$. It follows that $y = mz = x^n(A\cos(x) + B\sin(x))$.

Exercise 72. Consider the operator $Lu = u''/(1+2x)^2 - 2u'/(1+2x)^3$, and the eigenvalue problem $Lu = \lambda u$ with the Dirichlet boundary condition that $u = 0$ when $x = 0$ or $x = 1$.

(a) Convert L to Sturm-Liouville form.
(b) Suppose that $Lu = \lambda u$ and $Lv = \mu v$ with $\lambda \neq \mu$ (and that u and v satisfy the boundary conditions). Write down the orthogonality relation for u and v.
(c) Put $t = x + x^2$ and $\dot{u} = du/dt$. What is the relationship between u' and $\dot{u}$? What are the boundary conditions in terms of t?
(d) Rewrite L in terms of t and thus find the values of λ such that $Lu = \lambda u$ has a nontrivial solution.

Solution:

(a) We have $A = (1+2x)^{-2}$ and $B = -2(1+2x)^{-3}$ and $C = 0$ so $B/A = -2/(1+2x)$. It follows that $\int B/A\,dx = -\ln(1+2x)$, so $p = \exp\left(\int B/A\,dx\right) = (1+2x)^{-1}$. We then have $q = pC/A = 0$ and $r = p/A = 1+2x$. This gives

$$Lu = (u'/(1+2x))'/(1+2x).$$

(b) The standard orthogonality relation is $\int_a^b ruv = 0$. Our boundary conditions are at $a = 0$ and $b = 1$, and we have $r = 1+2x$, so the orthogonality relation is

$$\int_{x=0}^{1} (1+2x)u(x)v(x)\,dx = 0.$$

(c) We have $dt/dx = 1+2x$ and so

$$\dot{u} = \frac{du}{dt} = \frac{du}{dx}\Big/\frac{dx}{dt} u'/(1+2x).$$

The boundary conditions are at $x = 0$ and $x = 1$, which corresponds to $t = 0+0^2 = 0$ and $t = 1+1^2 = 2$.

(d) Note that (c) can be used twice to get

$$\ddot{u} = (\dot{u})'/(1+2x) = (u'/(1+2x))'/(1+2x) = Lu.$$

Thus, our problem is to find λ and u with $u \neq 0$, but with $\ddot{u} = \lambda u$, and $u = 0$ when $t = 0$ or $t = 2$. For this we can take

$$u = \sin(\tfrac{1}{2}n\pi t) = \sin(\tfrac{1}{2}n\pi(x+x^2))$$

and $\lambda = -n^2\pi^2/4$. The method used in Exercise 55 shows that there are no other possibilities.

Exercise 73. Consider the operator

$$Lu = \cos(x)^4 u'' - 2\sin(x)\cos(x)(1+\cos(x)^2)u' + 8u.$$

(a) Rewrite L in Sturm-Liouville form.
(b) Put $t = \tan(x)$ and $\dot{u} = du/dt$. What is the relationship between u' and $\dot{u}$?
(d) Rewrite L in terms of t, and show that it becomes one of the standard equations discussed in the notes. Find a solution that is a polynomial function of t.

Solution:

(a) We have $A = \cos(x)^4$ and $B = -2\sin(x)\cos(x)(1+\cos(x)^2)$ and $C = 8$. This gives

$$\begin{aligned}
B/A &= -2\frac{\sin(x)}{\cos(x)^3} - 2\frac{\sin(x)}{\cos(x)} \\
\int \frac{B}{A}\,dx &= -\cos(x)^{-2} + 2\ln(\cos(x)) \\
p &= \exp\left(\int B/A\,dx\right) = \exp(-\cos(x)^{-2})\cos(x)^2 \\
q &= pC/A = 8\exp(-\cos(x)^{-2})\cos(x)^{-2} \\
r &= p/A = \exp(-\cos(x)^{-2})\cos(x)^{-2},
\end{aligned}$$

so

$$Lu = \left((\exp(-\cos(x)^{-2})\cos(x)^2u')' + 8\exp(-\cos(x)^{-2})\cos(x)^{-2}u\right)\exp(\cos(x)^{-2})\cos(x)^2.$$

(b) We have We have $dt/dx = \tan'(x) = \cos(x)^{-2}$, so

$$\dot{u} = \frac{du}{dt} = \frac{du}{dx}\Big/\frac{dt}{dx} = \cos(x)^2u'.$$

(c) We now have

$$\begin{aligned}
\ddot{u} &= \cos(x)^2(\dot{u})' = \cos(x)^2(\cos(x)^2u')' \\
&= \cos(x)^4u'' - 2\cos(x)^3\sin(x)u' \\
Lu - \ddot{u} &= -2\sin(x)\cos(x)u' + 8u = -2\tan(x)\cos(x)^2u' + 8u \\
&= -2\tan(x)\dot{u} + 8u = -2t\dot{u} + 8u \\
Lu &= \ddot{u} - 2t\dot{u} + 8u.
\end{aligned}$$

This means that the equation $Lu = 0$ is equivalent to the Hermite equation (for u as a function of t) with $n = 4$. We saw in Exercise 59 that there is a solution which is a polynomial of degree 4, involving only even powers of t. This means that $u = a_0 + a_2t^2 + a_4t^4$ say. This gives

$$\begin{aligned}
\ddot{u} &= 2a_2 + 12a_4t^2 \\
-2t\dot{u} &= -4a_2t^2 - 8a_4t^4 \\
8u &= 8a_0 + 8a_2t^2 + 8a_4t^4 \\
\ddot{u} - 2t\dot{u} + 8u &= (8a_0 + 2a_2) + (4a_2 + 12a_4)t^2
\end{aligned}$$

Thus, to have $Lu = 0$ we need $a_2 = -4a_0$ and $a_4 = -a_2/3$. We can choose $a_0 = 3$ and then we get $a_2 = -12$ and $a_4 = 4$, giving

$$u = 3 - 12t^2 + 4t^4 = 3 - 12\tan(x)^2 + 4\tan(x)^4.$$

Exercise 74. Consider the operator $Lu = x^{-2}u'' - x^{-3}u'$, and the eigenvalue problem $Lu = \lambda u$ with the Dirichlet boundary condition that $u = 0$ when $x = 0$ or $x = 1$.

(a) Convert L to Sturm-Liouville form.
(b) Suppose that $Lu = \lambda u$ and $Lv = \mu v$ with $\lambda \neq \mu$ (and that u and v satisfy the boundary conditions). Write down the orthogonality relation for u and v.
(c) Use power series to find a function $y = f(x)$ with $Ly = -4\pi^2 y$, where $y = \pi x^2 + O(x^4)$. Find a simple formula for x in terms of standard functions, and thus check that y satisfies the boundary conditions.
(d) For the function $f(x)$ in (c), show that the function $z_n = f(\sqrt{n}x)$ also satisfies the boundary conditions and is an eigenfunction of L, for any integer $n > 0$.
(e) Check the orthogonality relation for z_n and z_m (where $n \neq m$) by direct calculation.

Solution:

(a) We have $A = x^{-2}$ and $B = -x^{-3}$ and $C = 0$ so $B/A = -1/x$. It follows that $\int B/A\,dx = -\ln(x)$, so $p = \exp\left(\int B/A\,dx\right) = x^{-1}$. We then have $q = pC/A = 0$ and $r = p/A = x^{-1}/x^{-2} = x$. This gives

$$Lu = (u'/x)'/x.$$

(b) The standard orthogonality relation is $\int_a^b ruv = 0$. Our boundary conditions are at $a = 0$ and $b = 1$, and we have $r = x$, so the orthogonality relation is

$$\int_{x=0}^{1} x\,u(x)v(x)\,dx = 0.$$

(c) We want $y = \sum_k a_kx^k$ with $a_0 = a_1 = a_3 = 0$ and $a_2 = \pi$ and $Ly = -4\pi^2 y$. This gives

$$\begin{aligned}
y'/x &= \sum_k ka_kx^{k-2} \\
(y'/x)'/x &= \sum_k (k-2)ka_kx^{k-4} = \sum_j (j+2)(j+4)a_{j+4}x^j \\
-4\pi^2 y &= \sum_j -4\pi^2 a_jx^j.
\end{aligned}$$

Thus, we need $a_{j+4} = -4\pi^2 a_j/((j+2)(j+4))$. In particular, if $a_j = 0$ then $a_{j+4} = 0$. As $a_0 = a_1 = a_3 = 0$ we see that $a_j = 0$ unless j has the form $j = 4i+2$ for some $i \geq 0$. If we put $b_i = a_{4i+2}$ then we have $b_0 = \pi$ and $y = \sum_i b_ix^{4i+2}$, and the relation $a_{j+4} = -4\pi^2 a_j/((j+2)(j+4))$ becomes

$$b_{i+1} = \frac{-4\pi^2 b_i}{(4i+4)(4i+6)} = \frac{-\pi^2 b_i}{(2i+2)(2i+3)}.$$

Using this we can show by induction that $b_i = (-1)^i\pi^{2i+1}/(2i+1)!$. This gives

$$f(x) = y = \sum_i b_ix^{4i+2} = \sum_i (-1)^i \frac{(\pi x^2)^{2i+1}}{(2i+1)!} = \sin(\pi x^2).$$

In particular, we have $f(0) = \sin(0) = 0$ and $f(1) = \sin(\pi) = 0$, so the boundary conditions are satisfied.

(d) Now put $z_n = f(\sqrt{n}x) = \sin(n\pi x^2)$. This is again zero when $x = 0$ or $x = 1$. We also have

$$\begin{aligned} z_n'/x &= 2n\pi x\cos(n\pi x^2)/x = 2n\pi\cos(n\pi x^2) \\ (z_n'/x)'/x &= 2n\pi \times (-2n\pi x\sin(n\pi x^2))/x \\ &= -4n^2\pi^2\sin(n\pi x^2) = -4n^2\pi^2 z_n. \end{aligned}$$

Thus, z_n is an eigenfunction, with eigenvalue $-4n^2\pi^2$.

(e) The orthogonality relation for z_n and z_m says that the integral

$$I = \int_{x=0}^{1} x\,\sin(n\pi x^2)\,\sin(m\pi x^2)\,dx$$

is zero. To check this directly, just substitute $t = x^2$, so $x\,dx = \frac{1}{2}dt$, and the endpoints $x = 0, 1$ are just the same as $t = 0, 1$. This gives

$$\begin{aligned} I &= \frac{1}{2}\int_{t=0}^{1}\sin(n\pi t)\sin(m\pi t)\,dt = \frac{1}{4}\int_{t=0}^{1}\cos((n-m)\pi t) - \cos((n+m)\pi t)\,dt \\ &= \left[\frac{\sin((n-m)\pi t)}{4(n-m)} - \frac{\sin((n-m)\pi t)}{4(n+m)}\right]_0^1 = 0. \end{aligned}$$

Exercise 75. Consider the equations

(A) $$y'' + x^2 y = 0$$

(B) $$x^2 z'' + xz' + (x^4 - \tfrac{1}{4})z = 0$$

(C) $$t^2\frac{d^2z}{dt^2} + t\frac{dz}{dt} + (t^2 - \tfrac{1}{16})z = 0$$

Note that (C) is just a Bessel equation.

(a) Show that if we put $y = x^{1/2}z$ then A is equivalent to B.
(b) Show that if we put $t = x^2/2$ then B is equivalent to C.
(c) Hence give the general solution to A in terms of Bessel functions.

Solution:

(a) If $y = x^{1/2}z$ then

$$\begin{aligned} y' &= \tfrac{1}{2}x^{-1/2}z + x^{1/2}z' \\ y'' &= -\tfrac{1}{4}x^{-3/2}z + x^{-1/2}z' + x^{1/2}z'' \\ y'' + x^2 y &= -\tfrac{1}{4}x^{-3/2}z + x^{-1/2}z' + x^{1/2}z'' + x^{5/2}z \\ &= x^{-3/2}\left(x^2z'' + xz' + (x^4 - \tfrac{1}{4})z\right). \end{aligned}$$

This makes it clear that A is equivalent to B.

(b) Now put $t = x^2/2$, so $x = \sqrt{2t}$ and $\frac{dt}{dx} = x$. This gives

$$\begin{aligned} \frac{dz}{dt} &= \left(\frac{dt}{dx}\right)^{-1}\frac{dz}{dx} = x^{-1}z' \\ \frac{d^2z}{dt^2} &= \left(\frac{dt}{dx}\right)^{-1}\frac{d}{dx}\left(\frac{dz}{dt}\right) = x^{-1}(x^{-1}z')' \\ &= x^{-1}(-x^{-2}z' + x^{-1}z'') = x^{-2}z'' - x^{-3}z' \end{aligned}$$

so

$$\begin{aligned} &t^2\frac{d^2z}{dt^2} + t\frac{dz}{dt} + (t^2 - \tfrac{1}{16})z \\ =&\frac{x^4}{4}(x^{-2}z'' - x^{-3}z') + \frac{x^2}{2}x^{-1}z' + \frac{x^4}{4}z - \frac{1}{16}z \\ =&\frac{1}{4}\left(x^2z'' - xz' + 2xz' + x^4z - \tfrac{1}{4}z\right) = \frac{1}{4}\left(x^2z'' + xz' + (x^4 - \tfrac{1}{4})z\right). \end{aligned}$$

This makes it clear that C is equivalent to B.

(c) Equation C is the Bessel equation with $n = 1/4$, so the solutions are $z = AJ_{1/4}(t)+BY_{1/4}(t)$, with A and B constant. Here $t = x^2/2$, so this can be rewritten as $z = AJ_{1/4}(x^2/2) + BJ_{1/4}(x^2/2)$. It follows in turn that

$$y = x^{1/2}z = Ax^{1/2}J_{1/4}(x^2/2) + Bx^{1/2}J_{1/4}(x^2/2).$$

Exercise 76. Consider the equations

$$\text{(A)} \qquad y'' + x^4 y = 0$$

$$\text{(B)} \qquad x^2 z'' + xz' + (x^6 - \tfrac{1}{4})z = 0$$

$$\text{(C)} \qquad t^2 \frac{d^2 z}{dt^2} + t\frac{dz}{dt} + (t^2 - \tfrac{1}{36})z = 0$$

Note that (C) is just a Bessel equation.

(a) Show that if we put $y = x^{1/2}z$ then A is equivalent to B.
(b) Show that if we put $t = x^3/3$ then B is equivalent to C.
(c) Hence give the general solution to A in terms of Bessel functions.

Solution: This is essentially the same as the previous exercise except that the numbers are different.

(a) If $y = x^{1/2}z$ then

$$\begin{aligned} y' &= \tfrac{1}{2}x^{-1/2}z + x^{1/2}z' \\ y'' &= -\tfrac{1}{4}x^{-3/2}z + x^{-1/2}z' + x^{1/2}z'' \\ y'' + x^4 y &= -\tfrac{1}{4}x^{-3/2}z + x^{-1/2}z' + x^{1/2}z'' + x^{9/2}z \\ &= x^{-3/2}\left(x^2 z'' + xz' + (x^6 - \tfrac{1}{4})z\right). \end{aligned}$$

This makes it clear that A is equivalent to B.

(b) Now put $t = x^3/3$, so $\frac{dt}{dx} = x^2$. This gives

$$\begin{aligned} \frac{dz}{dt} &= \left(\frac{dt}{dx}\right)^{-1}\frac{dz}{dx} = x^{-2}z' \\ \frac{d^2 z}{dt^2} &= \left(\frac{dt}{dx}\right)^{-1}\frac{d}{dx}\left(\frac{dz}{dt}\right) = x^{-2}(x^{-2}z')' \\ &= x^{-2}(-2x^{-3}z' + x^{-2}z'') = x^{-4}z'' - 2x^{-5}z' \end{aligned}$$

so

$$\begin{aligned} & t^2\frac{d^2 z}{dt^2} + t\frac{dz}{dt} + (t^2 - \tfrac{1}{36})z \\ =& \frac{x^6}{9}(x^{-4}z'' - 2x^{-5}z') + \frac{x^3}{3}x^{-2}z' + \frac{x^6}{9}z - \frac{1}{36}z \\ =& \frac{1}{9}\left(x^2 z'' - 2xz' + 3xz' + x^6 z - \tfrac{1}{4}z\right) = \frac{1}{9}\left(x^2 z'' + xz' + (x^6 - \tfrac{1}{4})z\right). \end{aligned}$$

This makes it clear that C is equivalent to B.

(c) Equation C is the Bessel equation with $n = 1/6$, so the solutions are $z = AJ_{1/6}(t)+BY_{1/6}(t)$, with A and B constant. Here $t = x^3/3$, so this can be rewritten as $z = AJ_{1/6}(x^3/3) + BJ_{1/6}(x^3/3)$. It follows in turn that

$$y = x^{1/2}z = Ax^{1/2}J_{1/6}(x^3/3) + Bx^{1/2}J_{1/6}(x^3/3).$$

Exercise 77. Consider the Cauchy-Euler equation

$$r^2\frac{d^2 R}{dr^2} + r\frac{dR}{dr} - 4R = 0.$$

Rewrite this in terms of the variable $x = \ln(r)$. Hence find the solution that satisfies $R = 0$ when $r = 2$ and $R = 1$ when $r = 4$.

Solution: Note that $r = e^x$, so $r' = dr/dx = e^x = r$ and $dx/dr = r^{-1}$.

$$\begin{aligned}
\frac{dR}{dr} &= \frac{dx}{dr}\frac{dR}{dx} = r^{-1}R' \\
\frac{d^2R}{dr^2} &= \frac{d}{dr}(r^{-1}R') = \frac{dx}{dr}(r^{-1}R')' = r^{-1}(-r^{-2}r'R' + r^{-1}R'') \\
&= r^{-1}(-r^{-1}R' + r^{-1}R'') = r^{-2}(R'' - R').
\end{aligned}$$

This gives

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - 4R = (R'' - R') + R' - 4R = R'' - 4R,$$

so the original equation is equivalent to $R'' = 4R$, which has solutions $R = Ae^{2x} + Be^{-2x}$. Here $e^x = r$, so the solution is $R = Ar^2 + Br^{-2}$. When $r = 2$ we have $R = 4A + \frac{1}{4}B$, but we want $R = 0$, so we must have $B = -16A$ and $R = A(r^2 - 16r^{-2})$. Now when $r = 4$ we have $R = A(16 - 16/16) = 15A$, but we want $R = 1$, so we must have $A = 1/15$. Thus, the solution is $R = (r^2 - 16/r^2)/15$.

Exercise 78. Suppose that y satisfies $y'' + xy = 0$ (which is similar to the Airy equation $y'' - xy = 0$, except that the sign is different). Put $t = \frac{2}{3}x^{3/2}$ and $\dot{u} = du/dt$. Show that the function $z = x^{-1/2}y$ satisfies

$$t^2\ddot{z} + t\dot{z} + (t^2 - \tfrac{1}{9})z = 0$$

(which is Bessel's equation with $n = 1/3$).

Solution: First note that for any function u we have

$$u' = \frac{du}{dx} = \frac{dt}{dx}\frac{du}{dt} = x^{1/2}\dot{u}.$$

Note also that $x^{3/2} = \frac{3}{2}t$, so $x^{-3/2} = \frac{2}{3}t^{-1}$. We thus have

$$\begin{aligned}
y &= x^{1/2}z \\
y' &= x^{1/2}z' + \tfrac{1}{2}x^{-1/2}z \\
&= x^{1/2}(x^{1/2}\dot{z}) + \tfrac{1}{2}x^{-1/2}z = x(\dot{z} + \tfrac{1}{2}x^{-3/2}z) \\
&= x(\dot{z} + \tfrac{1}{3}t^{-1}z) \\
y'' &= (\dot{z} + \tfrac{1}{3}t^{-1}z) + x(\dot{z} + \tfrac{1}{3}t^{-1}z)' \\
&= \dot{z} + \tfrac{1}{3}t^{-1}z + x^{3/2}(\dot{z} + \tfrac{1}{3}t^{-1}z)^{\cdot} \\
&= \dot{z} + \tfrac{1}{3}t^{-1}z + \tfrac{3}{2}t(\ddot{z} + \tfrac{1}{3}t^{-1}\dot{z} - \tfrac{1}{3}t^{-2}z) \\
&= \tfrac{3}{2}t\ddot{z} + \tfrac{3}{2}\dot{z} - \tfrac{1}{6}t^{-1}z = \tfrac{3}{2}t^{-1}(t^2\ddot{z} + t\dot{z} - \tfrac{1}{9}z) \\
y'' + xy &= \tfrac{3}{2}t^{-1}(t^2\ddot{z} + t\dot{z} - \tfrac{1}{9}z) + x^{3/2}z = \tfrac{3}{2}t^{-1}(t^2\ddot{z} + t\dot{z} - \tfrac{1}{9}z) + \tfrac{3}{2}tz \\
&= \tfrac{3}{2}t^{-1}(t^2\ddot{z} + t\dot{z} + (t^2 - \tfrac{1}{9})z).
\end{aligned}$$

The claim is clear from this.