Pure Mathematics Core

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Contents

Chapter 1: Introduction

This is a core course, designed to review and reinforce some fundamental ideas that are used in all areas of mathematics. The main topics are as follows:

- Manipulation of algebraic expressions and inequalities.
- Sets of numbers, vectors and other mathematical objects; geometric figures as sets of points; sets of solutions to equations or systems of equations.
- General theory of functions between sets. Composing and inverting functions; finding ranges of functions.
- Particular special functions, such as sin, cos, exp and log.
- Special classes of functions, such as polynomials, rational functions, periodic functions, bell curves, decaying oscillations, and so on.
- Differentiation: the meaning of derivatives, and general techniques for calculating them. Derivatives of particular special functions, and of special classes of functions.
- Integration: the meaning of integrals, and general techniques for calculating them. Integrals of particular special functions, and of special classes of functions.
- Vectors and matrices, emphasising the link with systems of linear equations.
- Complex numbers are discussed briefly in an appendix, and are used occasionally in the main body of the notes.

Many of you will have met many of these topics at school already. However, in this course, we will take a slightly different point of view. We will stand back a little from the detailed calculations, look for patterns and common features, and try to understand a little more deeply why things work the way they do.

This course will also provide some pointers to other areas of mathematics that you may end up studying over the next few years. All the examples in this course are carefully chosen, and most have some kind of story behind them. Many are related to particular applications of mathematics, or they involve functions with special properties or geometric meaning, or they will arise naturally from a topic in some other module. The notes will contain brief accounts of some of these stories; it is up to you to decide how far to follow them.

There are many exercises, some of which you will be asked to prepare for discussion in tutorials. Solutions are given at the end of the notes for about half of the exercises. Solutions for the remaining exercises will be released towards the end of the course, to help you with revision. More challenging exercises are marked with a star.

Finally, the flavour of this course is influenced by the availability of computer algebra systems (although we will not be using them systematically at this stage). These systems (such as *Maple* and *Mathematica*) can carry out many kinds of manipulations automatically, such as expanding out complicated expressions, plotting graphs, differentiating and integrating, and so on. This makes it more important for humans to understand the larger picture and learn how to formulate problems and extract information.

Chapter 2: Algebraic manipulation

In this section, we will recall and practice the rules for manipulating algebraic expressions. All the methods should be familiar to you. The examples may be larger than those you have handled before, but they are all chosen to work out in neat and interesting ways. To keep control of these larger calculations, you should try to be systematic. In the worked examples, you will see various places where we have laid out our formulae in particular ways to make it easier to see cancellations and so on.

Skills:

You should make sure that you can do the following:

- Expand out products and powers of algebraic expressions.
- Factor algebraic expressions in some simple cases.
- Manipulate powers (using the rules $a^n a^m = a^{n+m}$ and $(a^n)^m = a^{nm}$).
- Manipulate and simplify algebraic fractions.
- Convert rational functions to partial fraction form.

Some mistakes to avoid:

- $(a + b)^n$ is not the same as $a^n + b^n$. For example, $(a + b)^2 = a^2 + b^2 + 2ab \neq a^2 + b^2$. Usually, people only make this mistake when a and b are already complicated expressions: for example, they convert $(\sin(\alpha+\beta+\gamma)+\sin(\alpha-\beta+\gamma))^2$ to $\sin(\alpha+\beta+\gamma)^2+\sin(\alpha-\beta+\gamma)^2$, which is of course not correct.
- You cannot simplify $(a+b)/(c+d)$ to $a/c+b/d$, or $1/(a+b)$ to $1/a+1/b$. Again, these mistakes are most commonly made when a, b, c and d are themselves complicated expressions.
- You should take care with brackets. Often people start with an expression like $(a+b)-(c-d)$, and somewhere along the line they lose the brackets to get $a + b - c - d$, whereas of course they should have $a + b - c + d$.
- For essentially the same reason, I would always recommend writing $\sin(2x)$ (with brackets) rather that $\sin 2x$. If you get in the habit of leaving out brackets, you can inadvertently change something like $\sin(x + \pi)$ to $\sin(x) + \pi$, which is completely different.
- It is easy to use the wrong rules for powers by mistake. The main points are that $(a^n)^m = a^{nm}$ and $a^n a^m = a^{n+m}$, and the most common mistake is to think that $(a^n)^m = a^{n+m}$. For example, $((1 + x)^3)^4$ is $(1 + x)^{12}$, not $(1 + x)^7$.
- Remember that $f(x)^n$ always means $(f(x))^n$, not $f(x^n)$. For example, $\sin(x)^2$ means $\sin(x)\times\sin(x)$, not $\sin(x^2)$.

2.1 Expansion

Exercise 2.1.1. Check the following basic identities:

$$\begin{aligned}(a + b)(a - b) &= a^2 - b^2\\ (a + b)^2 &= a^2 + 2ab + b^2\\ (a - b)^2 &= a^2 - 2ab + b^2.\end{aligned}$$

Solution:

$$(a+b)(a-b) = a^2 - ab + ba - b^2 = a^2 - b^2$$
$$(a+b)^2 = (a+b)(a+b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = (a-b)(a-b) = a^2 - ab - ba + b^2 = a^2 - 2ab + b^2.$$

Exercise 2.1.2. Expand out the expression

$$d + x(c + x(b + x(a + x))).$$

(This is essentially *Horner's rule* .)

Solution: We can either work from the inside outwards, or from the outside inwards. The first approach gives

$$\begin{aligned} d + x(c + x(b + x(a + x))) &= d + x(c + x(b + ax + x^2)) \\ &= d + x(c + bx + ax^2 + x^3) \\ &= d + cx + bx^2 + ax^3 + x^4, \end{aligned}$$

and the second gives

$$\begin{aligned} d + x(c + x(b + x(a + x))) &= d + cx + x^2(b + x(a + x)) \\ &= d + cx + bx^2 + x^3(a + x) \\ &= d + cx + bx^2 + ax^3 + x^4. \end{aligned}$$

Background: This exercise is relevant when one wants to evaluate a polynomial by computer very quickly, for a very large number of different values of x, particularly when one needs the result to be accurate to a large number of decimal places. Addition is very much faster than multiplication, so it can be ignored. The original expression involves only three multiplications, as you can see if we put them in explicitly:

$$d + x\times(c + x\times(b + x\times(a + x))).$$

Written in the more obvious form, the expression involves nine multiplications, and is thus much slower:

$$d + c\times x + b\times x\times x + a\times x\times x\times x + x\times x\times x\times x$$

Exercise 2.1.3. (a) Expand out $(q-1)(q^3 + q^2 + q + 1)$.

(b) Expand out $(q-1)(q^5 + q^4 + q^3 + q^2 + q + 1)$.

(c) What is the general rule? Try to make a complete, self-contained statement that could be used and understood by someone who had not read the question.

Solution:

(a)
$$\begin{aligned} (q-1)(q^3 + q^2 + q + 1) = \; & q^4 \quad + q^3 + q^2 + q \\ & \qquad - q^3 - q^2 - q \quad -1 \\ = \; & q^4 - 1. \end{aligned}$$

(b)
$$\begin{aligned} (q-1)(q^5 + q^4 + q^3 + q^2 + q + 1) = \; & q^6 \quad + q^5 + q^4 + q^3 + q^2 + q \\ & \qquad - q^5 - q^4 - q^3 - q^2 - q \quad -1 \\ = \; & q^6 - 1. \end{aligned}$$

(c) The general rule is that for any q and any integer $n \geq 0$, we have

$$(q-1)(1+q+\cdots+q^n) = (q-1)\sum_{k=0}^{n} q^k = q^{n+1}-1;$$

this is called the *geometric progression formula*.

Exercise 2.1.4. Expand out the following expressions:

(a) $(a+b)^3$

(b) $(a+b+c)^2$

(c) $(x-a)(x-b)(x-c)$

(d) $(a-b)(a^3+a^2b+ab^2+b^3)$

Solution:

(a) By direct expansion, we have $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$. This can also be done by the binomial formula, which is covered in detail in PMA111 (Numbers and Polynomials).

(b) $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$.

(c) $(x-a)(x-b)(x-c) = x^3-(a+b+c)x^2+(ab+bc+ca)x-abc$.

(d) a^4-b^4.

Exercise 2.1.5*. Let a, b, c, u, v, w be numbers such that no two of a, b and c are the same. Put

$$f(x) = u\frac{(x-b)(x-c)}{(a-b)(a-c)} + v\frac{(x-a)(x-c)}{(b-a)(b-c)} + w\frac{(x-a)(x-b)}{(c-a)(c-b)}.$$

(a) Simplify $f(a)$, $f(b)$ and $f(c)$.

(b) Convince yourself that $f(x) = px^2+qx+r$ for some (rather complicated) expressions p, q and r, which do not depend on x. You need not work out these expressions explicitly.

Solution:

(a) When we put $x = a$, we have $x-a = 0$ so the second and third terms in $f(x)$ become zero. The first term becomes $u(a-b)(a-c)/((a-b)(a-c))$, which is just u. Thus, we have $f(a) = u$. By the same method, we have $f(b) = v$ and $f(c) = w$.

(b) First note that $(x-b)(x-c)$ is a quadratic polynomial in x. As $u/((a-b)(a-c))$ is independent of x, we see that $u((x-b)(x-c))/((a-b)(a-c))$ is also a quadratic polynomial in x. Similarly, the other two terms in $f(x)$ are quadratic polynomials, so when we add them all together, we still have a quadratic polynomial in x. The gory details:

$$\begin{aligned} p &= (uc-ub+va-vc+wb-wa)/\Delta \\ q &= (ub^2-uc^2+vc^2-va^2+wa^2-wb^2)/\Delta \\ r &= (ubc(c-b)+vca(a-c)+wab(b-a))/\Delta, \end{aligned}$$

where $\Delta = (a-b)(b-c)(c-a)$.

Background: The basic point is as follows: if you give me a, b, c, u, v and w, then I can always find a quadratic polynomial $f(x)$ such that $f(a) = u$, and $f(b) = v$, and $f(c) = w$. There are a number of different ways to find f; the expression in the exercise (known as the *Lagrange interpolation formula*) is the most explicit way (but not very efficient computationally). More generally, given distinct numbers $a_1, \ldots, a_n$ and arbitrary numbers $u_1, \ldots, u_n$, there is a unique polynomial $f(x)$ of degree at most n such that $f(a_i) = u_i$ for $i = 1, \ldots, n$.

Exercise 2.1.6. Let a, b and c be numbers with $a \neq 0$. Expand out the expression

$$a\left(\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-4ac}{4a^2}\right)\right).$$

Can you use your answer to derive the standard formula for the roots of a quadratic?

Solution:

$$\begin{aligned} a\left(\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-4ac}{4a^2}\right)\right) &= a\left(x^2+\frac{bx}{a}+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{4ac}{4a^2}\right) \\ &= ax^2+bx+c. \end{aligned}$$

This means that

$$\begin{aligned} & ax^2+bx+c=0 \\ \iff & a\left(\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-4ac}{4a^2}\right)\right)=0 \\ \iff & \left(x+\frac{b}{2a}\right)^2=\left(\frac{b^2-4ac}{4a^2}\right) \\ \iff & x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a} \\ \iff & x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}. \end{aligned}$$

This is of course the usual formula for the roots of a quadratic.

The "difference of squares" identity

$$a^2-b^2=(a+b)(a-b)$$

is often useful as an ingredient in other simplifications.

Example 2.1.7. Suppose we want to simplify

$$(w+x+y+z)^2-(x+y+z)^2.$$

If we just expand out in the obvious way, we get 25 terms, many of which need to be cancelled or collected together. Alternatively, we can put $a=w+x+y+z$ and $b=x+y+z$, giving

$$\begin{aligned} (w+x+y+z)^2-(x+y+z)^2 &= a^2-b^2=(a+b)(a-b) \\ &= (w+2x+2y+2z)w=w^2+2wx+2wy+2wz, \end{aligned}$$

which is much simpler. □

Example 2.1.8. Suppose we want to simplify

$$(w+x+y+z)(w-x+y-z).$$

If we take $a=w+y$ and $b=x+z$ then we have

$$\begin{aligned} (w+x+y+z)(w-x+y-z) &= (a+b)(a-b) \\ &= a^2-b^2 \\ &= w^2+2wy+y^2-x^2-2xz-z^2. \quad \square \end{aligned}$$

Exercise 2.1.9. Simplify the following expressions:

(a) $9(a+1)^2 - 9(a-1)^2$

(b) $(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})$

(c) $(a^2+ab+b^2)(a^2-ab+b^2)$

Solution:

(a) $36a$

(b) $x-y$

(c) $a^4+a^2b^2+b^4$.

Exercise 2.1.10. Let a, b and c be numbers with $a \neq 0$ and $b^2 \geq 4ac$. Put

$$\begin{aligned} d &= \sqrt{b^2-4ac} \\ u &= (-b-d)/(2a) \\ v &= (-b+d)/(2a). \end{aligned}$$

Simplify the following expressions:

(a) $u+v$

(b) uv

(c) $a(x-u)(x-v)$.

(In part (c), you should use parts (a) and (b), rather than just substituting the definitions of u and v.)

Solution: We have $u+v=-b/a$ and

$$\begin{aligned} uv &= (-b-d)(-b+d)/(4a^2) \\ &= (b^2-d^2)/(4a^2) \\ &= (b^2-b^2+4ac)/(4a^2) = c/a. \end{aligned}$$

It follows that

$$a(x-u)(x-v) = ax^2 - a(u+v)x + auv = ax^2+bx+c.$$

Using this we see that $ax^2+bx+c=0$ if and only if $x=u$ or $x=v$, or in other words $x=(-b\pm\sqrt{b^2-4ac})/(2a)$; so we have again proved the familiar formula for the roots of a quadratic.

Exercise 2.1.11. By expanding out both sides, check that

$$(x^2+y^2+z^2)(u^2+v^2+w^2) = (xu+yv+zw)^2+(xv-yu)^2+(yw-zv)^2+(zu-xw)^2.$$

Solution: The left hand side expands to

$$x^2u^2+x^2v^2+x^2w^2+y^2u^2+y^2v^2+y^2w^2+z^2u^2+z^2v^2+z^2w^2.$$

On the right hand side, we have

$$\begin{aligned} (xu+yv+zw)^2 &= x^2u^2+y^2v^2+z^2w^2+2xyuv+2xzuw+2yzvw \\ (xv-yu)^2 &= x^2v^2-2xyuv+y^2u^2 \\ (yw-zv)^2 &= y^2w^2-2yzvw+z^2v^2 \\ (zu-xw)^2 &= z^2u^2-2xzuw+x^2w^2 \end{aligned}$$

If we add these four terms together, we get the same as on the left hand side.

Here is another way to formulate this identity, which you can ignore if you are not familiar with vector algebra. Consider the vectors

$$\begin{aligned}\mathbf{p} &= (x, y, z)\\ \mathbf{q} &= (u, v, w).\end{aligned}$$

Note that

$$\begin{aligned}\|\mathbf{p}\|^2 &= \mathbf{p}.\mathbf{p} = x^2 + y^2 + z^2\\ \|\mathbf{q}\|^2 &= \mathbf{q}.\mathbf{q} = u^2 + v^2 + w^2\\ \mathbf{p}.\mathbf{q} &= xu + yv + zw\\ \mathbf{p} \times \mathbf{q} &= (xv - yu, yw - zv, zu - xw)\\ \|\mathbf{p} \times \mathbf{q}\|^2 &= (xv - yu)^2 + (yw - zv)^2 + (zu - xw)^2.\end{aligned}$$

Thus, the original equation can be rewritten as

$$\|\mathbf{p}\|^2\|\mathbf{q}\|^2 = (\mathbf{p}.\mathbf{q})^2 + \|\mathbf{p} \times \mathbf{q}\|^2.$$

In particular, because $\|\mathbf{p} \times \mathbf{q}\|^2$ is certainly greater than or equal to zero, we have

$$\|\mathbf{p}\|^2\|\mathbf{q}\|^2 \geq (\mathbf{p}.\mathbf{q})^2$$

and thus

$$|\mathbf{p}.\mathbf{q}| \leq \|\mathbf{p}\|\|\mathbf{q}\|.$$

This is called the *Cauchy-Schwartz inequality* .

Exercise 2.1.12. By expanding out both sides, check that

$$(ad - bc)(ps - rq) = (ap + br)(cq + ds) - (aq + bs)(cp + dr).$$

Solution: The right hand side is

$$\begin{aligned}(ap + br)(cq + ds) - (aq + bs)(cp + dr) &= (acpq + adps + bcqr + bdrs) -\\ &\quad\;(acpq + adqr + bcps + bdrs)\\ &= adps + bcqr - adqr - bcps,\end{aligned}$$

and this is easily seen to be the same as $(ad - bc)(ps - rq)$.

Note that in each term we have written the variables in alphabetical order (eg $acpq$ rather than $apcq$); this makes it easier to see which terms cancel. **Background:** This identity has a natural interpretation in terms of matrices. (We will study matrices in Chapter 8; if you do not already know about them, you can ignore these comments.)

If we put $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right]$ then

$$\begin{aligned}\det(A) &= ad - bc\\ \det(B) &= ps - rq\\ AB &= \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}\\ \det(AB) &= (ap + br)(cq + ds) - (aq + bs)(cp + dr).\end{aligned}$$

Thus, the equation just says that $\det(A)\det(B) = \det(AB)$. We have therefore proved the relation $\det(AB) = \det(A)\det(B)$ for 2×2 matrices. You may well be aware that this relation actually holds for matrices of any size (but the general proof is much less direct).

Exercise 2.1.13. By expanding out both sides, check that

$$(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2.$$

Solution:

$$\begin{aligned}(ac-bd)^2 + (ad+bc)^2 &= a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2 \\ &= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2 \\ &= (a^2+b^2)(c^2+d^2).\end{aligned}$$

Background: If you are familiar with complex numbers (see Appendix A), there is an easy way to understand the above equation. If we put $z = a+bi$ and $w = c+di$ then

$$\begin{aligned}|z|^2 &= a^2+b^2 \\ |w|^2 &= c^2+d^2 \\ zw &= (ac-bd) + (ad+bc)i \\ |zw|^2 &= (ac-bd)^2 + (ad+bc)^2.\end{aligned}$$

Thus, the equation reads $|z|^2|w|^2 = |zw|^2$.

Exercise 2.1.14. Simplify the following expressions:

$$\begin{aligned}&(x+1)^2 - 2(x+1) + 1 \\ &(x+1)^3 - 3(x+1)^2 + 3(x+1) - 1 \\ &(x+1)^4 - 4(x+1)^3 + 6(x+1)^2 - 4(x+1) + 1.\end{aligned}$$

What is the pattern? Can you see a simple explanation?

Solution: We start with the last expression; the first two are similar, but easier. If we just expand everything out, we get

$$\begin{array}{lclllll} (x+1)^4 & = & x^4 & +4x^3 & +6x^2 & +4x & +1 \\ -4(x+1)^3 & = & & -4x^3 & -12x^2 & -12x & -4 \\ +6(x+1)^2 & = & & & +6x^2 & +12x & +6 \\ -4(x+1) & = & & & & -4x & -4 \\ +1 & = & & & & & +1 \end{array}$$

When we add these expressions together, almost everything cancels out, leaving

$$(x+1)^4 - 4(x+1)^3 + 6(x+1)^2 - 4(x+1) + 1 = x^4.$$

There is of course a simple explanation for this. For any number y, we have the binomial expansion

$$(y-1)^4 = y^4 - 4y^3 + 6y^2 - 4y + 1.$$

Now take $y = x+1$, so that $y-1 = x$: we get

$$x^4 = (x+1)^4 - 4(x+1)^3 + 6(x+1)^2 - 4(x+1) + 1,$$

as before. By the same method, we have

$$\begin{aligned}(x+1)^2 - 2(x+1) + 1 &= x^2 \\ (x+1)^3 - 3(x+1)^2 + 3(x+1) - 1 &= x^3.\end{aligned}$$

Exercise 2.1.15. Consider the following functions:

$$\begin{aligned} \phi_1(x) &= x-1 & \phi_5(x) &= x^4+x^3+x^2+x+1 \\ \phi_2(x) &= x+1 & \phi_6(x) &= x^2-x+1 \\ \phi_3(x) &= x^2+x+1 & \phi_{12}(x) &= x^4-x^2+1 \\ \phi_4(x) &= x^2+1 \end{aligned}$$

(These are called *cyclotomic polynomials* . They are very important in *Number Theory*, which means the study of prime numbers, divisibility and so on.)

Expand out the following products:

$$\begin{aligned} &\phi_1(x)\phi_2(x) \\ &\phi_1(x)\phi_3(x) \\ &\phi_1(x)\phi_2(x)\phi_4(x) \\ &\phi_1(x)\phi_5(x) \\ &\phi_1(x)\phi_2(x)\phi_3(x)\phi_6(x). \end{aligned}$$

Can you see the pattern? Can you guess what is the corresponding equation involving $\phi_{12}(x)$?

Solution:

$$\begin{aligned} \phi_1(x)\phi_2(x) &= (x-1)(x+1) = x^2-1 \\ \phi_1(x)\phi_3(x) &= (x-1)(x^2+x+1) = x^3-1 \\ \phi_1(x)\phi_2(x)\phi_4(x) &= (x-1)(x+1)(x^2+1) \\ &= (x^2-1)(x^2+1) = x^4-1 \\ \phi_1(x)\phi_5(x) &= (x-1)(x^4+x^3+x^2+x+1) = x^5-1 \\ \phi_1(x)\phi_2(x)\phi_3(x)\phi_6(x) &= (x-1)(x+1)(x^2+x+1)(x^2-x+1) \\ &= (x^2-1)(x^4+x^2+1) = x^6-1. \end{aligned}$$

The general rule so far is as follows: for any number n, we take all the numbers d that divide n, take the corresponding polynomials $\phi_d(x)$, multiply them together, and we get x^n-1. (Consider the case $n=6$, for example: the divisors of 6 are 1, 2, 3 and 6, the corresponding polynomials are $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$ and $\phi_6(x)$, and we saw above that if we multiply these together, we get x^6-1.) This suggests that we should have

$$\phi_1(x)\phi_2(x)\phi_3(x)\phi_4(x)\phi_6(x)\phi_{12}(x) = x^{12}-1.$$

If we feed in the case $n=6$ which we have already worked out, the left hand side becomes $(x^6-1)\phi_4(x)\phi_{12}(x)$. It is easy to check that $\phi_4(x)\phi_{12}(x) = x^6+1$, so the left hand side becomes $(x^6-1)(x^6+1)$ which is $x^{12}-1$ as expected.

Exercise 2.1.16. Simplify the expression

$$(x+y+z)^3 - 3(x+y+z)(xy+yz+zx) + 3xyz$$

Solution: We have

$$\begin{aligned} (x+y+z)^3 &= (x+y+z)(x+y+z)(x+y+z) \\ &= (x+y+z)(x^2+y^2+z^2+2xy+2yz+2xz) \\ &= x^3+xy^2+xz^2+2x^2y+2xyz+2x^2z+ \\ &\quad x^2y+y^3+yz^2+2xy^2+2y^2z+2xyz+ \\ &\quad x^2z+y^2z+z^3+2xyz+2yz^2+2xz^2 \\ &= x^3+y^3+z^3+3(xy^2+xz^2+yx^2+yz^2+zx^2+zy^2)+6xyz \end{aligned}$$

and

$$\begin{aligned}(x+y+z)(xy+yz+zx) &= x^2y + xyz + zx^2 + \\ &\quad xy^2 + y^2z + xyz + \\ &\quad xyz + yz^2 + z^2x \\ &= (xy^2 + xz^2 + yx^2 + yz^2 + zx^2 + zy^2) + 3xyz.\end{aligned}$$

Putting these together, we find that

$$(x+y+z)^3 - 3(x+y+z)(xy+yz+zx) + 3xyz = x^3 + y^3 + z^3.$$

Exercise 2.1.17. Check that

$$(u+u^{-1})^6 - 6(u+u^{-1})^4 + 9(u+u^{-1})^2 - 2 = u^6 + u^{-6}.$$

Can you find a similar formula expressing $u^4 + u^{-4}$ in terms of powers of $u + u^{-1}$? (This question is related to the *Chebyshev polynomials* $T_n(x)$, and in particular to the polynomial $2T_6(x/2) = x^6 - 6x^4 + 9x^2 - 2$.)

Solution: Using the binomial expansion, we find that

$$\begin{array}{rclllllll}(u+u^{-1})^6 &=& u^6 & +6u^4 & +15u^2 & +20 & +15u^{-2} & +6u^{-4} & +u^{-6} \\ -6(u+u^{-1})^4 &=& & -6u^4 & -24u^2 & -36 & -24u^{-2} & -6u^{-4} & \\ 9(u+u^{-1})^2 &=& & & +9u^2 & +18 & +9u^{-2} & & \\ -2 &=& & & & -2 & & & \end{array}$$

When we add all this together, all the columns cancel out except for the first and last ones, leaving

$$(u+u^{-1})^6 - 6(u+u^{-1})^4 + 9(u+u^{-1})^2 - 2 = u^6 + u^{-6}$$

as claimed.

To find a similar formula for $u^4 + u^{-4}$, we start by calculating

$$(u+u^{-1})^4 = u^4 + 4u^2 + 6 + 4u^{-2} + u^{-4}.$$

We can cancel out the unwanted term of $4u^2$ by subtracting the expression $4(u+u^{-1})^2 = 4u + 8 + 4u^{-1}$. This gives

$$(u+u^{-1})^4 - 4(u+u^{-1})^2 = u^4 - 2 + u^{-4}.$$

As a final correction, we add 2 to both sides, giving the required formula

$$(u+u^{-1})^4 - 4(u+u^{-1})^2 + 2 = u^4 + u^{-4}.$$

Exercise 2.1.18. Expand out the following expressions:

$$\begin{aligned}&(1+x)^2\\ &(1+x+x^2)^2\\ &(1+x+x^2+x^3)^2.\end{aligned}$$

Now guess what you get by expanding out

$$(1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8)^2.$$

Can you justify your guess? Can you write down the general rule for the expansion of $(\sum_{k=0}^n x^k)^2$? Try to write down a precise, self-contained statement that would make sense to someone who had not read or attempted this exercise.

Solution: By direct expansion, we have

$$\begin{aligned}(1+x)^2 &= 1+2x+x^2\\ (1+x+x^2)^2 &= 1+2x+3x^2+2x^3+x^4\\ (1+x+x^2+x^3)^2 &= 1+2x+3x^2+4x^3+3x^4+2x^5+x^6.\end{aligned}$$

We now want to formulate a general rule for $(1+x+\ldots+x^n)^2$. The general pattern is clear: the coefficients of the powers of x start at 1, then increase by one each time for a while, then they decrease by one each time until they get back down to one. What is the term in the middle, with the maximum coefficient? In the case $n=2$, we are looking at $(1+x+x^2)^2$; from the equations above, we see that the middle term is $3x^2$. In the case $n=3$, the middle term is $4x^3$. In general, it seems that the middle term is $(n+1)x^n$. Similarly, when $n=2$ the last term is x^4, when $n=3$ the last term is x^6, and in general it seems that the last term is x^{2n}.

This suggests the following general rule:

$$\begin{aligned}(1+x+\ldots+x^n)^2 =& 1+2x+3x^2+\ldots+nx^{n-1}+(n+1)x^n+\\ & nx^{n+1}+(n-1)x^{n+2}+\ldots+2x^{2n-1}+x^{2n}.\end{aligned}$$

Alternatively, we can use $\sum$ notation:

$$\left(\sum_{k=0}^{n} x^k\right)^2 = \sum_{k=0}^{n}(k+1)x^k + \sum_{k=1}^{n}(n-k+1)x^{n+k}.$$

We now explain why this rule is correct. We will illustrate the explanation with a picture for the case $n=6$, but it should be clear that everything works in the same way for all n. You should first note that the sum $1+x+\ldots+x^n$ actually contains $n+1$ terms (because of the 1 at the beginning), not just n terms. If we expand out $(1+x+\ldots+x^n)^2$ without collecting together like terms, we therefore get $(n+1)^2$ terms, which can be arranged in a square, which looks like this when $n=6$:

$$\begin{array}{ccccccc}
1 & x & x^2 & x^3 & x^4 & x^5 & x^6\\
x & x^2 & x^3 & x^4 & x^5 & x^6 & x^7\\
x^2 & x^3 & x^4 & x^5 & x^6 & x^7 & x^8\\
x^3 & x^4 & x^5 & x^6 & x^7 & x^8 & x^9\\
x^4 & x^5 & x^6 & x^7 & x^8 & x^9 & x^{10}\\
x^5 & x^6 & x^7 & x^8 & x^9 & x^{10} & x^{11}\\
x^6 & x^7 & x^8 & x^9 & x^{10} & x^{11} & x^{12}
\end{array}$$

The four corners of the square are the terms $1\times 1=1$, $x^n\times 1=x^n$, $1\times x^n=x^n$, and $x^n\times x^n=x^{2n}$. Looking near the top left corner, we see that we have 2 copies of x, 3 copies of x^2 and so on, so the expansion starts $1+2x+3x^2+\ldots$. All the entries on the diagonal from the top right to the bottom left are equal to x^n. There are $n+1$ such entries, so the middle term in our expansion is $(n+1)x^n$. From then on, the coefficents decrease, ending up with x^{2n} as claimed.

Exercise 2.1.19. Put

$$\begin{aligned}b_3(x) &= x(x-1)(x-2)/6\\ b_4(x) &= x(x-1)(x-2)(x-3)/24\\ b_5(x) &= x(x-1)(x-2)(x-3)(x-4)/120.\end{aligned}$$

- Simplify $b_4(x+1)-b_4(x)$. If you do this in the right way, it will take only a few simple steps; if you do it the wrong way, you will have to work much harder. Do not expand anything out if you do not have to.
- Simplify $b_5(x+1)-b_5(x)$.

- What is the general pattern? (Your answer should include a definition of $b_n(x)$ for all n.)

Solution: We have

$$\begin{aligned} b_4(x+1) &= (x+1)(x+1-1)(x+1-2)(x+1-3)/24 \\ &= (x+1)x(x-1)(x-2)/24 \\ b_4(x) &= x(x-1)(x-2)(x-3)/24, \end{aligned}$$

so

$$\begin{aligned} b_4(x+1) - b_4(x) &= ((x+1)-(x-3))x(x-1)(x-2)/24 \\ &= 4x(x-1)(x-2)/24 \\ &= x(x-1)(x-2)/6 = b_3(x). \end{aligned}$$

Similarly,

$$\begin{aligned} b_5(x+1) &= (x+1)x(x-1)(x-2)(x-3)/120 \\ b_5(x) &= x(x-1)(x-2)(x-3)(x-4)/120, \end{aligned}$$

so

$$\begin{aligned} b_5(x+1) - b_5(x) &= ((x+1)-(x-4))x(x-1)(x-2)(x-3)/120 \\ &= x(x-1)(x-2)(x-3)/24 = b_4(x). \end{aligned}$$

The general pattern is that $b_n(x+1) - b_n(x) = b_{n-1}(x)$, where

$$b_n(x) = x(x-1)\dots(x-n+1)/n!.$$

2.2 Powers

The rules for manipulating powers are as follows:

$$\begin{array}{llll} a^n a^m & = a^{n+m} & \qquad (a^n)^m & = a^{nm} \\ a^n b^n & = (ab)^n & \qquad a^n/b^n & = (a/b)^n = a^n b^{-n} \\ (a+b)^n & \neq a^n + b^n & \qquad (a+b)^n & = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k} \end{array}$$

The last equation here is the binomial expansion formula, which is included for the sake of completeness only; it will be reviewed in detail in PMA111 (Numbers and Polynomials).

Example 2.2.1.

$$\begin{aligned} a^2a^3 &= (a\times a)\times(a\times a\times a) = a^{2+3} = a^5 \\ (a^2)^3 &= (a\times a)^3 = (a\times a)\times(a\times a)\times(a\times a) = a^{2\times 3} = a^6. \quad \square \end{aligned}$$

Example 2.2.2. Consider the equation $4^n = (2^2)^n = 2^{2n}$. DNA consists of a string of "bases", each of which can be one of four possible molecules (adenine, guanine, thymine, or cytosine). There are thus $4^n = 2^{2n}$ possibilities for a DNA sequence of n bases. This is the same as the number of possible states of a block of computer memory $2n$ bits long. Thus, a sequence of n bases contains $2n$ bits of information. $\square$

Exercise 2.2.3. We have $10^3 = 1000 \approx 1024 = 2^{10}$. Deduce similar approximations for 10^9 and 8×10^9 as powers of 2.

Solution:

$$\begin{aligned} 10^9 &= 10^{3\times 3} = (10^3)^3 \approx (2^{10})^3 = 2^{10\times 3} = 2^{30} \\ 8\times 10^9 &= 2^3 \times 10^9 \simeq 2^3 \times 2^{30} = 2^{33}. \end{aligned}$$

(The exact numbers are $2^{30} = 1073741824$ and $2^{33} = 8589934592$.)

According to the usual metric conventions, a "gigabyte" of memory should contain 10^9 bytes. Because digital electronics is based on binary numbers, it is easier and more natural to build memory chips with 2^{30} bytes of capacity, so that is what a gigabyte means in practice. A byte is 8 bits, so the capacity is $8 \times 2^{30} = 2^{33} \approx 8 \times 10^9$ bits.

Exercise 2.2.4. Put $u = 10^{1/10} \approx 1.25893$. An amplifier is said to have a *gain* of n decibels if the ratio of the output power to the input power is u^n. If this ratio is equal to 100, what is the gain?

Solution: First note that $u^{10} = 10^{(1/10)\times 10} = 10^1 = 10$. The ratio is $100 = 10^2 = (u^{10})^2 = u^{20}$, so the gain is 20 decibels.

Exercise 2.2.5. Put $a = x^2$ and $b = a^3$ and $c = b^4$ and $d = c^5$. Express $\frac{ad}{bc}$ as a power of x.

Solution:

$$\begin{aligned} a &= x^2 \\ b &= a^3 = (x^2)^3 = x^{2\times 3} = x^6 \\ c &= b^4 = (x^6)^4 = x^{6\times 4} = x^{24} \\ d &= c^5 = (x^{24})^5 = x^{24\times 5} = x^{120} \\ \frac{ad}{bc} &= \frac{x^2 x^{120}}{x^6 x^{24}} = \frac{x^{122}}{x^{30}} = x^{122-30} = x^{92}. \end{aligned}$$

Exercise 2.2.6. Put

$$\begin{aligned} G &= M^{-1}L^3T^{-2} \\ F &= GM^2L^{-2} \\ a &= FM^{-1} \\ E &= M(aT)^2. \end{aligned}$$

Simplify this to write E in terms of M, L and T. (If you know some physics, you may recognize this as a dimensional analysis of the Newton's equations of gravity and motion.)

Solution:

$$\begin{aligned} F &= GM^2L^{-2} = M^{-1}L^3T^{-2}M^2L^{-2} = MLT^{-2} \\ a &= FM^{-1} = LT^{-2} \\ E &= M(LT^{-2}T)^2 = M(LT^{-1})^2 = ML^2T^{-2}. \end{aligned}$$

Exercise 2.2.7. Note that

$$\left(\left(3^3\right)^3\right)^3 = \left(3^{3\times 3}\right)^3 = 3^{3\times 3\times 3} = 3^{27}.$$

Express the following numbers as powers of 3 in the same way:

(a) $\left(3^3\right)^{\left(3^3\right)}$

(b) $3^{\left(3^{\left(3^3\right)}\right)}$

You will need the fact that $3^{27} = 7625597484987$.

Solution:

(a) We have $3^3 = 27$, so

$$\begin{aligned}\left(3^3\right)^{\left(3^3\right)} &= \left(3^3\right)^{27} \\ &= 3^{3\times 27} \\ &= 3^{81}.\end{aligned}$$

(b) Here we have

$$\begin{aligned}3^{\left(3^{\left(3^3\right)}\right)} &= 3^{\left(3^{27}\right)} \\ &= 3^{7625597484987}.\end{aligned}$$

Exercise 2.2.8*. Write $f_n(x) = (1 + 2^{-n}x)^{2^n}$. Simplify $f_{n+1}(2x)f_n(x)^{-2}$.

Solution: First, we have

$$f_{n+1}(2x) = (1 + 2^{-(n+1)}.2x)^{2^{n+1}},$$

but $2^{-(n+1)}.2 = 2^{-n-1}.2^1 = 2^{-n}$, so

$$f_{n+1}(2x) = (1 + 2^{-n}x)^{2^{n+1}}.$$

On the other hand, we have

$$\begin{aligned}f_n(x)^2 &= \left((1 + 2^{-n}x)^{2^n}\right)^2 \\ &= (1 + 2^{-n}x)^{2^n\times 2} \\ &= (1 + 2^{-n}x)^{2^{n+1}},\end{aligned}$$

which is just the same. Thus $f_{n+1}(2x)f_n(x)^{-2} = f_{n+1}(2x)/f_n(x)^2 = 1$.

It is an important fact that $f_n(x)$ tends to e^x as n tends to infinity. The proof is not too hard, and may be explained in PMA113 (Introduction to Analysis). If we let n tend to infinity in our relation $f_{n+1}(2x) = f_n(x)^2$, we get $e^{2x} = (e^x)^2$. Of course we knew that anyway, but it is nice to see how it all fits together.

2.3 Manipulation of algebraic fractions

By an *algebraic fraction*, we mean an expression of the form a/b, where a and b are algebraic expressions, which may themselves have a complicated structure. The basic rules for manipulating algebraic fractions are as follows:

$$\begin{array}{llllll}\frac{a}{b} + \frac{c}{d} & = \frac{ad+bc}{bd} & \frac{a}{b}\cdot\frac{c}{d} & = \frac{ac}{bd} & \left(\frac{a}{b}\right)^n & = \frac{a^n}{b^n} \\ \frac{a}{b} - \frac{c}{d} & = \frac{ad-bc}{bd} & \frac{a}{b}\Big/\frac{c}{d} & = \frac{ad}{bc} & \left(\frac{a}{b}\right)^{-n} & = \frac{b^n}{a^n}\end{array}$$

Exercise 2.3.1. For any real number t, check that

$$\left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2 = 1.$$

Solution:

$$\begin{aligned}\left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2 &= \frac{(1-t^2)^2+4t^2}{(1+t^2)^2}\\ &= \frac{1-2t^2+t^4+4t^2}{1+2t^2+t^4}\\ &= \frac{1+2t^2+t^4}{1+2t^2+t^4} = 1.\end{aligned}$$

Background: This says that the point $P = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$ lies on the unit circle. Let θ be the angle between P and the x-axis; it can be shown that $t = \tan(\theta/2)$.

Exercise 2.3.2*. For any four different numbers w, x, y and z, their *cross ratio* is defined to be the number

$$\chi(w,x,y,z) = \frac{(z-w)(y-x)}{(z-x)(y-w)}.$$

Check the following identities:

(a) $\chi(a+\lambda, b+\lambda, c+\lambda, d+\lambda) = \chi(a,b,c,d)$.

(b) $\chi(\mu a, \mu b, \mu c, \mu d) = \chi(a,b,c,d)$.

(c) $\chi(1/a, 1/b, 1/c, 1/d) = \chi(a,b,c,d)$.

(d) $(d-a)(c-b) + (b-a)(d-c) = (d-b)(c-a)$.

(e) $\chi(a,b,c,d) + \chi(b,c,d,a) = 1$.

Solution: The first two parts are easy:

$$\begin{aligned}\chi(a+\lambda, b+\lambda, c+\lambda, d+\lambda) &= \frac{(d+\lambda-a-\lambda)(c+\lambda-b-\lambda)}{(d+\lambda-b-\lambda)(c+\lambda-a-\lambda)}\\ &= \frac{(d-a)(c-b)}{(d-b)(c-a)}\\ &= \chi(a,b,c,d)\\ \chi(\mu a, \mu b, \mu c, \mu d) &= \frac{(\mu d-\mu a)(\mu c-\mu b)}{(\mu d-\mu b)(\mu c-\mu a)}\\ &= \frac{\mu^2(d-a)(c-b)}{\mu^2(d-b)(c-a)}\\ &= \chi(a,b,c,d).\end{aligned}$$

For part (c), we have

$$\begin{aligned}\chi(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}) &= \frac{\left(\frac{1}{d}-\frac{1}{a}\right)\left(\frac{1}{c}-\frac{1}{b}\right)}{\left(\frac{1}{d}-\frac{1}{b}\right)\left(\frac{1}{c}-\frac{1}{a}\right)}\\ &= \frac{\frac{a-d}{ad}\,\frac{b-c}{bc}}{\frac{b-d}{bd}\,\frac{a-c}{ac}}\\ &= \frac{(a-d)(b-c)/(abcd)}{(b-d)(a-c)/(abcd)}\\ &= \frac{--\,(d-a)(c-b)}{--\,(d-b)(c-a)}\\ &= \chi(a,b,c,d).\end{aligned}$$

For part (d), we have

$$\begin{aligned}(d-a)(c-b)+(b-a)(d-c) &= cd - bd - ac + ab + \\ &\quad bd - bc - ad + ac \\ &= dc - bc - da + ba = (d-b)(c-a).\end{aligned}$$

Next note that $\chi(b,c,d,a)$ is obtained from $\chi(a,b,c,d)$ by replacing a by b, replacing b by c, replacing c by d and replacing d by a. This means that

$$\chi(b,c,d,a) = \frac{(a-b)(d-c)}{(a-c)(d-b)} = \frac{(b-a)(d-c)}{(d-b)(c-a)}.$$

(We have made a slight rearrangement to get the same denominator as for $\chi(a,b,c,d)$.) It follows that

$$\begin{aligned}\chi(a,b,c,d)+\chi(b,c,d,a) &= \frac{(d-a)(c-b)}{(d-b)(c-a)} + \frac{(b-a)(d-c)}{(d-b)(c-a)} \\ &= \frac{(d-a)(c-b)+(b-a)(d-c)}{(d-b)(c-a)} \\ &= \frac{(d-b)(c-a)}{(d-b)(c-a)} = 1.\end{aligned}$$

(The third equality uses part (d)).

Exercise 2.3.3. Simplify the expression

$$\frac{1}{1+\dfrac{1}{1+\dfrac{1}{1+x}}}.$$

(This is an example of a *continued fraction* .)

Solution:

$$1+\frac{1}{1+x} = \frac{2+x}{1+x}$$

$$\frac{1}{1+\dfrac{1}{1+x}} = 1/\frac{2+x}{1+x} = \frac{1+x}{2+x}$$

$$1+\frac{1}{1+\dfrac{1}{1+x}} = 1+\frac{1+x}{2+x} = \frac{3+2x}{2+x}$$

$$\frac{1}{1+\dfrac{1}{1+\dfrac{1}{1+x}}} = 1/\frac{3+2x}{2+x} = \frac{2+x}{3+2x}.$$

Exercise 2.3.4*. Let q be a variable, and put

$$\pi(n) = (q^n-1)(q^{n-1}-1)\dots(q-1),$$

so for example

$$\pi(5) = (q^5 - 1)(q^4 - 1)(q^3 - 1)(q^2 - 1)(q - 1).$$

Then put

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{\pi(n)}{\pi(k)\pi(n-k)}.$$

(This is called the *Gaussian binomial coefficient* ; one can show that it becomes the ordinary binomial coefficient in the limit as $q \to 1$.)

(a) Factorize $q^5 - 1$ and $q^4 - 1$. (You may want to refer to an earlier exercise.)

(b) Simplify the expression $\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q$. Using part (a), you should be able to cancel out so many things that your final answer is just a polynomial in q, *NOT* an algebraic fraction.

(c) Put $q = 1$ in your answer to (b), and check that you get the ordinary binomial coefficient $\binom{5}{2} = \frac{5!}{2!3!}$.

Solution:

(a)

$$\begin{aligned} q^5 - 1 &= (q-1)(q^4 + q^3 + q^2 + q + 1) \\ q^4 - 1 &= (q^2 - 1)(q^2 + 1) = (q-1)(q+1)(q^2+1). \end{aligned}$$

(b) The relevant terms $\pi(n)$ are

$$\begin{aligned} \pi(5) &= (q^5 - 1)(q^4 - 1)(q^3 - 1)(q^2 - 1)(q - 1) \\ \pi(3) &= (q^3 - 1)(q^2 - 1)(q - 1) \\ \pi(2) &= (q^2 - 1)(q - 1). \end{aligned}$$

Thus,

$$\begin{aligned} \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q &= \frac{\pi(5)}{\pi(2)\pi(3)} \\ &= \frac{(q^5-1)(q^4-1)(q^3-1)(q^2-1)(q-1)}{(q^2-1)(q-1)(q^3-1)(q^2-1)(q-1)} \\ &= \frac{(q^5-1)(q^4-1)}{(q^2-1)(q-1)} \\ &= \frac{(q-1)(q^4+q^3+q^2+q+1)(q^2-1)(q^2+1)}{(q^2-1)(q-1)} \qquad \text{(by part (a))} \\ &= (q^4+q^3+q^2+q+1)(q^2+1) = q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1. \end{aligned}$$

(c) Putting $q = 1$ gives $1 + 1 + 2 + 2 + 2 + 1 + 1 = 10$. On the other hand, we have

$$\binom{5}{2} = \frac{5!}{2!\,3!} = \frac{120}{2 \times 6} = 10,$$

as required.

Exercise 2.3.5*. Suppose that $p, q > 0$. If $w = (pu + qv)/(p + q)$, check that

$$\frac{pu + qv + rw}{p + q + r} = w$$

for any $r > 0$.

Solution: First, we have

$$\begin{aligned}
pu+qv+rw &= pu+qv+r\frac{pu+qv}{p+q} \\
&= ((p+q)pu+(p+q)qv+rpu+rqv)/(p+q) \\
&= (p^2u+pqu+pqv+q^2v+rpu+rqv)/(p+q) \\
&= ((p^2+pq+pr)u+(pq+q^2+qr)v)/(p+q) \\
&= (p+q+r)(pu+qv)/(p+q) \\
&= (p+q+r)w.
\end{aligned}$$

We known that $p, q, r > 0$ so $p+q+r > 0$, so it is legitimate to divide both sides by $p+q+r$; this gives

$$\frac{pu+qv+rw}{p+q+r} = w$$

as claimed.

2.4 Partial fractions

Definition 2.4.1. A *rational function* is a function $f(x)$ that can be written using constants, addition, multiplication, division and integer powers, but without any roots, fractional powers, logarithms, trigonometric or exponential functions, and so on. For example, the following functions are rational:

$$\frac{1+x+x^2}{1-x+x^2}\ ,\ \frac{1+2x+3x^2+4x^3}{1+x+x^2+x^3+x^4}\ ,\ \frac{1}{x}+\frac{\pi}{x-1}+\frac{\pi^2}{x-2}\ ,\ x^2+x+1+x^{-1}+x^{-2}$$

but the following are not:

$$e^{-x}\sin(x)\ ,\ \sqrt{1-x^2}\ ,\ \frac{\log(x)}{1+x}\ ,\ \frac{\arctan(x)}{2\pi}.$$

It is an important and useful fact that any rational function can be rewritten in terms of *partial fractions*. We give some examples to illustrate what this means.

$$\begin{aligned}
\frac{x^2+1}{x^2-1} &= 1+\frac{1}{x-1}-\frac{1}{x+1} \\
\frac{360}{(x^2-25)(x^2-16)} &= \frac{4}{x-5}-\frac{4}{x+5}-\frac{5}{x-4}+\frac{5}{x+4} \\
\left(\frac{x+1}{x-1}\right)^3 &= 1+\frac{6}{x-1}+\frac{12}{(x-1)^2}+\frac{8}{(x-1)^3} \\
\frac{1-x^3+x^4}{x^2-2x+1} &= x^2+x+1+\frac{1}{x-1}+\frac{1}{(x-1)^2} \\
\frac{x^4}{(x-1)(x-2)} &= x^2+3x+7-\frac{1}{x-1}+\frac{16}{x-2}.
\end{aligned}$$

(You can check each of these equations if you wish: just convert the right hand side to a single fraction using the rule $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$, and what you get should be visibly the same as the left hand side. Alternatively, you can just accept that the equations are true, and this should enable you to understand the following discussion. Later in this section we shall explain how such equations are found.)

In each of these equations, the left hand side is a rational function. Each of the terms on the right hand side is either a constant, or a multiple of a power of x, or a multiple of a power of $1/(x-a)$ for some a. (Note that this includes things like $1/(x+7)$, because $1/(x+7)=1/(x-(-7))$.) An expression consisting of terms like this is called a *partial fraction decomposition*; for example, the first equation above says that the partial fraction decomposition of $(x^2+1)/(x^2-1)$ is $1+\frac{1}{x-1}-\frac{1}{x+1}$. Such decompositions are very useful if you want to integrate or differentiate a rational function, or understand its properties.

Definition 2.4.2. If the partial fraction decomposition of a function $f(x)$ contains a power of $1/(x-a)$, we say that $f(x)$ has a *pole* at $x=a$. The highest power that occurs is called the *order* of the pole.

Example 2.4.3. The decomposition

$$\left(\frac{x+1}{x-1}\right)^3 = 1+\frac{6}{x-1}+\frac{12}{(x-1)^2}+\frac{8}{(x-1)^3}$$

shows that the function $((x+1)/(x-1))^3$ has a pole of order 3 at the point $x=1$ (and no other poles). □

Example 2.4.4. The decomposition

$$\frac{x^4}{(x-1)(x-2)} = x^2+3x+7-\frac{1}{x-1}+\frac{16}{x-2}$$

shows that the function $f(x)=x^4/((x-1)(x-2))$ has a pole of order one at $x=1$, another pole of order one at $x=2$, and no other poles. □

Example 2.4.5. Consider the function

$$f(x)=\frac{x^5+6x^3-7x+8}{x(x+1)^2(x+2)^3}.$$

The decomposition

$$\begin{aligned} f(x) &= \frac{1}{x}-\frac{8}{(x+1)^2}+\frac{29}{(x+2)^3} \\ &= \frac{1}{(x-0)^1}-\frac{8}{(x-(-1))^2}+\frac{29}{(x-(-2))^3} \end{aligned}$$

shows that $f(x)$ has a pole of order 1 at $x=0$, a pole of order 2 at $x=-1$, and a pole of order 3 at $x=-2$. □

Finding poles and orders

It is possible to find the poles and their orders of a rational function $f(x)$ without knowing the partial fraction decomposition:

Method 2.4.6. (i) Write $f(x)$ in the form $p(x)/q(x)$, where p and q are polynomials.

(ii) Check to see whether $p(x)$ and $q(x)$ have any common factors, and if so, cancel them out. This gives a new expression $f(x)=r(x)/s(x)$, where r and s are polynomials without any common factors.

(iii) Factorise $s(x)$ completely, as a product of terms of the form $(x-\alpha)^n$ or just $x-\alpha$. The numbers α in the factorisation are the poles, and the numbers n are the corresponding orders.

Example 2.4.7. Consider the function $f(x)=\frac{1+3/x}{x^2-1}+\frac{2}{x^4-x^2}$. We first combine the two terms to get $f(x)=\frac{x^2+3x+2}{x^4-x^2}$. We then factorise this and cancel common factors to get

$$f(x)=\frac{(x+1)(x+2)}{x^2(x+1)(x-1)}=\frac{x+2}{x^2(x-1)}=\frac{x+2}{(x-0)^2(x-1)^1}$$

Thus, we have a pole of order 2 at $x=0$, and a pole of order 1 at $x=1$. □

Finding decompositions

If we work with *complex* numbers, then every rational function has a strict decomposition. For example, the decomposition of $2x/(x^2+1)$ is the expression $1/(x-i)+1/(x+i)$, which involves the imaginary number $i = \sqrt{-1}$. If we want to avoid complex numbers (as is sometimes convenient) then we must allow some other terms in the decomposition. Specifically, we must allow terms like $1/(x^2+bx+c)^n$ or $x/(x^2+bx+c)^n$, where x^2+bx+c is a quadratic with $b^2 > 4c$ (so that there are no real roots). A decomposition into terms like this is called a *quadratic partial fraction decomposition*. Any rational function with real coefficients can be decomposed like this, with no need to introduce any imaginary numbers. For example, we have

$$\frac{1}{x^3+x} = \frac{1}{x} - \frac{x}{x^2+1}$$
$$\frac{x^5+2x^3+7x-1}{(x^3-1)^2} = \frac{1}{(x-1)^2} + \frac{x}{x^2+x+1} - \frac{2}{(x^2+x+1)^2}.$$

We now discuss the method for finding partial fraction decompositions. There are two steps: we first find the general form, and then solve some linear equations to determine the coefficients. We give an example of the second step first.

Example 2.4.8. Consider $f(x) = 6x/((x-2)(x-1)(x+1)(x+2))$. A general rule, which will be described in detail later, says that the partial fraction decomposition has the form

$$f(x) = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{x+1} + \frac{D}{x+2}$$

for some numbers A, B, C and D; the only problem is to find these numbers. If we multiply both sides by $(x-2)(x-1)(x+1)(x+2)$, we find that

$$\begin{aligned} 6x =& A(x-1)(x+1)(x+2) + B(x-2)(x+1)(x+2)+ \\ & C(x-2)(x-1)(x+2) + D(x-2)(x-1)(x+1). \end{aligned}$$

This holds for all x; in particular, it holds for $x = -2$. If we put $x = -2$ then $x+2$ becomes zero, so most of the terms on the right hand side vanish. We are left with the equation

$$6 \times (-2) = 0 + 0 + 0 + D \times (-2-2) \times (-2-1) \times (-2+1),$$

or equivalently $-12 = -12D$, which gives $D = 1$. Similarly, we can put $x = -1$ to get

$$6 \times (-1) = 0 + 0 + C \times (-1-2) \times (-1-1) \times (-1+2) + 0,$$

and thus $-6 = 6C$, so $C = -1$. By the same method, we get $B = -1$ (by putting $x = 1$) and $A = 1$ (by putting $x = 2$). Putting this back into our expression for $f(x)$, we get

$$f(x) = \frac{1}{x+2} - \frac{1}{x+1} - \frac{1}{x-1} + \frac{1}{x-2}. \quad \square$$

Remark 2.4.9. As we will explain in Chapter 8, a system of linear equations may have no solution at all, or it may have a unique solution, or it may have infinitely many solutions. It turns out that the equations arising in the above method always have a unique solution, so every rational function has a unique partial fraction decomposition. You should be aware that this is an important part of the theoretical background, although we will not discuss it further.

The method for finding the general form is as follows:

Method 2.4.10. Let $f(x)$ be a rational function.

(i) Write $f(x)$ as $p(x)/q(x)$, where p and q are polynomials.

(ii) If degree($p(x)$) $\geq$ degree($q(x)$), then put $m = \text{degree}(p(x)) - \text{degree}(q(x))$. In this case, the partial fraction decomposition of $f(x)$ will include multiples of x^k for $k = 0, 1, \ldots, m$. (Here a "multiple of x^0" just means a constant.)

(iii) Find the poles of $f(x)$ and their orders, using Method 2.4.6.

(iv) For each pole α, the decomposition will include multiples of $(x - \alpha)^{-k}$ for $k = 1, \ldots, n$, where n is the order of the pole.

Example 2.4.11. Consider the function

$$f(x) = \frac{4x^2(x+2)^2(x+4)^2}{(x+1)^2(x+3)^2}.$$

In step (i) of the method we have $p(x) = 4x^2(x+2)^2(x+4)^2$, and $q(x) = (x+1)^2(x+3)^2$. Thus degree($p(x)$) = 6 and degree($q(x)$) = 4, so in step (ii) we have $m = 2$. We therefore need to include multiples of 1, x and x^2 in the decomposition. Next, we clearly have poles of order 2 at $x = -1$ and $x = -3$, so we must include multiples of $(x+1)^{-1}$, $(x+1)^{-2}$, $(x+3)^{-1}$ and $(x+3)^{-2}$. The decomposition thus has the form

$$f(x) = Ax^2 + Bx + C + \frac{D}{x+1} + \frac{E}{(x+1)^2} + \frac{F}{x+3} + \frac{G}{(x+3)^2}.$$

In fact, it works out that

$$f(x) = 4x^2 + 16x - 8 + \frac{3}{x+1} + \frac{9}{(x+1)^2} - \frac{3}{x+3} + \frac{9}{(x+3)^2}. \quad \square$$

Exercise 2.4.12. Write $f(x) = 4x/(x^2-1)^2$ in terms of partial fractions.

Solution: We first note that $x^2 - 1 = (x-1)(x+1)$, so $(x^2-1)^2 = (x-1)^2(x+1)^2$. We can rewrite $f(x)$ as $4x/((x-1)^2(x+1)^2)$, so there are poles of order two at $x = 1$ and $x = -1$. Thus, the partial fraction decomposition will involve $(x-1)^{-1}$, $(x-1)^{-2}$, $(x+1)^{-1}$ and $(x+1)^{-2}$.

The degree of $4x$ is 1, which is strictly less that the degree of the $(x-1)^2(x+1)^2$ (which is 4), so we do not need any terms of the form x^k.

Thus, the general form of the decomposition is

$$\frac{4x}{(x^2-1)^2} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{(x+1)^2} + \frac{D}{x-1}.$$

We multiply both sides by $(x-1)^2(x+1)^2$ and expand out to get

$$\begin{aligned} 4x &= A(x+1)^2 + B(x-1)(x+1)^2 + C(x-1)^2 + D(x-1)^2(x+1) \\ &= A(x^2+2x+1) + B(x^3+x^2-x-1) + C(x^2-2x+1) + D(x^3-x^2-x+1) \\ &= (B+D)x^3 + (A+B+C-D)x^2 + (2A-B-2C-D)x + (A-B+C-D). \end{aligned}$$

By equating coefficients, we find that

$$B + D = 0 \quad (1)$$
$$A + B + C - D = 0 \quad (2)$$
$$2A - B - 2C - D = 4 \quad (3)$$
$$A - B + C - D = 0. \quad (4)$$

By adding (1) and (4), we see that $A + C = 0$, so $C = -A$; inserting this in (2) gives $B = D$, which we combine with (1) to see that $B = D = 0$. Equation (3) now reads $4A = 4$, so $A = 1$, so $C = -1$. In conclusion, we have

$$\frac{4x}{(x^2-1)^2} = \frac{1}{(x-1)^2} - \frac{1}{(x+1)^2}.$$

Exercise 2.4.13. Find the partial fraction decomposition of $f(x) = \frac{x^2+2}{(x-1)^3}$.

Solution: As degree$(x^2+2) <$ degree$((x-1)^3)$, we do not have any powers of x in the decomposition. There is clearly a pole of order 3 at $x = 1$, so we have multiples of $(x-1)^{-1}$, $(x-1)^{-2}$ and $(x-1)^{-3}$. The general form is thus

$$\frac{x^2+2}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}.$$

We multiply by $(x-1)^3$ to get

$$\begin{aligned} x^2+2 &= A(x-1)^2 + B(x-1) + C \\ &= Ax^2 + (-2A+B)x + (A-B+C), \end{aligned}$$

so (by equating coefficients)

$$\begin{aligned} A &= 1 \\ -2A+B &= 0 \\ A-B+C &= 2. \end{aligned}$$

It follows easily that $A = 1$, $B = 2$ and $C = 3$, so

$$f(x) = \frac{x^2+2}{(x-1)^3} = \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{3}{(x-1)^3}.$$

Exercise 2.4.14. What is the general form of the partial fraction decomposition of $x^6/(x^2-1)^2$? (You need not work out the coefficients.)

Solution: Firstly, x^6 is a polynomial of degree 6 and $(x^2-1)^2$ is a polynomial of degree 4, so we need terms in x^k for k up to $6-4=2$. Secondly, the denominator factors as $(x-1)^2(x+1)^2$, so we have poles of order 2 at $x=-1$ and $x=+1$, so we must include multiples of $(x-1)^{-1}$, $(x-1)^{-2}$, $(x+1)^{-1}$ and $(x+1)^{-2}$ in the decomposition. Thus, the partial fraction representation has the form

$$\frac{x^6}{(x^2-1)^2} = Ax^2 + Bx + C + \frac{D}{x-1} + \frac{E}{(x-1)^2} + \frac{F}{x+1} + \frac{G}{(x+1)^2}.$$

The coefficients were not asked for, but here they are anyway , for the record:

$$\frac{x^6}{(x^2-1)^2} = x^2 + 2 + \frac{5/4}{x-1} + \frac{1/4}{(x-1)^2} - \frac{5/4}{x+1} + \frac{1/4}{(x+1)^2}.$$

Exercise 2.4.15. Find the partial fraction decomposition of the function

$$f(x) = \frac{\dfrac{1}{x-1} + \dfrac{1}{x-3}}{\dfrac{1}{x-2} + \dfrac{1}{x-4}}.$$

Solution: We must first write $f(x)$ in the form $p(x)/q(x)$. We have

$$\begin{aligned} f(x) &= \frac{2x-4}{(x-1)(x-3)} \Big/ \frac{2x-6}{(x-2)(x-4)} \\ &= \frac{2(x-2)}{(x-1)(x-3)} \, \frac{(x-2)(x-4)}{2(x-3)} \\ &= \frac{(x-2)^2(x-4)}{(x-1)(x-3)^2}. \end{aligned}$$

This gives degree$(p(x)) = $ degree$(q(x)) = 2$, so degree$(p(x)) - $degree$(q(x)) = 0$. We therefore need to include a multiple of x^0, or in other words a constant, in the decomposition. There is a pole of order 1 at $x = 1$ (so we need a multiple of $(x-1)^{-1}$) and a pole of order 2 at $x = 3$ (so we need multiples of $(x-3)^{-1}$ and and $(x-3)^{-2}$). The general form is thus

$$f(x) = \frac{(x-2)^2(x-4)}{(x-1)(x-3)^2} = A + \frac{B}{x-1} + \frac{C}{x-3} + \frac{D}{(x-3)^2}.$$

Multiplying this by $(x-1)(x-3)^2$ gives

$$(x-2)^2(x-4) = A(x-1)(x-3)^2 + B(x-3)^2 + C(x-1)(x-3) + D(x-1).$$

If we substitute $x = 1$, we get

$$(-1)^2.(-3) = A.0 + B.(-2)^2 + C.0 + D.0 = 4B,$$

which gives $B = -3/4$. If instead we substitute $x = 3$, we get

$$1^2.(-1) = A.0 + B.0 + C.0 + D.2 = 2D,$$

so $D = -1/2$. Next, it is easy to see that the coefficient of x^3 on the left hand side is just 1, whereas the coefficient on the right hand side is A, so $A = 1$. Finally, we substitute $x = 2$ to get

$$\begin{aligned} 0 &= A.1.(-1)^2 + B.(-1)^2 + C.1.(-1) + D.1 \\ &= A + B - C + D \\ &= 1 + (-3/4) - C + (-1/2) = -C - 1/4, \end{aligned}$$

so $C = -1/4$. We conclude that

$$f(x) = 1 - \frac{3}{4}\frac{1}{x-1} - \frac{1}{4}\frac{1}{x-3} - \frac{1}{2}\frac{1}{(x-3)^2}.$$

Chapter 3: Sets

Key ideas:

- Mathematical objects can and should be gathered into sets: the set of all real numbers, the set of all 2-dimensional vectors, the set of 3×3 matrices, and so on.
- Geometric figures (such as lines, rectangles or circles) can be thought of as subsets of $\mathbb{R}^2$ or $\mathbb{R}^3$.
- Given an equation or a system of equations, you should think about the set of all solutions. "Solving" a system of equations means finding the solution set.

Skills:

You should make sure you know how to do the following:

- Understand and use the basic notation of set theory
- Show that one set is a subset of another
- Show that two sets are the same
- Find the intersection or union of two sets
- Find the complement of one set in another.

3.1 Basic definitions

Definition 3.1.1. A *set* is an (unordered) collection of mathematical objects. The objects in the collection are called the *elements* of the set; they could be numbers, vectors, matrices or functions, for example. The *order* of a set is the number of different elements (which might be infinite). We write $|A|$ for the order of the set A. We write $x \in A$ to indicate that an object x is an element of the set A, and we write $y \notin A$ to indicate that y is not an element of A.

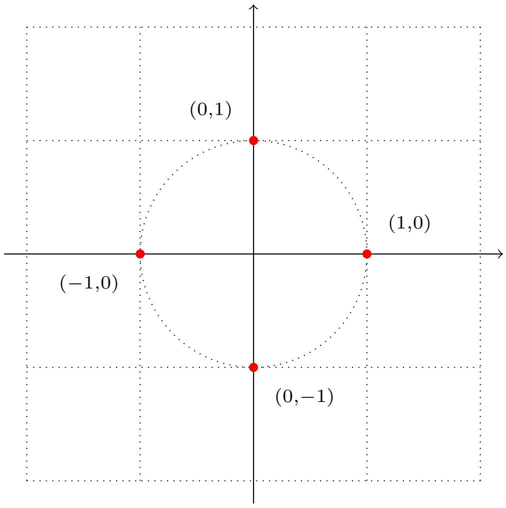
Example 3.1.2. Let A be the set of whole numbers n such that $0 < n < 10$. This can also be described by just listing the elements:

$$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} = \{9, 8, 7, 6, 5, 4, 3, 2, 1\}.$$

Note that a set is an *unordered* collection, so the two different ways of listing the elements actually specify the same set. Clearly we have $|A| = 9$ and $2 \in A$ but $-2 \notin A$. □

Example 3.1.3. Let B be the set of all two-dimensional vectors whose entries are whole numbers, and whose length is one. It is not hard to see that

$$B = \{(1, 0),\ (-1, 0),\ (0, 1),\ (0, -1)\}.$$



Thus, B has four different elements, i.e. $|B| = 4$.

Note that the numbers 0, 1 and -1 are *not* elements of B. The elements of B are the four vectors listed, not the numbers that occur in those vectors. In symbols, we have $(0,1) \in B$ but $0 \notin B$. □

Example 3.1.4. We can define a set C of matrices as follows:

$$C = \left\{ \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0\\0&0&1\\0&1&0 \end{bmatrix}, \begin{bmatrix} 0&1&0\\1&0&0\\0&0&1 \end{bmatrix}, \begin{bmatrix} 0&1&0\\0&0&1\\1&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\1&0&0\\0&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&1&0\\1&0&0 \end{bmatrix} \right\}.$$

Clearly we have $|C| = 6$. □

Background:eg-permmat A *permutation matrix* is a square matrix such that

- Each row contains a single 1, and all other entries in the row are zero.
- Each column contains a single 1, and all other entries in the column are zero.

The elements of the set C are precisely the 3×3 permutation matrices. Permutation matrices are essentially the same thing as permutations, which are studied in PMA112 (Algebra).

Example 3.1.5. The simplest set is the *empty set*, written $\{\}$ or $\emptyset$, which has no elements at all. This is useful for the same sort of reason that the number zero is useful. Clearly we have $|\emptyset| = 0$.

Next, given any object x, we can consider the set $\{x\}$ whose only element is x (so $|\{x\}| = 1$). This kind of set is called a *singleton*. Strictly speaking, the set $\{x\}$ is different from the object x. For example, 5 is number, but $\{5\}$ is a set, so $5 \neq \{5\}$. However, we will gloss over this distinction when it is harmless to do so.

Note also that the set $\{1,2,3\}$ is not the same thing as the vector $(1,2,3)$. In particular, the set $\{1,2,3\}$ is the same as $\{3,2,1\}$, but the vectors $(1,2,3)$ and $(3,2,1)$ are different. □

Example 3.1.6. There are standard symbols for the most important classes of numbers:

$$\begin{aligned}
\mathbb{N} &= \{\text{natural numbers}\} \\
&= \{\text{integers } n \text{ such that } n > 0\ \} \\
&= \{1,2,3,4,\dots\} \\
\mathbb{Z} &= \{\text{all integers}\} \\
&= \{\dots,-3,-2,-1,0,1,2,3,\dots\} \\
\mathbb{Q} &= \{\ \text{all rational numbers}\ \} \\
\mathbb{R} &= \{\ \text{all real numbers}\ \} \\
\mathbb{C} &= \{\ \text{all complex numbers}\ \}.
\end{aligned}$$

Note that not all books agree about the definition of $\mathbb{N}$; some people start with 1 (as we have done) and others start with 0. There is no dispute about $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, however.

More details of the various kinds of numbers are given in PMA111 (Numbers and Polynomials). Here we just mention some key points:

- Infinity is not an element of any of the sets mentioned. Mathematicians understand a great deal about different notions of infinity, but there are many traps and pitfalls, so infinity must be considered separately from the standard number systems. The first steps towards understanding infinity are taken in PMA113 (Introduction to Analysis).
- The set $\mathbb{R}$ consists of all possible numbers that do not involve infinity or the square root of minus one. Some popular elements of $\mathbb{R}$ include π, e, $\sqrt{2}$, 10^{100} and so on. However, there is no way to give a list of all elements of $\mathbb{R}$ (see the section on "countability" in PMA112 (Algebra)).

Definition 3.1.7. Let A be a set. A *subset* of A is a set B such that every element of B is also an element of A. In other words, a subset of A is a set B consisting of some (but usually not all) of the elements of A. We write $B \subseteq A$ to indicate that B is a subset of A.

Example 3.1.8.

- $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R} \subseteq \mathbb{C}$
- $\{2, 4, 6, 8\} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \subseteq \mathbb{N}$
- For any set A we have $\emptyset \subseteq A \subseteq A$. □

3.2 Describing sets

A set can be described by:

- Listing the elements (eg $A = \{-2, -1, 0, 1, 2\}$)
- Describing the elements (eg $B = \{$ all prime numbers $\}$)
- Cutting down a larger set (eg $C = \{n \in A \mid n < 0\} = \{-1, -2\}$)
- Applying an operation to the elements of another set (eg $D = \{n^2 + 2 \mid n \in A\} = \{(-2)^2 + 2, (-1)^2 + 2, 0^2 + 2, 1^2 + 2, 2^2 + 2\} = \{2, 3, 6\}$)
- Using standard names for standard sets (as in Example 3.1.6 and Example 3.2.2).
- Using intersections, unions and complements (as discussed in Section 3.4).

We further explain the notation by giving some more detailed examples.

- The notation $U = \{n \in \mathbb{Z} \mid n^2 < 9\}$ means the subset of all elements n of $\mathbb{Z}$ having the property that $n^2 < 9$. The only integers n for which $n^2 < 9$ are -2, -1, 0, 1 and 2, so we have
$$U = \{n \in \mathbb{Z} \mid n^2 < 9\} = \{-2, -1, 0, 1, 2\}.$$
- The notation $V = \{n \in \mathbb{N} \mid n - 1 \text{ is even }\}$ means the subset of natural numbers n such that $n - 1$ is even. Of course, $n - 1$ is even if and only if n is odd, so
$$V = \{\text{positive odd numbers}\} = \{1, 3, 5, \dots\}.$$
- Consider the set $W = \{-2\pi, -\pi, 0, \pi, 2\pi\}$. Suppose we want to take the cosine of each element, and collect together the results to form a new set X. The notation for this is $X = \{\cos(x) \mid x \in W\}$. The elements of X are $\cos(-2\pi)$, $\cos(-\pi)$, $\cos(0)$, $\cos(\pi)$ and $\cos(2\pi)$, or in other words, 1, -1, 1, -1 and 1. This means that $X = \{1, -1\}$ (the repetitions in our list do not make any difference to this answer). In particular, we have $|X| = 2$.

- Suppose we take all integers and double them; the resulting set is of course just the set of even integers. In symbols, we have

$$\{2n \mid n \in \mathbb{Z}\} = \{m \in \mathbb{Z} \mid m \text{ is even }\}.$$

Exercise 3.2.1. Put

$$\begin{aligned} A &= \{n \in \mathbb{N} \mid 12 \text{ is divisible by } n\} \\ B &= \{n \in A \mid n \text{ is prime }\} \\ C &= \{(n-1)(n-2)(n-3) \mid n \in A\} \\ D &= \{12/n \mid n \in A\}. \end{aligned}$$

For each of these sets, list the elements and state the order of the set. (You will need to remember the following details of the terminology: any number m is considered to be divisible by 1 and by itself, and 1 is not considered to be a prime number.)

Solution: First, the divisors of 12 are 1, 2, 3, 4, 6 and 12, so $A = \{1, 2, 3, 4, 6, 12\}$, giving $|A| = 6$.
Next, the only numbers in A that are prime are 2 and 3. Thus $B = \{2, 3\}$ and $|B| = 2$.
We now tabulate the values of $(n-1)(n-2)(n-3)$ and $12/n$ for $n \in A$:

n	1	2	3	4	6	12
$n(n-1)(n-2)$	0	0	0	6	60	990
$12/n$	12	6	4	3	2	1

This shows that $C = \{0, 6, 60, 990\}$, so $|C| = 4$. Similarly, we have

$$D = \{12, 6, 4, 3, 2, 1\} = \{1, 2, 3, 4, 6, 12\} = A,$$

so $|D| = |A| = 6$.
(It is not too hard to see that $D = A$, even without listing the elements.)

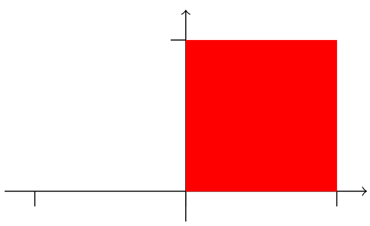
Example 3.2.2. Let a and b be real numbers with $a \leq b$. The following notation is standard:

$$\begin{aligned} (a,b) &= \{x \in \mathbb{R} \mid a < x < b\} & [a,b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\} \\ (a,b] &= \{x \in \mathbb{R} \mid a < x \leq b\} & [a,b) &= \{x \in \mathbb{R} \mid a \leq x < b\} \\ (-\infty,b) &= \{x \in \mathbb{R} \mid x < b\} & (-\infty,b] &= \{x \in \mathbb{R} \mid x \leq b\} \\ (a,\infty) &= \{x \in \mathbb{R} \mid a < x\} & [a,\infty) &= \{x \in \mathbb{R} \mid a \leq x\}. \end{aligned}$$

All these sets are called *intervals*. Those of the form $[a, b]$ are called *closed intervals*, those of the form (a, b) or (a, ∞) or $(-\infty, b)$ are called *open intervals*, and the remainder are called *half-open intervals*. □

Example 3.2.3. We write $\mathbb{R}^2$ for the set of all two-dimensional vectors (there are some reminders about vectors in Chapter 8). Thus, for example, we have $(1, 2) \in \mathbb{R}^2$ and $(-3, -4) \in \mathbb{R}^2$ but $1 \notin \mathbb{R}^2$ and $2^2 = 4 \notin \mathbb{R}^2$ and $(1, 2, 3) \notin \mathbb{R}^2$.

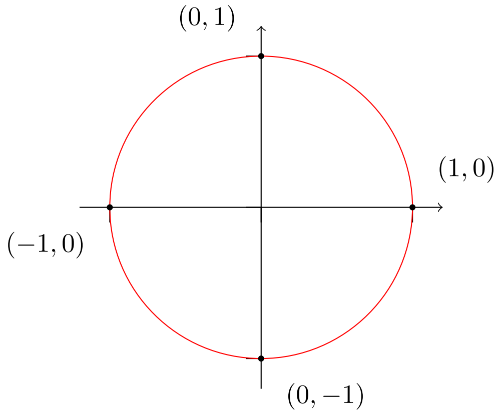
Any geometric figure in the plane can be thought of as a set of points, and thus as a subset of $\mathbb{R}^2$. For example, let A be the square region shown in the picture:



Clearly, a point (x, y) lies in the square A if and only if $0 \leq x \leq 1$ and $0 \leq y \leq 1$. In other words, we have

$$A = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}.$$

Similarly, let C be the circle of radius one centred at the origin.



It is well-known that the equation for C is $x^2 + y^2 = 1$. What this means is that the points on C are precisely the vectors (x, y) for which $x^2 + y^2 = 1$, or in other words that

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

It is equally well-known that the parametric equation for C is $(x, y) = (\cos(\theta), \sin(\theta))$. This means that C can also be described as

$$C = \{(\cos(\theta), \sin(\theta)) \mid \theta \in \mathbb{R}\}. \quad \square$$

Example 3.2.4. We write $\mathbb{R}^3$ for the set of all three-dimensional vectors. Thus, for example, we have $(1, 2, 3) \in \mathbb{R}^3$ and $(-3, -4, -5) \in \mathbb{R}^3$ but $(1, 1) \not\in \mathbb{R}^3$ and $2^3 = 8 \not\in \mathbb{R}^3$ and $(1, 2, 3, 4) \not\in \mathbb{R}^3$. Any three-dimensional geometric figure can be thought of as a subset of $\mathbb{R}^3$. For example, the set

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid |x| + |y| + |z| \leq 1\}$$

is a solid octahedron with vertices at $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$. $\square$

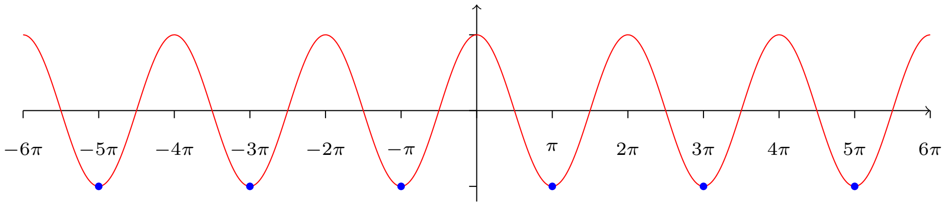
3.3 Sets of solutions

When we solve an equation, or a system of equations, what we are really doing is to find the *set* of solutions. In good cases there will be precisely one solution, but in other cases there may be no solutions (so the solution set is empty) or there may be many solutions (in which case we may need to find a creative way to describe the solution set).

Example 3.3.1. Suppose we are asked to solve the equation $x^2 - 5x + 6 = 0$. By factoring or by the quadratic formula, we see that the solutions are $x = 2$ or $x = 3$. The conclusion is best expressed as an equation between sets:

$$\{x \in \mathbb{R} \mid x^2 - 5x + 6 = 0\} = \{2, 3\}. \quad \square$$

Example 3.3.2. Suppose we are asked to solve the equation $\cos(\theta) = -1$. You should be familiar with the fact that $\cos(\theta) = -1$ if and only if θ is an odd multiple of π:



This fact is best expressed as an equation between sets:

$$\{\theta \in \mathbb{R} \mid \cos(\theta) = -1\} = \{\ldots, -5\pi, -3\pi, -\pi, \pi, 3\pi, 5\pi, \ldots\} = \{(2n+1)\pi \mid n \in \mathbb{Z}\}. \quad \square$$

Example 3.3.3. Suppose we are asked to solve the equations

$$\begin{aligned} x + y + z &= 1 \\ x - y + z &= -1. \end{aligned}$$

In other words, we must find the set of all vectors (x, y, z) for which these two equations are true.

By subtracting the two equations we get $y = 1$, and we can substitute this back into the first equation to get $z = -x$; this means that $(x, y, z) = (x, 1, -x)$ (and x can be anything). In terms of sets, the conclusion is as follows:

$$\{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\ ,\ x - y + z = -1\} = \{(x, 1, -x) \mid x \in \mathbb{R}\}. \quad \square$$

3.4 Boolean operations

- The *intersection* of A and B is $A \cap B = \{x \mid x \in A \text{ and also } x \in B\}$
- The *union* of A and B is $A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ (or both) }\}$
- The *relative complement* of B in A is $A \setminus B = \{x \in A \mid x \notin B\}$
- Often all sets under consideration are subsets of some fixed set E (typically $E = \mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{R}^3$). We then write A^c for $E \setminus A$, and call this the *complement* of A. Note that $A^{cc} = A$.

Example 3.4.1. Consider the sets

$$\begin{aligned} A &= \{1, 2, 3, 4, 5\} \\ B &= \{4, 5, 6, 7\}. \end{aligned}$$

Then

$$\begin{aligned} A \cap B &= \{4, 5\} \\ A \cup B &= \{1, 2, 3, 4, 5, 6, 7\} \\ A \setminus B &= \{1, 2, 3\} \\ B \setminus A &= \{6, 7\}. \quad \square \end{aligned}$$

Example 3.4.2. Consider the sets

$$\begin{aligned} A &= \{2n + 1 \mid n \in \mathbb{Z}\} && = \{\ldots, -7, -5, -3, -1, 1, 3, 5, 7, \ldots\} \\ B &= \{3n \mid n \in \mathbb{Z}\} && = \{\ldots, -12, -9, -6, -3, 0, 3, 6, 9, 12, \ldots\} \\ C &= \{6n \mid n \in \mathbb{Z}\} && = \{\ldots, -18, -12, -6, 0, 6, 12, 18, \ldots\} \\ D &= \{6n + 3 \mid n \in \mathbb{Z}\} && = \{\ldots, -15, -9, -3, 3, 9, 12, 15, \ldots\} \end{aligned}$$

(a) Every element of B lies either in C or in D, but not both; the sets C and D cover B with no overlap. This is easy to see if we write them out like this:

$$
\begin{array}{lllllllllllll}
B = \{\dots, & -15, & -12, & -9, & -6, & -3, & 0, & 3, & 6, & 9, & 12, & 15, & \dots\} \\
C = \{\dots, & & -12, & & -6, & & 0, & & 6, & & 12, & & \dots\} \\
D = \{\dots, & -15, & & -9, & & -3, & & 3, & & 9, & & 15, & \dots\}.
\end{array}
$$

We therefore have $C \cup D = B$ and $C \cap D = \emptyset$.

(b) Next, note that $B \cap A$ is the set of numbers that lie in B and also in A. Of course, A is just the set of odd numbers, so $B \cap A$ is the set of odd numbers in B. As

$$B = \{\dots, -18, -15, -12, -9, -6, -3, 0, 3, 6, 9, 12, 15, 18, \dots\},$$

we see that

$$B \cap A = \{\dots, -15, -9, -3, 3, 9, 15, \dots\},$$

so $B \cap A = D$.

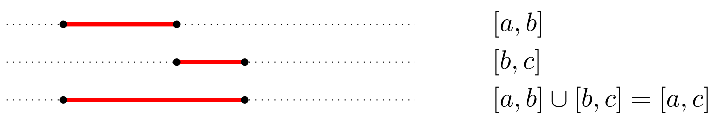
(c) Similarly, the set $B \setminus A$ consists of all numbers in B that do not lie in A, or in other words, all even numbers in B. We thus see that

$$B \setminus A = \{\dots, -18, -12, -6, 0, 6, 12, 18, \dots\},$$

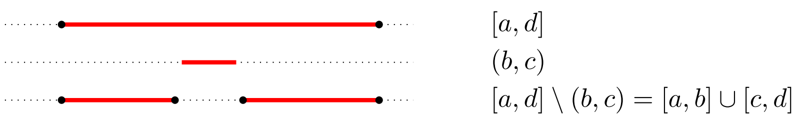
so $B \setminus A = C$. □

Example 3.4.3. Suppose we have real numbers a, b, c and d with $a < b < c < d$. We then have

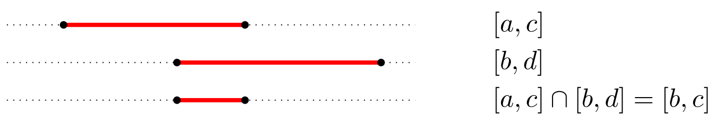
- $[a, b] \cup [b, c] = [a, c]$



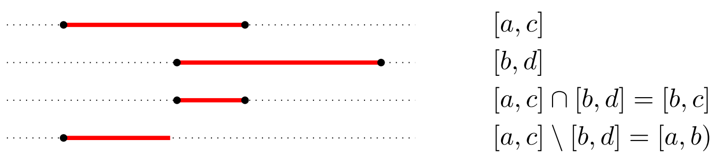
- $[a, b] \cup [c, d] = [a, d] \setminus (b, c)$



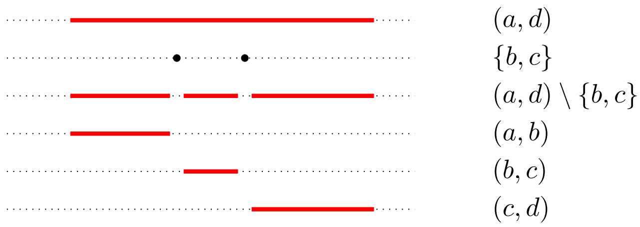
- $[a, c] \cap [b, d] = [b, c]$



- $[a, c] \cap [b, d] = [b, c]$ and $[a, c] \setminus [b, d] = [a, b)$



- $(a,b) \cup (b,c) \cup (c,d) = (a,d) \setminus \{b,c\}$



Example 3.4.4. Let A, B and C be the planes in $\mathbb{R}^3$ with equations $x+y=2$, $y+z=2$ and $z+x=2$. Then $A \cap B \cap C$ consists of the points where all three of these equations hold, in other words

$$A \cap B \cap C = \{(x,y,z) \in \mathbb{R}^3 \mid x+y=y+z=z+x=2\}.$$

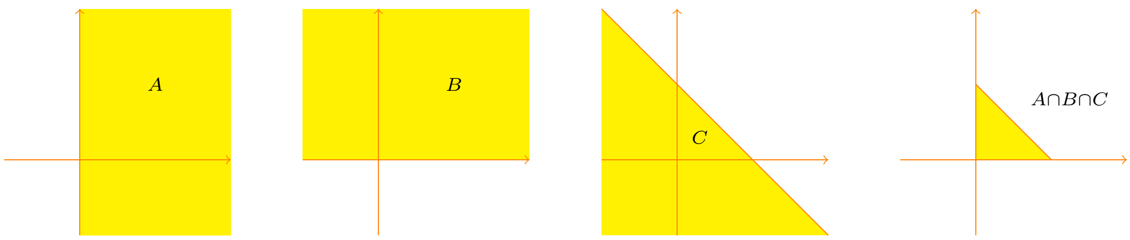
By adding the equations $x+y=2$ and $y+z=2$ and subtracting the equation $z+x=2$, we see that y must be equal to one. Similarly, we must have $x=z=1$. Thus, if we put $P=(1,1,1)$, then the three planes meet only at P, so $A \cap B \cap C = \{P\}$. □

Exercise 3.4.5. Put

$$A = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0\}$$
$$B = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$$
$$C = \{(x,y) \in \mathbb{R}^2 \mid x+y \leq 1\}.$$

Draw A, B, C and $A \cap B \cap C$. What is $A \cup B \cup C$?

Solution:



The sets A, B and C together cover the whole plane, so $A \cup B \cup C = \mathbb{R}^2$.

Exercise 3.4.6. Put

$$A = \{n \in \mathbb{Z} \mid n \text{ is odd}\}$$
$$B = \{4m-1 \mid m \in \mathbb{Z}\}$$
$$C = \{4m+1 \mid m \in \mathbb{Z}\}$$
$$D = \{n^2 \mid n \in A\}.$$

(a) List the elements of A, B, C and D. (Of course these sets are infinite, but you can use notation like this: $\mathbb{Z} = \{\dots,-2,-1,0,1,2,\dots\}$.)

(b) Describe $B \cap C$ and $B \cup C$.

(c) Which, if any, of the following are true: (i) $C \subseteq D$ (ii) $D \subseteq C$ (iii) $D = C$?

(d) Can you justify your answer to (c)?

Solution:

(a)
$$\begin{aligned} A &= \{\dots,-7,-5,-3,-1,1,3,5,7,9,11,13,15,\dots\} \\ B &= \{\dots,-13,-9,-5,-1,3,7,11,15,19,23,\dots\} \\ C &= \{\dots,-11,-7,-3,1,5,9,13,17,21,25,29,\dots\} \\ D &= \{1,9,25,49,\dots\}. \end{aligned}$$

(b) The sets B and C do not overlap, so $B \cap C = \emptyset$. We also have $B \cup C = A$, as can be seen from the following picture:

$$\begin{array}{lllllllllllll} A = \{\dots, & -9, & -7, & -5, & -3, & -1, & 1, & 3, & 5, & 7, & 9, & 11, & \dots\} \\ B = \{\dots, & -9, & & -5, & & -1, & & 3, & & 7, & & 11, & \dots\} \\ C = \{\dots, & & -7, & & -3, & & 1, & & 5, & & 9, & & \dots\}. \end{array}$$

(c) (i) and (iii) are false, because C contains many negative numbers, whereas all numbers in D are positive. All the listed elements of D lie in C, so it appears that $D \subseteq C$. To check that this is really right, note that any element $p \in D$ is the square of an odd number, so it has the form $p = (2n+1)^2$ for some $n \in \mathbb{Z}$. This can be rewritten as $p = 4(n^2+n)+1$, so $p = 4m+1$ for some $m \in \mathbb{Z}$, so $p \in C$.

Exercise 3.4.7. Put $A = (-2,1]$ and $B = [-1,2)$. What are $A \cap B$, $A \cup B$, $A \setminus B$ and $B \setminus A$?

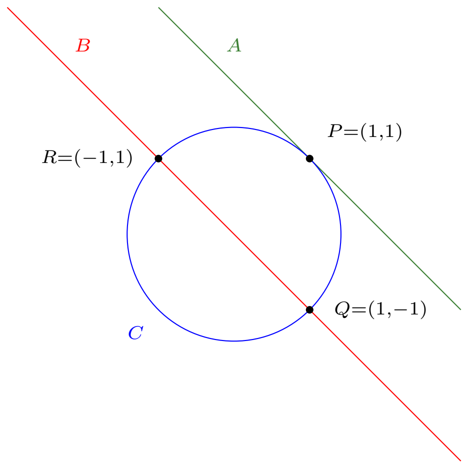
Solution:
$$\begin{aligned} A \cap B &= [-1,1] & A \cup B &= (-2,2) \\ A \setminus B &= (-2,-1) & B \setminus A &= (1,2) \end{aligned}$$

Exercise 3.4.8. Put

$$A = \text{ the line with equation } x+y=2 \tag{3.1}$$
$$B = \text{ the line with equation } x+y=0 \tag{3.2}$$
$$C = \text{ the circle with equation } x^2+y^2=2. \tag{3.3}$$

Find $A \cap B$, $B \cap C$ and $C \cap A$.

Solution: The picture is as follows:



It is clear from this that $A \cap B = \emptyset$ (because the lines A and B do not cross), that $B \cap C = \{Q,R\}$, and that $C \cap A = \{P\}$. This can also be seen from the equations:

(a) It is obviously not possible to have $x+y=2$ and $x+y=0$ at the same time, so $A \cap B = \emptyset$.

(b) Suppose that $x+y=0$ (so $y=-x$) and $x^2+y^2=2$. We can substitute $y=-x$ in the second equation to get $2x^2=2$, so $x=\pm 1$. If $x=1$ then $y=-1$ so we have the point Q. If $x=-1$ then $y=1$ so we have the point R. Thus $B \cap C = \{Q,R\}$.

(c) Suppose that $x + y = 2$ (so $y = 2 - x$) and $x^2 + y^2 = 2$. Putting $y = 2 - x$ in the second equation gives $x^2 + (2 - x)^2 = 2$, which gives $2x^2 - 4x + 4 = 2$, which simplifies to $(x - 1)^2 = 0$. The only solution is $x = 1$, which gives $y = 2 - x = 1$, so we have the point P. Thus $C \cap A = \{P\}$.

Exercise 3.4.9*. Put $A = \{2, 3, 4, 5, \dots\} = \{n \in \mathbb{Z} \mid n > 1\}$. Consider the set

$$B = A \setminus \{ab \mid a, b \in A\}.$$

Describe this in a simpler way.

Solution: B is just the set of prime numbers. To explain this, recall that a number n is *composite* if it can be written as $n = ab$, where a and b are integers with $a, b > 1$. In other words, the set C of composite numbers is $\{ab \mid a, b \in A\}$. A natural number p is *prime* if it is greater than one and not composite, so the set P of primes is $P = A \setminus C = A \setminus \{ab \mid a, b \in A\}$, or in other words $P = B$.

Exercise 3.4.10. Suppose that $B \subseteq A$. What are $A \cap B$ and $A \cup B$?

Solution: We have $A \cap B = B$ and $A \cup B = A$.

Exercise 3.4.11. Give an example of sets A, B and C where $A \subseteq B \cup C$ but $A \not\subseteq B$ and $A \not\subseteq C$.

Solution: The simplest example is where $A = \{0, 1\}$ and $B = \{0\}$ and $C = \{1\}$. Alternatively, we can take $A = [-1, 1]$ and $B = (-\infty, 0]$ and $C = [0, \infty)$.

3.5 De Morgan's laws

Just as there are systematic rules for simplifying algebraic expressions, there are also rules for simplifying expressions in the language of set theory. The rules for algebra involve the operations of addition, subtraction, multiplication and division. The rules for set theory involve unions, intersections, and (relative) complements. Among the most important rules are *De Morgan's laws*, which we now explain.

We fix a set E (sometimes called the *universal set*) and consider only subsets of E. (Often E will be $\mathbb{Z}$ or $\mathbb{R}$ or $\mathbb{R}^2$ or $\mathbb{R}^3$.) As usual, we write

$$A^c = E \setminus A = \{x \in E \mid x \notin A\},$$

and call this the *complement* of A.

The first thing to note is that $A^{cc} = A$ for any set $A \subseteq E$. Indeed, A^c consists of the things that are not in A, so the things that are *not* in A^c are precisely the things that *are* in A. For example, take $E = \mathbb{Z}$ and $A = \{\text{even integers}\}$. Then

$$\begin{aligned}
A^c &= \{\text{ integers that are not in } A \ \} \\
&= \{\text{ integers that are not even }\} \\
&= \{\text{ odd integers}\} \\
A^{cc} &= \{\text{ integers that are not in } A^c \ \} \\
&= \{\text{ integers that are not odd }\} \\
&= \{\text{ even integers }\} = A.
\end{aligned}$$

Proposition 3.5.1 (De Morgan's laws)**.** *For any subsets A and B of E, we have*

$$\begin{aligned}
(A \cap B)^c &= A^c \cup B^c \\
(A \cup B)^c &= A^c \cap B^c.
\end{aligned}$$

In words: the complement of the intersection is the union of the complements, and the complement of the union is the intersection of the complements.

You may find that the easiest way to understand this is to do the examples. Alternatively, here is a logical argument. Note that x lies in $A \cap B$ if and only if x lies in A and also in B. Thus, if x does not lie in $A \cap B$, then it must either lie outside A, or lie outside B. In other words, it must either lie in A^c, or in B^c. Equivalently, it must lie in $A^c \cup B^c$. In symbols, we have

$$\begin{aligned} x \in (A \cap B)^c &\Leftrightarrow (x \notin A) \text{ or } (x \notin B) \\ &\Leftrightarrow (x \in A^c) \text{ or } (x \in B^c) \\ &\Leftrightarrow x \in (A^c \cup B^c). \end{aligned}$$

This shows that $(A \cap B)^c = A^c \cup B^c$. You can show in a similar way that $(A \cup B)^c = A^c \cap B^c$.

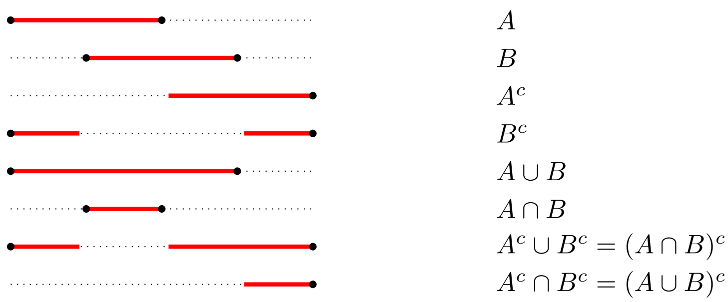
Exercise 3.5.2. In each of the following cases, find A^c, B^c, $A \cup B$, $A \cap B$, $A^c \cup B^c$ and $A^c \cap B^c$, and check that De Morgan's laws hold.

(a) $E = [0, 4]$, $A = [0, 2]$ and $B = [1, 3]$

(b) $E = \{1, 2, 3, \dots, 6\}$ and $A = \{n \in E \mid n \text{ is even }\}$, and $B = \{n \in E \mid n \text{ is divisible by } 3\}$

(c) $E = \{u, v, w, x\}$, $A = \{u, v\}$, $B = \{u, w\}$. (Here u, v, w and x could be just about anything, but you should assume that they are all different.)

Solution:

(a) 

(b)
$$\begin{array}{rlcccccccl} A = & \{ & & 2, & & 4, & & 6 & \} & \\ B = & \{ & & & 3, & & & 6 & \} & \\ A^c = & \{ & 1, & & 3, & & 5 & & \} & \\ B^c = & \{ & 1, & 2, & & 4, & 5 & & \} & \\ A \cup B = & \{ & & 2, & 3, & 4, & & 6 & \} & \\ A \cap B = & \{ & & & & & & 6 & \} & \\ A^c \cup B^c = & \{ & 1, & 2, & 3, & 4, & 5 & & \} & = (A \cap B)^c \\ A^c \cap B^c = & \{ & 1, & & & & 5 & & \} & = (A \cup B)^c. \end{array}$$

(c)
$$\begin{array}{rlcccccl} A = & \{ & u, & v & & & \} & \\ B = & \{ & u, & & w & & \} & \\ A^c = & \{ & & & w, & x & \} & \\ B^c = & \{ & & v, & & x & \} & \\ A \cup B = & \{ & u, & v, & w & & \} & \\ A \cap B = & \{ & u & & & & \} & \\ A^c \cup B^c = & \{ & & v, & w, & x & \} & = (A \cap B)^c \\ A^c \cap B^c = & \{ & & & & x & \} & = (A \cup B)^c. \end{array}$$

3.6 Inequalities

Sets are often described using *inequalities*. We have seen many examples of this already, most obviously the intervals $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ and so on. In this section review the rules for working with inequalities, and give some further examples.

Terminology:

- The notation $x > y$ means that x is greater than y, and not equal to y. It is thus true that $5 > 4$ and $-2 > -3$, but false that $2 > 2$.
- The notation $x \geq y$ means that x is either greater than y, or equal to y. It is thus true that $2 \geq 2$ and $-5 \geq -7$, but false that $-1 \geq 1$.
- A real number x is *strictly positive* if $x > 0$.
- A real number x is *weakly positive* or *nonnegative* if $x \geq 0$. Thus 0 is weakly positive, but not strictly positive.
- The *absolute value* of a real number x is

$$\operatorname{abs}(x) = |x| = \text{the distance between 0 and } x,$$

  so if $t \geq 0$ then $|t| = |-t| = t$.
- The distance between two real numbers x and y is $|x - y|$.

Warning 3.6.1. Inequalities do not really make sense for complex numbers; numbers such as $1 + i$ are neither positive nor negative. If z is a number that might be complex, then the notation $z \geq 0$ actually means that z does *not* have an imaginary part after all, it is a nonnegative real number. All numbers considered in this section will be real.

The algebraic rules for working with inequalities are as follows:

(1) For any real number x we have $x^2 \geq 0$.

(2) For any c, the inequalities $x < y$ and $x + c < y + c$ are equivalent.

(3) If $u < v$ and $x < y$ then $u + x < v + y$.

(4) For any $c > 0$, the inequalities $x < y$ and $cx < cy$ are equivalent.

(5) If $c < 0$ then the inequality $x < y$ is equivalent to $cy < cx$ (note order reversed). In particular, $x < y \quad\Leftrightarrow\quad -y < -x$.

In slightly different language:

(1) All squares are weakly positive.

(2) You can add any real number to both sides of an inequality.

(3) You can add two inequalities together.

(4) You can multiply an inequality by any strictly positive number.

(5) You can multiply an inequality by any strictly negative number, provided that you also reverse the inequality.

Now suppose that u, v, x and y are all strictly positive.

(6) If $u < v$ and $x < y$ then $ux < vy$.

(7) If $x < y$ then $1/y < 1/x$ (note order reversed).

(8) If $a > 0$ then the inequality $x < y$ is equivalent to $x^a < y^a$.
In particular, $x < y \quad\Leftrightarrow\quad x^2 < y^2 \quad\Leftrightarrow \sqrt{x} < \sqrt{y}$.

Arguments with inequalities tend to have more subtle logic than arguments with equations.

Exercise 3.6.2. All the following statements are *false*:

(a) Let x and y be real numbers with $x^2 < y^2$; then $x < y$.

(b) Let a, b, c and d be real numbers such that b and d are nonzero and $\frac{a}{b} > \frac{c}{d}$; then $ad > bc$.

(c) Let x, y, a and b be real numbers with $x < y$ and $a < b$; then $ax < by$

(d) Let θ and ϕ be real numbers with $\theta < \phi$; then $\sin(\theta) < \sin(\phi)$.

To see that (a) is false, consider $x = 2$ and $y = -3$; than $x^2 = 4$ and $y^2 = 9$, so $x^2 < y^2$, but it is *not* true that $x < y$. This is a "typical" counterexample; the simplest possible counterexample is where $x = 0$ and $y = -1$. Find counterexamples for the other three statements. Try make your answers as simple as possible.

Solution:

(b) Put $a = b = -1$, $c = 0$ and $d = 1$. Then $a/b = 1$ and $c/d = 0$, so $a/b > c/d$. However, $ad = -1$ and $bc = 0$, so it is *not* true that $ad > bc$.

(c) Put $x = a = -1$ and $y = b = 0$. Then $x < y$ and $a < b$, but $ax = 1$ and $by = 0$, so it is *not* true that $ax < by$.

(d) Put $\theta = \pi/2$ and $\phi = \pi$. Then $\theta < \phi$ but $\sin(\theta) = 1$ and $\sin(\phi) = 0$ so it is *not* true that $\sin(\theta) < \sin(\phi)$.

Exercise 3.6.3*. Show that when $a, b > 0$ we have $\frac{a+b}{2} \geq \sqrt{ab}$. Hints:

(i) What do we know about the sign of $(a - b)^2$?

(ii) Deduce that $a^2 + 2ab + b^2 \geq 4ab$.

(iii) What do the rules say about taking square roots?

For which values of a and b do we have $\frac{a+b}{2} = \sqrt{ab}$ (so that the inequality is actually an equality)?

Solution: The square of any real number is weakly positive, so $(a - b)^2 \geq 0$. By expanding this out, we see that $a^2 - 2ab + b^2 \geq 0$. Rule (2) says that we can add $4ab$ to both sides of this inequality, so $a^2 + 2ab + b^2 \geq 4ab$. We can rewrite this as $(a + b)^2 \geq 4ab$. Both sides of this inequality are positive, so rule (8) allows us to take square roots, giving $a + b \geq \sqrt{4ab} = 2\sqrt{ab}$. Rule (4) says that we can multiply both sides by the positive number $\frac{1}{2}$, giving $\frac{a+b}{2} \geq \sqrt{ab}$ as claimed.

Now suppose that this inequality is actually an equality, so $\frac{a+b}{2} = \sqrt{ab}$. We can square both sides and multiply by 4 to see that $a^2 + 2ab + b^2 = 4ab$. We then subtract $4ab$ from both sides and factorize to see that $(a - b)^2 = 0$, so $a - b = 0$, so $a = b$. Thus, $\frac{a+b}{2} = \sqrt{ab}$ if and only if $a = b$. **Background:** The number $\frac{a+b}{2}$ is of course the *average* (or *mean*) of a and b; more precisely, it is called the *arithmetic mean*. The *geometric mean* of a and b is $\sqrt{ab}$. We have thus shown that the arithmetic mean is greater than or equal to the geometric mean; this is known as the *AMGM inequality*. More generally, given numbers $c_1, \ldots, c_n > 0$, the arithmetic mean A and the geometric mean G are defined by

$$A = (c_1 + \cdots + c_n)/n$$
$$G = (c_1 \cdots c_n)^{1/n}.$$

It is still true that $A \geq G$, but the proof is more complicated.

Exercise 3.6.4. Write the set $A = \{x \in \mathbb{R} \mid x^3 \geq x\}$ as a union of intervals.

Solution: The inequality $x^3 \geq x$ reduces to $x^3 - x \geq 0$, or equivalently $x(x + 1)(x - 1) \geq 0$. In other words, we have

$$A = \{x \in \mathbb{R} \mid x(x + 1)(x - 1) \geq 0\}.$$

There are four cases to consider:

(a) If $x < -1$ then $x < 0$ and $x + 1 < 0$ and $x - 1 < 0$. The product of three strictly negative numbers is strictly negative, so $x(x + 1)(x - 1) < 0$, so $x \notin A$.

(b) If $-1 \leq x \leq 0$ then $x \leq 0$ and $x - 1 < 0$ and $x + 1 \geq 0$. The product of two weakly negative numbers is weakly positive, so $x(x - 1) \geq 0$. We can now multiply by $x + 1$ (noting that $x + 1 \geq 0$, so there is no reversal) to get $x(x - 1)(x + 1) \geq 0$. This shows that $x \in A$.

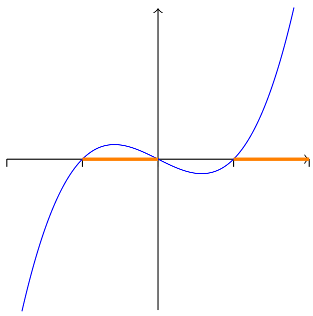
(c) If $0 < x < 1$ then $x - 1 < 0$ and $x, x + 1 > 0$ so $x(x + 1)(x - 1) < 0$, so $x \notin A$.

(d) If $x \geq 1$ then $x - 1 \geq 0$ and $x > 0$ and $x + 1 > 0$, so $x(x + 1)(x - 1) \geq 0$, so $x \in A$.

We conclude that

$$\begin{aligned} A &= \{x \in \mathbb{R} \mid (-1 \leq x \leq 0) \text{ or } (x \geq 1)\} \\ &= [-1, 0] \cup [1, \infty). \end{aligned}$$

This can also be read off from the graph of $y = x^3 - x$:



Exercise 3.6.5. Find the set $B = \{x \in \mathbb{R} \mid x^4 < x^2\}$.

Solution: We first rewrite the inequality as $x^4 - x^2 < 0$, or equivalently $x^2(x - 1)(x + 1) < 0$. There are four cases to consider:

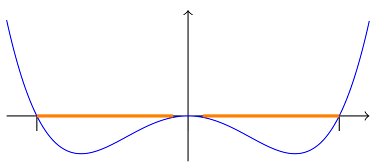
(a) If $x \leq -1$ then $x - 1 \leq 0$ and $x + 1 \leq 0$, so $(x - 1)(x + 1) \geq 0$. We also have $x^2 \geq 0$, so $x^2(x - 1)(x + 1) \geq 0$, so $x \notin B$.

(b) Now suppose that $-1 < x < 1$ and $x \neq 0$. Then $x - 1 < 0$ and $x + 1 > 0$ and $x^2 > 0$, so $x^2(x - 1)(x + 1) < 0$ (because we have one strictly negative number multiplied by two strictly positive numbers). Thus $x \in B$.

(c) We have an exception at $x = 0$: it is clearly *not* true that $0^4 < 0^2$ (because this is a *strict* inequality), so $0 \notin B$.

(d) If $x \geq 1$ then all three factors are weakly positive, so $x^2(x - 1)(x + 1) \geq 0$, so $x \notin B$.

Thus $B = (-1, 1) \setminus \{0\} = (-1, 0) \cup (0, 1)$. This can also be read off from the graph of $y = x^4 - x^2$:



Exercise 3.6.6. Find the set $C = \{x \in \mathbb{R} \mid x^3 + 3x > 3x^2 + 1\}$.

Solution: We first rewrite the inequality as $x^3 - 3x^2 + 3x - 1 > 0$, or equivalently $(x - 1)^3 > 0$. This clearly holds if and only if $x > 1$, so $C = (1, \infty)$.

Exercise 3.6.7. Find the set $D = \{x \in \mathbb{R} \mid f(x) < 0\}$, where

$$f(x) = x(x-1)(x-2)(x-3)(x-4)(x-5).$$

Solution: When $x < 0$, we have a product of six strictly negative terms, which is strictly positive. Each time we pass through one of the roots of $f(x)$, one of the six signs changes, so the overall sign changes. This means that $f(x)$ is strictly positive for $0 < x < 1$, then strictly negative for $1 < x < 2$ and so on. After $x = 5$ there are no more sign changes, and $f(x)$ is always strictly positive. It follows that $f(x) < 0$ precisely when $0 < x < 1$ or $2 < x < 3$ or $4 < x < 5$, so

$$D = (0,1) \cup (2,3) \cup (4,5).$$

Exercise 3.6.8*. Let a, b and c be three different real numbers, and put $f(x) = (x-a)(x-b)$ and $g(x) = (x-b)(x-c)$. Find the set

$$E = \{x \in \mathbb{R} \mid f(x)^2 + g(x)^2 \leq 0\}.$$

(This requires logic, not calculation.)

Solution: For any x we have $f(x)^2 \geq 0$ and $g(x)^2 \geq 0$, so the only way that we can have $f(x)^2 + g(x)^2 \leq 0$ is if $f(x)^2 = g(x)^2 = 0$. This in turn means that $f(x) = 0$ (so $x \in \{a, b\}$) and $g(x) = 0$ (so $x \in \{b, c\}$). It follows that x must be equal to b. Thus, we have $E = \{b\}$.

3.7 Proving relations between sets

Given sets A and B, it is important to be able to check whether A is a subset of B (or whether A is actually equal to B).

(a) To show that A is a subset of B, you must check that every element of A is also an element of B.

(b) To show that A is *not* a subset of B, it is enough to find a single example of an element of A that does not lie in B.

(c) To show that A is equal to B, you must show that every element of A is also an element of B, *and* that every element of B is also an element of A.

(d) Alternatively, to show that A is equal to B, you can show that every element of A is an element of B, and that every element that is not in A is also not in B.

(e) To show that A is *not* equal to B, it is enough to find *either* an element that lies in A but not in B, *or* an element that lies in B but not in A.

Example 3.7.1. I claim that $\{1 + x^2 \mid x \in \mathbb{R}\} \subseteq (1/2, \infty)$. To check this, we must show that every element of the first set is also an element of the second set. In other words, we must show that every number of the form $1 + x^2$ (with $x \in \mathbb{R}$) satisfies $1 + x^2 > 1/2$. This is clear, because $x^2 \geq 0$, so $1 + x^2 \geq 1 > 1/2$. □

Example 3.7.2. I claim that $\{n + \frac{1}{2} \mid n \in \mathbb{Z}\} \subseteq \mathbb{R} \setminus \mathbb{Z}$. To check this, we must show that for every integer n, the number $n + \frac{1}{2}$ lies in $\mathbb{R} \setminus \mathbb{Z}$. In other words, we must show that $n + \frac{1}{2}$ is a real number that is not an integer; and this is clear. □

Example 3.7.3. Consider the sets

$$A = \{(x,y) \in \mathbb{Z}^2 \mid x^2 + y^2 = 1\}$$
$$B = \{(x,y) \in \mathbb{R}^2 \mid xy = 0\}.$$

I claim that $A \subseteq B$. Indeed, as we saw in Example 3.1.3, the set A has only four elements, given by $(x,y) = (1,0)$, $(x,y) = (0,1)$, $(x,y) = (-1,0)$ and $(x,y) = (0,-1)$. For each of these four points we obviously have $xy = 0$, so each of the four points in A also lie in B, so $A \subseteq B$. □

Exercise 3.7.4. Which of the following cases is it true that $B \subseteq A$? Justify your answers.

(a) $B = \mathbb{Z}$ and $A = \{x \in \mathbb{R} \mid \cos(2\pi x) = 0\}$

(b) $B = \{x \in \mathbb{R} \mid \cos(\pi x/2) = 0\}$ and $A = \mathbb{Z}$

(c) $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ and $A = \{(x, y) \in \mathbb{R}^2 \mid x, y \in [-1, 1]\}$

(d) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ and $A = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \in \mathbb{Z}\}$.

(e) $B = \{x^3 \mid 0 \leq x \leq 2\}$ and $A = \{x^5 \mid 0 \leq x \leq 2\}$.

Solution:

(a) Here $0 \in B$ but $0 \notin A$ (because $\cos(2\pi \times 0) = \cos(0) = 1 \neq 0$) so $B \not\subseteq A$.

(b) Here $B \subseteq A$. Indeed, we see from the graph that $\cos(\theta) = 0$ precisely when θ has the form $(2k+1)\pi/2$ for some integer k. This means that $\cos(\pi x/2) = 0$ precisely when $\pi x/2$ has the form $(2k+1)\pi/2$, or equivalently x has the form $2k+1$, or equivalently x is an odd integer. Thus B is the set of odd integers, which is a subset of the set A of all integers.

(c) Here $B \subseteq A$. Indeed, if $(x, y) \in B$ then $x^2 + y^2 \leq 1$, and y^2 is nonnegative so we must have $x^2 \leq 1$, which means that $-1 \leq x \leq 1$, or in other words $x \in [-1, 1]$. By a similar argument we also have $y \in [-1, 1]$, so $(x, y) \in A$. This shows that every element of B is also an element of A, so $B \subseteq A$.

(d) Here the vector $(1/\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2})$ lies in B but not in A, so $B \not\subseteq A$.

(e) Here it is not hard to see that $B = [0, 8]$ and $A = [0, 32]$ so $B \subseteq A$.

Chapter 4: General theory of functions

Key ideas:

- The notion of a function
- The domain, codomain and range of a function
- The composite of two functions
- The inverse of a function

Skills:

You should make sure that you can do the following:

- Decide whether a function definition is valid or not
- Find the range of a function
- Find the composite of two functions
- Find the inverse of a function

4.1 Definitions

Definition 4.1.1. Let A and B be sets. A *function* f from A to B means an unambiguous rule giving an element $f(a) \in B$ for every element $a \in A$. We write $f\colon A \to B$ to indicate that f is function from A to B. The set A is called the *domain* of f, and the set B is called the *codomain*.

Example 4.1.2. We can define a function $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = 1 + x + x^2 + x^3$. The domain of this function is the set $\mathbb{R}$, and the codomain is the same set. The function associates to each number $x \in \mathbb{R}$ the number $1 + x + x^2 + x^3$. □

Example 4.1.3. We can define a function $g\colon [1,2] \to [10,20]$ by $g(x) = 10 + x$. Here the domain is the set $[1,2]$ and the codomain is the set $[10,20]$. Note that the values of $g(x)$ all lie in the interval $[11,12]$, so they do not fill up all of the codomain; this is allowed. □

Remark 4.1.4. Here are some pitfalls.

(a) The element $f(a)$ specified by the rule, must *always* lie in B for *all* $a \in A$. For example, we cannot define a map $f\colon (0,1) \to (0,2)$ by $f(x) = 1 + x + x^2 + x^3$, because $2/3 \in (0,1)$ but $f(2/3) = 65/27 \notin (0,2)$.

(b) The rule must define an element $f(a) \in B$ for *every* element $a \in A$. We cannot define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = (1+x)/(1-x)$, because $f(1)$ is not defined. (If you insist on saying that $f(1) = \infty$, then we have problem (a) again, because $f(1) \notin \mathbb{R}$.) To cure this problem we consider the set $\mathbb{R} \setminus \{1\}$ obtained by removing 1 from $\mathbb{R}$. The stated formula does give a function $f\colon \mathbb{R} \setminus \{1\} \to \mathbb{R}$.

(c) The rule must be unambiguous. For example, one might try to define a function $f\colon \mathbb{R} \to \mathbb{Z}$ by

$$f(x) = \{\text{ the nearest integer to } x\}.$$

Most of the time, this is fine: for example, we have

$$\begin{aligned} f(0.4) &= 0 \\ f(1000.1) &= 1000 \\ f(-12.34) &= -12 \end{aligned}$$

and so on. However, the integers 2 and 3 are both equally close to 2.5, so our rule does not specify unambiguously what $f(2.5)$ should be. We cannot consider f as a well-defined function until the rule has been clarified to resolve this ambiguity.

(d) The domain and codomain are part of the specification of the function. For example, we can define functions f_1, f_2 and f_3 by

$$\begin{array}{ll} f_1\colon \mathbb{R} \to \mathbb{R} & f_1(x) = x^2 \\ f_2\colon \mathbb{R} \to [0,\infty) & f_2(x) = x^2 \\ f_3\colon [0,1] \to [0,1] & f_3(x) = x^2 \end{array}$$

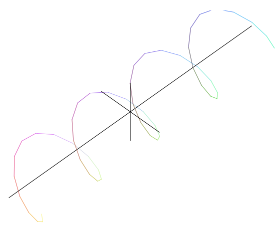
Strictly speaking, these are three different functions, even though they all have the same formula. In certain contexts this distinction will be important, but it will be glossed over when it is harmless to do so.

Example 4.1.5. For many of the functions that we consider, the domain and codomain will be subsets of $\mathbb{R}$. For example, we have the trigonometric functions $\sin, \cos\colon \mathbb{R} \to [-1,1]$, the exponential function $\exp\colon \mathbb{R} \to (0,\infty)$, the logarithm $\log\colon (0,\infty) \to \mathbb{R}$, and so on. These standard functions will be discussed in more detail in Chapter 5. □

Example 4.1.6. It is also common to consider functions whose domain or codomain is $\mathbb{R}^2$ or $\mathbb{R}^3$, or a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$, rather than $\mathbb{R}$. For example, given constants a, b and ω, we can define $f\colon \mathbb{R} \to \mathbb{R}^3$ by

$$f(t) = (at, b\cos(\omega t), b\sin(\omega t)).$$

This describes the position at time t of a charged particle moving in a uniform magnetic field. (The constants a, b and ω depend on the strength of the field, the charge of the particle, and its initial position and velocity; we will not discuss the details.) The particle curls around the x-axis as it moves along it, as illustrated in the following picture:



Similarly, given a constant c, we can define

$$g\colon \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}^3$$

by

$$g(x,y,z) = \left(\frac{-cx}{(x^2+y^2+z^2)^{3/2}}, \frac{-cy}{(x^2+y^2+z^2)^{3/2}}, \frac{-cz}{(x^2+y^2+z^2)^{3/2}} \right)$$

This again has a physical interpretation: if there is a charged particle at the origin, and another charged particle at the point (x,y,z), then the electric force felt by the second particle is $g(x,y,z)$. □

Example 4.1.7. There are many interesting and important functions involving matrices. For example, consider the set M of all 2×2 matrices (with entries in $\mathbb{R}$). An element of M is a matrix, say $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. The determinant of such a matrix is the number $\det(A) = ad - bc \in \mathbb{R}$. Thus, we have a rule that assigns an element $\det(A) \in \mathbb{R}$ to each element $A \in M$. In other words, we have a function $\det\colon M \to \mathbb{R}$.

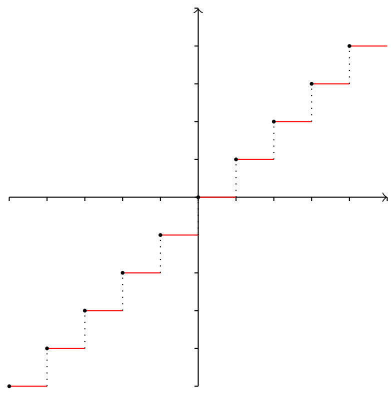
Similarly, the *trace* of a matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is defined to be the number $\mathrm{trace}(A) = a + d$. This gives another function $\mathrm{trace}\colon M \to \mathbb{R}$.

Finally, the *transpose* of a matrix A as above is the matrix $A^T = \left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$. We can define a function $\mathrm{trans}\colon M \to M$ by $\mathrm{trans}(A) = A^T$. □

Example 4.1.8. A function can be defined by a rule that is more complicated than a single formula. For example, we can define a function $f\colon \mathbb{R} \to \mathbb{Z}$ by the following rule: $f(x)$ is the largest integer n such that $n \leq x$. (This is often written as $\operatorname{floor}(x)$ or $\lfloor x \rfloor$.) For example, we have

$$\begin{aligned} f(-10.5) &= -11 \\ f(-0.99) &= -1 \\ f(-0.01) &= -1 \\ f(\pi) &= 3 \\ f(9.999) &= 9. \end{aligned}$$

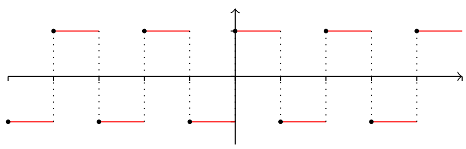
The graph is as follows:



Next, we can define $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(x) = (-1)^{\lfloor x \rfloor} = \begin{cases} +1 & \text{if } \lfloor x \rfloor \text{ is even} \\ -1 & \text{if } \lfloor x \rfloor \text{ is odd.} \end{cases}$$

The graph of $g(x)$ is as follows:



This is called a *square wave* . The CPU of a computer contains a "clock circuit", which generates a square wave, which is used to synchronise the actions of the various other parts of the chip. The advertised speed of the chip is the frequency of this square wave; in a 2GHz chip, the voltage jumps up and down 2×10^9 times each second. □

Example 4.1.9. If A is a finite set, then one can specify a function $f\colon A \to B$ simply by listing the values $f(a)$ for each element $a \in A$. For example, we can define a function

$$\operatorname{gray}\colon \{0, 1, \ldots, 7\} \to \{0, 1, \ldots, 7\}$$

by the rule

$$\begin{aligned} \operatorname{gray}(0) &= 0 & \qquad \operatorname{gray}(4) &= 6 \\ \operatorname{gray}(1) &= 1 & \operatorname{gray}(5) &= 7 \\ \operatorname{gray}(2) &= 3 & \operatorname{gray}(6) &= 5 \\ \operatorname{gray}(3) &= 2 & \operatorname{gray}(7) &= 4 \end{aligned}$$

This table of values is in itself a complete and acceptable definition of the function. There might or might not be some hidden reason why I chose to write down these particular values, but you do not need to know about that; the values themselves are all that you need in order to work with the function. □

Background: This function is called the *Gray code* , and is not as random as it might seem. It is easiest to understand if we rewrite the values in binary:

$$\begin{array}{ll} \text{gray}(0) = 000_2 & \text{gray}(4) = 110_2 \\ \text{gray}(1) = 001_2 & \text{gray}(5) = 111_2 \\ \text{gray}(2) = 011_2 & \text{gray}(6) = 101_2 \\ \text{gray}(3) = 010_2 & \text{gray}(7) = 100_2 \end{array}$$

The rule is as follows. You start with $\text{gray}(0) = 0$, and then define $\text{gray}(1)$ in terms of $\text{gray}(0)$, then define $\text{gray}(2)$ in terms of $\text{gray}(1)$, and so on. To define $\text{gray}(n)$, you look at the binary expansion of $\text{gray}(n-1)$ and flip a single bit from 0 to 1 or vice-versa. The bit to be flipped is taken as far to the right as possible, subject to the condition that $\text{gray}(n)$ must be different from $\text{gray}(0), \dots, \text{gray}(n-1)$. For example, to get $\text{gray}(4)$, we must flip one of the bits in $\text{gray}(3) = 010_2$. If we flip the rightmost bit, we get 011_2, which has already occurred as $\text{gray}(2)$, so this is disallowed. If we flip the middle bit, we get 000_2, which has already occurred as $\text{gray}(0)$, which is again disallowed. We thus have to flip the leftmost bit to get $\text{gray}(4) = 110_2$.

This function is useful if you want a digital system to measure the position of a wheel, for example. The most obvious kind of measurement device risks being wildly wrong for an instant as it changes from one state to the next, but this problem is cured by the Gray code, because of the fact that $\text{gray}(n)$ and $\text{gray}(n-1)$ differ in only one place.

Exercise 4.1.10*. Put $A = \{1, 2, 3\}$ and $B = \{\pi, -\pi\}$.

(a) Write down a function $f_1 : A \to B$. Any function will do.

(b) There are precisely eight different functions from A to B. Write down the other seven (you could call them $f_2, \dots, f_8$).

(c) Can you explain in a simple way why there are exactly eight different functions?

(d) Suppose more generally that we have finite sets C and D, with $|C| = n$ and $|D| = m$. How many different functions are there from C to D?

Solution:

(a) The simplest answer is the constant function defined by $f_1(1) = f_1(2) = f_1(3) = \pi$.

(b) The full list is as follows:

$$\begin{array}{lll} f_1(1) = \pi & f_1(2) = \pi & f_1(3) = \pi \\ f_2(1) = \pi & f_2(2) = \pi & f_2(3) = -\pi \\ f_3(1) = \pi & f_3(2) = -\pi & f_3(3) = \pi \\ f_4(1) = \pi & f_4(2) = -\pi & f_4(3) = -\pi \\ f_5(1) = -\pi & f_5(2) = \pi & f_5(3) = \pi \\ f_6(1) = -\pi & f_6(2) = \pi & f_6(3) = -\pi \\ f_7(1) = -\pi & f_7(2) = -\pi & f_7(3) = \pi \\ f_8(1) = -\pi & f_8(2) = -\pi & f_8(3) = -\pi \end{array}$$

(c) There are 2 ways to choose $f(1)$, then two ways to choose $f(2)$, then two ways to choose $f(3)$, making $2 \times 2 \times 2 = 8$ ways to choose a function $f : A \to B$ altogether.

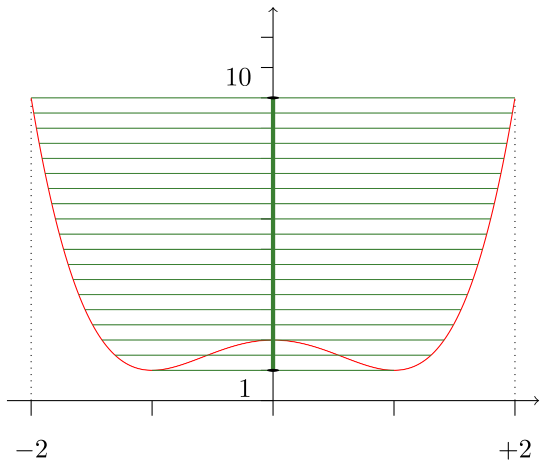
(d) As $|C| = n$, we can list the elements of C as $\{c_1, \dots, c_n\}$ say (without any repetitions). We then have m ways to choose $f(c_i)$ for each i, making $m \times \dots \times m = m^n$ ways to choose a function $f : C \to D$. Thus, there are precisely m^n different functions from C to D.

4.2 The range

Given a function $f\colon A \to B$, the *range* of f is the set of all $b \in B$ that can be written in the form $b = f(a)$ for some $a \in A$. We write $\mathrm{range}(f)$ or $f(A)$ for this set.

If A and B are intervals in $\mathbb{R}$, then we can display this graphically. We draw the graph of f (with the x-axis covering only the set A), then squash the graph sideways onto the y-axis. The resulting set on the y-axis is the range of f.

Example 4.2.1. Consider the function $f\colon [-2,2] \to \mathbb{R}$ given by $f(x) = (x^2-1)^2+1 = x^4-2x^2+2$. The graph is as follows:



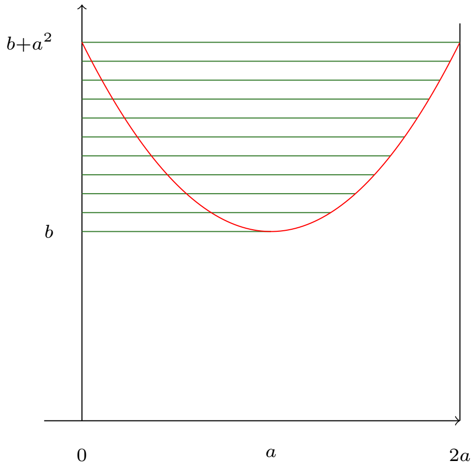
As x runs from -2 to 2, the function $f(x)$ runs from 10 down to 1, then back up to 2, back down to 1, and then back up to 10. It is thus clear that the range is $[1,10]$. □

Warning 4.2.2. Please avoid the **standard mistake**. Given a function $f\colon [a,b] \to \mathbb{R}$, you **cannot** find the range by working out $f(a)$ and $f(b)$ and saying that the range is $[f(a), f(b)]$. In the above example, we have a function $f\colon [-2,+2] \to \mathbb{R}$ with $f(-2) = f(2) = 10$. The incorrect rule would say that the range is the interval $[10,10]$, or in other words the set $\{10\}$, which is obviously wrong.

Example 4.2.3. Consider $f\colon [0,\infty) \to [0,\infty)$ given by $f(a) = a+1$ (so $A = B = [0,\infty)$). It is easy to see that the range is $[1,\infty)$. □

Exercise 4.2.4. Let a and b be strictly positive constants, and define $f\colon [0,2a] \to \mathbb{R}$ by $f(x) = (x-a)^2+b$. What is the range of f? (You may want to choose some random values for a and b, and sketch the graph, to see what is going on. However, your final answer should be given in a way that works for all a and b.)

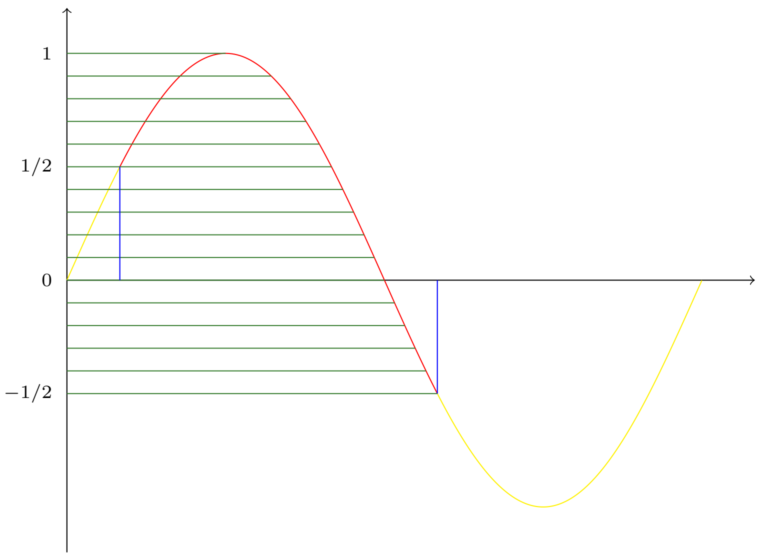
Solution: The picture is as follows:



As x runs from 0 to $2a$, the function $f(x)$ runs from $b+a^2$ down to b and then up to $b+a^2$ again. This makes it clear that the range is $[b, b+a^2]$.

Exercise 4.2.5. Define $f\colon [\pi/6, 7\pi/6] \to \mathbb{R}$ by $f(x) = \sin(x)$. What is the range of f?

Solution: The graph is as follows:



As x runs from $\pi/6$ to $7\pi/6$, the function $f(x)$ runs from $1/2$ up to 1 and then down to $-1/2$. This shows that the range is $[-1/2, 1]$.

Exercise 4.2.6**. Define $f\colon \mathbb{R} \to \mathbb{R}^2$ by $f(t) = (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$. (Note that this is meaningful for all t, because $1+t^2$ is never zero.) What is the range of f? (Hint: An earlier exercise almost gives the answer; you should at least get that far. One small detail needs to be adjusted, and then everything needs to be properly justified, which is quite intricate.)

Solution: Let C be the unit circle, or in other words

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

We saw in Exercise 2.3.1 that

$$\left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2 = 1,$$

so that $f(t)$ always lies in C, so $\text{range}(f) \subseteq C$. The obvious guess, which you could support with Maple or a graphing calculator, is that $\text{range}(f) = C$. However, this is not quite right.

Put $p = (-1, 0) \in C$; I claim that $f(t)$ is never equal to p, so that p does *not* lie in $\text{range}(f)$. Indeed, if $f(t)$ were equal to p we would have $(1-t^2)/(1+t^2) = -1$ and $2t/(1+t^2) = 0$. Multiplying the first equation by $1+t^2$ and then adding t^2 to both sides, we get $1 = -1$, which is impossible. Thus, we cannot have $f(t) = p$, so the range of f is not all of C. More precisely, if we put

$$C' = C \setminus \{p\} = \{(x, y) \in C \mid x > -1\},$$

we have shown that the range of f is a subset of C'. I claim that the range is actually all of C'. Indeed, suppose we have a point $(x, y) \in C'$, so that $x^2 + y^2 = 1$ and $x > -1$. This means that $1 + x$ is nonzero, so we can meaningfully put $t = y/(1 + x)$. Using the equation $y^2 = 1 - x^2 = (1 - x)(1 + x)$, we see that

$$\begin{aligned}
t^2 &= \frac{y^2}{(1+x)^2} = \frac{(1-x)(1+x)}{(1+x)^2} = \frac{1-x}{1+x} \\
1+t^2 &= 1 + \frac{1-x}{1+x} = \frac{2}{1+x} \\
1-t^2 &= 1 - \frac{1-x}{1+x} = \frac{2x}{1+x} \\
\frac{1-t^2}{1+t^2} &= \frac{2x}{1+x} \Big/ \frac{2}{1+x} = x \\
\frac{2t}{1+x} &= \frac{2y}{1+x} \Big/ \frac{2}{1+x} = y \\
f(t) &= \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) = (x, y).
\end{aligned}$$

This shows that (x, y) lies in the range of f, for any point (x, y) in C', so $C' \subseteq \text{range}(f)$. We have already seen that $\text{range}(f) \subseteq C'$, so in fact $\text{range}(f) = C'$ as claimed.

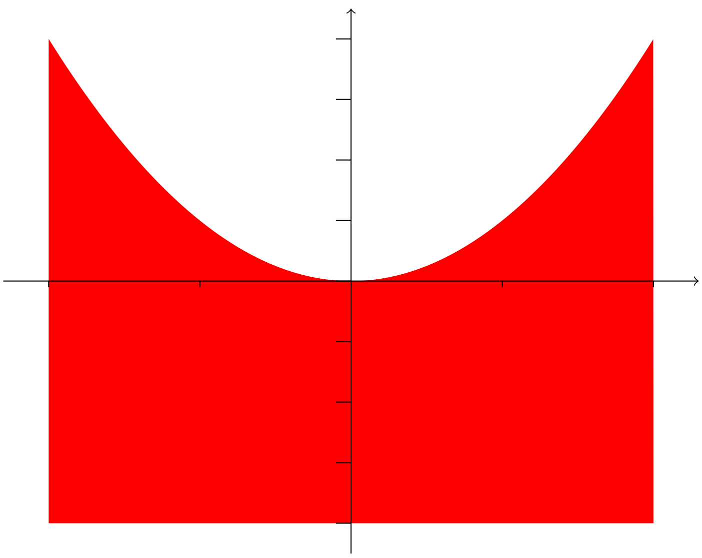
Exercise 4.2.7****.** Define a map $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(u, v) = (u + v, 4uv).$$

Consider the set

$$Q = \{(x, y) \in \mathbb{R}^2 \mid y \leq x^2\},$$

or in other words, the region below the parabola $y = x^2$:



Show that Q is the range of f.

Solution: We must first show that $\text{range}(f) \subseteq Q$, or in other words that $(u+v, 4uv)$ always lies in Q, or in other words that $4uv \leq (u+v)^2$ for all real numbers u and v. To see this, note that

$$\begin{aligned}(u+v)^2 - 4uv &= u^2 + 2uv + v^2 - 4uv \\ &= u^2 - 2uv + v^2 = (u-v)^2 \geq 0,\end{aligned}$$

so $(u+v)^2 \geq 4uv$ as claimed.

Secondly, we must show that $Q \subseteq \text{range}(f)$. Equivalently, given a point $(x, y) \in \mathbb{R}^2$ with $y \leq x^2$, we must find a point $(u, v) \in \mathbb{R}^2$ such that $u + v = x$ and $4uv = y$. For this, we note that $x^2 - y \geq 0$ by assumption, so it is meaningful to put $r = \sqrt{x^2 - y}$. We then put $u = (x + r)/2$ and $v = (x - r)/2$. It is clear that $u + v = x$, and

$$4uv = (x + r)(x - r) = x^2 - r^2 = x^2 - (x^2 - y) = y,$$

so

$$f(u, v) = (u + v, 4uv) = (x, y)$$

as required.

Exercise 4.2.8*. Show that there are well-defined functions $f\colon (0, \infty) \to (0, 1)$ and $g\colon (0, 1) \to (0, \infty)$, given by

$$\begin{aligned}f(x) &= x/\sqrt{1 + x^2} \\ g(y) &= y/\sqrt{1 - y^2}.\end{aligned}$$

Show that f and g are inverse to each other. (First just do the algebra to check that $f(g(x)) = x$ and $g(f(y)) = y$; then give a careful argument to show that everything is well-defined and that there are no problems with signs.)

Solution: I first claim that we can define a function $f\colon (0, \infty) \to (0, 1)$ by $f(x) = x/\sqrt{1 + x^2}$. Indeed, when $x > 0$ we certainly have $1 + x^2 > 0$ so $\sqrt{1 + x^2}$ is defined and is nonzero, so $f(x) = x/\sqrt{1 + x^2}$ is a well-defined number. However, if we want to consider f as a function from $(0, \infty)$ to $(0, 1)$, we need to show that $f(x)$ always lies in $(0, 1)$, or equivalently that $0 < x/\sqrt{1 + x^2} < 1$ for all $x > 0$. For this, we note that

$$0 < x^2 < 1 + x^2.$$

Everything here is positive, so it is legitimate to take square roots, and we deduce that

$$0 < x < \sqrt{1 + x^2}.$$

As $\sqrt{1 + x^2} > 0$, it is legitimate to divide by this quantity without changing the inequality signs, and we deduce that

$$0 < x/\sqrt{1 + x^2} < 1,$$

as required.

I now claim that we can define a function $g\colon (0, 1) \to (0, \infty)$ by $g(y) = y/\sqrt{1 - y^2}$. Indeed, if $y \in (0, 1)$ then $1 - y^2 > 0$, so $\sqrt{1 - y^2}$ is defined and is strictly positive, so $y/\sqrt{1 - y^2}$ is also defined and strictly positive.

Finally, I claim that f and g are inverses of each other, or in other words that $f(g(y)) = y$ for all $y \in (0, 1)$, and $g(f(x)) = x$ for all $x \in (0, \infty)$. To see this, note that

$$\begin{aligned}1 + g(y)^2 &= 1 + \frac{y^2}{1 - y^2} = \frac{1}{1 - y^2} \\ \sqrt{1 + g(y)^2} &= 1/\sqrt{1 - y^2} \\ f(g(y)) &= \frac{g(y)}{\sqrt{1 + g(y)^2}} = \frac{y}{\sqrt{1 - y^2}} / \frac{1}{\sqrt{1 - y^2}} = y.\end{aligned}$$

The proof that $g(f(x)) = x$ is similar.

4.3 Composition

Definition 4.3.1. Suppose we have functions $f\colon A \to B$ and $g\colon B \to C$. We then define a function $h = g \circ f\colon A \to C$ by $h(a) = g(f(a))$.

$$\begin{array}{ccccc} A & \xrightarrow{\quad f \quad} & B & \xrightarrow{\quad g \quad} & C \\ a & \longmapsto & f(a) & \longmapsto & g(f(a)) \end{array}$$

This is called the *composite* of f and g.

Example 4.3.2. Consider $f, g, h\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x + 1$ and $g(x) = 2x$. I claim that $f \circ f \circ g = g \circ f$. Indeed, we have

$$\begin{aligned} (f \circ f \circ g)(x) &= f(f(g(x))) = f(f(2x)) = f(2x+1) = 2x+2 \\ (g \circ f)(x) &= g(f(x)) = g(x+1) = 2(x+1) = 2x+2. \end{aligned}$$

This holds for all x, so $f \circ f \circ g = g \circ f$. □

Example 4.3.3. Consider the functions $\mathbb{R} \xrightarrow{f} \mathbb{R}^3 \xrightarrow{g} \mathbb{R}$ and $\mathbb{R} \xrightarrow{h} \mathbb{R}$ given by

$$\begin{aligned} f(t) &= (1, t, t^2) \\ g(x, y, z) &= x + 2y + z \\ h(t) &= (1+t)^2. \end{aligned}$$

I claim that $g \circ f = h$. Indeed, we have

$$\begin{aligned} (g \circ f)(t) &= g(f(t)) \\ &= g(1, t, t^2) \\ &= 1 + 2t + t^2 \\ &= (1+t)^2 = h(t). \end{aligned}$$

This holds for all $t \in \mathbb{R}$, so $h = g \circ f$. □

Exercise 4.3.4. Let M be the set of all 2×2 matrices. (Matrices will be explained in Chapter 8; you should ignore this exercise if you are not already familiar with them.)

Define functions $\mathbb{R} \xrightarrow{f} M \xrightarrow{g} M \xrightarrow{h} \mathbb{R}$ by

$$\begin{aligned} f(t) &= \left[\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right] \\ g(A) &= AA^T \\ h(B) &= \operatorname{trace}(B). \end{aligned}$$

What are the domain and codomain of the function $k = h \circ g \circ f$? What is the formula for this function?

Solution: The composite $k = h \circ g \circ f$ is a function from $\mathbb{R}$ to $\mathbb{R}$, so $\operatorname{domain}(k) = \operatorname{codomain}(k) = \mathbb{R}$.

The formulae are as follows:

$$\begin{aligned}
(g\circ f)(t) = g(f(t)) &= g\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\\
&= \begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}^T\\
&= \begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ t & 1\end{bmatrix}\\
&= \begin{bmatrix}1+t^2 & t\\ t & 1\end{bmatrix}\\
h\begin{bmatrix}p & q\\ r & s\end{bmatrix} &= p+s\\
k(t) = h(g(f(t))) = h\begin{bmatrix}1+t^2 & t\\ t & 1\end{bmatrix} &= (1+t^2)+1 = 2+t^2.
\end{aligned}$$

Exercise 4.3.5. Put $A = \mathbb{R}\setminus\{0,1\}$, and define $f, g, h\colon A \to A$ by

$$\begin{aligned}
f(x) &= 1-x\\
g(x) &= 1/x\\
h(x) &= g(f(x)).
\end{aligned}$$

Check that this is all meaningful, and simplify $(h\circ h)(x)$ and $(h\circ h\circ h)(x)$.

Solution: First, if $x\in\mathbb{R}$ and $x\neq 0,1$ then $1-x$ is again a real number and $1-x\neq 0,1$. This means that the equation $f(x) = 1-x$ really does give a function $f\colon A\to A$.

Similarly, if $x\in\mathbb{R}$ and $x\neq 0,1$ then $1/x$ is a well-defined real number (because we have excluded the problematic case where $x=0$) and we see easily that $1/x\neq 0,1$. Thus, the rule $g(x)=1/x$ defines another function $g\colon A\to A$. It follows in turn that the rule $h(x) = g(f(x))$ defines a well-defined function $g\colon A\to A$.

Next, we have

$$\begin{aligned}
h(x) &= 1-1/x = (x-1)/x\\
(h\circ h)(x) &= h(\frac{x-1}{x}) = 1-\frac{x}{x-1} = \frac{1}{1-x}\\
(h\circ h\circ h)(x) &= h\left(\frac{1}{1-x}\right) = 1-\frac{1-x}{1} = x.
\end{aligned}$$

Exercise 4.3.6. Given numbers λ and μ, we put

$$\begin{aligned}
f_\lambda(t) &= t+\lambda\\
g_\mu(t) &= \mu t\\
h(t) &= 1/t.
\end{aligned}$$

Let a, b, c and d be numbers with $c\neq 0$ and $ad\neq bc$, and put

$$k = f_{a/c}\circ h\circ g_{c^2/(bc-ad)}\circ f_{d/c}.$$

Simplify $k(t)$. (You may wish to consider the intermediate variables $u = f_{d/c}(t)$, $v = g_{c^2/(bc-ad)}(u)$ and $w = h(v)$.)

This is relevant to the theory of *Möbius transformations* (otherwise known as *linear fractional transformations*) . You need not discuss the domain of definition of any of the functions used.

Solution: We must find $k(t) = f_{a/c}(h(g_{c^2/(bc-ad)}(f_{d/c}(t))))$. We can rephrase this as follows: we put $u = f_{d/c}(t)$ and $v = g_{c^2/(bc-ad)}(u)$ and $w = h(v)$ and $x = f_{a/c}(w)$, so $k(t) = x$.

$$
\begin{aligned}
u &= t + \frac{d}{c} = \frac{ct+d}{c} \\
v &= \frac{c^2}{bc-ad} u = \frac{c^2}{bc-ad}\,\frac{ct+d}{c} = \frac{c(ct+d)}{bc-ad} \\
w &= h(u) = 1/u = \frac{bc-ad}{c(ct+d)} \\
x &= w + \frac{a}{c} = \frac{bc-ad}{c(ct+d)} + \frac{a(ct+d)}{c(ct+d)} \\
&= \frac{bc - ad + act + ad}{c(ct+d)} = \frac{at+b}{ct+d}
\end{aligned}
$$

The conclusion is that $k(t) = (at+b)/(ct+d)$.

Exercise 4.3.7. Define $f, g \colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$
\begin{aligned}
f(x,y) &= (1-y, 1-x) \\
g(x,y) &= (-1-y, -1-x).
\end{aligned}
$$

Find $f \circ f$, $f \circ g$, $g \circ f$ and $g \circ g$.

Solution:

$$
\begin{aligned}
(f\circ f)(x,y) &= f(1-y,1-x) = (1-(1-x), 1-(1-y)) = (x,y) \\
(f\circ g)(x,y) &= f(-1-y,-1-x) = (1-(-1-x), 1-(-1-y)) = (x+2, y+2) \\
(g\circ f)(x,y) &= g(1-y,1-x) = (-1-(1-x), -1-(1-y)) = (x-2, y-2) \\
(g\circ g)(x,y) &= g(-1-y,-1-x) = (-1-(-1-x), -1-(-1-y)) = (x,y).
\end{aligned}
$$

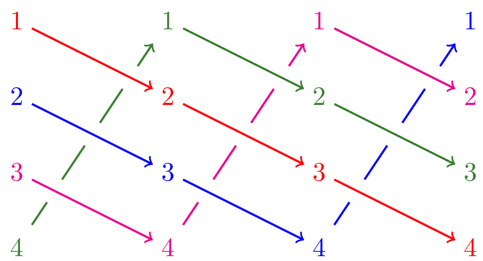
Geometrically, it works out that $f(x,y)$ is the reflection of x across the line $x+y=1$, and $g(x,y)$ is the reflection of (x,y) across the line $x+y=-1$. It is also possible to work out the composites from this information. For example, if you reflect twice in succession across the same line then that is the same as doing nothing, which explains why $(f\circ f)(x,y) = (x,y) = (g\circ g)(x,y)$.

Exercise 4.3.8. Put $A = \{1,2,3,4\}$, and define $f\colon A \to A$ by $f(1)=2$, $f(2)=3$, $f(3)=4$ and $f(4)=1$. Describe the function $g = f\circ f\circ f\colon A \to A$.

Solution: We can list the values as follows:

n	1	2	3	4
$f(n)$	2	3	4	1
$f(f(n))$	3	4	1	2
$g(n) = f(f(f(n)))$	4	1	2	3

This can also be displayed as follows:



The conclusion is that $g(1) = 4$, $g(2) = 1$, $g(3) = 2$ and $g(4) = 3$.

Exercise 4.3.9*. There are plausible (but crude) models of population growth that look like this: if there are x beasts per square metre this year, then there will be $f(x)$ beasts per square metre next year, where $f(x) = 3x(1 - x)$. (The idea is that if the population is small then it will grow rapidly, but if the population approaches one beast per square metre then many will starve through overcrowding and the population will plummet.) The population in two years' time will then be $f^2(x) = f(f(x))$, in three years' time it will be $f^3(x) = f(f(f(x)))$ and so on.

- Expand out $f(x)$.
- Expand out $f(f(x))$.
- Use Maple to expand out $f^3(x)$.
- How many terms are there in each of the answers you have obtained? What are the first and last terms? What do you get if you factor the coefficients in the first and last terms?
- Can you guess how many terms there will be in $f^{10}(x)$? What will the first and last terms be?

This exercise should convince you that explicit calculation of $f^{10}(x)$ would give an expression that is much to large and unwieldy to tell us anything very useful. Nonetheless, there is an elaborate and interesting theory of models of this type, which tells us a great deal about $f^n(x)$ for all n, by more indirect methods. Some of these ideas are covered in PMA324 (Chaos) and PMA443 (Fractals).

Solution: We have

$$
\begin{aligned}
f(x) &= 3x - 3x^2 \\
f^2(x) &= 3f(x) - 3f(x)^2 \\
&= 3(3x - 3x^2) - 3(3x - 3x^2)^2 \\
&= 9x - 9x^2 - 3(9x^2 - 18x^3 + 9x^4) \\
&= 9x - 36x^2 + 54x^3 - 27x^4 \\
&= 3^2x - \cdots - 3^3x^4 \\
f^3(x) &= 27x - 351x^2 + 2106x^3 - 6885x^4 + 13122 * x^5 - 14580x^6 + 8748x^7 - 2187x^8 \\
&= 3^3x - \cdots - 3^7x^8
\end{aligned}
$$

The general rule is that $f^n(x)$ has 2^n terms, starting with $3^n x$, and ending with $-3^{2^n-1}x^{2^n}$. (Can you explain why?) In particular,

$$f^{10}(x) = 3^{10}x - \cdots - 3^{1023}x^{1024}.$$

4.4 Inverse functions

Definition 4.4.1. Suppose we have a function $f\colon A \to B$. An *inverse* for f is a function $g\colon B \to A$ such that $g(f(a)) = a$ for *all* $a \in A$, *and* $f(g(b)) = b$ for *all* $b \in B$. Once we have shown that g has these properties, we sometimes use the notation $f^{-1}(b)$ for $g(b)$.

Example 4.4.2. Define $f, g\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x + 1$ and $g(y) = (y - 1)/2$. Then

$$
\begin{aligned}
g(f(x)) &= g(2x + 1) = ((2x + 1) - 1)/2 = x \\
f(g(y)) &= f((y - 1)/2) = 2(y - 1)/2 + 1 = y.
\end{aligned}
$$

These equations hold for all x and y in $\mathbb{R}$, so g is inverse to f. □

Example 4.4.3. As discussed in more detail in Section 5.1, the function $\exp\colon \mathbb{R} \to (0,\infty)$ is inverse to the function $\log\colon (0,\infty) \to \mathbb{R}$. $\square$

Warning 4.4.4. It is a **standard mistake** to confuse $f^{-1}(x)$ with the function $f(x)^{-1} = 1/f(x)$. These two functions are completely different. For example, $\exp^{-1}(e) = \log(e) = 1$, but $\exp(e)^{-1} = e^{-e} \approx 0.065988$.

Warning 4.4.5. Another **standard mistake** is to check only one of the two equations $g(f(x)) = x$ and $f(g(y)) = y$. *Both* of these must be true for g to be inverse to f.

Example 4.4.6. Put $A = \{0,1,2,3\}$ and $B = \{1,2,4,8\}$, and define functions $A \xrightarrow{f} B \xrightarrow{g} A$ as follows:

n	0	1	2	3
$f(n)$	8	4	2	1

m	1	2	4	8
$g(m)$	3	2	1	0

We then have

$$\begin{aligned} g(f(0)) &= g(8) = 0 & f(g(1)) &= f(3) = 1 \\ g(f(1)) &= g(4) = 1 & f(g(2)) &= f(2) = 2 \\ g(f(2)) &= g(2) = 2 & f(g(4)) &= f(1) = 4 \\ g(f(3)) &= g(1) = 3 & f(g(8)) &= f(0) = 8. \end{aligned}$$

Thus $g(f(n)) = n$ for all $n \in A$ and $f(g(m)) = m$ for all $m \in B$, so g is inverse to f. $\square$

Example 4.4.7. Define $f\colon \mathbb{R} \to [0,\infty)$ by $f(x) = x^2$. Next, define $g\colon [0,\infty) \to \mathbb{R}$ by

$$g(y) = \sqrt{y} = \text{ the nonnegative square root of } y.$$

It is natural to think that g is inverse to f, but this is not quite right. To see this, note that $f(-1) = (-1)^2 = 1$, so $g(f(-1)) = g(1) = \sqrt{1} = 1$, so $g(f(-1)) \neq -1$.

This can be fixed if we specify the domains and codomains of our functions differently. Put $A = [0,\infty)$ and let f_0 be the function *from A to A* with formula $f_0(x) = x^2$, and let g_0 be the function *from A to A* with formula $g_0(y) = \sqrt{y}$. As explained in Remark 4.1.4(d), f_0 and g_0 are, strictly speaking, different functions from f and g. For any $x \in A$ we have $x \geq 0$ and so $x = \sqrt{x^2} = (\sqrt{x})^2$, or in other words $x = g_0(f_0(x)) = f_0(g_0(x))$. This means that g_0 is inverse to f_0, even though g is not inverse to f. $\square$

Many functions do not have inverses. For $f\colon A \to B$ to have an inverse:

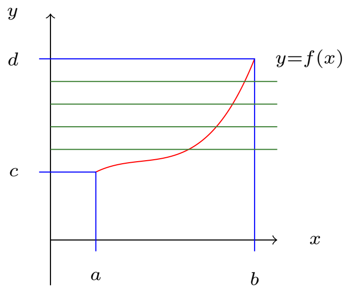
- f must not merge distinct values together: if $a_0 \neq a_1$, then $f(a_0) \neq f(a_1)$.
- f must take every value in its codomain: if $b \in B$, then there is some $a \in A$ such that $b = f(a)$.

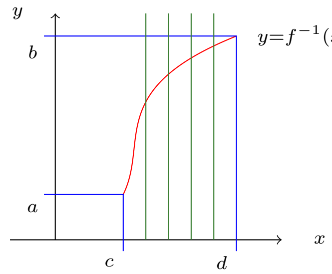
However, if $f\colon A \to B$ does not have an inverse, then one can often modify the domain and codomain to get a new function $f_0\colon A_0 \to B_0$ (with the same "formula" as f) such that f_0 *does* have an inverse. Example 4.4.7 is an instance of this.

Example 4.4.8.
- The function $\sin\colon \mathbb{R} \to [-1,1]$ does not have an inverse, because it merges some distinct values together: for example, $0 \neq \pi$, but $\sin(0) = \sin(\pi)$.
- The function $\sin\colon [-\pi/2, \pi/2] \to \mathbb{R}$ does not have an inverse, because it does not take every value in its codomain. Indeed, we took the codomain to be all of $\mathbb{R}$, but the values of $\sin(x)$ are always between -1 and 1.

- The function $\sin\colon [-\pi/2, \pi/2] \to [-1, 1]$ *does* have an inverse (given by the function $\arcsin(y)$, which is discussed in more detail in Section 5.6). □

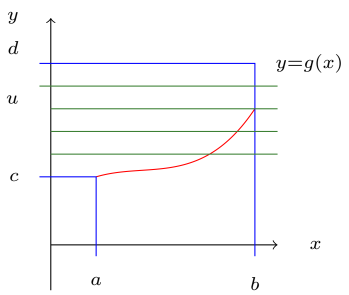
We will often consider functions $f\colon A \to B$ where A and B are intervals in $\mathbb{R}$, say $A = [a, b]$ and $B = [c, d]$. In this case, f will have an inverse if and only if the graph of f crosses every horizontal line between height c and height d precisely once, as in the example shown on the left below:

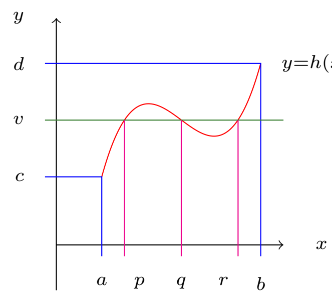




Equivalently, as x runs from a to b, $f(x)$ must either run upwards from c to d without ever decreasing, or it must run downwards from d to c without ever increasing. In either case, the graph of $f^{-1}(x)$ is obtained by flipping over the graph of $f(x)$, so as to exchange the axes; an example is shown above on the right.

We next give examples of two functions $g, h\colon [a, b] \to [c, d]$ that *do not* have inverses.





The function $g(x)$ shown on the left does not have an inverse, because the graph does not cross the horizontal line $y = u$. If there were an inverse, then $g^{-1}(u)$ would be a number $x \in [a, b]$ such that $g(x) = u$; but you can see from the graph that there is no such x.

The function $h(x)$ shown on the right does not have an inverse either, because the graph crosses the line $y = v$ in three places, where $x = p$, $x = q$ and $x = r$. This makes it unclear whether $h^{-1}(v)$ should be p, q or r, so h^{-1} is not unambiguously defined.

Method 4.4.9. Suppose we are given a function $f\colon A \to B$, and we want to find its inverse (if it has one).

(a) Write $y = f(x)$, then rearrange this to express x in terms of y.

(b) Suppose that this can be done in a way that is meaningful, unambiguous, and contained in A, for all $y \in B$; then $y = g(x)$, where $g\colon B \to A$ is probably inverse to f. (However, this should still be checked, bearing in mind the kind of issues we saw in Example 4.4.7.)

(c) Otherwise, $f\colon A \to B$ does not have an inverse.

(d) This method gives a formula for $f^{-1}(y)$ in terms of y. Often it is convenient to give a formula for $f^{-1}(x)$ in terms of x instead. To do this, simply change all the y's to x's in the formula.

Example 4.4.10. Consider $f\colon (0, \infty) \to (0, \infty)$ given by $f(x) = e^x - 1$.

- If $y = f(x) = e^x - 1$ then $y + 1 = e^x$ so $x = \log(y + 1)$.
- If $y \in (0, \infty)$ then $y + 1 > 1$, so $\log(y + 1)$ is defined and strictly greater than $\log(1) = 0$. Thus, $g(y) = \log(y + 1)$ gives a well-defined function $g\colon (0, \infty) \to (0, \infty)$.
- We have $g(f(x)) = \log((e^x - 1) + 1) = x$ for all $x > 0$, and $f(g(y)) = e^{\log(y+1)} - 1 = y$ for all $y > 0$; so g is inverse to f.
- We conclude that $f^{-1}(y) = \log(y + 1)$, or equivalently $f^{-1}(x) = \log(x + 1)$. □

Exercise 4.4.11. Let a and b be constants with $a \neq 0$. Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = ax + b$. Find the inverse of f.

Solution: Put $y = f(x) = ax + b$. Then $ax = y - b$ and $f^{-1}(y) = x = (y - b)/a$. Equivalently, we have $f^{-1}(x) = (x - b)/a$.

Exercise 4.4.12. Consider the function $f\colon (0, \infty) \to (0, 1)$ given by $f(x) = (x + 1)^{-3}$. You may assume that this is well-defined and that it has an inverse. Give a formula for $f^{-1}(x)$.

Solution: Put $y = f(x) = (x+1)^{-3}$, so $x = f^{-1}(y)$. We then have $y^{-1/3} = x+1$, so $x = y^{-1/3}-1$, or in other words $f^{-1}(y) = y^{-1/3} - 1$. We can now rename the variable to get $f^{-1}(x) = x^{-1/3} - 1$.

Exercise 4.4.13. Consider the function $f\colon (0, 1/2) \to (0, \infty)$ given by $f(x) = \pi/\tan(\pi x)$. You may assume that this is well-defined and that it has an inverse. Give a formula for $f^{-1}(x)$.

(The function $\tan\colon (0, \pi/2) \to (0, \infty)$ has an inverse function $\arctan\colon (0, \infty) \to (0, \pi/2)$; your answer should be given in terms of this.)

Solution: Put $y = f(x) = \pi/\tan(\pi x)$, so $x = f^{-1}(y)$. We then have $\pi/y = \tan(\pi x)$, so $\pi x = \arctan(\pi/y)$, so $x = \arctan(\pi/y)/\pi$. In other words we have $f^{-1}(y) = \arctan(\pi/y)/\pi$. We can now rename the variable to get $f^{-1}(x) = \arctan(\pi/x)/\pi$.

Exercise 4.4.14. Put $A = \{0, 1, 2, 3, 4, 5, 6, 7\}$, and define $f\colon A \to A$ as follows:

n	0	1	2	3	4	5	6	7
$f(n)$	1	2	3	0	7	4	5	6

Find the inverse of f.

Solution: $f^{-1}(0)$ is the number n such that $f(n) = 0$. By inspecting the table we see that $n = 3$ is the only number with $f(n) = 0$, so $f^{-1}(0) = 3$ (and this is well-defined). Similarly, $f^{-1}(1)$ is the number n such that $f(n) = 1$, and $n = 0$ is the only possibility, so $f^{-1}(1) = 0$. Proceeding in this way, we find the following table of values:

n	0	1	2	3	4	5	6	7
$f^{-1}(n)$	3	0	1	2	5	6	7	4

Exercise 4.4.15. Define $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $f(a, b, c) = (b + 1, c + 1, a + 1)$. Find the inverse of f.

Solution: Put $(u, v, w) = f(a, b, c) = (b+1, c+1, a+1)$, so $f^{-1}(u, v, w) = (a, b, c)$. We then have $a = w - 1$ and $b = u - 1$ and $c = v - 1$, so

$$f^{-1}(u, v, w) = (w - 1, u - 1, v - 1).$$

If we wish, we can rename the variables and rewrite this as

$$f^{-1}(a, b, c) = (c - 1, a - 1, b - 1).$$

Exercise 4.4.16. Define $f\colon (0,1) \to (0,1)$ by $f(x) = 2x/(1+x)$. You may assume that this function is well-defined and has an inverse. Find the formula for $f^{-1}(x)$.

Solution: Put $y = f(x) = 2x/(1+x)$, so $x = f^{-1}(y)$. Then $2x = y(1+x) = y + xy$, so $x(2-y) = 2x - xy = y$, so $x = y/(2-y)$. This means that $f^{-1}(y) = y/(2-y)$, or equivalently $f^{-1}(x) = x/(2-x)$.

Exercise 4.4.17. Let a, b, c and d be constants with $ad \neq bc$, and put $f(x) = (ax+b)/(cx+d)$. Find the inverse of f.

(You may ignore details about the domain and codomain. For the record, they should both be taken to be $\mathbb{R} \cup \{\infty\}$, with the convention that $f(\infty) = a/c$ and $f(-d/c) = \infty$.)

Solution: Put $y = f(x) = (ax+b)/(cx+d)$, so $x = f^{-1}(y)$. Then $ax + b = (cx+d)y = cxy + dy$, so $b - dy = cxy - ax = (cy-a)x$, so $x = (b-dy)/(cy-a)$. We thus have $f^{-1}(y) = (b-dy)/(cy-a)$, or equivalently $f^{-1}(x) = (b-dx)/(cx-a)$.

Exercise 4.4.18*. Define $f\colon (0,1) \to (0,\infty)$ by $f(x) = 2x/(1-x^2)$, and define $g\colon (0,\infty) \to (0,1)$ by $g(y) = ((1+y^2)^{1/2} - 1)/y$. Check that g is inverse to f.

Solution: First, consider $x \in (0,1)$ and put $y = f(x) = 2x/(1-x^2)$. Then

$$\begin{aligned}
1 + y^2 &= 1 + \frac{4x^2}{(1-x^2)^2} = \frac{(1-x^2)^2 + 4x^2}{(1-x^2)^2} \\
&= \frac{1 - 2x^2 + x^4 + 4x^2}{(1-x^2)^2} = \frac{1 + 2x^2 + x^4}{(1-x^2)^2} = \frac{(1+x^2)^2}{(1-x^2)^2} \\
(1+y^2)^{1/2} - 1 &= \frac{1+x^2}{1-x^2} - 1 = \frac{2x^2}{1-x^2} \\
g(y) &= ((1+y^2)^{1/2} - 1)/y \\
&= \frac{2x^2}{1-x^2} \Big/ \frac{2x}{1-x^2} = x.
\end{aligned}$$

As y was $f(x)$, this shows that $g(f(x)) = x$. Note that in the third line, we implicitly used the fact that $0 < x < 1$ to see that there was no problem with the sign of the square root.

We next check that $f(g(y)) = y$ for all $y > 0$. We put $x = g(y) = (\sqrt{1+y^2} - 1)/y$, so that

$$\begin{aligned}
x^2 &= (\sqrt{1+y^2} - 1)^2/y^2 = ((1+y^2) - 2\sqrt{1+y^2} + 1)/y^2 \\
&= (2 + y^2 - 2\sqrt{1+y^2})/y^2 \\
1 - x^2 &= (y^2 - (2 + y^2 - 2\sqrt{1+y^2}))/y^2 = 2(\sqrt{1+y^2} - 1)/y^2 \\
f(x) &= 2x/(1-x^2) = \frac{2(\sqrt{1+y^2} - 1)}{y} \Big/ \frac{2(\sqrt{1+y^2} - 1)}{y^2} \\
&= y.
\end{aligned}$$

As x was $g(y)$, this gives $f(g(y)) = y$, as required.

Chapter 5: Special functions

We will call the following functions the *primary special functions*:

$$\exp, \log, \sin, \cos, \tan, \arcsin, \arccos, \arctan$$

You should make sure you know the properties of the primary special functions. You should be familiar with their domains, their ranges and their inverses, and the shape of their graphs. You should also know properties such as $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ or $\cos(-x) = \cos(x)$ or $\log(\exp(x)) = x$; a detailed list will be given later. The derivatives and integrals of all special functions will be recalled in later sections; you should know them as well.

We also refer to the following as *secondary special functions*:

$$\begin{gathered} \sec, \csc, \cot, \\ \sinh, \cosh, \tanh, \operatorname{sech}, \operatorname{csch}, \coth, \\ \operatorname{arcsinh}, \operatorname{arccosh}, \operatorname{arctanh}. \end{gathered}$$

You should know how these are defined in terms of the primary functions (for example, $\sinh(x) = (\exp(x) - \exp(-x))/2$, and $\sec(x) = 1/\cos(x)$). You should either remember the properties of the secondary functions, or be able to derive them from the properties of the primary functions.

5.1 The exponential and the logarithm

Probably the most important special function in all of mathematics is the exponential function $\exp(x)$. It is given by the formula

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

(This is a sum with infinitely many terms, and such things generally need very careful treatment, but we will gloss over such points in this course. The proper treatment of infinite sums starts in PMA113 (Introduction to Analysis).)

In particular, we put

$$e = \exp(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \approx 2.71828.$$

This is perhaps the most important special number in all of mathematics, with the possible exception of π.

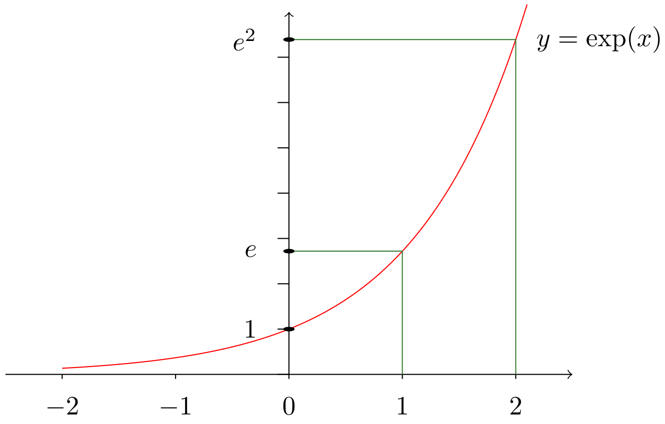
The exponential function has the following crucial properties:

$$\begin{aligned} \exp(x+y) &= \exp(x)\exp(y) \\ \exp(x-y) &= \exp(x)/\exp(y) \\ \exp(0) &= 1 \\ \exp(-x) &= 1/\exp(x) \\ \exp(nx) &= \exp(x)^n \\ \exp(x) &= e^x. \end{aligned}$$

In reading this, you should remember that $\exp(x)^n$ means the n'th power of $\exp(x)$, which is different from $\exp(x^n)$, and you should bear in mind the rules for powers as discussed in Section 2.2.

Exercise 5.1.1. If you know the first of the above properties, then the remaining properties follow automatically.

The graph is as follows:



Note that the function is always strictly larger than zero, it decays rapidly towards zero as x moves off towards $-\infty$, and it increases very rapidly as x moves towards $+\infty$.

The inverse function of $\exp(x)$ is the natural logarithm function, written $\ln(y)$ or $\log(y)$, which is defined only when $y > 0$. (If we allow complex numbers then we can define logarithms for negative numbers, for example $\log(-e) = 1 - i\pi$. However, there are still many subtleties, leading to beautiful and powerful ideas in Complex Analysis; see PMA214 and PMA316.) Because log is the inverse of exp, we have

$$\begin{aligned} \log(e^x) &= \log(\exp(x)) = x && \text{for all } x \\ e^{\log(y)} &= \exp(\log(y)) = y && \text{for all } y > 0. \end{aligned}$$

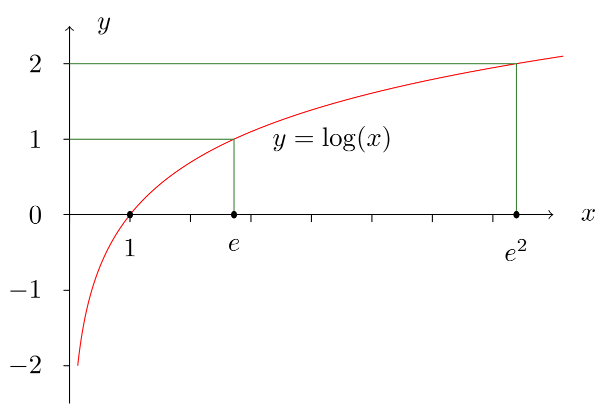
As usual, we must remember that log is the inverse of exp in the sense implied by the above equations, which does *not* mean that $\log(x) = 1/\exp(x)$. The basic rules for the logarithm are as follows:

$$\begin{aligned} \log(xy) &= \log(x) + \log(y) \\ \log(x/y) &= \log(x) - \log(y) \\ \log(1) &= 0 \\ \log(1/y) &= -\log(y) \\ \log(y^n) &= n\log(y). \end{aligned}$$

One other very important fact is the integration formula, which will be reexamined in Chapter 7:

$$\log(y) = \int_{t=1}^{y} \frac{dt}{t}.$$

The graph is obtained by flipping the graph of exp:



For any $a > 1$, one can also define the "base a" logarithm $\log_a(y)$ by the formula

$$\log_a(y) = \log(y)/\log(a) = \text{ the number } x \text{ such that } a^x = y \ .$$

(To reconcile the two halves of this definition, note that $a = e^{\log(a)}$, so

$$a^{\log_a(y)} = e^{\log(a)\log_a(y)} = e^{\log(y)} = y.$$

Note also that $\log_e(y) = \log(y)/\log(e) = \log(y)$.)

For example, we have

$$\begin{aligned}
\log_{10}(1000) &= \log_{10}(10^3) = 3 \\
\log_2(1024) &= \log_2(2^{10}) = 10 \\
\log_{1024}(2) &= \log_{1024}(1024^{1/10}) = 1/10 \\
\log_{12}(144) &= \log_{12}(12^2) = 2.
\end{aligned}$$

The number $\log_{10}(x)$ is roughly the number of digits in x. More precisely, if $n - 1 \leq \log_{10}(x) < n$ for some $n \in \mathbb{N}$, then x has n digits to the left of the decimal point. The function $\log_{10}$ is used in measuring the gain of amplifiers, for example: if the ratio of output power to input power is r, then the gain is $\log_{10}(r)$ bels, or $10\log_{10}(r)$ decibels. Historically, base 10 logarithms were used as an aid to hand calculation before calculators and computers were available. Because of this, you will still find some books that use the name log to mean $\log_{10}$, and reserve the symbol ln for what we call log. The function $\log_2$ is useful in computer science as a measure of information content.

Exercise 5.1.2. What is the largest possible real domain for the function $f(x) = \log(\log(x))$? What about $g(x) = \log(\log(\log(x)))$?

Solution: Recall that $\log(x)$ is only defined for $x > 0$. To define $\log(\log(x))$, we must have $x > 0$ (so that the inner log is defined) and $\log(x) > 0$ (so that the outer log is defined). This means that we must have $x > \exp(0) = 1$; so the largest possible domain for $f(x)$ is the set $(1, \infty)$.

Next, note that $g(x) = \log(\log(\log((x)))$. For this to be defined, we must have $x > 1$ (so that $\log(\log(x))$ is defined) and $\log(\log(x)) > 0$ (so that the outer log is defined). The inequality $\log(\log(x)) > 0$ is equivalent to $\log(x) > \exp(0) = 1$, or to $x > \exp(1) = e$. Thus, the largest possible domain for $g(x)$ is (e, ∞).

5.2 Hyperbolic functions

The hyperbolic functions are defined by the formulae

$$\begin{aligned}
\sinh(x) &= (e^x - e^{-x})/2 \\
\cosh(x) &= (e^x + e^{-x})/2 \\
\tanh(x) &= \sinh(x)/\cosh(x) = (e^x - e^{-x})/(e^x + e^{-x}) \\
\operatorname{csch}(x) &= 1/\sinh(x) = 2/(e^x - e^{-x}) \\
\operatorname{sech}(x) &= 1/\cosh(x) = 2/(e^x + e^{-x}) \\
\coth(x) &= 1/\tanh(x) = (e^x + e^{-x})/(e^x - e^{-x}).
\end{aligned}$$

The notation $\operatorname{cosech}(x)$ is also used in place of $\operatorname{csch}(x)$.

Almost anything that you might want to know about hyperbolic functions can quickly be deduced from the properties of the exponential function. However, they have an important relationship with trigonometric functions, and they turn up in many other contexts, so it is worth committing some of their properties to memory. Here is a sample:

$$\begin{aligned}
\cosh(x)^2 - \sinh(x)^2 &= 1 \\
\operatorname{sech}(x)^2 + \tanh(x)^2 &= 1 \\
\sinh(x+y) &= \sinh(x)\cosh(y) + \cosh(x)\sinh(y) \\
\cosh(x+y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \\
\sinh(2x) &= 2\sinh(x)\cosh(x) \\
\cosh(2x) &= 2\cosh(x)^2 - 1.
\end{aligned}$$

To check these, it is convenient to write $u = e^x$, so $u^{-1} = e^{-x}$. We then have $\sinh(x) = (u - u^{-1})/2$ and $\cosh(x) = (u + u^{-1})/2$, so

$$\begin{aligned}
\cosh(x)^2 - \sinh(x)^2 &= \frac{(u+u^{-1})^2}{4} - \frac{(u-u^{-1})^2}{4} \\
&= \frac{(u^2 + 2 + u^{-2}) - (u^2 - 2 + u^{-2})}{4} \\
&= (2 - (-2))/4 = 1.
\end{aligned}$$

Exercise 5.2.1. Check the identities $\cosh(2x) = 2\cosh(x)^2 - 1$ and $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$.

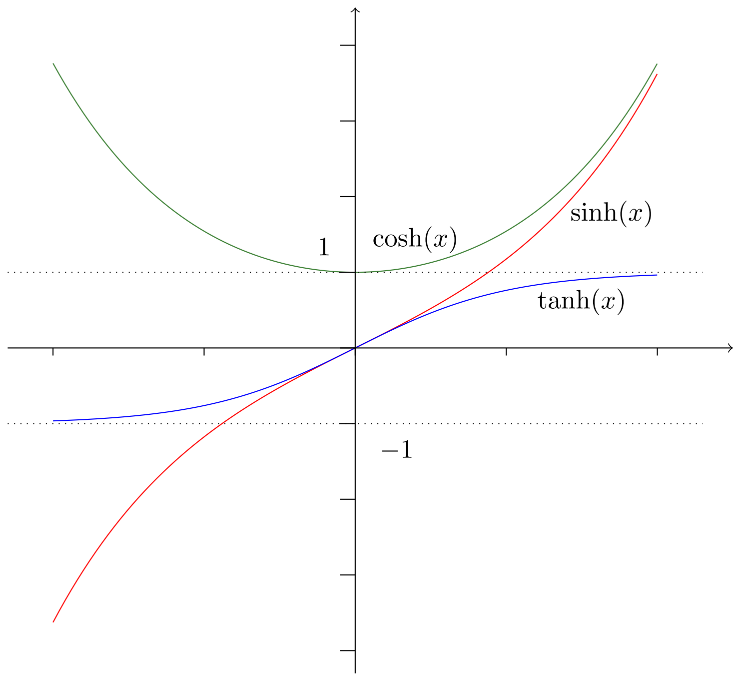
Solution: First, we put $u = e^x$, so $u^2 = e^{2x}$. We also have $\sinh(x) = (u - u^{-1})/2$ and $\cosh(x) = (u + u^{-1})/2$, so

$$\begin{aligned}
2\cosh(x)^2 - 1 &= 2\left(\frac{u+u^{-1}}{2}\right)^2 - 1 = 2\frac{u^2 + 2 + u^{-2}}{4} - 1 \\
&= \frac{u^2 + 2 + u^{-2} - 2}{2} = \frac{u^2 + u^{-2}}{2} \\
&= \frac{e^{2x} + e^{-2x}}{2} = \cosh(2x).
\end{aligned}$$

Next, we put $v = e^y$, so $uv = e^{x+y}$. We also have $\sinh(y) = (v - v^{-1})/2$ and $\cosh(y) = (v + v^{-1})/2$, so

$$\begin{aligned}
\sinh(x)\cosh(y) + \cosh(x)\sinh(y) &= \frac{u-u^{-1}}{2}\,\frac{v+v^{-1}}{2} + \frac{u+u^{-1}}{2}\,\frac{v-v^{-1}}{2} \\
&= \frac{1}{4}\left(uv + uv^{-1} - u^{-1}v - u^{-1}v^{-1} + uv - uv^{-1} + u^{-1}v - u^{-1}v^{-1}\right) \\
&= \frac{1}{4}(2uv - 2u^{-1}v^{-1}) = \frac{e^{x+y} - e^{-x-y}}{2} \\
&= \sinh(x+y).
\end{aligned}$$

The graphs of the hyperbolic functions are as follows:



If a chain hangs freely with its ends fixed, then it will have the same shape as the graph of the function $\cosh(x)$, which is called a *catenary* .

5.3 Inverse hyperbolic functions

From the picture shown earlier, we see that the graph of $\sinh(x)$ crosses every horizontal line precisely once, so there is a well-defined inverse function. We will use the notation $\operatorname{arcsinh}(y)$ for this inverse function, so $y = \sinh(x)$ is equivalent to $x = \operatorname{arcsinh}(y)$. (In some other books, you will see the notation $\sinh^{-1}(y)$ instead of $\operatorname{arcsinh}(y)$. We prefer to avoid this, to eliminate any possibility of confusion with the function $\sinh(x)^{-1} = 1/\sinh(x) = \operatorname{csch}(x)$.)

It turns out that there is a formula for $\operatorname{arcsinh}(y)$ in terms of logarithms:

$$\operatorname{arcsinh}(y) = \log(y + \sqrt{1 + y^2}).$$

Proof. Suppose that $y = \sinh(x)$; we must show that $x = \log(y + \sqrt{1 + y^2})$. We have $1 + y^2 = 1 + \sinh(x)^2 = \cosh(x)^2$, so $\sqrt{1 + y^2} = \cosh(x)$, so

$$y + \sqrt{1 + y^2} = \sinh(x) + \cosh(x) = \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} = e^x,$$

so $\log(y + \sqrt{1 + y^2}) = \log(e^x) = x$ as required.

There is a small point to note here about signs and square roots. It is clear that $1 + y^2$ is always positive, so $\sqrt{1 + y^2}$ is always well-defined. By definition, $\sqrt{1 + y^2}$ means the positive square root of $1 + y^2$, which is thus the positive square root of $\cosh(x)^2$. We also see from the graph of $\cosh(x)$ that $\cosh(x)$ is always positive, so we can conclude that $\sqrt{1 + y^2} = \cosh(x)$. If we did not know that $\cosh(x)$ is always positive, we would have to consider the possibility that $\sqrt{1 + y^2} = -\cosh(x)$ as well. □

We next consider the inverse of $\cosh(x)$. From the graph we see that when x runs from 0 to ∞, the value $\cosh(x)$ runs from 1 to ∞, without repetitions. This means that there is an inverse function $\operatorname{arccosh}(y)$ which is defined for $y \geq 1$ and is always positive. It has the properties that

$$\begin{aligned} \operatorname{arccosh}(\cosh(x)) &= x && \text{for } x \geq 0 \\ \cosh(\operatorname{arccosh}(y)) &= y && \text{for } y \geq 1. \end{aligned}$$

Similarly, as x runs from $-\infty$ to ∞, the value of $\tanh(x)$ runs from -1 to $+1$, without any repetitions. This means that there is an inverse function $\operatorname{arctanh}(y)$ defined for $-1 < y < 1$, such that

$$\begin{aligned} \operatorname{arctanh}(\tanh(x)) &= x && \text{for all } x \\ \tanh(\operatorname{arctanh}(y)) &= y && \text{for } -1 < y < 1. \end{aligned}$$

Exercise 5.3.1. Show that

$$\begin{aligned} \operatorname{arccosh}(y) &= \log(y + \sqrt{y^2 - 1}) && \text{for all } y \geq 1 \\ \operatorname{arctanh}(y) &= \tfrac{1}{2} \log\left(\tfrac{1+y}{1-y}\right) && \text{for } -1 < y < 1 \end{aligned}$$

First just do the algebra, then give a careful argument to show that everything is well-defined and the signs are right.

Solution:

(a) First suppose that $x = \operatorname{arccosh}(y)$, so $y = \cosh(x)$. Recall that arccosh was constructed as a function from $[1, \infty)$ to $[0, \infty)$, so we must have $x \geq 0$ and $y \geq 1$.

Using the identity $\cosh(x)^2 - \sinh(x)^2 = 1$, we see that $y^2 - 1 = \sinh(x)^2$, so $\sqrt{y^2 - 1} = \pm\sinh(x)$. However, $\sqrt{y^2 - 1} \geq 0$ (because $\sqrt{u}$ means the nonnegative square root of u, by definition) and $\sinh(x) \geq 0$ (because $x \geq 0$) so we must have $\sqrt{y^2 - 1} = +\sinh(x)$. Thus, we have

$$y + \sqrt{y^2 - 1} = \sinh(x) + \cosh(x) = \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} = e^x,$$

so $\log(y + \sqrt{y^2 - 1}) = x$ as required.

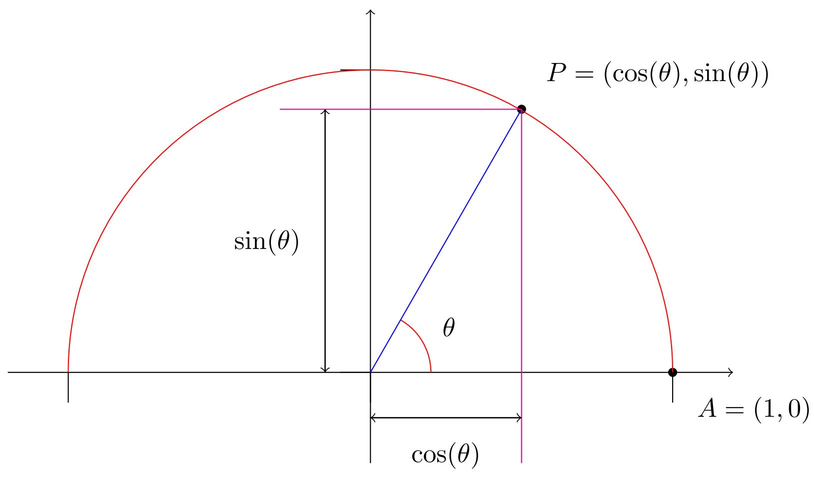
(b) Now suppose instead that $-1 < y < 1$ and $x = \operatorname{arctanh}(y)$. Then

$$\begin{aligned} y = \tanh(x) &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ 1 + y = 1 + \frac{e^x - e^{-x}}{e^x + e^{-x}} &= \frac{2e^x}{e^x + e^{-x}} \\ 1 - y = 1 - \frac{e^x - e^{-x}}{e^x + e^{-x}} &= \frac{2e^{-x}}{e^x + e^{-x}} \\ \frac{1+y}{1-y} &= \frac{2e^x}{e^x + e^{-x}} \Big/ \frac{2e^{-x}}{e^x + e^{-x}} \\ &= (2e^x)/(2e^{-x}) = e^{2x} \\ \frac{1}{2}\log\left(\frac{1+y}{1-y}\right) &= \frac{1}{2}\log(e^{2x}) = \frac{1}{2}.2x = x, \end{aligned}$$

as required.

5.4 Trigonometric functions

The function $\sin(\theta)$ is, as usual, defined to be the y-coordinate of a point P one unit away from the origin, at an angle of θ to the x-axis. Similarly, $\cos(\theta)$ is the x-coordinate of P.



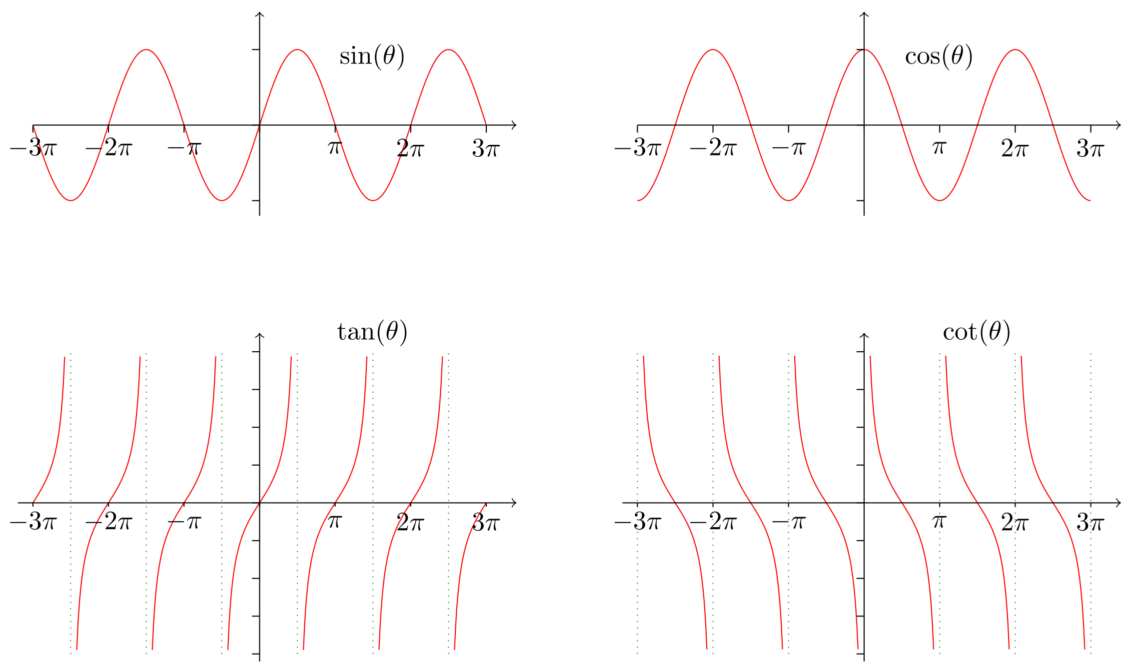
Here we are using radian measure for angles; this is set up so that the length of the curved arc from A to P is precisely θ. (Radians will be used exclusively in all courses.)

We next define

$$\begin{aligned}\tan(\theta) &= \sin(\theta)/\cos(\theta)\\ \cot(\theta) &= \cos(\theta)/\sin(\theta)\\ \sec(\theta) &= 1/\cos(\theta)\\ \csc(\theta) &= 1/\sin(\theta).\end{aligned}$$

The notation $\operatorname{cosec}(\theta)$ is also used in place of $\csc(\theta)$.

You should make sure that you are familiar with the graphs of these functions, including the points were they are zero or where they blow up to infinity, the intervals in which they are positive and negative, and so on.



The following properties can be seen from the graphs:

$$\begin{aligned} \sin(\pi/2+x) &= \cos(x) & \cos(\pi/2+x) &= -\sin(x) \\ \sin(\pi+x) &= -\sin(x) & \cos(\pi+x) &= -\cos(x) \\ \sin(2\pi+x) &= \sin(x) & \cos(2\pi+x) &= \cos(x) \\ \sin(-x) &= -\sin(x) & \cos(-x) &= \cos(x). \end{aligned}$$

Indeed, the equation $\sin(\pi/2+x) = \cos(x)$ says that if you shift the graph of $\sin(x)$ to the left by $\pi/2$, then you get the graph of $\cos(x)$, and this is visibly true. The remaining equations can be seen in the same way. Note that the equations $\sin(\pi/2+x) = \cos(x)$ and $\cos(\pi/2+x) = -\sin(x)$ allow us to convert freely between sin and cos.

5.4.1 Trigonometric identities

You should also be familiar with various identities, of which the most basic are the following:

$$\begin{aligned} \sin(x \pm y) &= \sin(x)\cos(y) \pm \cos(x)\sin(y) \\ \cos(x \pm y) &= \cos(x)\cos(y) \mp \sin(x)\sin(y) \\ \sin(x) - \sin(y) &= 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right). \end{aligned}$$

There are many other identities, and we shall not attempt to make a complete list. However, the following general picture is important:

Fact 5.4.1. A very large class of functions (called *finite Fourier series* or *trigonometric polynomials* can be rewritten (in a completely systematic way) as sums of multiples of functions like $\sin(nx)$ or $\cos(nx)$. □

For example:

$$\begin{aligned} \sin(x)^2 &= \tfrac{1}{2} - \tfrac{1}{2}\cos(2x) \\ \cos(x)^2 &= \tfrac{1}{2} + \tfrac{1}{2}\cos(2x) \\ \sin(x)^3 &= \tfrac{3}{4}\sin(x) - \tfrac{1}{4}\sin(3x) \\ \sin(x)\sin(2x)\sin(4x) &= -\sin(x)/4 + \sin(3x)/4 + \sin(5x)/4 - \sin(7x)/4 \\ \cos(x)\cos(2x)\cos(4x) &= \cos(x)/8 + \cos(3x)/8 + \cos(5x)/8 + \cos(7x)/8 \\ \sin(x)^4 + \cos(x)^4 &= \tfrac{3}{4} + \tfrac{1}{4}\cos(4x) \\ \sin(nx)\sin(mx) &= \tfrac{1}{2}\cos((n-m)x) - \tfrac{1}{2}\cos((n+m)x). \end{aligned}$$

Once a function has been rewritten in this form, it is very easy to differentiate it or integrate it.

The most efficient method for understanding these identities is to use complex numbers and De Moivre's theorem. This remarkable result links together the exponential and trigonometric functions, as follows:

$$\begin{aligned} \exp(ix) &= \cos(x) + \sin(x)i \\ \sin(x) &= (e^{ix} - e^{-ix})/(2i) = \sinh(ix)/i \\ \cos(x) &= (e^{ix} + e^{-ix})/2 = \cosh(ix) \\ \tan(x) &= \sin(x)/\cos(x) = \tanh(ix)/i. \end{aligned}$$

(The first of these equations is the main one; you can easily deduce the others from it.) In PMA111 (Numbers and Polynomials) you will learn more systematically how to derive trigonometric identities from De Moivre's theorem. For the moment, we will just go through one example.

Proposition 5.4.2. *$\cos(x)\cos(2x) = \cos(x)/2 + \cos(3x)/2$.*

Proof. Put $u = e^{ix}$, so

$$\begin{aligned} \cos(x) &= (e^{ix} + e^{-ix})/2 &&= (u + u^{-1})/2 \\ \cos(2x) &= (e^{2ix} + e^{-2ix})/2 &&= (u^2 + u^{-2})/2 \\ \cos(3x) &= (e^{3ix} + e^{-3ix})/2 &&= (u^3 + u^{-3})/2. \end{aligned}$$

If we multiply the first two of these equations together, we get

$$\begin{aligned} \cos(x)\cos(2x) &= (u + u^{-1})(u^2 + u^{-2})/4 \\ &= (u^3 + u + u^{-1} + u^{-3})/4 \\ &= \frac{1}{2}\frac{u^3 + u^{-3}}{2} + \frac{1}{2}\frac{u + u^{-1}}{2} \\ &= \frac{1}{2}\cos(3x) + \frac{1}{2}\cos(x). \end{aligned}$$

□

The general method is essentially the same: you just write everything in terms of the variable $u = \exp(ix)$ and expand it all out.

Exercise 5.4.3. Check that $\sin(3\theta) = 3\sin(\theta) - 4\sin(\theta)^3$ for all θ.

Solution: Put $u = e^{i\theta}$, so

$$\begin{aligned} u^{\pm 1} &= e^{\pm i\theta} = \cos(\theta) \pm \sin(\theta)i \\ u^{\pm 3} &= e^{\pm 3i\theta} = \cos(3\theta) \pm \sin(3\theta)i \\ \sin(\theta) &= (u - u^{-1})/(2i) \\ \sin(3\theta) &= (u^3 - u^{-3})/(2i). \end{aligned}$$

We then have

$$\begin{aligned} 3\sin(\theta) - 4\sin(\theta)^3 &= \frac{3u - 3u^{-1}}{2i} - \frac{4(u - u^{-1})^3}{8i^3} \\ &= \frac{3u - 3u^{-1}}{2i} + \frac{(u - u^{-1})^3}{2i} \\ &= (3u - 3u^{-1} + u^3 - 3u + 3u^{-1} - u^3)/(2i) \\ &= (u^3 - u^{-3})/(2i) = \sin(3\theta), \end{aligned}$$

as required.

Exercise 5.4.4. Show that $\sin(x)^4 + \cos(x)^4 = 1 - \frac{1}{2}\sin(2x)^2$ for all x.

Solution: We first do this by the standard method: put $u = e^{ix}$ and just expand everything out. On one side we have

$$\begin{aligned} \sin(x)^4 + \cos(x)^4 &= \left(\frac{u - u^{-1}}{2i}\right)^4 + \left(\frac{u + u^{-1}}{2}\right)^4 \\ &= \frac{u^4 - 4u^2 + 6 - 4u^{-2} + u^{-4}}{16} + \frac{u^4 + 4u^2 + 6 + 4u^{-2} + u^{-4}}{16} \\ &= \frac{2u^4 + 12 + 2u^{-4}}{16} \\ &= \frac{u^4 + 6 + u^{-4}}{8}. \end{aligned}$$

On the other side we have

$$\begin{aligned}1-\frac{1}{2}\sin(2x)^2 &= 1-\frac{1}{2}\left(\frac{u^2-u^{-2}}{2i}\right)^2\\ &= 1-\frac{u^4-2+u^{-4}}{-8}\\ &= \frac{8+u^4-2+u^{-4}}{8}\\ &= \frac{u^4+6+u^{-4}}{8},\end{aligned}$$

which is the same.

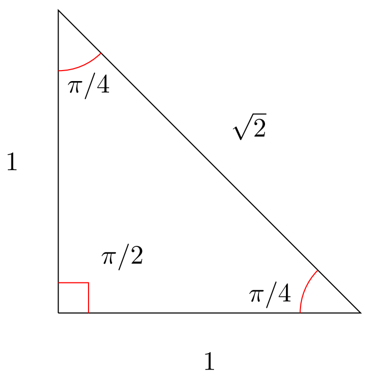
For this particular identity, there is another approach that is quicker but less systematic. If we square the identity $\sin(x)^2+\cos(x)^2=1$ we get $\sin(x)^4+2\sin(x)^2\cos(x)^2+\cos(x)^4=1$. On the other hand, we know that $\sin(2x)=2\sin(x)\cos(x)$, so $\sin(x)^2\cos(x)^2=\sin(2x)^2/4$. If we substitute this into the previous identity we get $\sin(x)^4+2\sin(2x)^2/4+\cos(x)^4=1$, which we can rearrange to get $\sin(x)^4+\cos(x)^4=1-\sin(2x)^2/2$ as claimed.

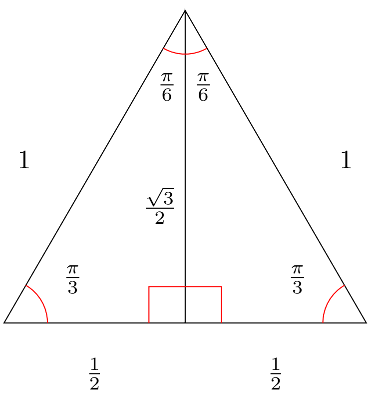
5.5 Special values of trigonometric functions

There are some angles θ for which one can write down an exact expression for $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ in terms of square roots. The really crucial ones are as follows:

θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$\pi/2$	1	0	∞
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$

Most of this can be proved by examining the triangles below.





The left hand triangle has angles of $\pi/2$ and $\pi/4$, and two sides of length 1. The third side has length $\sqrt{1^2+1^2}=\sqrt{2}$, by Pythagoras's theorem. Using the rules

$$\begin{aligned}\sin(\theta) &= \text{opposite}/\text{hypotenuse}\\ \cos(\theta) &= \text{adjacent}/\text{hypotenuse}\end{aligned}$$

we see that $\sin(\pi/4)=1/\sqrt{2}=\sqrt{2}/2$, and similarly that $\cos(\pi/4)=\sqrt{2}/2$. Of course it follows that $\tan(\pi/4)=\sin(\pi/4)/\cos(\pi/4)=1$.

Next, on the right we have an equilateral triangle of side 1, and height d say. It has been divided into two right-angled triangles with angles $\pi/2$, $\pi/3$ and $\pi/6$, and sides of length $1/2$, d and 1. Pythagoras tells us that $(1/2)^2 + d^2 = 1$, so $d^2 = 3/4$, so $d = \sqrt{3/4} = \sqrt{3}/2$, as marked on the diagram. The usual rules now give

$$\begin{aligned} \sin(\pi/6) &= 1/2 \\ \cos(\pi/6) &= \sqrt{3}/2 \\ \tan(\pi/6) &= (1/2)/(\sqrt{3}/2) = 1/\sqrt{3} \end{aligned} \qquad \begin{aligned} \sin(\pi/3) &= \sqrt{3}/2 \\ \cos(\pi/3) &= 1/2 \\ \tan(\pi/3) &= (\sqrt{3}/2)/(1/2) = \sqrt{3}. \end{aligned}$$

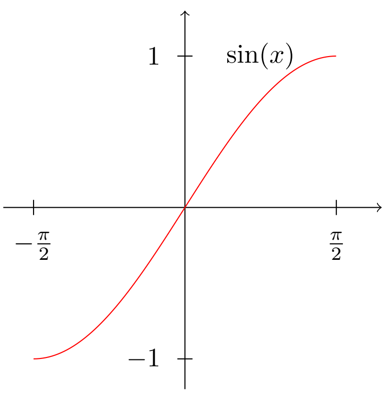
Using these and equations such as $\sin(x + \pi/2) = \cos(x)$, one can fill in many other values, as shown in the table below:

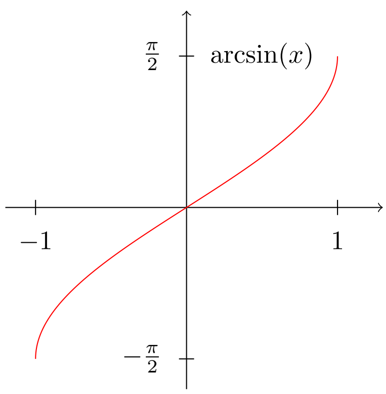
θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
0	0	1	0
$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	1	0	∞
$2\pi/3$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$
$3\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1
$5\pi/6$	$1/2$	$-\sqrt{3}/2$	$-\sqrt{3}/3$
π	0	-1	0

5.6 Inverse trigonometric functions

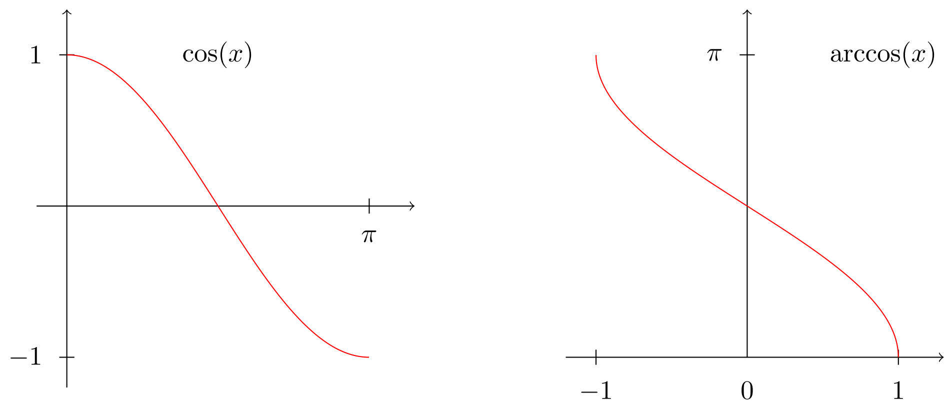
The trigonometric functions can only be inverted if one makes a careful and appropriate choice of domain and range.

Firstly, as x runs from $-\pi/2$ to $\pi/2$ (inclusive), the function $\sin(x)$ runs from -1 to $+1$ (inclusive), without repetitions. We can therefore define a function $\arcsin\colon [-1,1] \to [-\pi/2, \pi/2]$ that is inverse to the function $\sin\colon [-\pi/2, \pi/2] \to [-1,1]$. In other words, for any y with $-1 \leq y \leq 1$, we define $\arcsin(y)$ to be the unique angle x in the range $-\pi/2 \leq x \leq \pi/2$ for which $\sin(x) = y$.

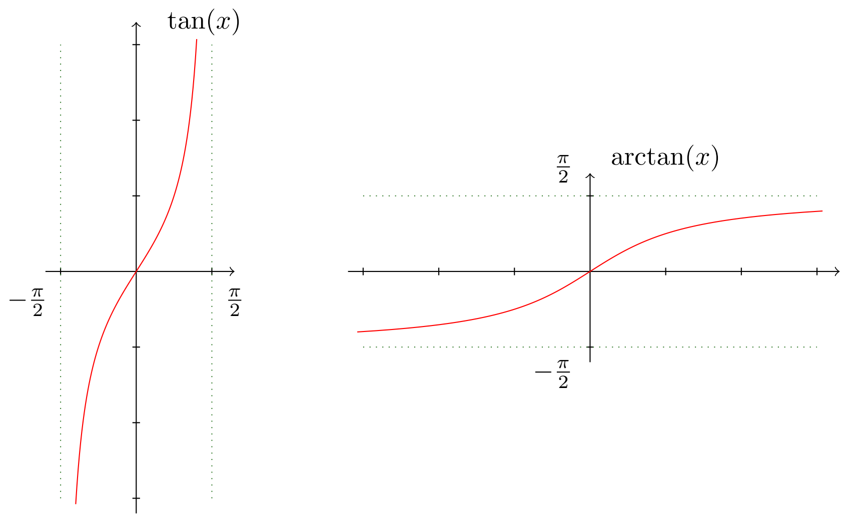




Similarly, as x runs from 0 to π (inclusive), the function $\cos(x)$ runs from 1 down to -1 (inclusive), without repetitions. We can therefore define a function $\arccos\colon [-1,1] \to [0,\pi]$ that is inverse to the function $\cos\colon [0,\pi] \to [-1,1]$. In other words, for any y with $-1 \leq y \leq 1$, we define $\arccos(y)$ to be the unique angle x in the range $0 \leq x \leq \pi$ for which $\cos(x) = y$.



Finally, as x runs from $-\pi/2$ to $\pi/2$ (not including the endpoints), the function $\tan(x)$ runs over the whole real line. We can therefore define an inverse function $\arctan\colon \mathbb{R} \to (-\pi/2, \pi/2)$, so $\arctan(y)$ is the unique angle in the range $-\pi/2 < x < \pi/2$ for which $\tan(x) = y$.



5.7 Trigonometric equations

We next consider trigonometric equations, such as $\sin(2\theta) = \cos(3\theta)$. This is a good exercise in using the properties of trigonometric functions, and also in using the notation of set theory.

The basic facts that you need to start with are as follows:

Proposition 5.7.1. *We have* $\sin(\theta) = 0$ *if and only if* θ *is an integer multiple of* π*. In other words,*

$$\{\theta \in \mathbb{R} \mid \sin(\theta) = 0\} = \mathbb{Z}\pi = \{n\pi \mid n \in \mathbb{Z}\}.$$

This is clear from the graph of $\sin(\theta)$, or from the geometric definition.

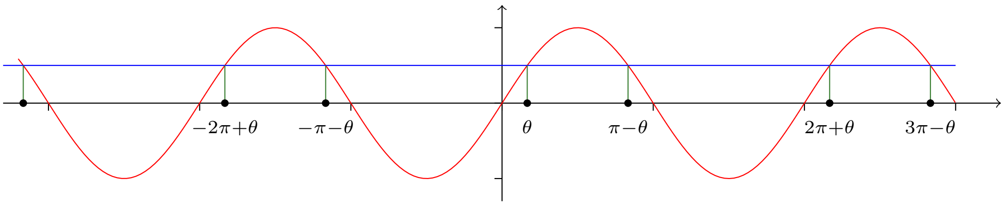
Proposition 5.7.2. *We have* $\sin(\theta) = \sin(\phi)$ *if and only if* ϕ *has the form* $2n\pi + \theta$ *for some integer* n*, or* ϕ *has the form* $2n\pi + \pi - \theta$ *for some integer* n*. In other words, we have*

$$\{\phi \mid \sin(\phi) = \sin(\theta)\} = \{2n\pi + \theta \mid n \in \mathbb{Z}\} \cup \{2n\pi + \pi - \theta \mid n \in \mathbb{Z}\}.$$

This can also be rewritten as follows (by putting $m = 2n$ *or* $m = 2n + 1$ *as appropriate):*

$$\{\phi \mid \sin(\phi) = \sin(\theta)\} = \{m\pi + (-1)^m\theta \mid m \in \mathbb{Z}\}.$$

There are various ways to see this. On the one hand, one can just inspect the graph. We draw a horizontal line at height $\sin(\theta)$, and then find all the places where it crosses the curve:



The x-coordinates of the crossing points are at $m\pi + (-1)^m\theta$ for all integers m, as claimed.

For a more formulaic argument, we can use the identity

$$\sin(\phi) - \sin(\theta) = 2\sin((\phi-\theta)/2)\cos((\phi+\theta)/2) = -2\sin((\phi-\theta)/2)\sin((\phi+\theta-\pi)/2).$$

This shows that $\sin(\phi) = \sin(\theta)$ if and only if (a) $\sin((\phi-\theta)/2) = 0$, or (b) $\sin((\phi+\theta-\pi)/2) = 0$. In case (a), the angle $(\phi-\theta)/2$ must be of the form $n\pi$ for some integer n, so $\phi = 2n\pi + \theta$. In case (b), the angle $(\phi+\theta-\pi)/2$ must be of the form $n\pi$ for some integer n, so $\phi = 2n\pi + \pi - \theta$.

We are now ready to solve some trigonometric equations.

Exercise 5.7.3. Solve the equation $\sin(3\theta) = \cos(2\theta)$.

Solution: We can rewrite the equation as $\sin(3\theta) = \sin(\pi/2 + 2\theta)$. The proposition tells us that either (a) $3\theta = 2n\pi + \pi/2 + 2\theta$ for some n, or (b) $3\theta = 2k\pi + \pi - (\pi/2 + 2\theta)$ for some k. In the second case, we have $5\theta = 2k\pi + \pi/2$, so $\theta = (4k+1)\pi/10$. In the first case, we have $\theta = 2n\pi + \pi/2 = (4n+1)\pi/2$. It turns out that for this particular equation, case (a) is redundant: if $\theta = (4n+1)\pi/2$ for some integer n, then the number $k = 5n+1$ is again an integer and

$$(4k+1)\pi/10 = (20n+5)\pi/10 = (4n+1)\pi/2 = \theta,$$

so θ is already covered by case (b). We therefore have

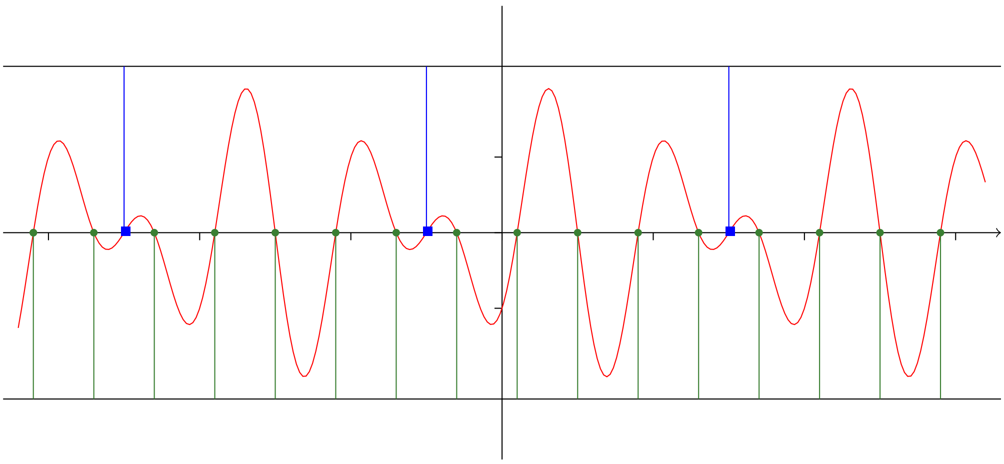
$$\{\theta \mid \sin(3\theta) = \cos(2\theta)\} = \{(4k+1)\pi/10 \mid k \in \mathbb{Z}\}.$$

Exercise 5.7.4. Solve the equation $\cos(3\theta) = \sin(2\theta)$.

Solution: We can rewrite the equation as $\sin(\pi/2 + 3\theta) = \sin(2\theta)$. The proposition tells us that either (a) $\pi/2 + 3\theta = 2n\pi + 2\theta$ for some n, or (b) $\pi/2 + 3\theta = 2n\pi + \pi - 2\theta$ for some n. In case (a), we have $\theta = 2n\pi - \pi/2 = (4n-1)\pi/2$. In case (b), we have $5\theta = 2n\pi + \pi - \pi/2 = (4n+1)\pi/2$, so $\theta = (4n+1)\pi/10$. We thus have

$$\{\theta \in \mathbb{R} \mid \cos(3\theta) = \sin(2\theta)\} = \{(4n-1)\pi/2 \mid n \in \mathbb{Z}\} \cup \{(4n+1)\pi/10 \mid n \in \mathbb{Z}\}.$$

To display this graphically, note that we are looking for the zeros of the function $f(\theta) = \cos(3\theta) - \sin(2\theta)$. The graph of this function is shown below:



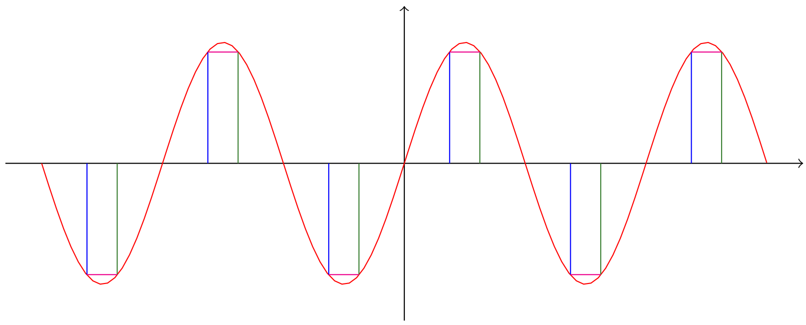
The zeros of the form $(4n-1)\pi/2$ are indicated by the blue bars above the x-axis, and those of the form $(4n+1)\pi/10$ are indicated by the green bars below the axis.

Exercise 5.7.5. Solve the equation $\sin(\theta+\pi/4)=\sin(\theta)$.

Solution: From the proposition, this holds if and only if (a) $\theta+\pi/4=2n\pi+\theta$ for some integer n, or (b) $\theta+\pi/4=2n\pi+\pi-\theta$ for some integer n. In case (a) we have $\pi/4=2n\pi$, so $n=1/8$, which is not an integer. This means that case (a) cannot happen, so we only have case (b). In that case, we have $2\theta=2n\pi+\pi-\pi/4=(8n+3)\pi/4$, so $\theta=(8n+3)\pi/8=(n+3/8)\pi$. It follows that

$$\{\theta\in\mathbb{R}\mid \sin(\theta+\pi/4)=\sin(\theta)\}=\{(n+3/8)\pi\mid n\in\mathbb{Z}\}.$$

We can illustrate this graphically as follows:



The graph shows the function $\sin(\theta)$. The blue bars mark the points where θ has the form $(n+3/8)\pi$, and the green bars are the corresponding points $\theta+\pi/4$. The graph meets the green and blue bars at the same level, showing that $\sin(\theta)=\sin(\theta+\pi/4)$ for these values of θ.

5.8 Advanced special functions

As a matter of general mathematical knowledge, you should be aware that there are many more advanced special functions that are used in various places in mathematics, physics and engineering. It used to be the case that there were hundreds of different functions whose properties had to be studied separately (Bessel functions, hypergeometric functions, Airy functions,...). The situation has now been greatly simplified, partly as a consequence of theoretical work behind the programming of systems such as Maple. Many of the old special functions can now be regarded as special cases of the *Meijer G-function* . Most of the remaining functions are related to *elliptic integrals* , for which there is also a systematic theory. You can explore the properties of all these functions at `http://functions.wolfram.com`.

Chapter 6: Differentiation

6.1 The meaning of differentiation

Consider two variables x and y that are related in some way, so that whenever x changes, there is an associated change in y. If x changes by a small amount δx, then we write δy for the associated change in y, and we consider the ratio $\delta y/\delta x$. This may depend on precisely how small δx is, so we let δx tend to zero and we write dy/dx for the limiting value of the ratio $\delta y/\delta x$. The process of taking limits will be examined a little more closely in Section 6.2, and also in PMA113 (Introduction to Analysis). The quantity dy/dx is called the *derivative* of y with respect to x.

Warning 6.1.1. Here δx is regarded as a single symbol; it is not something called δ multiplied by something called x. In particular, it is not valid to cancel δ's, so $\delta y/\delta x \neq y/x$.

If y is given explicitly as a function of x, say $y = f(x)$, then we write $f'(x)$ for dy/dx. We can give a kind of formula for this as follows. When x changes by δx, the value of f changes from $f(x)$ to $f(x + \delta x)$, so the change in $y = f(x)$ is $\delta y = f(x + \delta x) - f(x)$. We thus have $\delta y/\delta x = (f(x + \delta x) - f(x))/\delta x$. To get the quantity $f'(x) = dy/dx$, we take the limit as δx tends to zero. It is traditional and convenient to rename δx as h, giving the formula

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}.$$

Here are some examples of the meaning and use of derivatives.

Example 6.1.2. Many variables u depend on the time t at which they are measured. For example, u could be the position of a moving object, the temperature of the sun or the price of oil. The derivative du/dt is then the rate of change of the relevant variable. The rate of change of the position of an object is the velocity of that object. If u is a price, then du/dt, divided by u itself, is the rate of inflation for that price. □

Example 6.1.3. Many variables u depend on the place at which they are measured. For example, if you have an iron bar that is hotter at one end than the other, and you let u be the temperature, then u will depend on the distance x from the end of the bar to the point of measurement. If there is a large temperature change from one end to the other, then heat will flow very quickly along the bar; but if the temperature change is small, then the heat flow will be much slower. To state this relationship more precisely, we need to consider the rate of change of temperature with respect to position, or in other words du/dx.

It is of course more usual to consider variables that depend on all three coordinates (x, y and z), and possibly also on time. For example, if we consider gases flowing through a jet engine, then the temperature, pressure and concentration of carbon monoxide will all be variables of this type. There are a number of additional things to bear in mind when differentiating a function of several variables; these will be covered in PMA213 (Advanced Calculus) and AMA224 (Vectors and Fluids). □

Example 6.1.4. Suppose that you own a chemical factory that makes gloop. The public will buy g litres of gloop per day, where g depends on the price p (in pounds per litre). When you decide what to charge, you need to know how much the demand will drop in response to a price increase; if the drop is too large, it will wipe out the extra income that you get by charging more. Normally you will only be considering a small change δp in the price, so the key number that you need to know is the derivative $\frac{dg}{dp}$, which will allow you to estimate the resulting change δg in g. Economists generally prefer to talk in terms of *percentage* changes in price and demand, so they look at $\frac{\delta g}{g}/\frac{\delta p}{p}$, which tends in the limit to the quantity $e = \frac{p}{g}\frac{dg}{dp}$, which is called the *price elasticity of demand*. □

Example 6.1.5. Suppose that a point $P = (x, y)$ moves along a curve C in the plane. If we change x by a small amount δx, then y will also have to change by a small amount δy, in order for us to stay on the curve. The quantity $\delta y/\delta x$ is approximately the slope of the tangent line to C at P, and this approximation becomes arbitrarily exact as δx tends to zero. $\square$

6.2 Differentiation from first principles

In this section we show how a few functions can be differentiated directly from the definition.

Example 6.2.1. If $f(x) = x^2$, then $f'(x) = 2x$. Indeed, we have

$$\begin{aligned} f(x+h) &= (x+h)^2 = x^2 + 2xh + h^2 \\ \frac{f(x+h)-f(x)}{h} &= \frac{x^2+2xh+h^2-x^2}{h} = 2x+h \\ f'(x) &= \lim_{h\to 0}(2x+h) = 2x. \end{aligned}$$

Similarly, if $g(x) = x^3$ then

$$\begin{aligned} g(x+h) &= x^3 + 3x^2h + 3xh^2 + h^3 \\ \frac{g(x+h)-g(x)}{h} &= \frac{3x^2h+3xh^2+h^3}{h} \\ g'(x) &= \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2. \end{aligned}$$

The method can easily be extended to show that $\frac{d}{dx}x^n = nx^{n-1}$ for all $n \geq 0$. $\square$

Example 6.2.2. We now consider $f(x) = 1/x$. We have

$$\begin{aligned} f(x+h) - f(x) &= \frac{1}{x+h} - \frac{1}{x} = \frac{-h}{x(x+h)} \\ \frac{f(x+h)-f(x)}{h} &= \frac{-1}{x(x+h)} \\ f'(x) &= \lim_{h\to 0}\left(\frac{-1}{x(x+h)}\right) = \frac{-1}{x^2}. \quad \square \end{aligned}$$

Example 6.2.3. We next consider the exponential function

$$\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots .$$

We begin by showing that $\exp'(0) = 1$. We have

$$\begin{aligned} \exp(h) - \exp(0) &= \exp(h) - 1 = h + \frac{h^2}{2} + \frac{h^3}{3!} + \cdots \\ (\exp(h) - \exp(0))/h &= 1 + \frac{h}{2} + \frac{h^2}{3!} + \cdots \\ \exp'(0) &= \lim_{h\to 0}(1 + \frac{h}{2} + \frac{h^2}{3!} + \cdots) \\ &= 1 + 0 + 0 + \cdots = 1. \end{aligned}$$

At the last stage, we have glossed over some delicate points about limits and infinite sums; they turn out not to make a difference in this example.

We can now deduce that $\exp'(x) = \exp(x)$ for all x. Indeed, we have $\exp(x+h) = \exp(x)\exp(h)$, so

$$\frac{\exp(x+h)-\exp(x)}{h} = \exp(x)\frac{\exp(h)-1}{h}.$$

Taking the limit as h tends to zero, we get

$$\exp'(x) = \exp(x)\exp'(0) = \exp(x)$$

as claimed.

Another approach is simply to differentiate the definition. Recall that

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1,$$

which means that $n! = n \times (n-1)!$, so $n/(n!) = 1/(n-1)!$. In other words, we have $3/3! = 1/2!$, and $4/4! = 1/3!$, and so on. Using this, we have

$$\begin{aligned}
\exp(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \\
\exp'(x) &= 0 + 1 + \frac{2x}{2} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \frac{5x^4}{5!} + \cdots \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
&= \exp(x).
\end{aligned}$$

This glosses over some even more delicate issues about limits and infinite sums, but again they make no difference in this example. □

6.3 Derivatives of special functions

We next record the derivatives of the special functions considered in Chapter 5.

$$\begin{aligned}
\exp'(x) &= \exp(x) & \log'(x) &= 1/x \\
\sinh'(x) &= \cosh(x) & \operatorname{arcsinh}'(x) &= (1+x^2)^{-1/2} \\
\cosh'(x) &= \sinh(x) & \operatorname{arccosh}'(x) &= (x^2-1)^{-1/2} \\
\tanh'(x) &= \operatorname{sech}(x)^2 = 1 - \tanh(x)^2 & \operatorname{arctanh}'(x) &= (1-x^2)^{-1} \\
\sin'(x) &= \cos(x) & \arcsin'(x) &= (1-x^2)^{-1/2} \\
\cos'(x) &= -\sin(x) & \arccos'(x) &= -(1-x^2)^{-1/2} \\
\tan'(x) &= \sec(x)^2 = 1 + \tan(x)^2 & \arctan'(x) &= (1+x^2)^{-1}
\end{aligned}$$

In some sense, the first of these is all that you need. Once you know $\exp'(x)$ you can calculate $\sinh'(x)$, $\cosh'(x)$ and $\tanh'(x)$, because $\sinh(x)$, $\cosh(x)$ and $\tanh(x)$ are defined in a simple way in terms of $\exp(x)$. Next, we can use the relations $\sin(x) = \sinh(ix)/i$ and so on to deduce $\sin'(x)$, $\cos'(x)$ and $\tan'(x)$. We can then use the rules for inverse functions to find the derivatives of the remaining functions in the list. Some details of these deductions will be explained in the next section.

6.4 Rules for differentiation

In this section we state the main rules for finding derivatives: the product rule, the quotient rule, the power rule, the chain rule, and the inverse function rule. Here we give only brief examples, but there are many more examples in the following section.

6.4.1 The product rule

The rule can be stated in several different (but equivalent) ways, depending on the kind of notation being used.

(a) If u and v depend on x, then $\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$.

(b) If $f(x) = g(x)h(x)$ then $f'(x) = g'(x)h(x) + g(x)h'(x)$

(c) In case (a) we sometimes just write $(uv)' = u'v + uv'$.

Justification. Put $w = uv$. If x changes to $x + \delta x$, then u, v and w change to $u + \delta u$, $v + \delta v$ and $w + \delta w$ respectively, so

$$\begin{aligned} w + \delta w &= (u + \delta u)(v + \delta v) \\ &= uv + u\,\delta v + v\,\delta u + \delta u\,\delta v. \end{aligned}$$

If we subtract the equation $w = uv$, divide by δx, and rearrange slightly we get

$$\frac{\delta w}{\delta x} = u\frac{\delta v}{\delta x} + v\,\frac{\delta u}{\delta x} + \frac{\delta u}{\delta x}\frac{\delta v}{\delta x}\delta x.$$

We now take the limit as $\delta x \to 0$. The left hand side becomes w', and the first two terms on the right become $u'v + uv'$. For the last term, we have $\frac{\delta u}{\delta x} \to \frac{du}{dx}$ and $\frac{\delta v}{\delta x} \to \frac{dv}{dx}$ and $\delta x \to 0$ so

$$\frac{\delta u}{\delta x}\frac{\delta v}{\delta x}\delta x \to \frac{du}{dx}\frac{dv}{dx} \times 0 = 0.$$

We conclude that $w' = u'v + uv'$ as claimed. □

Remark 6.4.1. It is often convenient to rearrange the rule as follows: if $w = uv$, then

$$\frac{w'}{w} = \frac{u'v + uv'}{uv} = \frac{u'}{u} + \frac{v'}{v}.$$

Example 6.4.2. If $f(x) = \exp(x)\sin(x)$ then

$$\begin{aligned} f'(x) &= \exp'(x)\sin(x) + \exp(x)\sin'(x) \\ &= \exp(x)\sin(x) + \exp(x)\cos(x) \\ &= \exp(x)(\sin(x) + \cos(x)). \quad \square \end{aligned}$$

6.4.2 The quotient rule

(a) If u and v depend on x, then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}.$$

(b) If $f(x) = g(x)/h(x)$ then $f'(x) = (g'(x)h(x) - g(x)h'(x))/h(x)^2$.

(c) In case (a), we sometimes just write $(u/v)' = (u'v - uv')/v^2$.

Justification. Put $w = u/v$, so $u = vw$. The product rule then gives $u' = v'w + vw' = v'w + vw'$, which we rearrange to give

$$\begin{aligned} w' &= (u' - v'w)/v = (u'v - v'vw)/v^2 \\ &= (u'v - v'u)/v^2, \end{aligned}$$

as claimed. □

Remark 6.4.3. It is often convenient to rearrange the rule as follows: if $w = u/v$, then

$$\frac{w'}{w} = \frac{u'}{u} - \frac{v'}{v}.$$

To check this, note that

$$\frac{w'}{w} = \frac{u'v - uv'}{v^2}\,\frac{v}{u} = \frac{u'v^2}{uv^2} - \frac{uvv'}{uv^2} = \frac{u'}{u} - \frac{v'}{v}.$$

Example 6.4.4. We can derive the formula for $\tan'(x)$ from the definition $\tan(x) = \sin(x)/\cos(x)$ as follows:

$$\begin{aligned}
\tan'(x) &= (\sin'(x)\cos(x) - \sin(x)\cos'(x))/\cos(x)^2 \\
&= (\cos(x)\cos(x) - \sin(x).(-\sin(x)))/\cos(x)^2 \\
&= (\cos(x)^2 + \sin(x)^2)\cos(x)^{-2} \\
&= \cos(x)^{-2}.
\end{aligned}$$

(In the last step, we used the standard relation $\cos(x)^2 + \sin(x)^2 = 1$.) $\square$

6.4.3 The power rule

(a) If u depends on x and n does not, then $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$.

(b) If $f(x) = g(x)^n$ (where n is constant) then $f'(x) = ng(x)^{n-1}g'(x)$.

(c) In case (a), we sometimes just write $(u^n)' = nu^{n-1}u'$.

This is really a special case of the chain rule, as we will explain below.

Example 6.4.5. Consider the function $f(x) = \sqrt{1+x^2}$. This has the form $f(x) = g(x)^n$, where $n = 1/2$ and $g(x) = 1 + x^2$, so $g'(x) = 2x$. This means that

$$\begin{aligned}
f'(x) &= ng(x)^{n-1}g'(x) \\
&= \frac{1}{2}(1+x^2)^{\frac{1}{2}-1}2x \\
&= x(1+x^2)^{-1/2}. \quad \square
\end{aligned}$$

6.4.4 The chain rule

(a) If u depends on x, and y depends on u (and thus indirectly on x) then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$. In most applications of this rule, dy/du will initially be expressed as a function of u, and we will need to rewrite it as a function of x.

(b) If $f = g \circ h$ (so $f(x) = g(h(x))$ for all x) then $(g \circ h)'(x) = f'(x) = g'(h(x))h'(x)$.

Justification. If x changes by a small amount δx, then the corresponding changes in u and y are δu and δy. It is clear that

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u}\frac{\delta u}{\delta x},$$

and when we take the limit as $\delta x \to 0$ we get

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

$\square$

Remark 6.4.6. In the case $y = u^n$ we have $\frac{dy}{du} = nu^{n-1}$ and so $\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}$; so we recover the power rule.

Example 6.4.7. Consider the function $y = \cos(x^2)$. If we put $u = x^2$ then $y = \cos(u)$. We then have $dy/du = -\sin(u)$ as a function of u; we substitute $u = x^2$ to see that $dy/du = -\sin(x^2)$ as a function of x. We also have $du/dx = 2x$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\sin(x^2) \times 2x = -2x\sin(x^2). \quad \square$$

Example 6.4.8. Consider the function $f(x) = \log(\cos(x))$. We have

$$f'(x) = \log'(\cos(x))\cos'(x) = \frac{1}{\cos(x)} \times (-\sin(x)) = -\tan(x). \quad \square$$

6.4.5 The logarithmic rule

(a) If u depends on x and we know how to differentiate $\log(u)$, then du/dx can be calculated from the formula
$$\frac{du}{dx} = u\frac{d}{dx}\log(u).$$

(b) In alternative notation, we have $f'(x) = f(x)\,(\log\circ f)'(x)$.

Justification. Put $y = \log(u)$, so $\frac{dy}{du} = u^{-1}$. The chain rule then gives $\frac{d}{dx}\log(u) = u^{-1}\frac{du}{dx}$, which we can rearrange to give
$$\frac{du}{dx} = u\frac{d}{dx}\log(u).$$
□

Example 6.4.9. Put $u = x^x$. It is *not* correct to apply the power rule (which would give $du/dx = x.x^{x-1} = x^x$) because the exponent is not a constant. Instead, we recall the rule $\log(a^b) = b\log(a)$; by taking $a = b = x$ we see that that $\log(u) = x\log(x)$. This we can easily differentiate to give
$$\frac{d}{dx}\log(u) = 1.\log(x) + x.\frac{1}{x} = \log(x) + 1,$$
so the logarithmic rule gives
$$\frac{du}{dx} = u\frac{d}{dx}\log(u) = x^x(\log(x)+1). \quad \square$$

6.4.6 The inverse function rule

(a) Suppose that x and y are interdependent, so that a change in x causes a change in y and *vice-versa*. Then $\frac{dy}{dx}\frac{dx}{dy} = 1$. In most applications of this rule, we will want to rewrite x in terms of y, so as to get a formula that contains only y. Alternatively, we may rewrite y in terms of x, to get a formula containing only x.

(b) Suppose that f and g are inverses of each other, so that $f(g(y)) = y$ and $g(f(x)) = x$. Then $g'(y) = 1/f'(g(y))$, and $f'(x) = 1/g'(f(x))$.

Justification. If we have a small change δx in x, and an associated small change δy in y, it is clear that $\frac{\delta x}{\delta y}\frac{\delta y}{\delta x} = 1$. We now pass to the limit as δx and δy tend to zero, to get $\frac{dx}{dy}\frac{dy}{dx} = 1$. □

Example 6.4.10. Put $y = \sin(x)$, so $x = \arcsin(y)$. We have $dy/dx = \cos(x)$, so $dx/dy = 1/\cos(x)$. This is a perfectly valid equation, but it would be more useful if it were rewritten in terms of y alone. We note that $\cos(x)^2 + \sin(x)^2 = 1$, so $\cos(x) = (1-\sin(x)^2)^{1/2} = (1-y^2)^{1/2}$, so
$$\arcsin'(y) = \frac{dx}{dy} = \frac{1}{\cos(x)} = (1-y^2)^{-1/2}. \quad \square$$

Example 6.4.11. We will take it as given that $\tan'(x) = \sec(x)^2 = 1 + \tan(x)^2$, and derive a formula for $\arctan'(y)$. The arctan function is by definition the inverse of tan, so the rule says
$$\begin{aligned}\arctan'(y) &= 1/\tan'(\arctan(y)) \\ &= 1/(1+\tan(\arctan(y))^2) \\ &= 1/(1+y^2). \quad \square\end{aligned}$$

6.5 Examples of differentiation

Exercise 6.5.1. Consider the function

$$f(x) = \sqrt{2\pi}x^{x-1/2}e^{-x}$$

(This is interesting because it can be shown that $f(n+1)$ is a good approximation to $n!$ for large n; this is called *Stirling's approximation* .)

Calculate $f'(x)/f(x)$.

Solution: The easiest method is to use the logarithmic rule $f'(x) = f(x)\frac{d}{dx}\log(f(x))$. We have

$$\log(f(x)) = \log(\sqrt{2\pi}) + (x - 1/2)\log(x) - x,$$

so

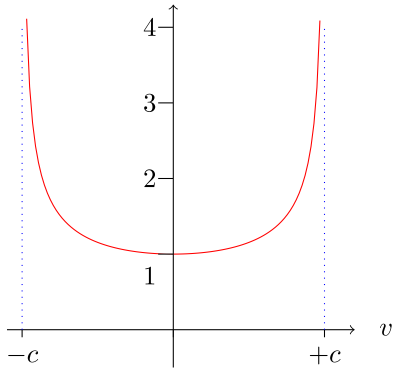
$$\begin{aligned} f'(x)/f(x) &= \frac{d}{dx}(\log(\sqrt{2\pi}) + (x - 1/2)\log(x) - x) \\ &= 0 + 1.\log(x) + (x - 1/2).x^{-1} - 1 \\ &= \log(x) - 1/(2x). \end{aligned}$$

Exercise 6.5.2. Let c be a positive constant, and put $g(v) = (1 - v^2/c^2)^{-1/2}$. Calculate $g'(v)$.

Solution:

$$\begin{aligned} g'(v) &= -\tfrac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2}\frac{d}{dv}\left(1 - \frac{v^2}{c^2}\right) && \text{(power rule)} \\ &= -\tfrac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2}(-2v/c^2) \\ &= vc^{-2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2}. \end{aligned}$$

Background: The graph of $g(v)$ is as follows:



Let c be the speed of light. The theory of relativity says that if you travel at speeds approaching c, you will observe all sorts of strange effects: lengths are distorted, clocks slow down, the colours of stationary objects change, and so on. If you move at speed v, then the strength of these effects is given by $g(v)$. Note that $g(0) = 1$, which mans that if you move at speed 0 then clocks are slowed by a factor of 1, or in other words they are not slowed at all, which makes sense. At 70 mph we have $v/c \approx 10^{-7}$, which gives $g(v) \approx 1 + 5 \times 10^{-15}$. When driving down the motorway your car clock is slowed by a factor of $g(v)$, which works out to about 0.15 microseconds per year. This kind of effect is measurable with atomic clocks. It is common for subatomic particles to move so fast that $g(v)$ is very large. The same is true for very remote galaxies, which gives rise to the "red shift" phenomenon, which allows astronomers to estimate interstallar distances.

Exercise 6.5.3. Let a, b, c and d be constants and put $y = (ax+b)/(cx+d)$. Calculate dy/dx.

Solution: Put $u = ax + b$ and $v = cx + d$, so $u' = a$ and $v' = c$. We then have

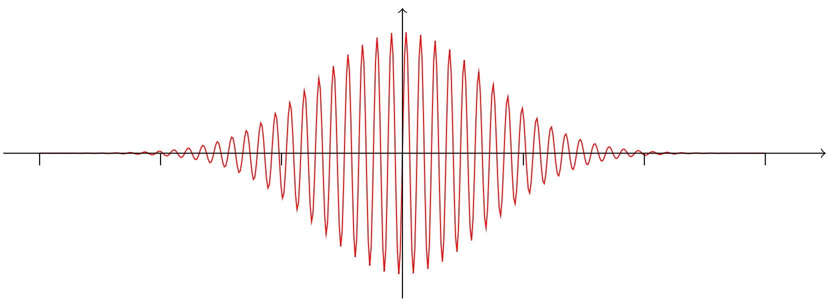
$$\begin{aligned} y' &= (u/v)' = (u'v - uv')/v^2 \\ &= ((acx + ad) - (acx + bc))/(cx + d)^2 \\ &= \frac{ad - bc}{(cx + d)^2}. \end{aligned}$$

Exercise 6.5.4. Calculate $\frac{d}{dx}\left(e^{-a^2x^2}\sin(\omega x)\right)$, where a and ω are constants.

Solution: Put $t = -a^2x^2$ and $u = \exp(t) = e^{-ax^2}$ and $v = \sin(\omega x)$ and $y = uv = e^{-a^2x^2}\sin(\omega x)$; we must find dy/dx. We have

$$\begin{aligned} \tfrac{dt}{dx} &= -2a^2x \\ \tfrac{du}{dt} &= \tfrac{d}{dt}e^t = e^t = e^{-a^2x^2} \\ \tfrac{du}{dx} &= \tfrac{dt}{dx}\tfrac{du}{dt} = -2a^2xe^{-a^2x^2} \\ \tfrac{dv}{dx} &= \omega\cos(\omega x) \\ \tfrac{dy}{dx} &= u\tfrac{dv}{dx} + \tfrac{du}{dx}v \\ &= e^{-a^2x^2}\omega\cos(\omega x) - 2a^2xe^{-a^2x^2}\sin(\omega x) \\ &= e^{-a^2x^2}(\omega\cos(\omega x) - 2a^2x\sin(\omega x)). \end{aligned}$$

(It is a common mistake for things like $2a$ to creep in. If a were a variable then we would have $\frac{d}{da}(a^2) = 2a$. However, a is in fact a constant, and we are using $\frac{d}{dx}$ rather than $\frac{d}{da}$, so the equation $\frac{d}{da}(a^2) = 2a$ is not relevant.) **Background:** The graph of y against x looks like this:



Functions of this type are important in quantum mechanics, where they are known as *wave packets*. Usually ω is much larger than a, which means that the hump contains many oscillations, as illustrated above.

Exercise 6.5.5. Calculate dy/dt, where $y = t/\sqrt{1+t^2}$. You should simplify the result as far as possible, which will take some algebraic manipulation. The final answer should have the form $(1+t^2)^c$ for some constant c.

Solution: First, put $u = t$ and $v = \sqrt{1+t^2}$, so $y = u/v$. We have $u' = 1$ and

$$v' = \tfrac{1}{2}(1+t^2)^{-1/2}.2t = t(1+t^2)^{-1/2} = t/v.$$

This gives

$$\begin{aligned} y' &= (u/v)' = (u'v - uv')/v^2 \\ &= (v - t.(t/v))/v^2 \\ &= (v^2 - t^2)/v^3. \end{aligned}$$

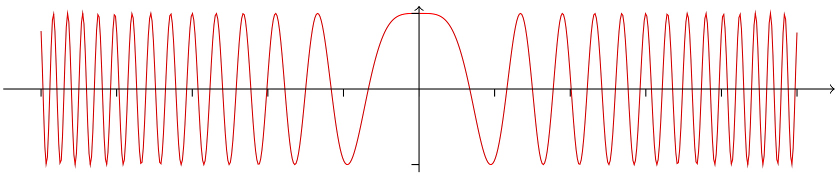
However, v^2 is just $1+t^2$, so $v^2-t^2=1$ and $y'=1/v^3=(1+t^2)^{-3/2}$.

We kept this calculation quite tidy by expressing things in terms of the intermediate variable v. This is a good trick, but not one that you would necessarily think of. Here is the calculation again, written in a more verbose way:

$$\begin{aligned} y' &= \frac{1.\sqrt{1+t^2} - t.t(1+t^2)^{-1/2}}{1+t^2} \\ &= \frac{\sqrt{1+t^2}}{\sqrt{1+t^2}} \, \frac{\sqrt{1+t^2} - t.t(1+t^2)^{-1/2}}{1+t^2} \\ &= \frac{\sqrt{1+t^2}\sqrt{1+t^2} - t^2}{(1+t^2)^{3/2}} \\ &= \frac{1+t^2-t^2}{(1+t^2)^{3/2}} \\ &= (1+t^2)^{-3/2}. \end{aligned}$$

Exercise 6.5.6. Calculate $f'(x)$, where $f(x)=\cos(x^2)$.

Solution: The chain rule gives $f'(x) = -2x\sin(x^2)$. **Background:** The graph is as follows:



When x is reasonable large, the angle x^2 varies rapidly with x, so $f(x)=\cos(x^2)$ oscillates wildly between $+1$ and -1. This means that in any short interval, the positive values will cancel out the negative values and the average value will be close to zero. Near the origin, however, the angle x^2 changes slowly, so $\cos(x^2)$ will generally not change sign in a short interval, and the average value over a short interval will be nonzero. This is essentially the *principle of stationary phase*, which is very important in quantum mechanics.

Exercise 6.5.7. Let α, ω, a and b be constants, and put

$$\begin{aligned} f(t) &= \sin((\omega + a\sin(\alpha t))t) \\ g(t) &= (1+b\sin(\alpha t))\sin(\omega t) \end{aligned}$$

Find $f'(t)$ and $g'(t)$.

Solution: Put $p(t) = (\omega + a\sin(\alpha t))t$, so $f(t)=\sin(p(t))$ and

$$p'(t) = \omega + a\sin(\alpha t) + a\alpha t\cos(\alpha t),$$

so

$$\begin{aligned} f'(t) &= \sin'(p(t))p'(t) = \cos(p(t))p'(t) \\ &= \cos((\omega + a\sin(\alpha t))t)(\omega + a\sin(\alpha t) + a\alpha t\cos(\alpha t)). \end{aligned}$$

We also have

$$g'(t) = b\alpha\cos(\alpha t)\sin(\omega t) + (1+b\sin(\alpha t))\omega\cos(\omega t).$$

Background: This function $f(t)$ is essentially the output of an FM radio transmitter broadcasting a pure tone of audio frequency $\alpha/2\pi$, where the frequency of the radio signal itself is $\omega/2\pi$. Here it is important that ω is very much bigger than α; typical values are $\omega \approx 10^9$ and $\alpha \approx 10^5$. It seems not to be possible to draw an illuminating plot of this function.

The function $g(t)$ is the output of an AM radio transmitter broadcasting the same signal at the same radio frequency.

Exercise 6.5.8*. Find $\frac{d}{dx}p(q(x)^b)^a$.

Solution: The answer is

$$ab\,p(q(x)^b)^{a-1}q(x)^{b-1}p'(q(x)^b)q'(x).$$

To see this, put

$$\begin{aligned} u &= q(x) \\ v &= u^b = q(x)^b \\ w &= p(v) = p(q(x)^b) \\ y &= w^a = p(q(x)^b)^a, \end{aligned}$$

so we must find dy/dx. We have

$$\begin{aligned} du/dx &= q'(x) \\ dv/du &= bu^{b-1} = bq(x)^{b-1} \\ dw/dv &= p'(v) = p'(q(x)^b) \\ dy/dw &= aw^{a-1} = ap(q(x)^b)^{a-1}, \end{aligned}$$

so

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dw}\,\frac{dw}{dv}\,\frac{dv}{du}\,\frac{du}{dx} \\ &= q'(x)bq(x)^{b-1}p'(q(x)^b)ap(q(x)^b)^{a-1} \\ &= ab\,p(q(x)^b)^{a-1}q(x)^{b-1}p'(q(x)^b)q'(x), \end{aligned}$$

as claimed.

Exercise 6.5.9. Suppose that $y = (pq)/(rs)$, where p, q, r and s all depend on x. Simplify y'/y. (You should write your answer as a sum of four terms, not as a single fraction.)

Solution: The most efficient method is as follows:

$$\begin{aligned} y'/y &= \log(y)' \\ &= (\log(p) + \log(q) - \log(r) - \log(s))' \\ &= p'/p + q'/q - r'/r - s'/s. \end{aligned}$$

Alternatively, we have

$$\begin{aligned} y' &= \frac{(pq)'rs - pq(rs)'}{(rs)^2} \\ &= \frac{p'qrs + pq'rs - pqr's - pqrs'}{r^2s^2}, \end{aligned}$$

so

$$\begin{aligned} \frac{y'}{y} &= \frac{p'qrs + pq'rs - pqr's - pqrs'}{r^2s^2}\,\frac{rs}{pq} \\ &= \frac{p'qrs + pq'rs - pqr's - pqrs'}{pqrs} \\ &= p'/p + q'/q - r'/r - s'/s. \end{aligned}$$

Exercise 6.5.10*. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be functions such that $f(1) = -1$ and $g(-1) = 1$. Show that $(f \circ g)'(-1) = (g \circ f)'(1)$.

Solution:

$$\begin{aligned}(f\circ g)'(-1) &= f'(g(-1))g'(-1) = f'(1)g'(-1)\\ (g\circ f)'(1) &= g'(f(1))f'(1) = g'(-1)f'(1) = f'(1)g'(-1).\end{aligned}$$

Exercise 6.5.11*. Let $f\colon \mathbb{R}\to\mathbb{R}$ be a function with $f(0)=0$. Put $g(x)=f(f(f(f(x))))$. Show that $g'(0)\geq 0$.

Solution: By the chain rule, we have

$$g'(x) = f'(f(f(f(x))))f'(f(f(x)))f'(f(x))f'(x).$$

Now put $x=0$, so $f(x)=f(0)=0$, so $f(f(x))=0$, so $f(f(f(x)))=0$. This gives

$$\begin{aligned}g'(0) &= f'(f(f(f(0))))f'(f(f(0)))f'(f(0))f'(0)\\ &= f'(0)f'(0)f'(0)f'(0) = f'(0)^4.\end{aligned}$$

The fourth power of any real number is nonnegative, so $g'(0)\geq 0$.

Exercise 6.5.12. Suppose we have functions $p(u)$, $q(x)$, $r(x)$ and $s(x)$, and we put $f(x) = p(q(x)+r(x)s(x))$. Find $f'(x)$ in terms of p, q, r, s and their derivatives.

Solution: Put $g(x) = q(x)+r(x)s(x)$, so that $g'(x) = q'(x)+r'(x)s(x)+r(x)s'(x)$. Then $f(x)=p(g(x))$, so

$$\begin{aligned}f'(x) &= p'(g(x))\,g'(x)\\ &= p'(q(x)+r(x)s(x))\,(q'(x)+r'(x)s(x)+r(x)s'(x)).\end{aligned}$$

Exercise 6.5.13. Put $y = a_1x+a_2x^2+a_3x^3+a_4x^4$.

(a) Find $x\frac{dy}{dx}$.

(b) Find $x\frac{d}{dx}\left(x\frac{dy}{dx}\right)$.

(c) Find $x\frac{d}{dx}\left(x\frac{d}{dx}\left(x\frac{dy}{dx}\right)\right)$.

(d) What is the general rule?

Solution: It is convenient to introduce the notation $Lz = x\frac{dz}{dx}$. The question then asks us to find Ly, LLy and $LLLy$. We have

$$\begin{aligned}Ly &= x\frac{dy}{dx} = x(a_1+2a_2x+3a_3x^2+4a_4x^3)\\ &= a_1x+2a_2x^2+3a_3x^3+4a_4x^4\\ LLy &= x(a_1+2\times 2a_2x+3\times 3a_3x^{+}4\times 4a_4x^3)\\ &= a_1x+4a_2x^2+9a_3x^3+16a_4x^4\\ &= a_1x+2^2a_2x^2+3^2a_3x^3+4^2a_4x^4\\ LLLy &= x(a_1+2\times 4a_2x+3\times 9a_3x^2+4\times 16a_4x^3)\\ &= a_1x+8a_2x^2+27a_3x^3+64a_4x^4\\ &= a_1x+2^3a_2x^2+3^3a_3x^3+4^3a_4x^4.\end{aligned}$$

The general rule is clearly that if $y=\sum_k a_kx^k$, then $L^ny = \sum_k k^na_kx^k$.

Exercise 6.5.14*. Consider the function $f(x) = (2x+3)/(x+2)$.

(a) Calculate f', f'' and f'''.

(b) The *Schwarzian derivative* of f is the function $Sf = f'''/f' - \frac{3}{2}(f''/f')^2$. (This is useful in the theory of population models discussed in Exercise 4.3.9*, for example.) Show that for this particular function f we have $Sf = 0$.

(c) Show more generally that $Sg = 0$ whenever $g(x)$ has the form $(ax+b)/(cx+d)$ for some constants a, b, c and d (with $ad - bc \neq 0$).

(In fact these are the only functions g for which $Sg = 0$, but we will not prove this here.)

Solution: We have

$$\begin{aligned}
f' &= \frac{2(x+2) - 1(2x+3)}{(x+2)^2} = \frac{1}{(x+2)^2} = (x+2)^{-2} \\
f'' &= -2(x+2)^{-3} \\
f''' &= 6(x+2)^{-4} \\
f'''/f' &= \frac{6(x+2)^{-4}}{(x+2)^{-2}} = 6(x+2)^{-2} \\
(f''/f')^2 &= \left(\frac{-2(x+2)^{-3}}{(x+2)^{-2}}\right)^2 = (-2(x+2)^{-1})^2 = 4(x+2)^{-2} \\
Sf &= 6(x+2)^{-2} - \tfrac{3}{2}4(x+2)^{-2} = 0.
\end{aligned}$$

More generally, consider $g(x) = (ax+b)/(cx+d)$. It will be convenient to write $\Delta = ad - bc$. We then have

$$g' = \frac{a(cx+d) - c(ax+b)}{(cx+d)^2} = \frac{acx + ad - acx - bc}{(cx+d)^2} = \Delta(cx+d)^{-2}.$$

If $\Delta = 0$ then the above formula gives $g' = 0$ and the Schwarzian derivative $Sg = g'''/g' - \frac{3}{2}(g''/g')^2$ is undefined. We therefore consider only the case where $\Delta \neq 0$. We then have

$$\begin{aligned}
g'' &= -2c\Delta(cx+d)^{-3} \\
g''' &= 6c^2\Delta(cx+d)^{-4} \\
g'''/g' &= \frac{6c^2\Delta(cx+d)^{-4}}{\Delta(cx+d)^{-2}} = 6c^2(cx+d)^{-2} \\
(g''/g')^2 &= \left(\frac{-2c(cx+d)^{-3}}{(cx+d)^{-2}}\right)^2 = (-2c(cx+d)^{-1})^2 = 4c^2(cx+d)^{-2} \\
Sg &= 6c^2(cx+d)^{-2} - \tfrac{3}{2}4c^2(cx+d)^{-2} = 0.
\end{aligned}$$

Exercise 6.5.15*. Given that y is a function of x, simplify the following expressions:

(a) $e^{-x}\frac{d}{dx}(e^x y)$

(b) $e^{-x}\frac{d^2}{dx^2}(e^x y)$

(c) $e^{-x}\frac{d^3}{dx^3}(e^x y)$

Can you guess the general rule? Can you prove it?

Solution: We first calculate the successive derivatives:

$$\begin{aligned}
\frac{d}{dx}(e^x y) &= e^x y + e^x y' \\
\frac{d^2}{dx^2}(e^x y) &= (e^x y + e^x y') + (e^x y' + e^x y'') \\
&= e^x y + 2e^x y' + e^x y'' \\
\frac{d^3}{dx^3}(e^x y) &= (e^x y + e^x y') + 2(e^x y' + e^x y'') + (e^x y'' + e^x y''') \\
&= e^x y + 3e^x y' + 3e^x y'' + e^x y'''.
\end{aligned}$$

It follows that

$$\begin{aligned}
e^{-x}\frac{d}{dx}(e^x y) &= y + y' \\
e^{-x}\frac{d^2}{dx^2}(e^x y) &= y + 2y' + y'' \\
e^{-x}\frac{d^3}{dx^3}(e^x y) &= y + 3y' + 3y'' + y'''.
\end{aligned}$$

You should recognize the numbers here as binomial coefficients; they are the same as in the formulae

$$\begin{aligned}
(1+t)^1 &= 1 + t \\
(1+t)^2 &= 1 + 2t + t^2 \\
(1+t)^3 &= 1 + 3t + 3t^2 + t^3.
\end{aligned}$$

The pattern seems to be that

$$\begin{aligned}
e^{-x}\frac{d^n}{dx^n}(e^x y) &= y + \binom{n}{1}\frac{dy}{dx} + \binom{n}{2}\frac{d^2y}{dx^2} + \cdots + \frac{d^n y}{dx^n} \\
&= \sum_{k=0}^{n} \binom{n}{k}\frac{d^k y}{dx^k}.
\end{aligned}$$

One way to prove this is by induction; we omit the details.

Here is another, more abstract approach; you can ignore it if you are not interested. Consider the operators $Dz = z'$ and $Lz = e^{-x}\frac{d}{dx}(e^x z)$. We have seen that $Lz = z + z' = (1+D)z$, so $L = 1 + D$, so

$$L^n = (1+D)^n = \sum_{k=0}^{n} \binom{n}{k} D^k,$$

so

$$L^n y = \sum_k \binom{n}{k} D^k y = \sum_k \binom{n}{k} \frac{d^k y}{dx^k}.$$

On the other hand, we have

$$\begin{aligned}
L^2 y = L(Ly) &= e^{-x}\frac{d}{dx}(e^x Ly) \\
&= e^{-x}\frac{d}{dx}\left(e^x\, e^{-x}\frac{d}{dx}(e^x y)\right) \\
&= e^{-x}\frac{d^2}{dx^2}(e^x y) \\
L^3 y &= e^{-x}\frac{d}{dx}\left(e^x\, e^{-x}\frac{d}{dx}\left(e^x\, e^{-x}\frac{d}{dx}(e^x y)\right)\right) \\
&= e^{-x}\frac{d^3}{dx^3}(e^x y)
\end{aligned}$$

and so on. It follows that

$$e^{-x}\frac{d^n}{dx^n}(e^x y) = L^n y = \sum_k \left(\begin{array}{c} n \\ k \end{array} \right) \frac{d^k y}{dx^k},$$

as claimed.

Exercise 6.5.16. Put $y = \exp(\exp(\exp(x)))$.

(a) Find dy/dx.

(b) Express x in terms of y.

(c) Working from (b), find dx/dy.

(d) Check that $\frac{dy}{dx}\frac{dx}{dy} = 1$.

Solution:

(a) The chain rule gives

$$\begin{aligned} dy/dx &= \exp'(\exp(\exp(x)))\ \exp'(\exp(x))\ \exp'(x) \\ &= \exp(\exp(\exp(x)))\ \exp(\exp(x))\ \exp(x). \end{aligned}$$

(b) We have $\log(y) = \exp(\exp(x))$, so $\log(\log(y)) = \exp(x)$, so $x = \log(\log(\log(y)))$.

(c) Using the chain rule again, we have

$$\begin{aligned} dx/dy &= \log'(\log(\log(y)))\ \log'(\log(y))\ \log'(y) \\ &= \log(\log(y))^{-1}\ \log(y)^{-1}\ y^{-1}. \end{aligned}$$

(d) Using the equations in (b), we can rewrite (a) as

$$dy/dx = y\ \log(y)\ \log(\log(y)).$$

When combined with (c), this clearly tells us that $(dy/dx).(dx/dy) = 1$.

Chapter 7: Integration

In Section 7.1, we will explore the meaning of integration, and the precise sense in which it is the reverse of differentiation.

7.1 The meaning of integration

Consider a function $f(x)$. There are two main ways to think about its integral:

(a) An *indefinite integral* of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. From this point of view, integration is just the reverse of differentiation. Indefinite integrals are also called *primitives* or *antiderivatives*, and we write $F(x) = \int f(x)\,dx$.

(b) The *definite integral* $\int_a^b f(x)\,dx$ is defined as follows. We divide up the interval $[a, b]$ into very short intervals, of length δx, say. For each short interval, we take the value of $f(x)$ on that interval, and multiply it by the length δx. We then add up all these contributions. The exact answer will depend slightly on the exact value of δx, so we let δx tend to zero and take the limiting answer; this is what we mean by $\int_a^b f(x)\,dx$.

Remark 7.1.1. As usual, we deal rather loosely with limits; some points will be treated more carefully in PMA113 (Introduction to Analysis).

Another imprecision in our definition of $\int_a^b f(x)\,dx$ comes from the phrase "the value of $f(x)$ on that interval". This is not well-defined, because $f(x)$ will have different values at different points of the interval. However, when the intervals are short, the value of $f(x)$ cannot change by much, so the imprecision is small, and becomes zero in the limit. Thus, there is no imprecision in the definition of $\int_a^b f(x)\,dx$ itself.

Remark 7.1.2. We should clarify some points about "arbitrary constants".

It is not strictly meaningful to talk about "*the* indefinite integral of $f(x)$", because any function has many different indefinite integrals, all differing from each other by a constant. For example, the functions $F_0(x) = (x+1)^2/2$ and $F_1(x) = x^2/2+x$ are both indefinite integrals of $f(x) = x+1$, with $F_0(x) - F_1(x) = 1/4$ for all x. If asked for *an* indefinite integral of $x+1$, you could correctly answer $x^2/2+x$ or $(x+1)^2/2$. If asked for *all* indefinite integrals, the answer is $\{x^2/2+x+c \mid c \in \mathbb{R}\}$ (or $\{(x+1)^2/2 + d \mid d \in \mathbb{R}\}$).

In some problems, it does not matter which indefinite integral you use. In particular, if you want to calculate $\int_a^b f(x)\,dx$ then you can find any indefinite integral $F(x)$ and the answer will be $F(b) - F(a)$. If you change $F(x)$ by a constant, then the constant will cancel out when you calculate $F(b) - F(a)$, so you get the same answer for $\int_a^b f(x)\,dx$.

In some applied problems, however, it is often important to find the "right" indefinite integral. For example, consider a particle moves with position $x(t)$ and velocity $v(t)$ at time t. Then $v(t)$ has many different indefinite integrals, including $x(t) - 5$ and $x(t) + 9$; but of course the indefinite integral that you really want to find and use is $x(t)$ itself.

In this course, we will not to need worry about arbitrary constants. We will therefore use the (slightly sloppy) notation $\int f(x)\,dx = F(x)$ to mean that "$F(x)$ is one of the indefinite integrals of $f(x)$".

There are many different applications in which one wants to add up an infinite number of very small contributions, as in (b).

Example 7.1.3. Consider a macroeconomic model of government tax revenue. Every day there are a very large number of payments, each very small compared to the total, so it is reasonable to idealise and imagine a continuous stream of payments. We let $p(t)$ be the rate of payments at time t, so the total amount paid between times t and $t+h$ is approximately $h\,p(t)$ when h is small. The total revenue R between time a and time b can be calculated by adding the contributions of a large number of small intervals between a and b. This means that $R = \int_a^b p(t)\,dt$. □

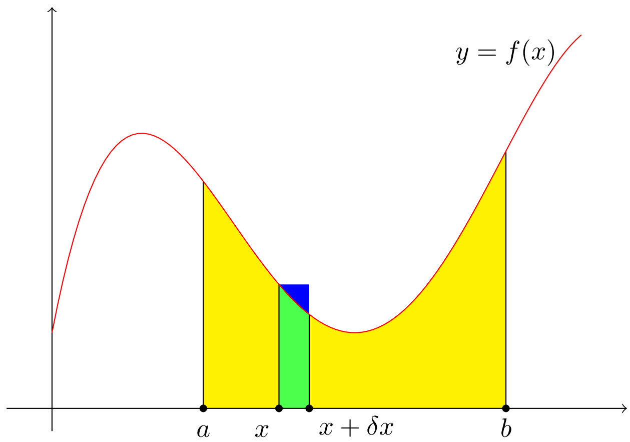
Example 7.1.4. Suppose we have a particle moving with velocity $v(t)$ at time t. This means that when h is small, the distance moved between time t and time $t+h$ is approximately $v(t)h$. If we want to know the distance moved between time a and time b, we divide the interval between a and b into very short intervals, use the above formula to find the distance moved in each short interval, and add up the results. The upshot is that the overall distance moved is $\int_a^b v(t)\,dt$.

Example 7.1.5. Suppose we have a wire carrying a current along the x-axis, and a charged particle moving somewhere nearby. The current generates a magnetic field, which exerts a force on the charged particle. There is a formula called the *Biot-Savart law* that gives the force exerted by a very short segment of wire. To find the total force exerted, we must add together the contributions of all the short segments, which again means that we must integrate.

Similarly, to find the gravitational force exerted by the earth on a satellite, we divide the earth into small chunks of rock, use Newton's formula for the force exerted by each chunk, and then add them all up. This, however, involves integration over all three coordinates (x, y and z), which involves a number of new issues. These are addressed in PMA213 (Advanced Calculus) and AMA224 (Vectors and Fluids). □

Example 7.1.6. Consider a randomly chosen person, with height x say (measured in metres). In statistics, one might want to know the probability that $1.8 \leq x \leq 1.9$. It is important to note that the probability that x is *exactly* 1.86 (for example) is essentially zero: everyone's height will be different from 1.86, even if only by a few microns. However, the probability that h lies in $[1.86, 1.86+h]$ will (for small h) be nonzero and approximately proportional to h, say $\phi(1.86)h$. This defines a function $\phi(x)$, called the *probability density*. To find the probability of having height between 1.8 and 1.9, we first divide the interval $[1.8, 1.9]$ into pieces so small that the above approximation is valid. We use the approximation to determine the probability that h lies in each of the short pieces, then add up the results to get the probability that $x \in [1.8, 1.9]$. This means that the relevant probability is just $\int_{1.8}^{1.9} \phi(x)\,dx$. □

Example 7.1.7. The traditional example, which we save until last, involves the area under a curve. Suppose we have a function $f(x)$ and numbers $a < b$, and we want to know the area of the region A enclosed by the lines $x = a$, $x = b$ and $y = 0$, and the curve $y = f(x)$. (We assume for simplicity that $f(x) \geq 0$ for $a \leq x \leq b$.)



This can be calculated by dividing A into thin strips of width δx and height $f(x)$; the area of such a strip is of course $f(x)\delta x$. The area of A is the sum of these contributions, which becomes $\int_a^b f(x)\,dx$ in the limit. □

Fact 7.1.8. The relationship between indefinite and definite integrals is called the *Fundamental Theorem of Calculus*:

(a) If $F(x)$ is an indefinite integral of $f(x)$, then for any a, b we have $\int_a^b f(x)\,dx = F(b) - F(a)$.

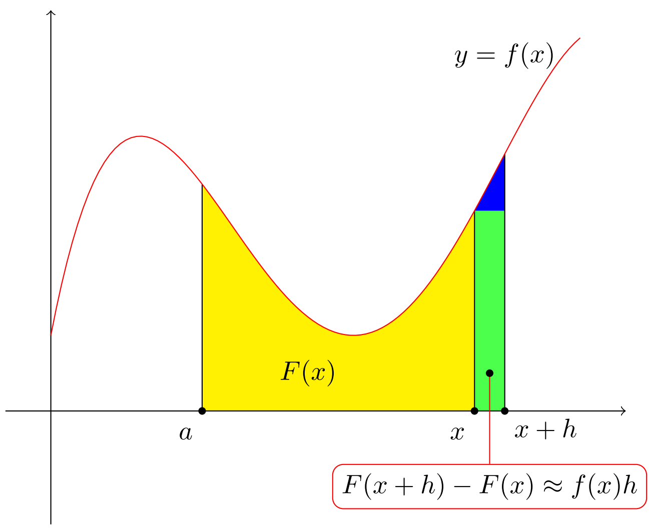
(b) Conversely, suppose we choose a fixed point a and define $F(x) = \int_a^x f(t)\,dt$. Then $F(x)$ is an indefinite integral of $f(x)$, so $F'(x) = f(x)$ for all x. □

We next outline why this is true. As usual, we will ignore many subtleties about limits.

(a) We divide the interval $[a, b]$ into many short intervals, of length h say. The change in F between a and b is the sum of the changes over each of these intervals. The change over the interval between x and $x + h$ is $F(x + h) - F(x)$, which is approximately $F'(x)h$. We are given that F is an indefinite integral of f, so $F'(x) = f(x)$, so the change in F between x and $x + h$ is approximately $f(x)h$. This means that $F(b) - F(a)$ is approximately the sum of these contributions $f(x)h$, which is approximately $\int_a^b f(x)\,dx$. All these approximations become exact in the limit.

(b) We have

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \\ &= \int_x^{x+h} f(t)\,dt. \end{aligned}$$



When h is very small, the interval from x to $x + h$ is already very short and we do not need to subdivide it further to evaluate the integral. The answer is just $\int_x^{x+h} f(t)\,dt \approx h\,f(x)$, so

$$(F(x + h) - F(x))/h \approx f(x).$$

This approximation becomes arbitrarily exact as h tends to zero, showing that

$$F'(x) = \lim_{h \to 0}(F(x + h) - F(x))/h = f(x).$$

7.2 Guessing and checking of integrals

Given a function $f(x)$, it is often difficult (or even provably impossible) to find a formula for the integral $\int f(x)\,dx$. However, you should bear the following firmly in mind:

Method 7.2.1. If you think that $\int f(x)\,dx = F(x)$, then this is easily checked: just find $F'(x)$ and see whether it is equal to $f(x)$.

Example 7.2.2. If asked to evaluate

$$\int \frac{1-3x^2}{2\sqrt{x}(1+x^2)^2}dx,$$

you might not know where to start. However, if someone tells you that the answer is $\sqrt{x}/(1+x^2)$, then it is easy to check that they are correct:

$$\begin{aligned}\frac{d}{dx}\frac{\sqrt{x}}{1+x^2} &= \frac{\frac{1}{2}x^{-1/2}.(1+x^2)-x^{1/2}.2x}{(1+x^2)^2} \\ &= \frac{\frac{1}{2}x^{-1/2}((1+x^2)-4x^2)}{(1+x^2)^2} \\ &= \frac{1-3x^2}{2\sqrt{x}(1+x^2)^2}. \quad \square\end{aligned}$$

Example 7.2.3. It is a common error to say that $\int \sin(x)^2\,dx = \sin(x)^3/3$. This cannot be right, because

$$\begin{aligned}\frac{d}{dx}\frac{\sin(x)^3}{3} &= \frac{3\sin(x)^2\cos(x)}{3} \\ &= \sin(x)^2\cos(x) \neq \sin(x)^2.\end{aligned}$$

(In fact, the correct answer is $(2x-\sin(2x))/4$, which can be obtained by writing $\sin(x)^2$ as $(1-\cos(2x))/2$.)

In examinations, students sometimes make errors like this out of desperation, because they cannot remember the correct method. You are much more likely to get credit by showing that you understand why the formula $\int \sin(x)^2\,dx = \sin(x)^3/3$ is incorrect, even if you cannot find the right answer. □

Example 7.2.4. It is another common error to write $\int e^{-x^2}\,dx = -e^{-x^2}/(2x)$. This cannot be right, because

$$\begin{aligned}\frac{d}{dx}\left(-\frac{e^{-x^2}}{2x}\right) &= \frac{e^{-x^2}2x.2x-(-e^{-x^2}).2}{(2x)^2} \\ &= e^{-x^2}+e^{-x^2}/(2x^2) \neq e^{-x^2}.\end{aligned}$$

(In fact, $\int e^{-x^2}\,dx = \sqrt{\pi}\mathrm{erf}(x)/2$, where $\mathrm{erf}(x)$ is a special function that cannot be expressed algebraically in terms of the functions discussed in this course.)

Exercise 7.2.5*. Check that

$$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \frac{2}{\sqrt{4ac-b^2}}\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right) & \text{if } 4ac > b^2 \\ \frac{-2}{\sqrt{b^2-4ac}}\operatorname{arctanh}\left(\frac{2ax+b}{\sqrt{b^2-4ac}}\right) & \text{if } 4ac < b^2 \end{cases}$$

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \log(2\sqrt{a^2x^2+abx+ac}+(2ax+b))/\sqrt{a}$$

Here are some hints. First, it is convenient to write $q(x) = ax^2+bx+c$, and to note that the expression $2ax+b$ occuring in the above formulae is just $q'(x)$. Assume for the moment that $4ac > b^2$ and put

$$\begin{aligned} d &= \sqrt{4ac-b^2} \\ F(x) &= 2\arctan(q'(x)/d)/d \\ r(x) &= 2\sqrt{a}\sqrt{q(x)}+q'(x) \\ G(x) &= \log(r(x))/\sqrt{a}. \end{aligned}$$

The claim is then that $\int q(x)^{-1}\,dx = F(x)$ and $\int q(x)^{-1/2}\,dx = G(x)$. It will be enough to check that $F'(x) = q(x)^{-1}$ and $G'(x) = q(x)^{-1/2}$. The following intermediate steps will help:

$$\begin{aligned} d^2 + q'(x)^2 &= 4a\,q(x) \\ r'(x) &= \sqrt{a}r(x)q(x)^{-1/2}. \end{aligned}$$

The case where $4ac < b^2$ is similar.

Solution:

(a) First, we have

$$\begin{aligned} d^2 + q'(x)^2 &= 4ac - b^2 + (2ax+b)^2 \\ &= 4ac - b^2 + 4a^2x^2 + 4abx + b^2 \\ &= 4a^2x^2 + 4abx + 4ac \\ &= 4a(ax^2+bx+c) = 4a\,q(x). \end{aligned}$$

We now use the chain rule and the facts that $\arctan'(u) = 1/(1+u^2)$ and $q''(x) = 2a$. This gives

$$\begin{aligned} F'(x) &= \frac{2}{d}\frac{q''(x)/d}{1+(q'(x)/d)^2} \\ &= \frac{2q''(x)}{d^2(1+(q'(x)/d)^2)} = \frac{4a}{d^2+q'(x)^2} \\ &= \frac{4a}{4a\,q(x)} = q(x)^{-1}. \end{aligned}$$

(b) The case where $4ac < b^2$ works exactly the same way except that we must take $d = \sqrt{b^2-4ac}$ and $F(x) = -2\,\mathrm{arctanh}(q'(x)/d)/d$.

(c) Next, we have

$$\begin{aligned} r'(x) &= 2\sqrt{a}.\frac{1}{2}q(x)^{-1/2}q'(x) + q''(x) \\ &= \sqrt{a}q'(x)q(x)^{-1/2} + 2a. \end{aligned}$$

Recall that $r(x)$ was defined to be $2\sqrt{a}\sqrt{q(x)} + q'(x)$, so $q'(x) = r(x) - 2\sqrt{a}\sqrt{q(x)}$. Substituting this in the above equation gives

$$\begin{aligned} r'(x) &= \sqrt{a}(r(x) - 2\sqrt{a}\sqrt{q(x)})q(x)^{-1/2} + 2a. \qquad = \sqrt{a}r(x)q(x)^{-1/2} - 2a + 2a \\ &= \sqrt{a}r(x)q(x)^{-1/2}. \end{aligned}$$

It follows that $r'(x)/r(x) = \sqrt{a}q(x)^{-1/2}$, so

$$\begin{aligned} G'(x) &= a^{-1/2}\frac{d}{dx}\log(r(x)) \\ &= a^{-1/2}r'(x)/r(x) = q(x)^{-1/2}. \end{aligned}$$

The following elaboration of Method 7.2.1 is often useful. (If it is not clear in the abstract, just look at the examples below.)

Method 7.2.6. Suppose that you know the general form of $\int f(x)\,dx$, but you do not know the precise values of all the coefficients. You can just write down the general with symbols as coefficients, differentiate it, and set the derivative equal to $f(x)$. This will give some equations for the coefficients, which you can solve to give a precise answer for $\int f(x)\,dx$.

This is essentially the method used by computer systems such as Maple. We will call it the *method of undetermined coefficients*. Later, we will discuss ways in which you can find the general form, in order to get this method started.

Example 7.2.7. Consider the function $f(x) = (x^2 + x + 1)e^{-x}$, and put $F(x) = \int f(x)\,dx$. A general rule to be discussed later tells us that $F(x) = (ax^2 + bx + c)e^{-x}$ for some constants a, b and c. We then have

$$\begin{aligned}(x^2 + x + 1)e^{-x} &= f(x) = F'(x)\\ &= (2ax + b)e^{-x} + (ax^2 + bx + c).(-e^{-x})\\ &= (-ax^2 + (2a - b)x + (b - c))e^{-x}.\end{aligned}$$

This holds for all x, so we can match up the coefficients:

$$\begin{aligned}1 &= -a && \text{(coefficient of } x^2e^{-x}\text{)}\\ 1 &= 2a - b && \text{(coefficient of } xe^{-x}\text{)}\\ 1 &= b - c && \text{(coefficient of } e^{-x}\text{).}\end{aligned}$$

Solving these equations gives $a = -1$, $b = -3$ and $c = -4$, so $F(x) = -(x^2 + 3x + 4)e^{-x}$. $\square$

Example 7.2.8. Consider the function $f(x) = e^{\lambda x}\sin(\omega x)$, and put $F(x) = \int f(x)\,dx$. Suppose you know that $F(x) = e^{\lambda x}(a\sin(\omega x) + b\cos(\omega x))$ for some constants a and b. We then have

$$\begin{aligned}e^{\lambda x}\sin(\omega x) &= f(x) = F'(x)\\ &= \lambda e^{\lambda x}(a\sin(\omega x) + b\cos(\omega x)) + e^{\lambda x}(a\omega\cos(\omega x) - b\omega\sin(\omega x))\\ &= e^{\lambda x}((a\lambda - b\omega)\sin(\omega x) + (b\lambda + a\omega)\cos(\omega x)).\end{aligned}$$

The only way this can hold for all x is if $a\lambda - b\omega = 1$ and $b\lambda + a\omega = 0$. These equations can be solved to give

$$\begin{aligned}a &= \lambda/(\lambda^2 + \omega^2)\\ b &= -\omega/(\lambda^2 + \omega^2),\end{aligned}$$

so

$$\int f(x)\,dx = F(x) = e^{\lambda x}(\lambda\sin(\omega x) - \omega\cos(\omega x))/(\lambda^2 + \omega^2).$$

Example 7.2.9. Consider the function $f(x) = e^{3x}\sin(4x)$, and put $F(x) = \int f(x)\,dx$. A general rule to be discussed later tells us that $F(x) = e^{3x}(a\sin(4x) + b\cos(4x))$ for some constants a and b. We then have

$$\begin{aligned}e^{3x}\sin(4x) &= f(x) = F'(x)\\ &= 3e^{3x}(a\sin(4x) + b\cos(4x)) + e^{3x}(4a\cos(4x) - 4b\sin(4x))\\ &= e^{3x}((3a - 4b)\sin(4x) + (3b + 4a)\cos(4x)).\end{aligned}$$

The only way this can hold for all x is if $3a - 4b = 1$ and $3b + 4a = 0$. These equations can be solved to give

$$\begin{aligned}a &= 3/25\\ b &= -4/25,\end{aligned}$$

so

$$\int f(x)\,dx = F(x) = e^{3x}(3\sin(4x) - 4\cos(4x))/25. \quad \square$$

Example 7.2.10. Consider the function $f(x) = \log(x)^3$, and put $F(x) = \int f(x)\,dx$. A general rule tells us that $F(x) = (a\log(x)^3 + b\log(x)^2 + c\log(x) + d)x$ for some constants a, b, c and d. We then have

$$\begin{aligned}\log(x)^3 &= f(x) = F'(x) \\ &= (3a\log(x)^2x^{-1} + 2b\log(x)x^{-1} + cx^{-1}).x + (a\log(x)^3 + b\log(x)^2 + c\log(x) + d).1 \\ &= a\log(x)^3 + (b+3a)\log(x)^2 + (c+2b)\log(x) + (d+c).\end{aligned}$$

Equating coefficients, we find that

$$\begin{aligned} a &= 1 \\ b + 3a &= 0 \\ c + 2b &= 0 \\ d + c &= 0, \end{aligned}$$

which gives $a = 1$, $b = -3$, $c = 6$ and $d = -6$, so

$$\int \log(x)^3\,dx = F(x) = (\log(x)^3 - 3\log(x)^2 + 6\log(x) - 6)\,x. \quad \square$$

7.3 Integrals of standard functions

Here are some standard integrals:

$$\int x^n\,dx = \begin{cases} x^{n+1}/(n+1) & \text{if } n \neq -1 \\ \log(x) & \text{if } n = -1. \end{cases}$$

$$\int \exp(x)\,dx = \exp(x)$$

$$\int a^x\,dx = a^x/\log(a)$$

$$\int \log(x)\,dx = x\log(x) - x$$

$$\int \sin(x)\,dx = -\cos(x)$$

$$\int \cos(x)\,dx = \sin(x)$$

$$\int \tan(x)\,dx = -\log(\cos(x))$$

$$\int \frac{dx}{1+x^2} = \arctan(x)$$

$$\int \frac{dx}{1-x^2} = \operatorname{arctanh}(x)$$

$$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \frac{2}{\sqrt{4ac-b^2}} \arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right) & \text{if } 4ac > b^2 \\ \frac{-2}{\sqrt{b^2-4ac}} \operatorname{arctanh}\left(\frac{2ax+b}{\sqrt{b^2-4ac}}\right) & \text{if } 4ac < b^2 \end{cases}$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \operatorname{arcsinh}(x)$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x)$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \operatorname{arccosh}(x)$$

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \log(2\sqrt{a^2x^2+abx+ac} + 2ax + b)/\sqrt{a}$$

$$\int f(ax+b)\,dx = F(ax+b)/a \qquad \left(\text{where } \int f(x)\,dx = F(x)\right)$$

Most of these can be checked using the derivatives recorded in Section 6.3. The remaining cases are either easy or covered by Exercise 7.2.5*.

You should note that some large classes of integrals can be reduced to the cases above by algebraic manipulation; examples of this will be given later. In particular, any rational function can be integrated by converting it to partial fraction form and then using the above table; this is discussed in Section 7.3.1.

Remark 7.3.1. The formula $\int x^{-1}\,dx = \log(x)$ must either be modified or interpreted very carefully if x might be negative. If $x < 0$ then $x = -|x|$ and so $\log(x) = \log(|x|) + \log(-1) = \log(|x|) + i\pi$. Here $i\pi$ is constant, and we do not worry about arbitrary constants, so we may as well use the indefinite integral $\int x^{-1}\,dx = \log(|x|)$ instead. This works for $x > 0$ as well, because then we have $|x| = x$ and so $\log(|x|) = \log(x)$.

Warning 7.3.2. Here is a strange consequence of the above discussion. Suppose we try to evaluate something like $\int_{-e}^{e} x^{-1}\,dx$, where the point $x = 0$ lies between the limits of the integral.

One approach gives

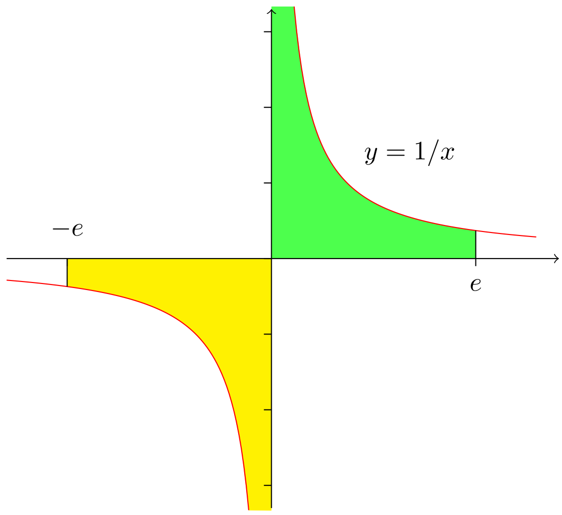
$$\int_{-e}^{e} x^{-1}\,dx = \big[\log(|x|)\big]_{-e}^{e} = \log(e) - \log(e) = 1 - 1 = 0,$$

but the other gives

$$\int_{-e}^{e} x^{-1}\,dx = \big[\log(x)\big]_{-e}^{e} = \log(e) - \log(-e) = 1 - (i\pi + 1) = -i\pi.$$

Which is correct?

The graph of $y = 1/x$ is as follows:



There is an infinite area above the x-axis, and an infinite area below it, so $\int_{-e}^{e} x^{-1}\,dx$ has the form "$\infty - \infty$". Any question whose answer is of this form is not really meaningful. If you try to work naively with expressions involving ∞, you quickly run into nonsense like this: $5 + \infty = \infty$ and $6 + \infty = \infty$ so $5 = \infty - \infty = 6$.

Before we try to evaluate $\int_{-e}^{e} x^{-1}\,dx$, we must first do some serious work (with the theory of limits and so on) to specify exactly what the question means. One possible meaning (the "Cauchy principal value") gives the answer of zero; another possible meaning (using "indented contours") gives the value $-i\pi$.

7.3.1 Rational functions:

If $f(x)$ is a rational function, then we can rewrite it terms of partial fractions. This leaves a sum of multiples of terms like x^k or $(x-a)^{-k}$, which can be integrated using the rules

$$\int x^k\,dx = x^{k+1}/(k+1)$$
$$\int (x-a)^{-1}\,dx = \log(x-a)$$
$$\int (x-a)^{-k}\,dx = (x-a)^{1-k}/(1-k) \qquad \text{(for } k > 1\text{)}$$

Recall the following decompositions from Section 2.4:

$$\begin{aligned}
\frac{x^2+1}{x^2-1} &= 1 + \frac{1}{x-1} - \frac{1}{x+1} \\
\frac{360}{(x^2-25)(x^2-16)} &= \frac{4}{x-5} - \frac{4}{x+5} - \frac{5}{x-4} + \frac{5}{x+4} \\
\left(\frac{x+1}{x-1}\right)^3 &= 1 + \frac{6}{x-1} + \frac{12}{(x-1)^2} + \frac{8}{(x-1)^3} \\
\frac{1-x^3+x^4}{x^2-2x+1} &= x^2+x+1+\frac{1}{x-1}+\frac{1}{(x-1)^2} \\
\frac{x^4}{(x-1)(x-2)} &= x^2+3x+7-\frac{1}{x-1}+\frac{16}{x-2}.
\end{aligned}$$

We deduce the following integrals:

$$\begin{aligned}
\int \frac{x^2+1}{x^2-1}dx &= \int 1dx + \int \frac{dx}{x-1} - \int \frac{dx}{x+1} \\
&= x + \log(x-1) - \log(x+1) \\
&= x + \log\left(\frac{x-1}{x+1}\right) \\
\int \frac{360}{(x^2-25)(x^2-16)}dx &= 4\log(x-5) - 4\log(x+5) - 5\log(x-4) + 5\log(x+4) \\
\int \left(\frac{x+1}{x-1}\right)^3 dx &= x + 6\log(x-1) - 12(x-1)^{-1} - 4(x-1)^{-2} \\
\int \frac{1-x^3+x^4}{x^2-2x+1}dx &= x^3/3 + x^2/2 + x + \log(x-1) - (x-1)^{-1} \\
\int \frac{x^4}{(x-1)(x-2)}dx &= x^3/3 + 3x^2/2 + 7x - \log(x-1) + 16\log(x-2).
\end{aligned}$$

This method can only be guaranteed to work as described if we are prepared to use complex numbers. Otherwise, we may be left with terms like $(ux+v)(x^2+bx+c)^{-k}$ (as well terms of the form x^k or $(x-a)^{-k}$ as mentioned previously). The integral may then contain terms like $\log(x^2+ax+b)$ and $\arctan(rx+s)$ and $\operatorname{arctanh}(rx+s)$ as well as rational functions.

In the case $k=1$, we can write this down directly:

$$\int \frac{ux+v}{x^2+bx+c}dx = \begin{cases} \frac{u}{2}\log(x^2+bx+c) + \frac{2v-bu}{\sqrt{4c-b^2}}\arctan\left(\frac{2x+b}{\sqrt{4c-b^2}}\right) & \text{if } 4c > b^2 \\ \frac{u}{2}\log(x^2+bx+c) + \frac{bu-2v}{\sqrt{b^2-4c}}\operatorname{arctanh}\left(\frac{2x+b}{\sqrt{b^2-4c}}\right) & \text{if } 4c < b^2 \end{cases}$$

(To check this, use the same methods as in Exercise 7.2.5*.)

If $k>1$ and $4c>b^2$, then the answer has the form

$$\int \frac{ux+v}{(x^2+bx+c)^k}dx = \frac{f(x)}{(x^2+bx+c)^{k-1}} + m\arctan\left(\frac{2x+b}{\sqrt{4c-b^2}}\right),$$

where m is a constant and $f(x)$ is a polynomial of degree $2k-3$. The constant m and the coefficients of $f(x)$ can be found by differentiating and comparing with the required answer, along the lines described in Section 7.2. If $4c<b^2$ we replace $\arctan((2x+b)/\sqrt{4c-b^2})$ by $\operatorname{arctanh}((2x+b)/\sqrt{b^2-4c})$. We will not go into details.

7.3.2 Trigonometric polynomials

Recall from Section 5.4 that a large class of functions can be rewritten in terms of the functions $\sin(nx)$ and $\cos(nx)$. The following examples were given:

$$\begin{aligned}
\sin(x)^2 &= \tfrac{1}{2} - \tfrac{1}{2}\cos(2x) \\
\cos(x)^2 &= \tfrac{1}{2} + \tfrac{1}{2}\cos(2x) \\
\sin(x)^3 &= \tfrac{3}{4}\sin(x) - \tfrac{1}{4}\sin(3x) \\
\sin(x)\sin(2x)\sin(4x) &= -\sin(x)/4 + \sin(3x)/4 + \sin(5x)/4 - \sin(7x)/4 \\
\cos(x)\cos(2x)\cos(4x) &= \cos(x)/8 + \cos(3x)/8 + \cos(5x)/8 + \cos(7x)/8 \\
\sin(x)^4 + \cos(x)^4 &= \tfrac{3}{4} + \tfrac{1}{4}\cos(4x) \\
\sin(nx)\sin(mx) &= \tfrac{1}{2}\cos((n-m)x) - \tfrac{1}{2}\cos((n+m)x).
\end{aligned}$$

The first two identities give the following integrals:

$$\begin{aligned}
\int \sin(x)^2 dx &= \int \tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\,dx \\
&= \tfrac{1}{2}x - \tfrac{1}{4}\sin(2x) = (2x - \sin(2x))/4 \\
\int \cos(x)^2 dx &= \int \tfrac{1}{2} + \tfrac{1}{2}\cos(2x)\,dx \\
&= \tfrac{1}{2}x + \tfrac{1}{4}\sin(2x) = (2x + \sin(2x))/4.
\end{aligned}$$

You should memorise these formulae.

The remaining five identities also give integral formulae, which you should understand, but you need not remember them.

$$\begin{aligned}
\int \sin(x)^3 dx &= -\tfrac{3}{4}\cos(x) + \tfrac{1}{12}\cos(3x) \\
\int \sin(x)\sin(2x)\sin(4x) dx &= \cos(x)/4 - \cos(3x)/12 - \cos(5x)/20 + \cos(7x)/28 \\
\int \cos(x)\cos(2x)\cos(4x) dx &= \sin(x)/8 + \sin(3x)/24 + \sin(5x)/40 + \sin(7x)/56 \\
\int \sin(x)^4 + \cos(x)^4 dx &= \tfrac{3x}{4} + \tfrac{1}{16}\sin(4x) \\
\int \sin(nx)\sin(mx) dx &= \tfrac{1}{2(n-m)}\sin((n-m)x) - \tfrac{1}{2(n+m)}\sin((n+m)x).
\end{aligned}$$

Exercise 7.3.3. Express $\sin(x)^2\cos(x)^2$ as a trigonometric polynomial (as in Section 5.4.1), and thus find

$$\int \sin(x)^2\cos(x)^2\,dx.$$

Solution: We know that $\sin(2x) = 2\sin(x)\cos(x)$, so $\sin(x)\cos(x) = \sin(2x)/2$, so $\sin(x)^2\cos(x)^2 = \sin(2x)^2/4$. We also know that $\sin(y)^2 = \frac{1}{2} - \frac{1}{2}\cos(2y)$ for all y; taking $y = 2x$ gives $\sin(2x)^2 = \frac{1}{2} - \frac{1}{2}\cos(4x)$. Putting all this together, we get

$$\sin(x)^2\cos(x)^2 = \sin(2x)^2/4 = \tfrac{1}{8} - \tfrac{1}{8}\cos(4x).$$

We now integrate to get

$$\int \sin(x)^2\cos(x)^2\,dx = \tfrac{x}{8} - \tfrac{1}{32}\sin(4x).$$

For an alternative approach to the first part of the question, put $u = e^{ix}$. We then have

$$\begin{aligned}
\sin(x)^2\cos(x)^2 &= \left(\frac{u-u^{-1}}{2i}\right)^2\left(\frac{u+u^{-1}}{2}\right)^2 \\
&= \frac{(u^2-2+u^{-2})(u^2+2+u^{-2})}{4i^2.4} \\
&= -\tfrac{1}{16}(u^4+2u^2+1-2u^2-4-2u^{-2}+1+2u^{-2}+u^{-4}) \\
&= -\tfrac{1}{8}\frac{u^4+u^{-4}}{2}+\tfrac{1}{8} \\
&= \tfrac{1}{8}-\tfrac{1}{8}\cos(4x),
\end{aligned}$$

just as before.

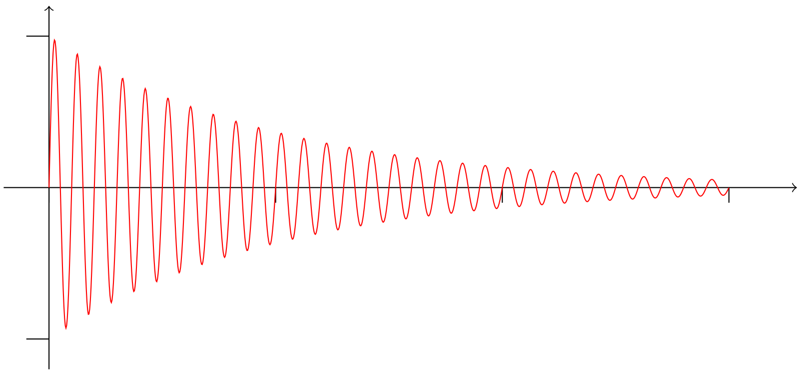
7.3.3 Exponential oscillations

Definition 7.3.4. An *exponential oscillation* means a function of the form

$$f(x) = (a\cos(\omega x) + b\sin(\omega x))e^{\lambda x},$$

where a, b, λ and ω are constants. We call λ the *growth rate* and ω the *frequency*.

If $\lambda < 0$ (the most common case) then this represents an oscillation that dies out exponentially, like the displacement of a guitar string after it has been plucked.



Such functions are extremely common in applied mathematics and electronics, for example.

Note that the following functions are special cases of exponential oscillations:

$$\begin{array}{lll}
f(x) & = a\cos(\omega x)e^{\lambda x} & (b=0) \\
f(x) & = a\cos(\omega x) & (b=\lambda=0) \\
f(x) & = a\cos(\omega x)+b\sin(\omega x) & (\lambda=0) \\
f(x) & = ae^{\lambda x} & (b=\omega=0).
\end{array}$$

The key point for us is as follows

Fact 7.3.5. The integral of an exponential oscillation is another exponential oscillation with the same growth rate and frequency. More explicitly, we have

$$\int (a\cos(\omega x)+b\sin(\omega x))e^{\lambda x}dx = (A\cos(\omega x)+B\sin(\omega x))e^{\lambda x},$$

where

$$\begin{aligned}
A &= (a\lambda - b\omega)/(\lambda^2+\omega^2) \\
B &= (a\omega + b\lambda)/(\lambda^2+\omega^2).
\end{aligned}$$

(These formulae can be neatly encoded in terms of complex numbers: $A+iB = (a+ib)/(\lambda - i\omega)$.) $\square$

To see that all this is correct, we first observe that

$$\begin{aligned}
A\lambda + B\omega &= ((a\lambda - b\omega)\lambda + (a\omega + b\lambda)\omega)/(\lambda^2 + \omega^2) \\
&= (a\lambda^2 - b\omega\lambda + a\omega^2 + b\omega\lambda)/(\lambda^2 + \omega^2) \\
&= a \\
B\lambda - A\omega &= ((a\omega + b\lambda)\lambda - (a\lambda - b\omega)\omega)/(\lambda^2 + \omega^2) \\
&= (a\omega\lambda + b\lambda^2 - a\omega\lambda + b\omega^2)/(\lambda^2 + \omega^2) \\
&= b.
\end{aligned}$$

We now just differentiate:

$$\begin{aligned}
\frac{d}{dx}(A\cos(\omega x) + B\sin(\omega x))e^{\lambda x} &= (-A\omega\sin(\omega x) + B\omega\cos(\omega x))e^{\lambda x} + \\
&\quad (A\cos(\omega x) + B\sin(\omega x))\lambda e^{\lambda x} \\
&= ((A\lambda + B\omega)\cos(\omega x) + (B\lambda - A\omega)\sin(\omega x))e^{\lambda x} \\
&= (a\cos(\omega x) + b\sin(\omega x))e^{\lambda x} f(x).
\end{aligned}$$

In particular examples, it may be easier to use undetermined coefficients instead of remembering the formulae for A and B.

Example 7.3.6. Consider $f(x) = \sin(x)e^{-x}$. In this case we have $a = 1$, $b = 0$, $\omega = 1$ and $\lambda = -1$. This gives

$$\begin{aligned}
A &= (1.(-1) - 0.1)/((-1)^2 + 1^2) = -1/2 \\
B &= (1.1 + 0.(-1))/((-1)^2 + 1^2) = 1/2,
\end{aligned}$$

so

$$\int f(x)\,dx = (-\cos(x) + \sin(x))e^{-x}/2. \quad \square$$

Example 7.3.7. Consider the function

$$f(x) = (20\pi\cos(10\pi x) - 2\sin(10\pi x))e^{-x}.$$

We know that the integral has the form

$$F(x) = (A\cos(10\pi x) + B\sin(10\pi x))e^{-x}$$

for some constants A and B. We have

$$\begin{aligned}
F'(x) &= (-10\pi A\sin(10\pi x) + 10\pi B\cos(10\pi x))e^{-x} + \\
&\quad (A\cos(10\pi x) + B\sin(10\pi x)).(-e^{-x}) \\
&= ((-A + 10\pi B)\cos(10\pi x) + (-B - 10\pi A)\sin(10\pi x))e^{-x}.
\end{aligned}$$

This must be the same as $f(x)$, so we must have

$$\begin{aligned}
-A + 10\pi B &= 20\pi \\
-B - 10\pi A &= -2.
\end{aligned}$$

After adding 10π times the second equation to the first, we see that $A = 0$; substituting this back in, we see that $B = 2$. We thus have

$$F(x) = 2\sin(10\pi x)e^{-x}. \quad \square$$

We next discuss a rather larger class of functions.

Definition 7.3.8. A *polynomial exponential oscillation* will mean a function of the form

$$f(x) = (a(x)\cos(\omega x) + b(x)\sin(\omega x))e^{\lambda x}.$$

The *degree* of $f(x)$ means the maximum of the degrees of the polynomials $a(x)$ and $b(x)$. We again refer to λ as the *growth rate*, and to ω as the *frequency*.

The key point for us is as follows.

Fact 7.3.9. The integral of a polynomial exponential oscillation is another polynomial exponential oscillation with the same degree, growth rate and frequency. More explicitly, if

$$f(x) = ((a_0 + a_1x + \cdots + a_nx^n)\cos(\omega x) + (b_0 + b_1x + \cdots + b_nx^n)\sin(\omega x))e^{\lambda x},$$

then

$$\int f(x)\,dx = ((c_0 + c_1x + \cdots + c_nx^n)\cos(\omega x) + (d_0 + d_1x + \cdots + d_nx^n)\sin(\omega x))e^{\lambda x}$$

for some system of coefficients c_j and d_j.

The details can be found by the method of undetermined coefficients. It would also be possible, but a little complicated, to give an explicit formula.

Example 7.3.10. Consider $f(x) = xe^{-x}\sin(x)$ and put $F(x) = \int f(x)\,dx$. Fact 7.3.9 says that $F(x)$ is a polynomial exponential oscillation of degree one, growth rate -1 and frequency 1. This means that

$$F(x) = (Ax + B)e^{-x}\cos(x) + (Cx + D)e^{-x}\sin(x)$$

for some constants A, B, C and D. This gives

$$\begin{aligned} F'(x) &= Ae^{-x}\cos(x) - (Ax+B)e^{-x}\cos(x) - (Ax+B)e^{-x}\sin(x) + \\ &\quad Ce^{-x}\sin(x) - (Cx+D)e^{-x}\sin(x) + (Cx+D)e^{-x}\cos(x) \\ &= (-A+C)xe^{-x}\cos(x) + (A-B+D)e^{-x}\cos(x) + \\ &\quad (-A-C)xe^{-x}\sin(x) + (-B+C-D)e^{-x}\sin(x). \end{aligned}$$

By comparing this with $f(x)$, we see that

$$\begin{aligned} -A+C &= 0 \\ A-B+D &= 0 \\ -A-C &= 1 \\ -B+C-D &= 0. \end{aligned}$$

The first and third equations together give $A = C = -1/2$. By adding the second and fourth equations, we get $2B = A + C = -1$ and so $B = -1/2$. The fourth equation then gives $D = C - B = 0$. This means that

$$F(x) = -((x+1)e^{-x}\cos(x) + xe^{-x}\sin(x))/2. \quad \square$$

Example 7.3.11. Consider $f(x) = x^3e^x$ and put $F(x) = \int f(x)\,dx$. Then $f(x)$ is a polynomial exponential oscillation of degree three, frequency zero and growth rate one, so $F(x)$ must be of the same type, say

$$F(x) = (Ax^3 + Bx^2 + Cx + D)e^x.$$

This gives

$$\begin{aligned} F'(x) &= (3Ax^2 + 2Bx + C)e^x + (Ax^3 + Bx^2 + Cx + D)e^x \\ &= (Ax^3 + (3A+B)x^2 + (2B+C)x + (C+D))e^x. \end{aligned}$$

Comparing this with $f(x)$ gives

$$\begin{aligned} A &= 1 \\ B + 3A &= 0 \\ C + 2B &= 0 \\ D + C &= 0, \end{aligned}$$

which in turn gives $A = 1$, $B = -3$, $C = 6$ and $D = -6$. It follows that

$$\int x^3 e^x \, dx = (x^3 - 3x^2 + 6x - 6)e^x. \quad \square$$

Example 7.3.12. Consider $f(x) = x^2 \sin(x)$ and put $F(x) = \int f(x)\, dx$. Fact 7.3.9 says that

$$F(x) = (Ax^2 + Bx + C)\cos(x) + (Px^2 + Qx + R)\sin(x)$$

for some constants A, B, C, P, Q and R. We can save ourselves some algebra by noting that $f(x)$ is an odd function (in the sense that $f(-x) = -f(x)$), so $F(x)$ is an even function, which easily implies that $B = P = R = 0$. We thus have

$$F(x) = (Ax^2 + C)\cos(x) + Qx\sin(x),$$

giving

$$\begin{aligned} F'(x) &= 2Ax\cos(x) - (Ax^2 + C)\sin(x) + Q\sin(x) + Qx\cos(x) \\ &= (2A + Q)x\cos(x) - Ax^2\sin(x) + (Q - C)\sin(x). \end{aligned}$$

When we compare this with $f(x)$, we see that

$$\begin{aligned} 2A + Q &= 0 \\ -A &= 1 \\ Q - C &= 0. \end{aligned}$$

This gives $A = -1$ and $C = Q = 2$, so

$$F(x) = (2 - x^2)\cos(x) + 2x\sin(x). \quad \square$$

7.4 Integration by parts

Method 7.4.1. Suppose we want to integrate a function $f(x)$ that can be written as a product, say $f(x) = g(x)h(x)$. Suppose also that we know how to integrate one of the factors, say $\int h(x)\, dx = H(x)$. We then have

$$\int f(x)\, dx = g(x)H(x) - \int g'(x)H(x)\, dx.$$

If we are lucky, we may also be able to integrate $g'(x)H(x)$, and this will give us the integral of $f(x)$.

To see that this method is valid, we must check that

$$\frac{d}{dx}\left(g(x)H(x) - \int g'(x)H(x)\, dx\right) = f(x).$$

This is easy, using the product rule for differentiation:

$$\begin{aligned}
\frac{d}{dx}(g(x)H(x)) &= g'(x)H(x) + g(x)H'(x) \\
&= g'(x)H(x) + g(x)h(x) \\
\frac{d}{dx}\int g'(x)H(x)dx &= g'(x)H(x) \\
\frac{d}{dx}\left(g(x)H(x) - \int g'(x)H(x)\,dx\right) &= (g'(x)H(x) + g(x)h(x)) - g'(x)H(x) \\
&= g(x)h(x) = f(x).
\end{aligned}$$

The rule is often written in a different notation. Put $u = g(x)$ (so $du/dx = g'(x)$) and $v = H(x)$ (so $dv/dx = H'(x) = h(x)$) and $y = f(x)$. In these terms, we factor y as $y = u\frac{dv}{dx}$, then integrate $\frac{dv}{dx}$ to find v, then apply the rule

$$\int y\,dx = \int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx}v\,dx.$$

Quite apart from its use in finding integrals, this rule is also important as a theoretical principle in physics.

In Section 7.3 we have already covered many examples that are more usually treated using integration by parts. Nonetheless, we shall discuss some of them again in this section.

Example 7.4.2. To integrate xe^{2x}, we put $u = x$ (so $\frac{du}{dx} = 1$) and $\frac{dv}{dx} = e^{2x}$ (so $v = e^{2x}/2$). We then have

$$\begin{aligned}
\int xe^{2x}\,dx &= xe^{2x}/2 - \int 1.e^{2x}/2\,dx \\
&= xe^{2x}/2 - e^{2x}/4 \\
&= (2x-1)e^{2x}/4.
\end{aligned}$$

Note that this is a polynomial exponential oscillation (of frequency zero) so it could also have been integrated by the method in Section 7.3. □

Example 7.4.3. We now consider the function x^2e^{2x}. We put $u = x^2$ (so $\frac{du}{dx} = 2x$) and $\frac{dv}{dx} = e^{2x}$ (so $v = e^{2x}/2$). This gives

$$\int x^2e^{2x}\,dx = x^2e^{2x}/2 - \int 2x.e^{2x}/2\,dx = x^2e^{2x}/2 - \int xe^{2x}\,dx.$$

We now substitute in the result of the previous example to get

$$\int x^2e^{2x}\,dx = x^2e^{2x}/2 - (2x-1)e^{2x}/4 = (2x^2 - 2x + 1)e^{2x}/4.$$

Clearly, this method can be repeated to find $\int x^3e^{2x}dx$, then $\int x^4e^{2x}\,dx$, and so on. □

Example 7.4.4. To integrate $x\sin(x)$, we put $u = x$ (so $\frac{du}{dx} = 1$) and $\frac{dv}{dx} = \sin(x)$ (so $v = -\cos(x)$). We find that

$$\begin{aligned}
\int x\sin(x)\,dx &= -x\cos(x) - \int 1.(-\cos(x))dx \\
&= -x\cos(x) + \int \cos(x)\,dx \\
&= -x\cos(x) + \sin(x).
\end{aligned}$$

Using the same method, we find that

$$\int x\cos(x)\,dx = x\sin(x) + \cos(x). \quad \square$$

Example 7.4.5. We next evaluate the integrals

$$I = \int e^x \cos(x)\, dx$$
$$J = \int e^x \sin(x)\, dx.$$

Integrating I by parts gives

$$I = e^x \cos(x) - \int e^x.(-\sin(x))\, dx = e^x \cos(x) + J,$$

so $I - J = e^x \cos(x)$. Similarly, we can integrate J by parts to get

$$J = e^x \sin(x) - \int e^x.\cos(x)\, dx = e^x \sin(x) - I,$$

so $I + J = e^x \sin(x)$. By adding and subtracting these two equations, we get

$$I = e^x(\sin(x) + \cos(x))/2$$
$$J = e^x(\sin(x) - \cos(x))/2. \quad \square$$

7.5 Integration by substitution

Suppose we want to calculate $\int f(x)dx$ for some function $f(x)$. It is often convenient to write x in terms of some other variable, say $x = g(u)$. This means that $\frac{dx}{du} = g'(u)$, or in other words $dx = g'(u)\, du$, so

$$\int f(x)\, dx = \int f(g(u))g'(u)\, du.$$

This rule is called *integration by substitution*; it is essentially the reverse of the chain rule for differentiation.

Remark 7.5.1. Purists will complain that the formula $dx = g'(u)du$ is suspect, because the symbol dx by itself is not meaningful; we have only defined things like dx/dt, which cannot properly be interpreted as a number called dx divided by a number called dt. Nonetheless, such formulae turn out to be harmless, if a little sloppy; it takes some rather abstract mathematics (the theory of "differential forms") to justify them properly, but at the end of the day, everything works out as expected.

Example 7.5.2. Consider the integral $\int (1 - x^2)^{-1/2}\, dx$. We put $x = \sin(u)$, so $dx/du = \cos(u)$, so $dx = \cos(u)\, du$. We then have $\sqrt{1 - x^2} = \cos(u)$ (because $\sin(u)^2 + \cos(u)^2 = 1$ for all u), so

$$\int (1 - x^2)^{-1/2}\, dx = \int \cos(u)^{-1}.\cos(u)\, du = \int 1\, du = u = \arcsin(x). \quad \square$$

Often it is convenient to express the relationship between u and x the other way around, say $u = h(x)$, where h is the inverse function of g. We then have $du = h'(x)\, dx$ and so $dx = h'(x)^{-1}du$, so

$$\int f(x)\, dx = \int f(x)h'(x)^{-1}\, du.$$

The expression on the right is not really usable as it stands: we must write the function $f(x)h'(x)$ in terms of u instead of x, either by inspection or by substituting $x = h^{-1}(u)$.

Example 7.5.3. Consider the integral $\int \sin(x)^4 \cos(x)\, dx$. Put $u = \sin(x)$, so $du/dx = \cos(x)$, so $\cos(x)\, dx = du$. The integral becomes $\int u^4 du$, which is just $u^5/5$, or in other words $\sin(x)^5/5$. The conclusion is that

$$\int \sin(x)^4 \cos(x)\, dx = \sin(x)^5/5.$$

Note that we could also have approached this using the identity

$$\sin(x)^4\cos(x) = (\cos(5x) - 3\cos(3x) + 2\cos(x))/16,$$

which gives

$$\int \sin(x)^4\cos(x)\,dx = \sin(5x)/80 - \sin(3x)/16 + \sin(x)/8.$$

The substitution method is less systematic, but in this particular case it gives a simpler answer. □

Exercise 7.5.4. Evaluate $\int xe^{-x^2}\,dx$ by substituting $u = x^2$.

Solution: If $u = x^2$ then $du/dx = 2x$, so $x\,dx = \frac{1}{2}du$. We thus have

$$\begin{aligned}\int xe^{-x^2}\,dx &= \int e^{-u}.\tfrac{1}{2}du \\ &= -\tfrac{1}{2}e^{-u} \\ &= -\tfrac{1}{2}e^{-x^2}.\end{aligned}$$

Exercise 7.5.5. Evaluate $\int \tan(x)\,dx$ by the substitution $u = \cos(x)$ (remembering that $\tan(x) = \sin(x)/\cos(x)$).

Solution: If $u = \cos(x)$ then $du/dx = -\sin(x)$, so $du = -\sin(x)\,dx$. Thus

$$\begin{aligned}\int \tan(x)\,dx &= \int \frac{\sin(x)\,dx}{\cos(x)} \\ &= \int \frac{-du}{u} = -\log(u) \\ &= -\log(\cos(x)).\end{aligned}$$

Exercise 7.5.6. For any function $f(x)$, evaluate

$$\int \frac{f'(x)}{f(x)}\,dx.$$

Solution: Put $u = f(x)$, so $du/dx = f'(x)$, so $f'(x)\,dx = du$. This gives

$$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{du}{u} = \log(u) = \log(f(x)).$$

One particularly common and important kind of substitution is as follows:

Fact 7.5.7. If $\int f(x)\,dx = F(x)$, and a and b are constants, then $\int f(ax+b)\,dx = F(ax+b)/a$. □

One way to see this is to differentiate: we have $F'(x) = f(x)$, so the chain rule gives $\frac{d}{dx}(F(ax+b)) = aF'(ax+b) = af(ax+b)$, so $\frac{d}{dx}(F(ax+b)/a) = f(ax+b)$ as required. An essentially equivalent approach is to substitute $u = ax+b$, so $dx = \frac{1}{a}du$; we then have

$$\int f(ax+b)\,dx = \int f(u)\frac{1}{a}\,du = \frac{1}{a}\int f(u)\,du = \frac{1}{a}f(u) = f(ax+b)/a.$$

Example 7.5.8.

$$\int \cos(2\pi x + \pi/4)\,dx = \sin(2\pi x + \pi/4)/(2\pi)$$
$$\int e^{3x+1}\,dx = e^{3x+1}/3$$
$$\int \frac{dx}{1+ax} = \log(1+ax)/a$$
$$\int (pt+q)^n\,dt = \frac{(pt+q)^{n+1}}{(n+1)p}. \quad \square$$

Chapter 8: Vectors and matrices

8.1 Vectors and dot products

For us, an *n-dimensional vector* is just a list of n numbers. For example:

- $(1,-1)$ and $(100,0)$ are two-dimensional vectors
- $(2,2,2)$ and $(7,8,9)$ are three-dimensional vectors
- For any $a \in \mathbb{R}$, we have a 6-dimensional vector $(1,a,a^2,a^3,a^4,a^5)$.

There is also the related concept of a *physical vector*, which is a "quantity with magnitude and direction", such as a force or velocity or electric field. Once we have chosen axes and units, we can convert a physical vector to a list of three numbers (or two numbers, for planar problems). We leave further discussion of this to applied mathematics courses. We will also postpone all consideration of the geometric meaning of vectors (although this is very important) and concentrate instead on their connection with linear equations.

Vectors are often written vertically, like this:

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \quad\text{or}\quad \begin{bmatrix}ab\\cd\end{bmatrix} \quad\text{or}\quad \begin{bmatrix}t+1\\t-1\\1-t\end{bmatrix}.$$

If we need to be precise about the distinction, we refer to horizontal vectors as *row vectors*, and to vertical vectors as *column vectors*. The obvious conversion between them is called *transposition*, and is indicated by the letter T: for example

$$(3,4,5)^T = \begin{bmatrix}3\\4\\5\end{bmatrix} \qquad\text{and}\qquad \begin{bmatrix}u\\v\end{bmatrix}^T = (u,v).$$

Often, however, there is little need to make this distinction and we write vectors vertically or horizontally as convenient.

We can add and subtract vectors of the same length, or multiply them by numbers, using the obvious rules:

$$(u,v,w)+(x,y,z) = (u+x,v+y,w+z)$$
$$a(x,y,z) = (ax,ay,az)$$

Exercise 8.1.1. Simplify the following vectors:

- $(1,-1,0,0)+(0,1,-1,0)+(0,0,1,-1)$
- $(1,-2,0,0)+2(0,1,-2,0)+4(0,0,1,-2)$
- $((0,1)-(2,3)+(4,5)-(6,7))/4$

Solution:

- $(1,0,0,-1)$
- $(1,0,0,-8)$
- $(-1,-1)$.

Given n-dimensional vectors $\mathbf{a}=(a_1,\dots,a_n)$ and $\mathbf{b}=(b_1,\dots,b_n)$, their *dot product* or *inner product* is the number

$$\mathbf{a}.\mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + \cdots a_n b_n.$$

We say that $\mathbf{a}$ and $\mathbf{b}$ are *orthogonal*, or that they *annihilate* each other, if $\mathbf{a}.\mathbf{b}=0$.

Exercise 8.1.2. Show that the following 4-dimensional vectors are all orthogonal to each other:

$$\begin{aligned}\mathbf{a} &= (1, -1, 1, -1)\\ \mathbf{b} &= (1, 1, -1, -1)\\ \mathbf{c} &= (1, 1, 1, 1).\end{aligned}$$

Solution:

$$\begin{aligned}\mathbf{a}.\mathbf{b} &= 1 \times 1 + (-1) \times 1 + 1 \times (-1) + (-1) \times (-1)\\ &= 1 - 1 - 1 + 1 = 0\\ \mathbf{a}.\mathbf{c} &= 1 \times 1 + (-1) \times 1 + 1 \times 1 + (-1) \times 1\\ &= 1 - 1 + 1 - 1 = 0\\ \mathbf{b}.\mathbf{c} &= 1 \times 1 + 1 \times 1 + (-1) \times 1 + (-1) \times 1\\ &= 1 + 1 - 1 - 1 = 0\end{aligned}$$

Exercise 8.1.3. Show that the following 5-dimensional vectors are all orthogonal to each other:

$$\begin{aligned}\mathbf{a} &= (1, 1, 0, 0, 0)\\ \mathbf{b} &= (1, -1, 1, 0, 0)\\ \mathbf{c} &= (1, -1, -2, 1, 0)\\ \mathbf{d} &= (1, -1, -2, -6, 1).\end{aligned}$$

Solution:

$$\begin{aligned}\mathbf{a}.\mathbf{b} &= 1 \times 1 + 1 \times (-1) + 0 + 0 + 0\\ &= 1 + (-1) = 0\\ \mathbf{a}.\mathbf{c} = \mathbf{a}.\mathbf{d} &= 0 \qquad\qquad \text{(similarly)}\\ \mathbf{b}.\mathbf{c} &= 1 \times 1 + (-1) \times (-1) + 1 \times (-2) + 0 + 0\\ &= 1 + 1 - 2 = 0\\ \mathbf{b}.\mathbf{d} &= 0 \qquad\qquad \text{(similarly)}\\ \mathbf{c}.\mathbf{d} &= 1 \times 1 + (-1) \times (-1) + (-2) \times (-2) + 1 \times (-6) + 0\\ &= 1 + 1 + 4 - 6 = 0.\end{aligned}$$

Exercise 8.1.4. Consider the following vectors:

$$\begin{aligned}\mathbf{a} &= (0, 0, 3)\\ \mathbf{b} &= (2\sqrt{2}, 0, -1)\\ \mathbf{c} &= (-\sqrt{2}, \sqrt{6}, -1)\\ \mathbf{d} &= (-\sqrt{2}, -\sqrt{6}, -1).\end{aligned}$$

(These are the vertices of a regular tetrahedron, or the positions of the hydrogen atoms in a molecule of methane.) Calculate $\mathbf{a}.\mathbf{b}$, $\mathbf{a}.\mathbf{c}$, $\mathbf{a}.\mathbf{d}$, $\mathbf{b}.\mathbf{c}$, $\mathbf{b}.\mathbf{d}$ and $\mathbf{c}.\mathbf{d}$.

Solution: If you believe the remark about the regular tetrahedron, it follows that all six of these dot products should be the same. We will check this algebraically. It is easy to see that $\mathbf{a}.\mathbf{b} = \mathbf{a}.\mathbf{c} = \mathbf{a}.\mathbf{d} = -3$. Next, we have

$$\begin{aligned}\mathbf{b}.\mathbf{c} &= (2\sqrt{2}).(-\sqrt{2}) + 0 + (-1).(-1) = -4 + 1 = -3\\ \mathbf{b}.\mathbf{d} &= (2\sqrt{2}).(-\sqrt{2}) + 0 + (-1).(-1) = -4 + 1 = -3\\ \mathbf{c}.\mathbf{d} &= (-\sqrt{2}).(-\sqrt{2}) + \sqrt{6}.(-\sqrt{6}) + (-1).(-1) = +2 - 6 + 1 = -3.\end{aligned}$$

The dot product has a well-known geometric meaning for physical vectors, which we will not discuss here. It is also useful in discussing linear equations, as we explain in Section 8.3.

8.2 Matrices

A *matrix* is just a rectangular table of numbers. For example:

$$A = \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$
$$B = \begin{bmatrix} u & v \\ u^2 & v^2 \\ u^3 & v^3 \end{bmatrix}$$
$$C = \begin{bmatrix} a+b & c+d \\ c+d & a+b \end{bmatrix}.$$

The *rows* of the matrix $A = \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$ above are the row vectors $(6, 5, 4)$ and $(3, 2, 1)$; the *columns* are the column vectors $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$. A matrix with n rows and m columns is said to be an *$n \times m$ matrix*; for example:

- A is a 2×3 matrix
- B is a 3×2 matrix
- C is a 2×2 matrix.

A matrix is *square* if it has the same number of rows as columns (so it is an $n \times n$ matrix for some n). Thus C is square but A and B are not.

We can add together matrices of the same shape in an obvious way, and we can multiply a matrix by a number:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$$
$$10 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 30 & 40 \end{bmatrix}$$
$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

However, we cannot add together matrices of different shapes: with the examples discussed above, the expressions $A + B$, $A + C$ and $B + C$ are not meaningful.

We can define a dot product for matrices and vectors, as follows. Suppose we have a matrix A and a vector $\mathbf{v}$, and that each row of A has the same length as $\mathbf{v}$. We can then take the dot products of all the rows of A with $\mathbf{v}$, giving a list of numbers, or in other words a vector, which is called $A\mathbf{v}$. (In this context, $\mathbf{v}$ should be written as a column vector.) The vector $A\mathbf{v}$ need not have the same length as $\mathbf{v}$. In fact, if A is an $n \times m$ matrix then $\mathbf{v}$ must be an m-dimensional vector (otherwise $A\mathbf{v}$ is undefined) and $A\mathbf{v}$ is an n-dimensional vector.

Example 8.2.1. Consider the 2×3 matrix $A = \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$ as before, and the 3-dimensional vector $\mathbf{v} = (x, y, z)$. The rows of A are the vectors $\mathbf{a}_1 = (6, 5, 4)$ and $\mathbf{a}_2 = (3, 2, 1)$. The entries in $A\mathbf{v}$ are the dot products

$$\mathbf{a}_1.\mathbf{v} = (6, 5, 4).(x, y, z) = 6x + 5y + 4z$$

and

$$\mathbf{a}_2.\mathbf{v} = (3,2,1).(x,y,z) = 3x + 2y + z.$$

We thus have

$$A\mathbf{v} = \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6x + 5y + 4z \\ 3x + 2y + z \end{bmatrix}. \quad \square$$

Example 8.2.2. Consider the 3×3 matrix $P = \begin{bmatrix} 0 & 0 & p \\ 0 & q & 0 \\ r & 0 & 0 \end{bmatrix}$, and the 3-dimensional vector $\mathbf{v} = (x,y,z)$. The rows of P are the vectors

$$\begin{aligned} \mathbf{p}_1 &= (0,0,p) \\ \mathbf{p}_2 &= (0,q,0) \\ \mathbf{p}_3 &= (r,0,0). \end{aligned}$$

The entries in $P\mathbf{v}$ are the dot products

$$\begin{aligned} \mathbf{p}_1.\mathbf{v} &= (0,0,p).(x,y,z) = pz \\ \mathbf{p}_2.\mathbf{v} &= (0,q,0).(x,y,z) = qy \\ \mathbf{p}_3.\mathbf{v} &= (r,0,0).(x,y,z) = rx, \end{aligned}$$

so

$$P\mathbf{v} = \begin{bmatrix} 0 & 0 & p \\ 0 & q & 0 \\ r & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} pz \\ qy \\ rx \end{bmatrix}. \quad \square$$

Example 8.2.3. In the following cases, the product $A\mathbf{v}$ is not defined, because the rows of A do not have the same length as $\mathbf{v}$.

- $A = \begin{bmatrix} 0 & 0 & p \\ 0 & q & 0 \end{bmatrix}$, $\mathbf{v} = (1,1)$.
- $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\mathbf{v} = (p,q,r)$. $\square$

Exercise 8.2.4. In the following cases, either evaluate $A\mathbf{v}$, or explain why it is undefined.

(a) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $\mathbf{v} = (x,y,z)$.

(b) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$ and $\mathbf{v} = (a,b,c,d)$.

(c) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $\mathbf{v} = (p,q)$.

(d) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$ and $\mathbf{v} = (1,-1)$.

Solution:

(a) $\begin{bmatrix} y \\ z \\ x \end{bmatrix}$ (or (y,z,x); either is an acceptable answer)

(b) $(a+b+c+d, a-b+c-d)$

(c) This is not defined, because the rows of A are 3-dimensional vectors, so we cannot dot them with the 2-dimensional vector $\mathbf{v}$.

(d) $(-1,-1,-1,-1)$.

Exercise 8.2.5. Put

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} \cos(\theta+\phi) \\ \sin(\theta+\phi) \end{bmatrix}$$

Check that $A\mathbf{u} = \mathbf{v}$.

Solution: The rows of A are the vectors

$$\begin{aligned} \mathbf{a}_1 &= (\cos(\theta), -\sin(\theta)) \\ \mathbf{a}_2 &= (\sin(\theta), \cos(\theta)). \end{aligned}$$

The entries in $A\mathbf{u}$ are thus the numbers

$$\begin{aligned} \mathbf{a_1}.\mathbf{u} &= (\cos(\theta), -\sin(\theta)).(\cos(\phi), \sin(\phi)) \\ &= \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) \\ &= \cos(\theta+\phi) \\ \mathbf{a_2}.\mathbf{u} &= (\sin(\theta), \cos(\theta)).(\cos(\phi), \sin(\phi)) \\ &= \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi) \\ &= \sin(\theta+\phi). \end{aligned}$$

Thus $A\mathbf{u} = (\cos(\theta+\phi), \sin(\theta+\phi)) = \mathbf{v}$.

Definition 8.2.6. The *identity matrix* I_n is the $n \times n$ matrix such that all the entries on the main diagonal are equal to one, and all other entries are zero. For example:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We will just write I instead of I_n if it is clear what n should be.

The identity matrix has the important property that $I_n\mathbf{v} = \mathbf{v}$ for any n-dimensional vector $\mathbf{v}$. For example, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (1,0,0).(x,y,z) \\ (0,1,0).(x,y,z) \\ (0,0,1).(x,y,z) \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

8.3 Linear equations

We have already seen some examples of systems of linear equations and their solutions, applied to the problem of finding integrals or partial fraction decompositions. We now explain the *Gaussian elimination method* for solving such systems. This is useful when

- The number of equations or variables is large (it could easily be several thousand in some engineering problems, for example)

- The numbers do not work out nicely
- One wants a systematic method of solution that works in all cases and can be programmed into a computer (even though a less mechanical approach may be easier in any particular case).

In practical cases the method is carried out by machine, but nonetheless you should understand it, for a number of reasons:

- There is a rich theory of systems of linear equations, which explains the possible types of solution set, the cases in which there may be no solutions, the behaviour of the solution set if the equations are changed slightly, and so on. This emphasises patterns and the larger picture rather than any particular system of equations; it is explained in the two Linear Mathematics courses (PMA211 and PMA212). If you have a good understanding of how and why Gaussian elimination works, you will find it much easier to follow this theoretical analysis.
- It is an important problem to develop computerised methods of answering mathematical questions. Gaussian elimination is an excellent example of a method that can easily be converted to a program, and so is a good introduction to the issues involved.

In outline, the Gaussian elimination method for solving systems of linear equations is as follows. The terms used will be explained later in this section.

(a) Write down the *augmented matrix* for the system of equations.

(b) Perform *row operations* to convert the matrix to *row reduced echelon form* (usually abbreviated *RREF*).

(c) Convert back to a system of linear equations, and *read off the solution*.

First, however, we do a calculation that will serve as an extended example.

Example 8.3.1. Consider the equations

$$x - y = 2 \tag{R$_1$}$$
$$w + 2x - y + 3z = 10 \tag{R$_2$}$$
$$w + 4x - 3y + 5z = 16. \tag{R$_3$}$$

We simplify these in six stages, as follows:

$\boxed{1}$ $w + 2x - y + 3z = 10$, $x - y = 2$, $w + 4x - 3y + 5z = 16$	$\boxed{2}$ $w + 2x - y + 3z = 10$, $x - y = 2$, $2x - 2y + 2z = 6$	$\boxed{3}$ $w + y + 3z = 6$, $x - y = 2$, $2x - 2y + 2z = 6$
$R_1 \leftrightarrow R_2$	$R_3 \mapsto R_3 - R_1$	$R_1 \mapsto R_1 - 2R_2$
$\boxed{4}$ $w + y + 3z = 6$, $x - y = 2$, $2z = 2$	$\boxed{5}$ $w + y + 3z = 6$, $x - y = 2$, $z = 1$	$\boxed{6}$ $w + y = 3$, $x - y = 2$, $z = 1$
$R_3 \mapsto R_3 - 2R_2$	$R_3 \mapsto R_3/2$	$R_1 \mapsto R_1 - 3R_3$

- At the first stage, we have exchanged the order of the equations R_1 and R_2 (as indicated by the annotation $R_1 \leftrightarrow R_2$). This ensures that the new R_1 (that is, the equation $w+2x-y+3z = 10$) depends on the variable w; we will see later why this is a Good Thing.
- We then subtract the new R_1 from R_3, as indicated by the annotation $R_3 \mapsto R_3 - R_1$. This ensures that *only* R_1 depends on w.

- We then subtract $2R_2$ from R_1 (written $R_1 \mapsto R_1 - 2R_2$) to ensure that the new R_1 (that is, the equation $w + y - z = 2$) no longer depends on x.
- We subtract $2R_2$ from R_3 (written $R_3 \mapsto R_3 - 2R_2$) to ensure that the new R_3 (that is, the equation $2z = 2$) no longer depends on x.
- We divide R_3 by 2 (written $R_3 \mapsto R_3/2$) so that the coefficient of z is just one.
- We subtract $3R_3$ from R_1 (written $R_1 \mapsto R_1 - 3R_3$) to ensure that R_1 no longer depends on z.

The solution can be read off directly from the last set of equations: the variable y can take any value, but w, x and z are given (in terms of the independent variable y) by

$$\begin{aligned} w &= 3 - y \\ x &= 2 + y \\ z &= 1 \quad \square \end{aligned}$$

8.3.1 Matrices for linear equations

We next explain how linear equations can be expressed in terms of vectors, matrices and dot products. The following exercises give the basic idea:

Exercise 8.3.2. Suppose that the vector (x, y, z) satisfies the equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}.$$

What can we say about x, y and z?

Solution: By working out the matrix product on the left hand side, we see that

$$\begin{bmatrix} x + 2y + 3z \\ y + 2z \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix},$$

so

$$\begin{aligned} x + 2y + 3z &= 6 && \text{(A)} \\ y + 2z &= 3 && \text{(B)} \\ z &= 1. && \text{(C)} \end{aligned}$$

Equation (C) says that $z = 1$, we can substitute this into (B) to see that $y = 1$, and then we can substitute $y = z = 1$ into (A) to see that $x = 1$.

Exercise 8.3.3. Rewrite the following system of equations using matrices:

$$\begin{aligned} u - v + w - x + y - z &= -4 \\ 2u + 2w + 2y &= 16 \\ 3v + 3x + 3z &= 36. \end{aligned}$$

(You need not actually *solve* the equations, although it is not too hard to do so if you wish to.)

Solution: Put $\mathbf{p} = (u, v, w, x, y, z)$; then the equations say

$$\begin{aligned} (1, -1, 1, -1, 1, -1).\mathbf{p} &= -4 \\ (2, 0, 2, 0, 2, 0).\mathbf{p} &= 16 \\ (0, 3, 0, 3, 0, 3).\mathbf{p} &= 36. \end{aligned}$$

Equivalently, if we form the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 2 & 0 & 2 & 0 & 2 & 0 \\ 0 & 3 & 0 & 3 & 0 & 3 \end{bmatrix}.$$

then the equations say that $A\mathbf{p} = (-4, 16, 36)$. Writing this out in full, the equations say that

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 2 & 0 & 2 & 0 & 2 & 0 \\ 0 & 3 & 0 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ x \\ y \\ z \end{bmatrix} = (-4, 16, 36).$$

Exercise 8.3.4. Rewrite the equations $A + B = B + C = C + D = D + A = 1$ using matrices.

Solution: The equations say

$$\begin{aligned} (1,1,0,0).(A,B,C,D) &= 1 \\ (0,1,1,0).(A,B,C,D) &= 1 \\ (0,0,1,1).(A,B,C,D) &= 1 \\ (1,0,0,1).(A,B,C,D) &= 1, \end{aligned}$$

or equivalently

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Remark 8.3.5. We can now explain exactly what we mean by a *linear* system of equations, which we have so far avoided doing. A system of equations is linear if it can be written as a matrix equation as described above. In particular, a linear equation in the variables x, y and z must have the form $ax + by + cz = d$ for some constants a, b, c and d. This means that equations such as $x^2 + y^2 + z^2 = 1$ or $xy = z$ or $(x + 1)/(x - 1) = x$ are *not* linear.

We can of course write all the systems of equations in Example 8.3.1 using matrices. For example, the original system

$$x - y = 2 \tag{R$_1$}$$
$$w + 2x - y + 3z = 10 \tag{R$_2$}$$
$$w + 4x - 3y + 5z = 16. \tag{R$_3$}$$

is equivalent to

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 2 & -1 & 3 \\ 1 & 4 & -3 & 5 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ 16 \end{bmatrix}.$$

It is convenient to condense this a little further: we leave out the names of the variables and the equality sign, and just write down all the numbers in a table like this:

$$\left[\begin{array}{cccc|c} 0 & 1 & -1 & 0 & 2 \\ 1 & 2 & -1 & 3 & 10 \\ 1 & 4 & -3 & 5 & 16 \end{array}\right].$$

This is called the *augmented matrix* for the system of equations.

Exercise 8.3.6. Write down the augmented matrix for the following system of equations:

$$\begin{aligned} v+w+x+y+z &= 5 \\ w+x+y &= 3 \\ x &= 1. \end{aligned}$$

Solution:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Exercise 8.3.7. Write down the augmented matrix for the following system of equations:

$$\begin{aligned} 9A+3B+C &= 1 \\ 16A+4B+C &= 4 \\ 25A+5B+C &= 10. \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 9 & 3 & 1 & 1 \\ 16 & 4 & 1 & 4 \\ 25 & 5 & 1 & 10. \end{array}\right]$$

We now write down the augmented matrices for all the stages in Example 8.3.1:

$$\left[\begin{array}{cccc|c} 0 & 1 & -1 & 0 & 2 \\ 1 & 2 & -1 & 3 & 10 \\ 1 & 4 & -3 & 5 & 16 \end{array}\right] \xrightarrow{1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 10 \\ 0 & 1 & -1 & 0 & 2 \\ 1 & 4 & -3 & 5 & 16 \end{array}\right] \xrightarrow{2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 10 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 2 & -2 & 2 & 6 \end{array}\right] \xrightarrow{3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 3 & 6 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 2 & -2 & 2 & 6 \end{array}\right]$$

$$\xrightarrow{4} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 3 & 6 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 2 & 2 \end{array}\right] \xrightarrow{5} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 3 & 6 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{6} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

The steps taken are as follows:

(1) We exchange the first two rows ($R_1 \leftrightarrow R_2$).

(2) We subtract the first row from the third row ($R_3 \mapsto R_3 - R_1$).

(3) We subtract twice the second row from the first row ($R_1 \mapsto R_1 - 2R_2$).

(4) We subtract twice the second row from the third row ($R_3 \mapsto R_3 - 2R_2$).

(5) We divide R_3 by 2 ($R_3 \mapsto R_3/2$).

(6) We subtract $3R_3$ from R_1 ($R_1 \mapsto R_1 - 3R_3$).

These are just the same as the steps explained in Example 8.3.1, stated in slightly different language. The steps are called *row operations*, and the final result is a matrix in *row reduced echelon form*. Our next task is to explain the general meaning of these terms.

8.3.2 Row operations and echelon form

Definition 8.3.8. Let A be a matrix. A *row operation* on A means any of the following:

- Reorder the rows of A (eg $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \mapsto \begin{bmatrix} 9 & 10 & 11 & 12 \\ 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \end{bmatrix}$, where we have exchanged R_1 and R_3, written $R_1 \leftrightarrow R_3$)
- Add a multiple of one row to another row (eg $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & 3 & 4 \\ 105 & 206 & 307 & 408 \end{bmatrix}$, where we have added $100R_1$ to R_2, written $R_2 \mapsto R_2 + 100R_1$)

- Multiply a row by a nonzero constant (eg $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \mapsto \begin{bmatrix} 10 & 20 & 30 & 40 \\ 5 & 6 & 7 & 8 \end{bmatrix}$, where we have multiplied R_1 by 10, written $R_1 \mapsto 10R_1$).

The point about row operations is as follows. Suppose we start with a system of linear equations, write down the augmented matrix, apply some row operations, then convert back to a system of equations. Then the new system of equations will be *equivalent* to the original one, so it will have exactly the same solutions.

Definition 8.3.9. Let A be a matrix. We say that A is in *row reduced echelon form* or *RREF* if

(a) If there are any rows of zeros, then they all occur at the bottom of the matrix.

(b) In every nonzero row, the first nonzero entry is 1. We refer to these 1's as the *pivots* of the matrix.

(c) All entries directly above or below a pivot are zero.

(d) The pivot in the $(k+1)$'st row is always further to the right than the pivot in the k'th row.

Example 8.3.10. (a) The matrix $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is not in RREF, because there is a row of zeros that is not at the bottom.

(b) The matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ is not in RREF, because the first nonzero entry in the second row is 2, not 1.

(c) The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ is not in RREF, because there is a nonzero entry directly above the pivot in the second row.

(d) The matrix $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ is not in RREF, because the pivot in the second row is further left than the pivot in the first row.

(e) The matrix

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in RREF. □

Exercise 8.3.11. Which of the following matrices are in RREF?

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution: The matrices A and C are in RREF. The matrix B is not in RREF because the pivot in the second row is to the left of the one in the first row. The matrix D is not in RREF because there is a nonzero entry above the second-row pivot. The matrix E is not in RREF because there is a row of zeros that is not at the bottom. The matrix F is not in RREF because the first nonzero entry in the first row is 2, not 1.

The point about RREF is this: given an augmented matrix in RREF, one can instantly read off the solution of the corresponding system of equations.

Example 8.3.12. Consider the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 5 & 6 \end{array}\right]$$

The corresponding system of equations (in variables w, x, y and z) is

$$\begin{aligned} w + 2x + 3z &= 4 \\ y + 5z &= 6, \end{aligned}$$

or equivalently

$$\begin{aligned} w &= 4 - 2x - 3z \\ y &= 6 - 5z. \end{aligned}$$

The variables x and z can take any value, and w and y are given in terms of x and z by the above formulae; this is the complete solution. The only work that we have had to do is to transfer some terms from the left hand side to the right hand side. □

In general, some of the variables will be *independent*, so they can take any value. The remaining (*dependent*) variables can then be expressed in terms of the independent ones. To work out which variables are independent, we proceed as follows:

Method 8.3.13. (a) If the problem involves n variables, then the augmented matrix will have n columns to the left of the line, and one extra column to the right of the line. Write the n variables underneath the first n columns.

(b) If there is a pivot in the extra column, then there are no solutions at all (so there is no meaningful distinction between dependent and independent variables).

(b) Otherwise, the variables underneath the pivots are dependent; any remaining variables are independent.

In Example 8.3.12, step (a) yields the matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 5 & 6 \\ \hline w & x & y & z & \end{array}\right].$$

There are pivots in the first and third columns, above w and y. This shows that x and z are independent, and w and y are dependent, just as we stated previously.

Exercise 8.3.14. Consider the system of equations (in variables x, y and z) whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$$

What is the solution?

Solution: The corresponding system of equations is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

but $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, so the solution is just $x = a$, $y = b$ and $z = c$.

Exercise 8.3.15. Consider the system of equations (in variables u, v, w, x, y and z) whose augmented matrix is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 & -1 & 4 \end{array}\right]$$

Write down the equations explicitly, and then write down the general solution.

Solution: The pivots are in the first, third and fourth columns, corresponding to the first, third and fourth variables, that is, u, w and x. These variables are thus the dependent ones. The remaining variables (v, y and z) are independent, so they can take any value. The equations are

$$\begin{aligned} u + y + z &= 2 \\ w - y + z &= 3 \\ x + y - z &= 4, \end{aligned}$$

or equivalently

$$\begin{aligned} u &= 3 - y - z \\ w &= 3 + y - z \\ x &= 4 - y + z. \end{aligned}$$

This gives the dependent variables in terms of the independent ones.

8.3.3 Gaussian elimination

Suppose again that we have a system of n linear equations, with augmented matrix A say. If we can convert this to RREF by row operations, then this will give an equivalent system of equations whose solution is very easy to write down. So all we now need is a systematic way of getting to RREF by row operations.

Method 8.3.16. (a) Find a nonzero entry that is as far to the left as possible.

(b) Reorder the rows if necessary to bring that entry to the top row.

(c) Divide the top row by that entry, so that the first nonzero entry in the new top row is 1. In other words, we have a pivot in the top row.

(d) Subtract multiples of the top row from the other rows, so that all entries below the top row pivot are zero.

(e) Ignore the top row, and apply steps (a),...,(d) to the rest of the matrix. Then ignore the top two rows, and apply (a),...,(d) to the remaining $n-2$ rows. Carry on in this way until there are no more rows left to consider, or until the remaining rows contain only zeros.

(f) Subtract a multiple of the second row from the first row, to ensure that the entry directly above the second-row pivot is zero.

(g) Subtract multiples of the third row from the first and second rows, to ensure that the entries above the third-row pivot are zero. Then clear the entries above the remaining pivots in the same way.

This process is called *row-reduction*. Note that the vertical line in the augmented matrix does not play any rôle, so the process can be applied to any matrix, whether or not it came from a system of linear equations. There are a number of applications of row-reduction that are only indirectly related to linear equations: this is discussed in the Linear Mathematics courses PMA211 and PMA212.

Example 8.3.17. Consider the equations

$$\begin{aligned} 2x - 2y - 2z &= -4 \\ 2x + 2y - 2z &= -4 \\ 2x + 2y + 2z &= 0 \end{aligned}$$

The associated augmented matrix is

$$A = \left[\begin{array}{ccc|c} 2 & -2 & -2 & -4 \\ 2 & 2 & -2 & -4 \\ 2 & 2 & 2 & 0 \end{array}\right]$$

This can be converted to RREF as follows:

$$\left[\begin{array}{ccc|c} 2 & -2 & -2 & -4 \\ 2 & 2 & -2 & -4 \\ 2 & 2 & 2 & 0 \end{array}\right] \xrightarrow{1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 2 & 2 & -2 & -4 \\ 2 & 2 & 2 & 0 \end{array}\right] \xrightarrow{2} \left[\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 0 & 4 & 0 & 0 \\ 0 & 4 & 4 & 4 \end{array}\right] \xrightarrow{3} \left[\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 4 & 4 \end{array}\right]$$

$$\xrightarrow{4} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 4 \end{array}\right] \xrightarrow{5} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{6} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The steps are as follows: (1): $R_1 \mapsto R_1/2$; (2): $R_2 \mapsto R_2 - 2R_1$ and $R_3 \mapsto R_3 - 2R_1$; (3): $R_2 \mapsto R_2/4$; (4): $R_1 \mapsto R_1 + R_2$ and $R_3 \mapsto R_3 - 4R_2$; (5): $R_3 \mapsto R_3/4$; (6): $R_1 \mapsto R_1 + R_3$.

This augmented matrix corresponds to the equations

$$\begin{aligned} x &= -1 \\ y &= 0 \\ z &= 1. \quad \square \end{aligned}$$

Exercise 8.3.18. Let a, b and c be given constants. Use Gaussian elimination to solve the equations

$$\begin{aligned} w - x + y - z &= 2a \\ w + x - y - z &= 2b \\ w + x + y + z &= 2c \end{aligned}$$

Solution: We write down the augmented matrix and row-reduce it as follows:

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 2a \\ 1 & 1 & -1 & -1 & 2b \\ 1 & 1 & 1 & 1 & 2c \end{array}\right] \xrightarrow{1} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 2a \\ 0 & 2 & -2 & 0 & 2b - 2a \\ 0 & 2 & 0 & 2 & 2c - 2a \end{array}\right] \xrightarrow{2} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 2a \\ 0 & 1 & -1 & 0 & b - a \\ 0 & 2 & 0 & 2 & 2c - 2a \end{array}\right] \xrightarrow{3}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & b + a \\ 0 & 1 & -1 & 0 & b - a \\ 0 & 0 & 2 & 2 & 2c - 2b \end{array}\right] \xrightarrow{4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & b + a \\ 0 & 1 & -1 & 0 & b - a \\ 0 & 0 & 1 & 1 & c - b \end{array}\right] \xrightarrow{5} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & b + a \\ 0 & 1 & 0 & 1 & c - a \\ 0 & 0 & 1 & 1 & c - b \end{array}\right]$$

The steps were as follows:

(1) $R_2 \mapsto R_2 - R_1$, $R_3 \mapsto R_3 - R_1$

(2) $R_2 \mapsto R_2/2$

(3) $R_1 \mapsto R_1 + R_2$, $R_3 \mapsto R_3 - 2R_2$

(4) $R_3 \mapsto R_3/2$

(5) $R_2 \mapsto R_2 + R_3$.

Our final matrix is in RREF. There are pivots in the first three columns, so the first three variables (w, x and y) are dependent, and the remaining variable (z) is independent. The final matrix corresponds to the equations

$$\begin{aligned} w - z &= b + a \\ x + z &= c - a \\ y + z &= c - b, \end{aligned}$$

so

$$\begin{aligned} w &= a + b + z \\ x &= -a + c - z \\ y &= -b + c - z. \end{aligned}$$

This gives the dependent variables in terms of the independent variable z and the given constants a, b and c.

Exercise 8.3.19. Let p be a given constant, and let a, b and c be variables. Use Gaussian elimination to solve the equations

$$\begin{aligned} b + c &= 2p \\ a + b &= 2p \\ a + c &= 2p. \end{aligned}$$

Solution: We write down the augmented matrix and row-reduce it as follows:

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 2p \\ 1 & 1 & 0 & 2p \\ 1 & 0 & 1 & 2p \end{array}\right] \xrightarrow{1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2p \\ 1 & 1 & 0 & 2p \\ 0 & 1 & 1 & 2p \end{array}\right] \xrightarrow{2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2p \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2p \end{array}\right] \xrightarrow{3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2p \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 2p \end{array}\right] \xrightarrow{4} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2p \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & p \end{array}\right] \xrightarrow{5} \left[\begin{array}{ccc|c} 1 & 0 & 0 & p \\ 0 & 1 & 0 & p \\ 0 & 0 & 1 & p \end{array}\right]$$

The steps were as follows:

(1) $R_1 \leftrightarrow R_3$

(2) $R_2 \mapsto R_2 - R_1$

(3) $R_3 \mapsto R_3 - R_2$

(4) $R_3 \mapsto R_3/2$

(5) $R_1 \mapsto R_1 - R_3$, $R_2 \mapsto R_2 + R_3$.

Our final matrix is in RREF. Each column to the left of the line contains a pivot, so there are no independent variables. The solution is just $x = y = z = p$.

Exercise 8.3.20. Use Gaussian elimination to show that the following equations have no solution.

$$\begin{aligned} A + 2B + 3C &= 2 \\ A + B + C &= 2 \\ 3A + 2B + C &= 2. \end{aligned}$$

Then give a quicker way to reach the same conclusion.

Solution: We write down the augmented matrix and row-reduce it as follows:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 1 & 1 & 1 & 2 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & -2 & 0 \\ 0 & -4 & -8 & -4 \end{array}\right] \xrightarrow{2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & -4 & -8 & -4 \end{array}\right] \xrightarrow{3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -4 \end{array}\right] \xrightarrow{4} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{5} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The steps were as follows:

(1) $R_2 \mapsto R_2 - R_1$, $R_3 \mapsto R_3 - 3R_1$

(2) $R_2 \mapsto -R_2$

(3) $R_1 \mapsto R_1 - 2R_2$, $R_3 \mapsto R_3 + 4R_2$

(4) $R_3 \mapsto -R_3/4$

(5) $R_1 \mapsto R_1 - 2R_3$.

Our final matrix is in RREF. There is a pivot in the last column, to the right of the line, so there are no solutions. This can also be seen directly from the original equations: if we add the equation $A + 2B + 3C = 2$ to the equation $3A + 2B + C = 2$ and then divide by 4 we get $A + B + C = 1$, but this is impossible because we are told that $A + B + C = 2$.

Exercise 8.3.21. Use Gaussian elimination to solve the equations

$$\begin{aligned} 2u + 2v + 2w &= 12 \\ u + 2v + 2w &= 11 \\ u + 2v + 3w &= 14. \end{aligned}$$

Solution: We write down the augmented matrix and row-reduce it as follows:

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 12 \\ 1 & 2 & 2 & 11 \\ 1 & 2 & 3 & 14 \end{array}\right] \xrightarrow{1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 2 & 11 \\ 1 & 2 & 3 & 14 \end{array}\right] \xrightarrow{2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & 2 & 8 \end{array}\right] \xrightarrow{3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{4} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

The steps were as follows:

(1) $R_1 \mapsto R_1/2$

(2) $R_2 \mapsto R_2 - R_1$, $R_3 \mapsto R_3 - R_1$

(3) $R_1 \mapsto R_1 - R_2$, $R_3 \mapsto R_3 - R_2$

(4) $R_2 \mapsto R_2 - R_3$.

Our final matrix is in RREF. Each column to the left of the line contains a pivot, so there are no independent variables. The solution is $u = 1$, $v = 2$ and $w = 3$.

Exercise 8.3.22. Use Gaussian elimination to solve the equations

$$\begin{aligned} x + 2y &= 1 \\ 3x + 4y &= 1 \\ 5x + 6y &= 1 \\ 7x + 8y &= 1. \end{aligned}$$

Solution: We write down the augmented matrix and row-reduce it as follows:

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 5 & 6 & 1 \\ 7 & 8 & 1 \end{array}\right] \xrightarrow{1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & -2 \\ 0 & -4 & -4 \\ 0 & -6 & -6 \end{array}\right] \xrightarrow{2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -4 & -4 \\ 0 & -6 & -6 \end{array}\right] \xrightarrow{3} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The steps were as follows:

(1) $R_2 \mapsto R_2 - 3R_1$, $R_3 \mapsto R_3 - 5R_1$, $R_4 \mapsto R_4 - 7R_1$

(2) $R_2 \mapsto -R_2/2$

(3) $R_1 \mapsto R_1 - 2R_2$, $R_3 \mapsto R_3 + 4R_2$, $R_4 \mapsto R_4 + 6R_2$.

Our final matrix is in RREF. Each column to the left of the line contains a pivot, so there are no independent variables. The solution is $x = -1$ and $y = 1$. (The two rows of zeros at the bottom correspond to the equation $0 = 0$, which is always true and tells us nothing about x or y, so it can be ignored.)

Exercise 8.3.23. Consider the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

Do you recognize the entries? Row-reduce the matrix.

Solution: The entries are binomial coefficients, so the matrix is a skewed copy of Pascal's triangle . It is possible to deduce the row-reduction painlessly from this by abstract linear algebra (which is introduced in PMA211). Here, however, we will just use the pedestrian method.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \\ 0 & 3 & 6 & 4 & 1 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 4 & 1 \end{bmatrix}$$
$$\xrightarrow{3} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

8.4 Matrix multiplication

Suppose we have three lists of variables, say (u, v, w), (p, q) and (x, y, z). Suppose also that we can express p and q in terms or x, y and z, and that we can express u, v and w in terms of p and q. For example, we might have

$$\begin{aligned} u &= -p + q \\ v &= -2p + q \\ w &= 2p + q \end{aligned}$$

and

$$\begin{aligned} p &= x - y + z \\ q &= 2x + 2y + 2z. \end{aligned}$$

We can then substitute the second lot of equations in the first lot to get

$$\begin{aligned} u &= -(x-y+z) + (2x+2y+2z) = x+3y+z \\ v &= -2(x-y+z) + (2x+2y+2z) = 4y \\ w &= 2(x-y+z) + (2x+2y+2z) = 4x+4z. \end{aligned}$$

We can rewrite all of this using matrices as follows:

$$\begin{aligned} \begin{bmatrix} u \\ v \\ w \end{bmatrix} &= \begin{bmatrix} -1 & 1 \\ -2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \\ \begin{bmatrix} p \\ q \end{bmatrix} &= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ \begin{bmatrix} u \\ v \\ w \end{bmatrix} &= \begin{bmatrix} 1 & 3 & 1 \\ 0 & 4 & 0 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \end{aligned}$$

In other words, if we put

$$\begin{aligned} \mathbf{a} &= \begin{bmatrix} u \\ v \\ w \end{bmatrix} & A &= \begin{bmatrix} -1 & 1 \\ -2 & 1 \\ 2 & 1 \end{bmatrix} \\ \mathbf{b} &= \begin{bmatrix} p \\ q \end{bmatrix} & B &= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \\ \mathbf{c} &= \begin{bmatrix} x \\ y \\ z \end{bmatrix} & C &= \begin{bmatrix} 1 & 3 & 1 \\ 0 & 4 & 0 \\ 4 & 0 & 4 \end{bmatrix} \end{aligned}$$

then $\mathbf{a} = A\mathbf{b}$, $\mathbf{b} = B\mathbf{c}$ and $\mathbf{a} = C\mathbf{c}$. On the other hand, we should be able to combine the equations $\mathbf{a} = A\mathbf{b}$ and $\mathbf{b} = B\mathbf{c}$ to say that $\mathbf{a} = AB\mathbf{c}$, and this suggests that AB should be equal to C. However, this does not yet make any sense: we have not defined what it means to multiply matrices together, so the symbol AB is meaningless. However, this train of thought does show that it would be desirable to define multiplication of matrices, and that the definition should work out in such a way that $AB = C$. We will next explain the relevant construction.

Definition 8.4.1. Let A be an $n \times p$ matrix, and let B be a $p \times m$ matrix. (This means that the rows of A have length p, which is the same as the length of the columns of B). We then define an $n \times m$ matrix AB as follows:

- We take the top row of A, and take its dot product with each of the m columns of B. This gives a list of m numbers, which forms the top row of AB.
- We take the second row of A, and take its dot product with each of the m columns of B. This gives a list of m numbers, which forms the second row of AB.
- and so on.

Another way to say the same thing is as follows: if the columns of B are $\mathbf{b}_1, \dots, \mathbf{b}_m$, then the columns of AB are $A\mathbf{b}_1, \dots, A\mathbf{b}_m$.

Example 8.4.2. Consider the matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

The entries in AB are the dot products of the rows of A with the columns of B. The rows of A are (a, b) and (c, d), and the columns of B are (p, r) and (q, s). We thus have

$$\begin{aligned} AB &= \left[\begin{array}{cc} a & b \\ \hline c & d \end{array}\right] \left[\begin{array}{c|c} p & q \\ r & s \end{array}\right] \\ &= \begin{bmatrix} (a,b).(p,r) & (a,b).(q,s) \\ (c,d).(p,r) & (c,d).(q,s) \end{bmatrix} \\ &= \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}. \quad \square \end{aligned}$$

Example 8.4.3. If A is an $n \times m$ matrix, then $I_n A = A = A I_m$, where I_n and I_m are the identity matrices of size n and m. For example, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

as you can check by calculating everything directly. (This is the case where $n = 3$ and $m = 2$.) $\square$

Warning 8.4.4. In general we *do not* have $AB = BA$.

- If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ then $AB = \begin{bmatrix} 3 & 3 & 3 \\ 7 & 7 & 7 \end{bmatrix}$, but BA is not even meaningful (because the rows of B do not have the same length as the columns of A).
- If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ then $AB = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$ and $BA = \begin{bmatrix} 3 & 3 \\ 6 & 6 \end{bmatrix}$, so AB and BA do not even have the same shape.
- If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ then $AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, so $AB \neq BA$.

Exercise 8.4.5. Consider the matrices

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Calculate AB and BA.

Solution:

$$AB = \begin{bmatrix} a & a & a \\ 0 & b & b \\ 0 & 0 & c \end{bmatrix} \qquad BA = \begin{bmatrix} a & b & c \\ 0 & b & c \\ 0 & 0 & c \end{bmatrix}$$

Exercise 8.4.6. Evaluate the following matrix products:

(a) $F(a)F(b)$, where $F(x)$ means $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$.

(b) $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Solution:

(a) $F(a)F(b) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = F(a+b)$.

(b) $\begin{bmatrix} ac & 0 \\ 0 & bd \end{bmatrix}$.

(c) $\begin{bmatrix} 0 & 0 \\ 4 & -4 \end{bmatrix}$.

(d) $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Exercise 8.4.7. Let a, b, c and d be numbers such that $a+b+c+d=4$. Calculate the product

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Solution: We have

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix},$$

so

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} &= \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a+b+c+d & a+b+c+d \\ a+b+c+d & a+b+c+d \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}. \end{aligned}$$

Exercise 8.4.8. Consider the matrices

$$A = \begin{bmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1+c^2 & -a-bc & -b+ac \\ a-bc & 1+b^2 & -c-ab \\ b+ac & c-ab & 1+a^2 \end{bmatrix}$$

Calculate the product AB.

Solution: The answer is just $(1+a^2+b^2+c^2)I$ (where I is the identity matrix).

8.5 Determinants and inverses

Suppose we have a system of n linear equations in m variables. As explained in Section 8.3.1, this can be rewritten in the form $A\mathbf{x} = \mathbf{c}$, where A is the (unaugmented) $n \times m$ matrix of coefficients, $\mathbf{x}$ is the m-dimensional vector containing all the variables, and $\mathbf{c}$ is the n-dimensional vector containing the right hand sides of all the equations. We have already seen how to solve such equations (or show that there are no solutions) by Gaussian elimination. We now explain an alternative method which is advantageous in some cases. However, it will only work when the number of variables is the same as the number of equations, so that $m = n$ and A is a *square* matrix. We will assume this for the rest of this section.

8.5.1 Determinants

A central rôle is played by a number called the *determinant* of A, written $\det(A)$. The definition will be given shortly, but first, for motivation, we explain one of the many reasons why determinants are useful. Consider an equation $A\mathbf{x} = \mathbf{c}$, where A is square. As we saw in our discussion of Gaussian elimination, this might or might not have a solution. If there is a solution, then it might involve some independent variables (in which case there are *infinitely many* different solutions) or it might not (in which case the solution is *unique*).

Fact 8.5.1.
- If $\det(A) \neq 0$, then there is a unique solution, with no independent variables.
- If $\det(A) = 0$ then there might or might not be a solution. If there are any solutions, then there will always be some independent variables, so there will actually be infinitely many different solutions. □

We next explain how $\det(A)$ is defined. The first few cases are as follows:

- A 1×1 matrix is just a number, and $\det([a]) = a$. This trivial case is needed to make various patterns fit together properly.
- For 2×2 matrices, we have
$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$
- For 3×3 matrices, we have
$$\begin{aligned}\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} &= a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix} \\ &= aei - afh - bdi + bfg + cdh - ceg.\end{aligned}$$

Definition 8.5.2. Let A be a square matrix of size n, and let x be an entry in A. The *associated minor* is the square matrix of size $n-1$ obtained from A by deleting the row and column containing x. For example, let A be the matrix on the left below, and let x be the 7 in the second row. Then the associated minor is the matrix B shown on the right.

$$A = \left[\begin{array}{cc|c|c} 1 & 2 & 3 & 4 \\ \hline 5 & 6 & \mathbf{7} & 8 \\ \hline 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{array}\right] \qquad B = \begin{bmatrix} 1 & 2 & 4 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{bmatrix}$$

The *cofactor* for x is the determinant of the associated minor. The *associated sign* for x is defined as follows: the sign for the top left entry is $+$, and the sign changes each time you move one step vertically or horizontally. The pattern of signs for a 4×4 matrix is therefore as follows:

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

The *determinant* of a matrix A is now defined as follows: take each entry in the top row, multiply it by its cofactor and its associated sign, and add up the results to get $\det(A)$.

We now check that this gives the same answer for 3×3 matrices as we wrote down earlier. Consider the matrix
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

- The first entry in the top row is a. The associated minor is $\begin{bmatrix} e & f \\ h & i \end{bmatrix}$, so the cofactor is $\det \begin{bmatrix} e & f \\ h & i \end{bmatrix} = ei - fh$. The associated sign is $+$, so this entry contributes $+a(ei - fh) = aei - afh$ to $\det(A)$.

- The second entry in the top row is b. The associated minor is $\begin{bmatrix} d & f \\ g & i \end{bmatrix}$, so the cofactor is $\det \begin{bmatrix} d & f \\ g & i \end{bmatrix} = di - fg$. The associated sign is $-$, so this entry contributes $-b(di - fg) = -bdi + bfg$ to $\det(A)$.

- The third entry in the top row is c. The associated minor is $\begin{bmatrix} d & e \\ g & h \end{bmatrix}$, so the cofactor is $\det \begin{bmatrix} d & e \\ g & h \end{bmatrix} = dh - eg$. The associated sign is $+$, so this entry contributes $+c(dh - eg) = cdh - ceg$ to $\det(A)$.

Putting these together, we get

$$\det(A) = aei - afh - bdi + bfg + cdh - ceg,$$

just as before.

There are a number of useful properties that make it easier to calculate determinants.

Fact 8.5.3. Let A be a square matrix, and let R be *any* row in A. Take each entry in R, multiply it by its associated sign and its cofactor, and add the results together: then the answer will be the same as $\det(A)$. In other words, we can expand the determinant along any row, not just the top row. □

Fact 8.5.4. Let A be a square matrix, and let C be any column in A. Take each entry in C, multiply it by its associated sign and its cofactor, and add the results together: then the answer will be the same as $\det(A)$. In other words, we can expand the determinant along a column instead of a row. □

If A has some entries that are zero, it makes sense to find a row or column with as many zeros as possible, and expand along that row or column; this can lead to great simplifications in the algebra.

Example 8.5.5. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 0 \\ 8 & 9 & 0 & 0 \\ 10 & 0 & 0 & 0 \end{bmatrix}$$

The most efficient way to calculate $\det(A)$ is to expand along the bottom row. Only the first entry ($= 10$) can contribute anything, as the other entries are zero. The associated sign is $-$, and the minor is the matrix

$$B = \begin{bmatrix} 2 & 3 & 4 \\ 6 & 7 & 0 \\ 9 & 0 & 0 \end{bmatrix}.$$

We thus have $\det(A) = -10 \det(B)$. To calculate $\det(B)$, it is again most efficient to expand along the bottom row. Only the first entry ($= 9$) can contribute anything, the sign is $+$, and the cofactor is $\det \begin{bmatrix} 3 & 4 \\ 7 & 0 \end{bmatrix} = 3 \times 0 - 4 \times 7 = -28$. We thus have $\det(B) = +9 \times (-28) = -252$, and so $\det(A) = -10 \times (-252) = 2520$. □

Fact 8.5.6. Let A be a square matrix, and let B be obtained from A by a single row operation. Then $\det(B)$ is related to $\det(A)$ as follows:

(a) If B is obtained by adding a multiple of one row to a different row, then $\det(A) = \det(B)$.

(b) If B is obtained by exchanging any two rows of A, then $\det(A) = -\det(B)$.

(c) If B is obtained by dividing one row of A by a constant c, then $\det(A) = c\det(B)$.

There are similar rules for column operations. □

As a consequence of this, it is often useful to start by applying some row or column operations to convert A to a simpler matrix B. We can then calculate $\det(B)$ directly, and work backwards using the above rules to find $\det(A)$.

Example 8.5.7. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix}.$$

If we subtract half of the bottom row from each of the other rows, we find obtain the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & 2 \end{bmatrix}.$$

Fact 8.5.6 tells us that $\det(A) = \det(B)$. It is easy to see that

$$\det(B) = 1 \times \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix} = 1 \times 1 \times \det \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} = 2,$$

and we conclude that $\det(A) = 2$ as well. This is much less work than expanding everything directly. □

Example 8.5.8. Consider the matrix $A = \begin{bmatrix} 0 & 0 & p \\ 0 & q & 0 \\ r & 0 & 0 \end{bmatrix}$, where p, q and r are nonzero. If we divide the first row by p, the second row by q and the third row by r, we get the matrix $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. If we then exchange the first and third rows, we get the identity matrix I_3. Rule (c) tells us that $\det(B) = -\det(I) = -1$, and rule (a) (applied three times) tells us that $\det(A) = pqr\det(B)$. It follows that $\det(A) = -pqr$ (which you can easily check directly).

Exercise 8.5.9. Consider the following matrix:

$$A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$

Show that $\det(A) = (a+b+c)(ab+bc+ca-a^2-b^2-c^2)$. (This is an example of a *circulant determinant* .)

Solution: We will do this in two different ways. Firstly, we can expand out the determinant directly:

$$\begin{aligned}\det(A) &= a(cb-a^2) - b(b^2-ca) + c(ba-c^2)\\ &= abc - a^3 - b^3 + abc + abc - c^3\\ &= 3abc - a^3 - b^3 - c^3.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(a+b+c)(ab+bc+ca-a^2-b^2-c^2) &= a^2b + abc + a^2c - a^3 - ab^2 - ac^2 +\\ &\quad ab^2 + b^2c + abc - a^2b - b^3 - bc^2 +\\ &\quad abc + bc^2 + ac^2 - a^2c - b^2c - c^3\\ &= 3abc - a^3 - b^3 - c^3,\end{aligned}$$

which is the same.

For a more cunning approach, let A' be the matrix obtained from A by adding the second and third rows to the top row, so

$$A' = \begin{bmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{bmatrix}$$

We know that this kind of row operation does not change the determinant, so

$$\begin{aligned}\det(A) &= \det(A')\\ &= (a+b+c)(bc-a^2) - (a+b+c)(b^2-ac) + (a+b+c)(ab-c^2)\\ &= (a+b+c)(bc-a^2-b^2+ac+ab-c^2)\\ &= (a+b+c)(ab+bc+ca-a^2-b^2-c^2)\end{aligned}$$

as before.

Exercise 8.5.10. Find the determinant of the following matrix:

$$A = \begin{bmatrix} t & a & b \\ -a & t & c \\ -b & -c & t \end{bmatrix}$$

Show that if a, b, c and t are all real numbers and $t \neq 0$ then $\det(A) \neq 0$.

Solution: First, we have

$$\begin{aligned}\det(A) &= t(t^2 - (-c^2)) - a(-at - (-b)c) + b((-a)(-c) - (-b)t)\\ &= t(t^2+c^2) - a(-at+bc) + b(ac+bt)\\ &= t^3 + a^2t + b^2t + c^2t\\ &= t(t^2+a^2+b^2+c^2).\end{aligned}$$

Now suppose that all the numbers are real and that $t \neq 0$. Then $t^2 > 0$ and $a^2, b^2, c^2 \geq 0$ so $t^2+a^2+b^2+c^2 > 0$. As t and $t^2+a^2+b^2+c^2$ are nonzero, we see that $\det(A)$ is also nonzero.

Exercise 8.5.11. Find the determinant of the following matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

Solution: It is most efficient to expand down the first column. In principle this gives four terms, with the first term multiplied by a_{11} and the remaining terms multiplied by zero. Thus, we need only consider the first term, which gives

$$\det(A) = a_{11} \det \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ 0 & a_{33} & a_{34} \\ 0 & 0 & a_{44} \end{bmatrix}.$$

By expanding out the 3×3 determinant down the first column in the same way, we find that

$$\det \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ 0 & a_{33} & a_{34} \\ 0 & 0 & a_{44} \end{bmatrix} = a_{22} \det \begin{bmatrix} a_{33} & a_{34} \\ 0 & a_{44} \end{bmatrix} = a_{22}a_{33}a_{44}.$$

Putting this together, we conclude that

$$\det(A) = a_{11}a_{22}a_{33}a_{44}.$$

It is easy to see that the same idea works for bigger matrices as well: if all the entries below the diagonal are zero, then the determinant is the product of all the entries on the diagonal.

Exercise 8.5.12*. Consider the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ a & 1 & c & 1 \\ b & a & d & c \\ 0 & b & 0 & d \end{bmatrix}$$

Check that

$$\det(A) = (d-b)^2 + (a-c)(ad-bc).$$

Solution: Expanding along the top row, we get

$$\det(A) = \det \begin{bmatrix} 1 & c & 1 \\ a & d & c \\ b & 0 & d \end{bmatrix} - 0.\det(\dots) + \det \begin{bmatrix} a & 1 & 1 \\ b & a & c \\ 0 & b & d \end{bmatrix} - 0.\det(\dots).$$

We also have

$$\begin{aligned} \det \begin{bmatrix} 1 & c & 1 \\ a & d & c \\ b & 0 & d \end{bmatrix} &= d^2 - c(ad-bc) + (-bd) \\ &= d(d-b) - c(ad-bc) \\ \det \begin{bmatrix} a & 1 & 1 \\ b & a & c \\ 0 & b & d \end{bmatrix} &= a(ad-bc) - bd + b^2 \\ &= -b(d-b) + a(ad-bc). \end{aligned}$$

Putting this together, we get

$$\begin{aligned} \det(A) &= [d(d-b) - c(ad-bc)] + [-b(d-b) + a(ad-bc)] \\ &= d(d-b) - b(d-b) + (a-c)(ad-bc) \\ &= (d-b)^2 + (a-c)(ad-bc) \end{aligned}$$

as required.

Exercise 8.5.13. Suppose that $x^2+y^2+z^2=1$ (make sure that you use this fact to simplify all your answers). Consider the matrix

$$A=\begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}$$

(a) What is $\det(A)$?

(b) What is $\operatorname{trace}(A)$?

(c) What is A^2-A?

Solution:

(a) We have $\det(A)=0$, by the following calculation:

$$\begin{aligned}\det(A)&=x^2(y^2z^2-(yz)^2)-xy(xyz^2-(xz)(yz))+xz((xy)(yz)-xzy^2)\\ &=x^2.0-xy.0+xz.0=0\end{aligned}$$

(b) $\operatorname{trace}(A)=x^2+y^2+z^2=1$.

(c) We first note that

$$\begin{aligned}A^2&=\begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}\begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}\\ &=\begin{bmatrix} x^4+x^2y^2+x^2z^2 & x^3y+xy^3+xyz^2 & x^3z+xy^2z+xz^3 \\ x^3y+xy^3+xyz^2 & x^2y^2+y^4+y^2z^2 & x^2yz+y^3z+yz^3 \\ x^3z+xy^2z+xz^3 & x^2yz+y^3z+yz^3 & x^2z^2+y^2z^2+z^4 \end{bmatrix}\\ &=\begin{bmatrix} (x^2+y^2+z^2)x^2 & (x^2+y^2+z^2)xy & (x^2+y^2+z^2)xz \\ (x^2+y^2+z^2)xy & (x^2+y^2+z^2)y^2 & (x^2+y^2+z^2)yz \\ (x^2+y^2+z^2)xz & (x^2+y^2+z^2)yz & (x^2+y^2+z^2)z^2 \end{bmatrix}\\ &=\begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}=A.\end{aligned}$$

(At the last stage, we have used the fact that $x^2+y^2+z^2=1$). Thus, A^2-A is just the zero matrix.

Finally, we state one more property of the determinant that is very important in a number of applications, although we shall not use it or justify it.

Fact 8.5.14. For any $n\times n$ matrices A and B we have $\det(AB)=\det(A)\det(B)$. □

We proved this for 2×2 matrices in Exercise 2.1.12, by simply expanding everything out. A less direct argument is needed for $n>2$.

8.5.2 The transpose

Given a matrix A, we can form a new matrix A^T by taking the rows of A to be the columns of A^T. For example

$$\left[\begin{array}{ccc} a & b & c \\ \hline d & e & f \end{array}\right]^T=\left[\begin{array}{c|c} a & d \\ b & e \\ c & f \end{array}\right]$$

$$\left[\begin{array}{ccc} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array}\right]^T=\left[\begin{array}{c|c|c} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{array}\right].$$

Exercise 8.5.15. Consider the following matrix (called a *Vandermonde matrix*):

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$$

(a) Check that $\det(A) = (a-b)(b-c)(c-a)$.

(b) Write down the transpose matrix A^T, then calculate AA^T. Can you see the pattern? You may want to introduce some new notation, so that you do not have to write the same expression several times over.

(c) Is AA^T the same as A^TA?

Solution:

(a) First, we have

$$\begin{aligned} \det(A) &= (bc^2 - b^2c) - (ac^2 - a^2c) + (ab^2 - a^2b) \\ &= bc^2 - b^2c - ac^2 + a^2c + ab^2 - a^2b \\ &= -a^2b + a^2c + ab^2 - ac^2 - b^2c + bc^2. \end{aligned}$$

(At the last stage, we have written the terms in "alphabetical order", for ease of comparison.) We also have

$$\begin{aligned} (a-b)(b-c)(c-a) &= (a-b)(bc - ba - c^2 + ca) \\ &= abc - a^2b - ac^2 + a^2c - b^2c + ab^2 + bc^2 - abc \\ &= -a^2b + a^2c + ab^2 - ac^2 - b^2c + bc^2, \end{aligned}$$

which is the same.

(b) We have

$$A^T = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$$

and so

$$AA^T = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = \begin{bmatrix} 3 & a+b+c & a^2+b^2+c^2 \\ a+b+c & a^2+b^2+c^2 & a^3+b^3+c^3 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^4+b^4+c^4 \end{bmatrix}.$$

It is convenient to introduce the notation $p_k = a^k + b^k + c^k$ (so in particular $p_0 = a^0 + b^0 + c^0 = 1 + 1 + 1 = 3$, and $p_1 = a + b + c$). We can then rewrite the answer as

$$AA^T = \begin{bmatrix} p_0 & p_1 & p_2 \\ p_1 & p_2 & p_3 \\ p_2 & p_3 & p_4 \end{bmatrix}.$$

In general, the (i,j)'th entry in AA^T is p_{i+j-2}.

(c) We next find that

$$\begin{aligned} A^TA &= \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} \\ &= \begin{bmatrix} 1+a^2+a^4 & 1+ab+a^2b^2 & 1+ac+a^2c^2 \\ 1+ab+a^2b^2 & 1+b^2+b^4 & 1+bc+b^2c^2 \\ 1+ac+a^2c^2 & 1+bc+b^2c^2 & 1+c^2+c^4. \end{bmatrix} \end{aligned}$$

For particular values of a, b and c, it can happen that $AA^T = A^TA$; it is easy to see that this happens when $a = b = c = 1$, and more generally it happens whenever $a = 1$ and $c = b^2$. (In these cases you can check that A^T is actually equal to A.) However, for typical values of a, b and c, the matrices AA^T and A^TA will be different. For example, when $a = b = c = 0$ we have

$$AA^T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

8.5.3 Inverses

Given an $n \times n$ matrix A, one can sometimes find an *inverse matrix* A^{-1} (also of size $n \times n$) such that $A^{-1}A = I_n$ (where I_n is the identity matrix, as usual). If we know A^{-1}, then it is very easy to solve equations of the form $A\mathbf{x} = \mathbf{c}$; we simply multiply both sides by A^{-1} to see that $\mathbf{x} = A^{-1}\mathbf{c}$.

Example 8.5.16. Consider the system of equations

$$\begin{aligned} 2x + y + 3z &= 3 \\ x + y + 2z &= 4 \\ x + y + 3z &= 5. \end{aligned}$$

This can be rewritten as $A\mathbf{x} = \mathbf{c}$, where

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}.$$

It turns out that A has an inverse matrix, given by

$$A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

(We will see shortly how this can be calculated.) The solution is thus given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix},$$

so $x = -2$, $y = 4$ and $z = 1$. □

Remark 8.5.17. If you just want to solve a single equation $A\mathbf{x} = \mathbf{c}$, then this method may not be efficient, because it takes time to find A^{-1}. However, if you need to solve many different equations of the form $A\mathbf{x} = \mathbf{c}$, with the same matrix A but different $\mathbf{c}$'s, then this is a good approach. You only need to find A^{-1} once, and then it is easy to calculate $\mathbf{x} = A^{-1}\mathbf{c}$ for each of the different $\mathbf{c}$'s.

Just as many functions have no inverse function (as discussed in Section 4.4), many matrices have no inverse matrix. Fortunately, however, there is a straightforward way to check whether an inverse exists.

Fact 8.5.18. Let A be a square matrix. If $\det(A) = 0$, then A does *not* have an inverse. If $\det(A) \neq 0$, then there is an inverse, given by the following procedure:

- Replace each entry in A by its associated cofactor.
- Multiply all the entries in the matrix of cofactors by their associated signs, and then take the transpose, to give a new matrix called the *adjugate* of A, written $\mathrm{adj}(A)$.

- Divide each entry in $\operatorname{adj}(A)$ by $\det(A)$ to get A^{-1}.

Note that you need the cofactors for the top row entries of A in order to calculate $\det(A)$; you should lay out your working in such a way that you avoid calculating these cofactors twice. If you are confident that $\det(A)$ is nonzero then you can just start by calculating $\operatorname{adj}(A)$, and then $\det(A)$ will be the dot product of the first row of A with the first column of $\operatorname{adj}(A)$. However, this will involve some wasted work if $\det(A)$ turns out to be zero. □

Warning 8.5.19. There is horrible confusion in the literature about conflicting uses of the terms "adjugate" and "adjoint". We will not give the details here, but you should be on your guard when reading other books.

Example 8.5.20. For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, this boils down to a simple formula: A has an inverse if and only if $ad - bc \neq 0$, in which case the inverse is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d/(ad-bc) & -b/(ad-bc) \\ -c/(ad-bc) & a/(ad-bc) \end{bmatrix}.$$

In other words, you swap a and d, change the signs of the other two entries, then divide by the determinant.

To see that this is correct, note that the matrix of cofactors is $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$. Multiplying by the associated signs gives $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$, transposing gives $\operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, and dividing by the determinant gives the answer stated above. □

Example 8.5.21. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. It is easy to see that $\det(A) = 0$, so A has no inverse. (Exercise: check that $\det(A) = 0$, either directly or by row operations.) □

Example 8.5.22. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

from Example 8.5.16. The matrix of cofactors is

$$\begin{bmatrix} \det\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} & \det\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} & \det\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ \det\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} & \det\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix} & \det\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \\ \det\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} & \det\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} & \det\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

We now multiply by the associated signs to get

$$\begin{bmatrix} +1 & -1 & +0 \\ -0 & +3 & -1 \\ +(-1) & -1 & +1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix}.$$

We take the transpose to get

$$\operatorname{adj}(A) = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

The determinant of A is the dot product of the first row of A with the first column of $\mathrm{adj}(A)$, which gives

$$\det(A) = (2,1,3).(1,-1,0) = 1.$$

Thus nothing changes when we divide the entries in $\mathrm{adj}(A)$ by $\det(A)$, so

$$A^{-1} = \mathrm{adj}(A) = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

just as we stated in Example 8.5.16. □

Exercise 8.5.23. Find the inverse of the matrix

$$A = \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ -b & 0 & a \end{bmatrix}$$

by the cofactor method.

Solution: The matrix of cofactors is

$$\begin{bmatrix} \det\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} & \det\begin{bmatrix} 0 & 0 \\ -b & a \end{bmatrix} & \det\begin{bmatrix} 0 & 1 \\ -b & 0 \end{bmatrix} \\ \det\begin{bmatrix} 0 & b \\ 0 & a \end{bmatrix} & \det\begin{bmatrix} a & b \\ -b & a \end{bmatrix} & \det\begin{bmatrix} a & 0 \\ -b & 0 \end{bmatrix} \\ \det\begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix} & \det\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} & \det\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a & 0 & b \\ 0 & a^2+b^2 & 0 \\ -b & 0 & a \end{bmatrix}.$$

We should now multiply each entry by the associated sign, but in this example, this has no effect, because the entries whose signs should be changed are all zero already. We then transpose the matrix to get

$$\mathrm{adj}(A) = \begin{bmatrix} a & 0 & -b \\ 0 & a^2+b^2 & 0 \\ b & 0 & a \end{bmatrix}.$$

The determinant is the dot product of the first row of A with the first column of $\mathrm{adj}(A)$, which gives

$$\det(A) = (a,0,b).(a,0,b) = a^2+b^2.$$

Dividing by this, we get

$$A^{-1} = \begin{bmatrix} a/(a^2+b^2) & 0 & -b/(a^2+b^2) \\ 0 & 1 & 0 \\ b/(a^2+b^2) & 0 & a/(a^2+b^2) \end{bmatrix}.$$

Exercise 8.5.24. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

by the cofactor method.

Solution: The matrix of cofactors is

$$\begin{bmatrix} \det\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & \det\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} & \det\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \det\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \\ \det\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} & \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \det\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

We now multiply by the associated signs and transpose to get

$$\operatorname{adj}(A) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The determinant is the dot product of the first row of A with the first column of $\operatorname{adj}(A)$, which gives

$$\det(A) = (1,1,0).(1,0,0) = 1.$$

Thus, it does not change anything to divide by $\det(A)$, and we have

$$A^{-1} = \operatorname{adj}(A) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Although the cofactor description of A^{-1} is interesting and theoretically useful, it turns out not to be very efficient for calculation. We now describe a different approach.

Method 8.5.25. Let A be an $n \times n$ matrix.

(a) Form a new $n \times 2n$ matrix B whose left half is A and whose right half is the identity matrix I_n. For example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad B = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right]$$

We call B the *augmented matrix*.

(b) Row-reduce B to get another matrix, which we call C.

(c) If the left-hand half of C is I_n, then A has an inverse, and A^{-1} is just the right-hand half of C.

(d) If the left-hand half of C is *not* equal to I_n, then A does not have an inverse.

Exercise 8.5.26. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

by the augmented matrix method.

Solution: We write down the augmented matrix and row-reduce it as follows:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right] \xrightarrow{3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

(Step 1 is $R_3 \mapsto R_3 + R_1$, step 2 is $R_3 \mapsto R_3/2$, and step 3 is $R_1 \mapsto R_1 - R_3$.)

At the final stage, the left hand block is I_3, so the right hand block is an inverse for A. The conclusion is that

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Exercise 8.5.27. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

by the augmented matrix method.

Solution: We write down the augmented matrix and row-reduce it as follows:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

(Step 1 is $R_2 \mapsto R_2 - R_3$, and step 2 is $R_1 \mapsto R_1 - R_2$.)

At the final stage, the left hand block is I_3, so the right hand block is an inverse for A. The conclusion is that

$$A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Exercise 8.5.28. Consider the following matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(a) Show that $A^2 = A + 2I$.

(b) Find A^{-1}.

Solution: For part (a), we just calculate directly:

$$A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = A + 2I.$$

It follows that $A(A-I)/2 = (A^2 - A)/2 = I$, so that $(A-I)/2$ is the inverse of A. Explicitly, we have

$$A^{-1} = (A-I)/2 = \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}.$$

One can of course find A^{-1} by the augmented matrix method instead:

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right) \xrightarrow{4} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right)$$

In step 1 we exchange R_1 and R_2, in step 2 we subtract R_1 and R_2 from R_3, in step 3 we multiply R_3 by $-1/2$, and in step 4 we subtract R_3 from R_1 and R_2. The conclusion is that

$$A^{-1} = \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix},$$

as before.

Exercise 8.5.29. Consider the following matrix (called a Jordan block):

$$A = \begin{bmatrix} -1/a & 1 & 0 & 0 \\ 0 & -1/a & 1 & 0 \\ 0 & 0 & -1/a & 1 \\ 0 & 0 & 0 & -1/a \end{bmatrix}$$

(a) Using an earlier exercise, find $\det(A)$.

(b) Find A^{-1}.

Solution:

(a) Using Exercise 8.5.11, we see that $\det(A) = (-1/a)^4 = a^{-4}$.

(b) We now use the augmented matrix method to find A^{-1}:

$$\left(\begin{array}{cccc|cccc} -1/a & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1/a & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1/a & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1/a & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{1} \left(\begin{array}{cccc|cccc} 1 & -a & 0 & 0 & -a & 0 & 0 & 0 \\ 0 & 1 & -a & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & 1 & -a & 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -a \end{array}\right)$$

$$\xrightarrow{2} \left(\begin{array}{cccc|cccc} 1 & 0 & -a^2 & 0 & -a & -a^2 & 0 & 0 \\ 0 & 1 & 0 & -a^2 & 0 & -a & -a^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -a & -a^2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -a \end{array}\right) \xrightarrow{3} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -a & -a^2 & -a^3 & -a^4 \\ 0 & 1 & 0 & 0 & 0 & -a & -a^2 & -a^3 \\ 0 & 0 & 1 & 0 & 0 & 0 & -a & -a^2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -a \end{array}\right)$$

In step 1, we multiply each row by $-a$. In step 2, we add aR_2 to R_1, and aR_3 to R_2, and aR_4 to R_3. Finally, in step 3, we add a^2R_3 to R_1, and a^2R_4 to R_2. The conclusion is that

$$A^{-1} = \begin{bmatrix} -a & -a^2 & -a^3 & -a^4 \\ 0 & -a & -a^2 & -a^3 \\ 0 & 0 & -a & -a^2 \\ 0 & 0 & 0 & -a \end{bmatrix}.$$

Exercise 8.5.30. Find the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Solution: We use the augmented matrix method:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{1} \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{2} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \end{array}\right)$$

In step 1 we subtracted R_2 from R_1, R_3 from R_2, and R_4 from R_3. In step 2, we reversed the order of the rows. The conclusion is that

$$A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

Appendix A: Complex numbers

Many of you will have met complex numbers at A level; almost everyone else will meet them in PMA111 (Numbers and Polynomials). Some topics become much simpler if one uses complex numbers, so we have mentioned them in those contexts, even though we do not treat them systematically. In this appendix we give a very brief summary.

A *complex number* is an expression of the form $z = x + iy$, where x and y are real numbers (called the *real part* and *imaginary part* of z). For example, $2 + \pi i$ is a complex number, with real part 2 and imaginary part π. Note also that any real number is also a special kind of complex number, with imaginary part equal to zero.

Clearly one can add together two complex numbers to get another complex number, for example $(2 + 3i) + (4 + 5i) = 6 + 8i$. Less obviously, we multiply complex numbers together by expanding out and using the rule $i^2 = -1$. This gives

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2\\ &= ac + (ad + bc)i + bd.(-1)\\ &= (ac - bd) + (ad + bc)i.\end{aligned}$$

We can also divide by nonzero complex numbers, using the rule

$$\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i.$$

The *modulus* or *absolute value* of a complex number $z = x + iy$ is the real number $|z| = \sqrt{x^2 + y^2}$. Note that this is never negative, and it is only zero if z is zero. It obeys the rule $|zw| = |z|\,|w|$.

It turns out that if we use complex numbers, then any polynomial can be factored completely as a constant multiplied by some factors of the form $x - \alpha$. For example:

$$x^4 + x^3 + x^2 + x + 1 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4),$$

where $\alpha_1 \approx -0.809017-0.587785i$, $\alpha_2 \approx 0.309017+0.951057i$, $\alpha_3 \approx 0.309017-0.951057i$ and $\alpha_4 \approx -0.809017 + 0.587785i$. This fact turns out to be incredibly useful; it is called the *Fundamental Theorem of Algebra*. There are many different proofs, making contact with many different areas of mathematics.

The other key fact about complex numbers is *De Moivre's Theorem*:

$$\exp(x + iy) = e^{x+iy} = e^x(\cos(x) + \sin(x)i).$$

There are a number of things to say about the meaning and logic of this formula (what does e^{x+iy} actually *mean*, after all?) but for this course we shall just accept it. For example, we have

$$\begin{aligned}e^{\log(2)/2+i\pi/4} &= (e^{\log(2)})^{1/2}(\cos(\pi/4) + \sin(\pi/4)i)\\ &= 2^{1/2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)\\ &= 1 + i.\end{aligned}$$

Appendix B: Maple

Maple is a computer program that can perform a wide range of mathematical operations automatically. For example, you can type in `expand((x+y)^4);`, and Maple will print out $x^4+4x^3y+6x^2y^2+4xy^3+y^4$. Some other examples:

What you type	Meaning	Maple's response
`diff(exp(-x^2),x);`	$\frac{d}{dx}e^{-x^2}$	$-2x\,e^{-x^2}$
`int(ln(x)^3,x);`	$\int \log(x)^3\,dx$	$\ln(x)^3x-3x\ln(x)^2+6x\ln(x)-6x$
`factor(x^8-1);`		$(x-1)(x+1)(x^2+1)(x^4+1)$
`solve(x^2-5*x+6=0,x);`		$2,3$

There are a number of other similar systems such as *Mathematica*, *Macsyma* and *AXIOM*; we concentrate on Maple because it is available at the University of Sheffield under a site license.

We will not use Maple systematically in this course, but you are free to try it out if you wish to. (There is also some teaching of Maple in AMA102 (Applied Mathematics Core)). The best way to start is to open the worksheet `mapleintro.mws`, which contains instructions and a tutorial. If everything is configured correctly, you can do this by visiting `http://www.shef.ac.uk/~puremath/PMA101/mapleintro.mws`. If that does not work, visit `http://www.shef.ac.uk/~puremath/PMA101` and follow the instructions there.

Appendix C: The Greek alphabet

In mathematics and the sciences, greek letters are often used (as well as the usual english alphabet) for variables. The full greek alphabet is shown below; the letters in green are very rarely used.

A	α	alpha	N	ν	nu
B	β	beta	Ξ	ξ	xi
Γ	γ	gamma	O	o	omicron
Δ	δ	delta	Π	π	pi
E	ϵ	epsilon	P	ρ	rho
Z	ζ	zeta	Σ	σ	sigma
H	η	eta	T	τ	tau
Θ	θ	theta	Υ	υ	upsilon
I	ι	iota	Φ	ϕ	phi
K	κ	kappa	X	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
M	μ	mu	Ω	ω	omega