Vector spaces and Fourier theory
Exam questions

(1) [Mock exam Q1]

(a) Let U be a finite-dimensional vector space, and let V and W be subspaces of V. Prove that

$$\dim(V+W) = \dim(V) + \dim(W) - \dim(V \cap W).$$

If you use a result about the existence of certain bases for V, W and $V \cap W$ then you should prove it. Other results may be quoted without proof. **(13 marks)**

(b) State the rank-nullity formula. **(5 marks)**

(c) Suppose that $U = V + W$ and that $\phi\colon U \to Z$ is a surjective linear map with $\ker(\phi) = V$. Suppose that $\dim(V) = 5$, $\dim(W) = 4$ and $\dim(V \cap W) = 3$. What is $\dim(Z)$? **(7 marks)**

Solution:

(a) We claim that there exist elements

$$u_1, \ldots, u_p, v_1, \ldots, v_q, w_1, \ldots, w_r$$

such that

- $u_1, \ldots, u_p$ is a basis for $V \cap W$
- $u_1, \ldots, v_1, \ldots, v_q$ is a basis for V
- $u_1, \ldots, w_1, \ldots, w_r$ is a basis for W
- $u_1, \ldots, v_1, \ldots, v_q, w_1, \ldots, w_r$ is a basis for $V + W$.

[**2**] Assuming this, we have $\dim(V \cap W) = p$ and $\dim(V) = p + q$ and $\dim(W) = p + r$ and

$$\dim(V+W) = p+q+r = (p+q)+(p+r)-p = \dim(V) + \dim(W) - \dim(V \cap W),$$

as required [**2**]. To prove the claim, we first choose a basis $u_1, \ldots, u_p$ for $V \cap W$ [**1**]. This list is then a linearly independent list in V, so can be extended to a basis $u_1, \ldots, u_p, v_1, \ldots, v_r$ for V [**1**]. Similarly, the list $u_1, \ldots, u_p$ is a linearly independent list in W, so it can be extended to a basis $u_1, \ldots, u_p, w_1, \ldots, w_r$ for W [**1**]. All that is left is to prove that the list $\mathcal{X} = u_1, \ldots, v_1, \ldots, v_q, w_1, \ldots, w_r$ is a basis for $V+W$ [**1**]. It is clear that $\mathcal{X} \subseteq V \cup W \subseteq V+W$ so $\text{span}(\mathcal{X}) \subseteq V + W$. Consider an element $x \in V + W$. We can then find $y \in V$ and $z \in W$ such that $x = y + z$. As $y \in V$ and $u_1, \ldots, u_p, v_1, \ldots, v_r$ is a basis for V, we have $y = \sum_i \lambda_i u_i + \sum_j \mu_j v_j$ for some constants $\lambda_i, \mu_j \in \mathbb{R}$. As $z \in W$ and $u_1, \ldots, u_p, w_1, \ldots, w_r$ is a basis for W, we have $z = \sum_i \lambda'_i u_i + \sum_k \nu_k w_k$ for some constants $\lambda'_i, \nu_k \in \mathbb{R}$. It follows that

$$x = y + z = \sum_i (\lambda_i + \lambda'_i) u_i + \sum_j \mu_j v_j + \sum_k w_k \in \text{span}(\mathcal{X}).$$

This proves that $V \subseteq \text{span}(\mathcal{X})$, so $\mathcal{X}$ spans V. Now suppose we have a linear relation

$$\sum_i \lambda_i u_i + \sum_j \mu_j v_j + \sum_k \nu_k w_k = 0.$$

Put $y = \sum_i \lambda_i u_i + \sum_j \mu_j v_j$ and $z = \sum_k \nu_k w_k$. It is clear that $y \in V$ and $z \in W$ but $z = -y$ so $z \in V \cap W$. We also know that $u_1, \dots, u_p$ is a basis for $V \cap W$, so $z = \sum_i \lambda'_i u_i = 0$. This means that

$$\sum_i \lambda'_i u_i + \sum_k \nu_k w_k = 0.$$

On the other hand, we know that the list $u_1, \dots, u_p, w_1, \dots, w_r$ is a basis for W and so has no nontrivial linear relations, so

$$\lambda'_1 = \cdots \lambda'_p = \nu_1 = \cdots \nu_r = 0.$$

This means that $z = 0$ but $y = -z$ so $y = z$, so

$$\sum_i \lambda_i u_i + \sum_j \mu_j v_j = 0.$$

This list $u_1, \dots, u_p, v_1, \dots, v_q$ is a basis for V and so has no nontrivial linear relations, so

$$\lambda_1 = \cdots = \lambda_p = \mu_1 = \cdots = \mu_q = 0.$$

This means that our original relation

$$\sum_i \lambda_i u_i + \sum_j \mu_j v_j + \sum_k \nu_k w_k = 0.$$

is the trivial relation. This means that the list $\mathcal{X}$ is linearly independent and so is a basis of $V + W$, as claimed.

(b) Let V and W be finite-dimensional vector spaces, and let $\phi\colon V \to W$ be a linear map. Then

$$\dim(V) = \dim(\ker(\phi)) + \dim(\mathrm{image}(\phi)).$$

(c) Part (a) gives

$$\dim(U) = \dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W) = 5 + 4 - 3 = 6.$$

As ϕ is surjective we have $\mathrm{image}(\phi) = Z$, and we are also given that $\ker(\phi) = V$. The rank-nullity formula therefore gives

$$6 = \dim(U) = \dim(\mathrm{image}(\phi)) + \dim(\ker(\phi)) = \dim(Z) + 5,$$

which gives $\dim(Z) = 1$.

(2) [0506 Q1]

(a) Let V and W be vector spaces over $\mathbb{R}$. Define what it means for a map $\alpha\colon V \to W$ to be linear. **(3 marks)**

(b) Which of the following maps are linear? Justify your answers briefly, giving specific counterexamples where appropriate. **(9 marks)**

(i) $\phi\colon M_2\mathbb{R} \to \mathbb{R}$, $\phi(A) = [1, 1]A \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$.

(ii) $\psi\colon \mathbb{R}[x] \to M_2\mathbb{R}$, $\psi(f) = \left[\begin{smallmatrix} f(0) & f(1) \\ f'(0) & f'(1)\end{smallmatrix}\right]$.

(iii) $\chi\colon M_2\mathbb{R} \to M_2\mathbb{R}$, $\chi(A) = A^T A$.

(iv) $\theta\colon \mathbb{R}^2 \to \mathbb{R}[x]$, $\theta \left[\begin{smallmatrix}a\\b\end{smallmatrix}\right] = ax^2 + b(1 - x)^2$.

(c) Consider the map $\alpha\colon M_2\mathbb{R} \to M_2\mathbb{R}$ given by $\alpha(A) = A^T - \mathrm{trace}(A)I$.

(i) Give a formula for $\alpha\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. **(2 marks)**

(ii) Give bases for the spaces

$$V = \{A \in M_2\mathbb{R} \mid \alpha(A) = A\}$$
$$W = \{A \in M_2\mathbb{R} \mid \alpha(A) = -A\}$$

(11 marks)

Solution:

(a) **Bookwork.** A map $\alpha\colon V \to W$ is linear iff for all $v, v' \in V$ and all $t, t' \in \mathbb{R}$ we have $\alpha(tv + t'v') = t\alpha(v) + t'\alpha(v')$. **[3]**

(b) **Similar to problem sheets**

(i) We have
$$\phi\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = [1, 1]\left[\begin{smallmatrix} a+b \\ c+d \end{smallmatrix}\right] = a + b + c + d.$$
This is linear **[1]**because if $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and $A' = \left[\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right]$ then

$$\phi(tA + t'A') = \phi\left[\begin{smallmatrix} ta+t'a' & tb+t'b' \\ tc+t'c' & td+t'd' \end{smallmatrix}\right] = ta + t'a' + tb + t'b' + tc + t'c' + td + t'd'$$
$$= t(a + b + c + d) + t'(a' + b' + c' + d') = t\phi(A) + t'\phi(A').\textbf{[1]}$$

(ii) The map ψ is also linear **[1]**, because if $f, g \in \mathbb{R}[x]$ and $s, t \in \mathbb{R}$ then

$$\psi(sf + tg) = \left[\begin{smallmatrix} sf(0)+tg(0) & sf(1)+tg(1) \\ sf'(0)+tg'(0) & sf'(1)+tg'(1) \end{smallmatrix}\right] = s\psi(f) + t\psi(g).\textbf{[1]}$$

(iii) The map χ is not linear **[1]**, because $\chi(I) = I$ and $\chi(-I) = I$, so $\chi((-1).I) \neq (-1).\chi(I)$. **[2]**

(iv) The map θ is linear **[1]**because

$$\theta\left(t\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] + t'\left[\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right]\right) = \theta\left[\begin{smallmatrix} ta+t'a' \\ tb+t'b' \end{smallmatrix}\right] = (ta + t'a')x^2 + (tb + t'b')(1 - x)^2$$
$$= t(ax^2 + b(1 - x)^2) + t'(a'x^2 + b'(1 - x)^2) = t\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] + t'\theta\left[\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right].\textbf{[1]}$$

(c) **Similar to problem sheets**

(i) $\alpha\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right] - (a + d)\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} -d & c \\ b & -a \end{smallmatrix}\right]$. **[2]**

(ii) Consider a matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. We have $\alpha(A) = A$ iff $-d = a$ and $c = b$ and $b = c$ and $d = -a$, **[2]** which means that A has the form

$$A = \left[\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right] = a\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] + b\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right].\textbf{[2]}$$

It follows that the list $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ is a basis for V **[2]**. Similarly, we have $\alpha(A) = -A$ iff $-d = -a$ and $c = -b$ and $b = -c$ and $-a = -d$ **[2]**, which means that A has the form

$$A = \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right] = a\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] + b\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right].\textbf{[1]}$$

It follows that the list $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ is a basis for W.**[2]**

(3) [0506R Q1]

(a) Let V be a finite-dimensional vector space over $\mathbb{R}$. Define what it means to say that W is a subspace of V. **(3 marks)**

(b) Let W_0 and W_1 be subspaces of V. State a formula relating $\dim(W_0+W_1)$ to the dimensions of various other spaces. **(3 marks)**

(c) In which of the following cases is W a subspace of V? Justify your answers briefly, giving specific counterexamples where appropriate. **(9 marks)**

(i) $V = M_2\mathbb{R}$, $W = \{A \in M_2\mathbb{R} \mid \det(A) = 0\}$.

(ii) $V = M_2\mathbb{R}$, $W = \{A \in M_2\mathbb{R} \mid \operatorname{trace}(A) \geq 0\}$.

(iii) $V = \mathbb{R}[x]_{\leq 3}$, $W = \{f \in V \mid f(1)^2 + f(-1)^2 = 0\}$.

(iv) $V = \mathbb{R}^3$, $W = \{v \in V \mid Av = 0\}$, where $A = \left[\begin{smallmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{smallmatrix}\right]$.

(d) Put $u = [1, 1, 1]^T$ and

$$\begin{aligned} U &= \{A \in M_3\mathbb{R} \mid Au = 0\} \\ V &= \{A \in M_3\mathbb{R} \mid A^T = A\} \\ W &= \{A \in M_3\mathbb{R} \mid A^T = -A\}. \end{aligned}$$

Find bases for $V \cap U$ and $W \cap U$. **(10 marks)**

Solution:

(a) **Bookwork.** A subspace of V is a subset $W \subseteq V$ such that (i) $0_V \in W$ (ii) for all $w, w' \in W$ we have $w + w' \in W$ and (iii) for all $w \in W$ and $t \in \mathbb{R}$ we have $tw \in W$. (Equivalently, one can combine (ii) and (iii) into the following condition: if $w, w' \in W$ and $t, t' \in \mathbb{R}$ then $tw + t'w' \in W$.) **[3]**

(b) $\dim(W_0 + W_1) = \dim(W_0) + \dim(W_1) - \dim(W_0 \cap W_1)$. **[3]**

(c) **Similar to problem sheets**

(i) This is not a subspace **[1]**, because $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] \in W$ but $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] \notin W$. **[1]**

(ii) This is not a subspace **[1]**, because $I \in W$ but $(-1).I \notin W$. **[1]**

(iii) This is a subspace **[1]**. To see this, we first note that $W = \{f \in V \mid f(1) = f(-1) = 0\}$ **[1]**. If $f, g \in W$ and $s, t \in \mathbb{R}$ and $h = sf + tg$ then $h(\pm 1) = sf(\pm 1) + tg(\pm 1) = s.0 + t.0 = 0$, so $h \in W$. It is also clear that $0 \in W$, so W is a subspace as claimed. **[1]**

(iv) This is a subspace **[1]**. Indeed, it is clear that $A0 = 0$, so $0 \in W$. Moreover, if $u, v \in W$ (so $Au = Av = 0$) and $s, t \in \mathbb{R}$ then $A(su + tv) = sAu + tAv = s.0 + t.0 = 0$, so $su + tv \in W$. This means that W is closed under taking linear combinations, so it is a subspace. **[1]**

(d) **Similar to problem sheets** A typical element of V has the form

$$A = \left[\begin{smallmatrix} a & b & c \\ b & d & e \\ c & e & f \end{smallmatrix}\right] \textbf{ [1] } \text{and so } Au = \left[\begin{smallmatrix} a+b+c \\ b+d+e \\ c+e+f \end{smallmatrix}\right]. \textbf{[1]}$$

Thus A lies in $V \cap U$ iff $c = -a - b$ and $e = -b - d$ and $f = -c - e = a + 2b + d$ **[1]**, in which case A has the form

$$A = \left[\begin{smallmatrix} a & b & -a-b \\ b & d & -b-d \\ -a-b & -b-d & a+2b+d \end{smallmatrix}\right] = a \left[\begin{smallmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{smallmatrix}\right] + b \left[\begin{smallmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 2 \end{smallmatrix}\right] + d \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{smallmatrix}\right]. \textbf{[1]}$$

From this it is clear that the matrices

$$A_1 = \left[\begin{smallmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{smallmatrix}\right], \qquad A_2 = \left[\begin{smallmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 2 \end{smallmatrix}\right], \qquad A_3 = \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{smallmatrix}\right]$$

give a basis for $V \cap U$. **[1]**

Similarly, a typical element of W has the form

$$B = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \textbf{[1]}, \text{ and so } Bu = \begin{bmatrix} a+b \\ c-a \\ -b-c \end{bmatrix}. \textbf{[1]}$$

Thus B lies in $W \cap U$ iff $b = -a$ and $c = a$ (and $-b - c = 0$, which follows automatically from the other two equations) **[1]**. If so then $B = aB_1$, where

$$B_1 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}. \textbf{[1]}$$

This means that B_1 is a basis for $W \cap U$.**[1]**

(4) [0607 Q1] Let V and W be vector spaces over $\mathbb{R}$.

(a) Define what it means for a map $\alpha\colon V \to W$ to be linear. **(3 marks)**

(b) Define what it means for a subset $U \subseteq V$ to be a subspace. **(3 marks)**

(c) Suppose that α is linear. Define the kernel of α, and prove that it is a subspace of V. **(5 marks)**

(d) Which of the following functions are linear? (You should justify your answers briefly.) **(6 marks)**

- (i) $\rho\colon M_2(\mathbb{R}) \to \mathbb{R}$ given by $\rho(A) = \text{trace}(A^2)$
- (ii) $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $\sigma \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y^3/(x^2+y^2) \\ x^3/(x^2+y^2) \end{bmatrix}$
- (iii) $\tau\colon M_2(\mathbb{R}) \to M_2(\mathbb{R})$ given by $\tau(A) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(e) Which of the following sets is a subspace? (You should justify your answers briefly.) **(8 marks)**

- (i) $U_0 = \{f \in \mathbb{R}[x] \mid f(1) \leq f(2) \leq f(3)\}$
- (ii) $U_1 = \{A \in M_2(\mathbb{R}) \mid \det(A + I) = \det(A - I)\}$
- (iii) $U_2 = \{A \in M_2(\mathbb{R}) \mid \text{trace}(A^2) = 0\}$
  (**Hint:** which elements of the standard basis for $M_2(\mathbb{R})$ lie in U_2?)

Solution:

(a) **Bookwork.** α is linear if $\alpha(tv + t'v') = t\alpha(v) + t'\alpha(v')$ for all $t, t' \in \mathbb{R}$ and $v, v' \in V$. **[3]**

(b) **Bookwork.** U is a subspace if (i) $0 \in U$ and (ii) for all $t, t' \in \mathbb{R}$ and all $u, u' \in U$ we have $tu + t'u' \in U$. **[3]**

(c) **Bookwork.** The kernel of α is the set $\{u \in V \mid \alpha(u) = 0\}$.**[1]**As α is linear we have $\alpha(0) = 0$, so $0 \in \ker(\alpha)$. **[1]**Now suppose that $t, t' \in \mathbb{R}$ and $u, u' \in \ker(\alpha)$. We then have $\alpha(u) = 0 = \alpha(u')$ **[1]**and also

$$\alpha(tu + t'u') = t\alpha(u) + t'\alpha(u') = t.0 + t'.0 = 0,$$

so $tu + t'u' \in \ker(\alpha)$. **[2]** This shows that $\ker(\alpha)$ is a subspace.

(d) **Similar to problem sheets**

- (i) We have $\rho(I) = 2 = \rho(-I)$, so $\rho(-I) \neq -\rho(I)$, so ρ is not linear. **[2]**

(ii) We have

$$\begin{aligned}\sigma([1,0]^T) &= [0^3/(1^2+0^2), 1^3/(1^2+0^2)]^T = [0,1]^T \\ \sigma([0,1]^T) &= [1^3/(1^2+0^2), 0^3/(1^2+0^2)]^T = [1,0]^T \\ \sigma([1,1]^T) &= [1^3/(1^2+1^2), 1^3/(1^2+1^2)]^T = [1/2,1/2]^T\end{aligned}$$

so $\sigma(\mathbf{e}_1+\mathbf{e}_2) \neq \sigma(\mathbf{e}_1)+\sigma(\mathbf{e}_2)$. Thus, σ is not linear. **[2]**

(iii) Put $Q = \left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$ for brevity. We have

$$\tau(tA+t'A') = Q(tA+t'A')Q = QtAQ+Qt'A'Q = tQAQ+t'QA'Q = t\tau(A)+t'\tau(A'),$$

so τ is linear. **[2]**

(e) **Similar to problem sheets**

(i) U_0 is not a subspace, because the function $f(x)=x$ lies in U_0, but $-f$ does not. **[2]**

(ii) Given a matrix $A = \left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]$ we have

$$\begin{aligned}\det(A+I) &= \det\left[\begin{smallmatrix}a+1&b\\c&d+1\end{smallmatrix}\right] = ad+a+d+1-bc \\ \det(A-I) &= \det\left[\begin{smallmatrix}a-1&b\\c&d-1\end{smallmatrix}\right] = ad-a-d+1-bc \\ \det(A+I)-\det(A-I) &= 2(a+d).\end{aligned}$$

This means that A lies in U_1 iff $a+d=0$ iff A has the form $\left[\begin{smallmatrix}a&b\\c&-a\end{smallmatrix}\right]$. Given this, it is clear that U_1 is a subspace. **[3]**

(iii) Given a matrix $A = \left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]$ we have $A^2 = \left[\begin{smallmatrix}a^2+bc&b(a+d)\\c(a+d)&d^2+bc\end{smallmatrix}\right]$, so $\mathrm{trace}(A^2) = a^2+d^2+2bc$. Using this we see that the matrices $\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]$ lie in U_2, but their sum does not. Thus, U_2 is not a subspace. **[3]**

(5) [0607R Q1] Let V and W be vector spaces over $\mathbb{R}$.

(a) Define what it means for a map $\alpha\colon V\to W$ to be linear. **(3 marks)**

(b) Define what it means for a subset $U\subseteq V$ to be a subspace of V. **(3 marks)**

(c) Suppose that α is linear. Define the image of α, and prove that it is a subspace of V. **(7 marks)**

(d) Which of the following functions are linear? (You should justify your answers.) **(6 marks)**

(i) $\rho\colon M_2(\mathbb{R})\to M_2(\mathbb{R})$ given by $\rho(A) = A^TA$

(ii) $\sigma\colon \mathbb{R}^2\to\mathbb{R}^2$ given by $\sigma\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] = \left[\begin{smallmatrix}x+y^2\\x^2+y\end{smallmatrix}\right]$

(iii) $\tau\colon \mathbb{R}[x]\to M_2(\mathbb{R})$ given by $\tau(f) = \left[\begin{smallmatrix}0&f(1)\\f'(1)&0\end{smallmatrix}\right]$

(e) Which of the following sets is a subspace? (You should justify your answers.) **(6 marks)**

(i) $U_0 = \{f\in\mathbb{R}[x] \mid f(1)=f(2)=f(3)\}$

(ii) $U_1 = \{A\in M_2(\mathbb{R}) \mid \mathrm{trace}(A)\geq 0\}$

(iii) $U_2 = \{A\in M_2(\mathbb{R}) \mid \det(A)=0\}$

Solution:

(a) **Bookwork.** α is linear if $\alpha(tv+t'v') = t\alpha(v)+t'\alpha(v')$ for all $t,t'\in\mathbb{R}$ and $v,v'\in V$. **[3]**

(b) **Bookwork.** U is a subspace if (i) $0 \in U$ and (ii) for all $t, t' \in \mathbb{R}$ and all $u, u' \in U$ we have $tu + t'u' \in U$. **[3]**

(c) **Bookwork.** The image of α is

$$\text{image}(\alpha) = \{\alpha(v) \mid v \in V\} = \{w \in W \mid w = \alpha(v) \text{ for some } v \in V\}.\mathbf{[2]}$$

As α is linear we have $0_W = \alpha(0_V)$, so $0_W \in \text{image}(\alpha)$. **[1]**

Now suppose we have elements $w, w' \in \text{image}(\alpha)$, and real numbers $t, t' \in \mathbb{R}$; we must show that the element $w'' = tw + t'w'$ lies in image(α). **[1]**By the definition of image(α), there must exist elements $v, v' \in V$ with $\alpha(v) = w$ and $\alpha(v') = w'$ **[1]**. Put $v'' = tv + t'v' \in V$. As α is linear we have

$$\alpha(v'') = t\alpha(v) + t'\alpha(v') = tw + t'w' = w''.\mathbf{[1]}$$

Thus w'' is α(something in V), so $w'' \in \text{image}(\alpha)$ as required. This shows that image(α) is a subspace of W. **[1]**

(d) **Similar to problem sheets and the June exam**

(i) We have $\rho(I) = I = \rho(-I)$, so $\rho(-I) \neq -\rho(I)$, so ρ is not linear. **[2]**

(ii) We have

$$\sigma([1,0]^T) = [1,1]^T$$
$$\sigma(-[1,0]^T) = [-1,1]^T \neq -\sigma([1,0]^T)$$

so σ is not linear. **[2]**

(iii) We have

$$\tau(sf+tg) = \begin{bmatrix} 0 & (sf+tg)(1) \\ (sf+tg)'(1) & 0 \end{bmatrix} = \begin{bmatrix} 0 & sf(1)+tg(1) \\ sf'(1)+tg'(1) & 0 \end{bmatrix}$$
$$= s\begin{bmatrix} 0 & f(1) \\ f'(1) & 0 \end{bmatrix} + t\begin{bmatrix} 0 & g(1) \\ g'(1) & 0 \end{bmatrix} = s\tau(f) + t\tau(g).$$

Thus, τ is linear. **[2]**

(e) **Similar to problem sheets and the June exam**

(i) U_0 is a subspace. Indeed, if $f, g \in U_0$ and $s, t \in \mathbb{R}$, then $f(1) = f(2) = f(3)$ (as $f \in U_0$) and $g(1) = g(2) = g(3)$ (as $g \in U_0$) so

$$sf(1) + tg(1) = sf(2) + tg(2) = sf(3) + tg(3),$$

or in other words $sf + tg \in U_0$. **[2]**

(ii) The set U_1 contains I but not $-I$, so it is not a subspace. **[2]**

(iii) The set U_2 contains the matrices $E_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $E_4 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ but not $E_1 + E_4$, so it is not a subspace. **[2]**

(6) [0708 Q1] In this question, X denotes the matrix $\left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$.

(a) Let V and W be vector spaces. Define what it means for a map $\phi\colon V \to W$ to be linear. **(3 marks)**

(b) Which of the following maps are linear? Justify your answers. **(8 marks)**

(i) $\phi_1 \colon M_2(\mathbb{R}) \to M_2(\mathbb{R})$ given by $\phi_1(A) = XAX$.

(ii) $\phi_2 \colon M_2(\mathbb{R}) \to M_2(\mathbb{R})$ given by $\phi_2(A) = AXA$.

(iii) $\phi_3\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}[x]_{\leq 4}$ given by $\phi_3(f(x)) = f(x)^2$.

(iv) $\phi_4\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}[x]_{\leq 4}$ given by $\phi_4(f(x)) = f(x^2)$ (eg $\phi_3(3x^2 + 4x + 5) = 3x^4 + 4x^2 + 5$).

(c) Define what it means for a linear map $\phi\colon V \to W$ to be (i) injective; (ii) surjective. **(5 marks)**

(d) Which of the following maps are injective? Justify your answers. **(9 marks)**

(i) $\psi_1\colon M_2(\mathbb{R}) \to M_2(\mathbb{R})$ given by $\psi_1(A) = XA - AX$.

(ii) $\psi_2\colon M_2(\mathbb{R}) \to M_2(\mathbb{R})$ given by $\psi_2(A) = XA$.

(iii) $\psi_3\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}^3$ given by $\psi_3(f(x)) = [f''(0), f'(1), f(2)]^T$.

Solution:

(a) ϕ is linear iff $\phi(tv + t'v') = t\phi(v) + t'\phi(v')$ for all $t, t' \in \mathbb{R}$ and $v, v' \in V$. [**3**] **Bookwork**

(b) **Similar to problem sheets, lecture notes and past papers**

(i) ϕ_1 is linear [**1**], because $\phi_1(tA + t'A') = X(tA + t'A')X = tXAX + t'XA'X = t\phi_1(A) + t'\phi_1(A')$ [**1**].

(ii) ϕ_2 is not linear [**1**], because $\phi_2(I) = IXI = X$ and $\phi_2(-I) = (-I)X(-I) = X \neq -\phi_2(I)$ [**1**].

(iii) ϕ_3 is not linear [**1**], because $\phi_3(-1) = \phi_3(1) = 1$, so $\phi_3(-1) \neq -\phi_3(1)$ [**1**].

(iv) ϕ_4 is linear [**1**]. To see this, suppose we have $f, g \in \mathbb{R}[x]_{\leq 2}$, say $f(x) = ax^2 + bx + x$ and $g(x) = px^2 + qx + r$, and $s, t \in \mathbb{R}$. Then

$$\begin{aligned}
\phi_4(sf(x) + tg(x)) &= \phi_3((sa + tp)x^2 + (sb + tq)x + (sc + tr)) \\
&= (sa + tp)x^4 + (sb + tq)x^2 + (sc + tr) \\
&= s(ax^4 + bx^2 + c) + t(px^4 + qx^2 + r) \\
&= s\phi_4(f(x)) + t\phi_4(g(x)).[\mathbf{1}]
\end{aligned}$$

(c) A linear map $\phi\colon V \to W$ is said to be *injective* if whenever $v, v' \in V$ and $\phi(v) = \phi(v')$ we have $v = v'$. [**3**] It is *surjective* if for each $w \in W$ there exists $v \in V$ with $\phi(v) = w$. [**2**] **Bookwork**

(d) **Similar to problem sheets, lecture notes and past papers**

(i) We have $\psi_1(I) = XI - IX = X - X = 0$, so $I \in \ker(\psi_1)$, so $\ker(\psi_1) \neq 0$, so ψ_1 is not injective [**3**]. Alternatively, we have $\psi_1(X) = X^2 - X^2 = 0$ and we can argue in the same way. For a more pedestrian approach, we have

$$\psi_1 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix} - \begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix} = \begin{bmatrix} 2c-3b & -2a-3b+2d \\ 3a+3c-3d & 3b-2c \end{bmatrix}.$$

This vanishes iff $2c - 3b = -2a - 3b + 2d = 3a + 3c - 3d = 3b - 2c = 0$, and these equations reduce to $c = 3b/2$ and $d = 3b/2 + a$, so

$$\ker(\psi_1) = \left\{ \begin{bmatrix} a & b \\ 3b/2 & 3b/2+a \end{bmatrix} \mid a, b \in \mathbb{R} \right\} \neq 0.$$

(ii) Note that $\det(X) = -2 \neq 0$ so X is invertible. If $\psi_2(A) = XA = 0$ then $A = X^{-1}XA = X^{-1}.0 = 0$, so $\ker(\psi_2) = 0$, so ψ_2 is injective [**3**]. For a more pedestrian approach, we have

$$\psi_2 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix}.$$

This vanishes iff $a+2c = 0$ (1) and $b+2d = 0$ (2) and $3a+4c = 0$ (3) and $3b+4d = 0$ (4). The equations (3) $-$ 2.(1) and (4) $-$ 2.(2) give $a = b = 0$, and by substituting these in (1) and (3) we get $c = d = 0$. This shows that $\ker(\psi_2) = 0$ as required.

(iii) We have $\psi_3(ax^2+bx+c) = [2a, 2a+b, 4a+2b+c]^T$. This can only be zero if $2a = 2a+b = 4a+2b+c = 0$, which easily implies that $a = b = c = 0$. Thus $\ker(\psi_3) = 0$ and ψ_3 is injective. **[3]**

(7) [0708R Q1]

(a) Let V be a finite-dimensional vector space over $\mathbb{R}$. Define what it means to say that W is a subspace of V. **(3 marks)**

(b) Let W_0 and W_1 be subspaces of V. State a formula relating $\dim(W_0+W_1)$ to the dimensions of various other spaces. **(3 marks)**

(c) In which of the following cases is W a subspace of V? Justify your answers briefly, giving specific counterexamples where appropriate. **(9 marks)**

(i) $V = M_2(\mathbb{R})$, $W = \{A \in M_2(\mathbb{R}) \mid \det(A) \geq 0\}$.

(ii) $V = M_2(\mathbb{R})$, $W = \{A \in M_2(\mathbb{R}) \mid \text{trace}(A) = 0\}$.

(iii) $V = \mathbb{R}[x]_{\leq 3}$, $W = \{f \in V \mid f(0)^4 + f(1)^4 + f(2)^4 = 0\}$.

(iv) $V = \mathbb{R}^3$, $W = \{v \in V \mid Av = 0\}$, where $A = \left[\begin{smallmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{smallmatrix}\right]$.

(d) Put $u = [1, 0, 1]$ and

$$\begin{aligned} U &= \{A \in M_3(\mathbb{R}) \mid uA = 0\} \\ V &= \{A \in M_3(\mathbb{R}) \mid A^T = A\} \\ W &= \{A \in M_3(\mathbb{R}) \mid A^T = -A\}. \end{aligned}$$

Find bases for $V \cap U$ and $W \cap U$. **(10 marks)**

Solution: This is a slight modification of a question from a past paper.

(a) **Bookwork.** A subspace of V is a subset $W \subseteq V$ such that (i) $0_V \in W$ (ii) for all $w, w' \in W$ we have $w + w' \in W$ and (iii) for all $w \in W$ and $t \in \mathbb{R}$ we have $tw \in W$. (Equivalently, one can combine (ii) and (iii) into the following condition: if $w, w' \in W$ and $t, t' \in \mathbb{R}$ then $tw + t'w' \in W$.) **[3]**

(b) $\dim(W_0 + W_1) = \dim(W_0) + \dim(W_1) - \dim(W_0 \cap W_1)$. **[3]**

(c) **Similar to problem sheets**

(i) This is not a subspace **[1]**, because $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right] \in W$ but $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right] \notin W$. **[1]**

(ii) This is a subspace **[1]**, because if $A, A' \in W$ and $t, t' \in \mathbb{R}$ then $\text{trace}(A) = \text{trace}(A') = 0$ so $\text{trace}(tA + t'A') = t\,\text{trace}(A) + t'\,\text{trace}(A') = t.0 + t'.0 = 0$, so $tA + t'A' \in W$.**[1]**

(iii) This is a subspace **[1]**. Indeed, as $f(n)^4 \geq 0$ for all n, we can only have $f(0)^4 + f(1)^4 + f(2)^4 = 0$ if $f(0) = f(1) = f(2) = 0$. Thus $W = \{f \in V \mid f(0) = f(1) = f(2) = 0\}$ **[1]**, and this is easily seen to be a subspace **[1]**.

(iv) This is a subspace **[1]**. Indeed, it is clear that $A0 = 0$, so $0 \in W$. Moreover, if $u, v \in W$ (so $Au = Av = 0$) and $s, t \in \mathbb{R}$ then $A(su + tv) = sAu + tAv = s.0 + t.0 = 0$, so $su + tv \in W$. This means that W is closed under taking linear combinations, so it is a subspace. **[1]**

(d) **Similar to problem sheets** A typical element of V has the form

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} [\mathbf{1}] \text{ and so } uA = [a+c,\ b+e,\ c+f][\mathbf{1}].$$

Thus A lies in $V \cap U$ iff $a = -c$ and $e = -b$ and $f = -c$ [**1**], in which case A has the form

$$A = \begin{bmatrix} -c & b & c \\ b & d & -b \\ c & -b & -c \end{bmatrix} = b \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} + c \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.[\mathbf{1}]$$

From this it is clear that the matrices

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

give a basis for $V \cap U$. [**1**]

Similarly, a typical element of W has the form

$$B = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} [\mathbf{1}], \text{ and so } uB = \begin{bmatrix} -b \\ a-c \\ b \end{bmatrix}.[\mathbf{1}]$$

Thus B lies in $W \cap U$ iff $b = 0$ and $c = a$ [**1**]. If so then $B = aB_1$, where

$$B_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.[\mathbf{1}]$$

This means that B_1 is a basis for $W \cap U$.[**1**]

(8) [**Mock exam Q2**] Let V be a finite-dimensional vector space over $\mathbb{R}$.

(a) Define what is meant by an *inner product* on V.

(b) State and prove the Cauchy-Schwartz inequality. (You need not discuss the case where it is actually an equality.)

(c) Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Prove that

$$\left(\int_0^1 x f(x)\, dx\right)^2 \leq \frac{1}{3} \int_0^1 f(x)^2\, dx.$$

(d) Find an orthogonal sequence u_1, u_2, u_3, u_4 in $\mathbb{R}^4$ such that $\operatorname{span}(u_1, \dots, u_i) = \operatorname{span}(v_1, \dots, v_i)$ for all i, where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Solution:

(a) An *inner product* on V is a rule that assigns a number $\langle u, v\rangle \in \mathbb{R}$ to each pair of elements $u, v \in V$ such that

(i) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$.

(ii) $\langle tu, v\rangle = t\langle u, v\rangle$ for all $u, v \in V$ and $t \in \mathbb{R}$.

(iii) $\langle u, v\rangle = \langle v, u\rangle$ for all $u, v \in V$.

(iv) We have $\langle u, u\rangle \geq 0$ for all $u \in V$, and $\langle u, u\rangle = 0$ iff $u = 0$.

(b) The Cauchy-Schwartz inequality says that for $u, v \in V$ we have $|\langle u, v\rangle| \leq \|u\|\|v\|$. To see this, first note that it is obviously true if $v = 0$, so we may assume that $v \neq 0$ and so $\|v\| > 0$. Put $x = \langle v, v\rangle u - \langle u, v\rangle v$. Then

$$\begin{aligned}\|x\|^2 &= \langle x, x\rangle \\ &= \langle v, v\rangle^2\langle u, u\rangle - 2\langle v, v\rangle\langle u, v\rangle\langle u, v\rangle + \langle u, v\rangle^2\langle v, v\rangle \\ &= \langle v, v\rangle(\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2).\end{aligned}$$

As $\langle v, v\rangle = \|v\|^2 > 0$, we can divide by this to get

$$\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2 = (\|x\|/\|v\|)^2.$$

This is a square, so it must be nonnegative, so $\langle u, u\rangle\langle v, v\rangle \geq \langle u, v\rangle^2$. As both sides are nonnegative this inequality reains valid when we take square roots. After noting that $\sqrt{t^2} = |t|$ for all $t \in \mathbb{R}$, we conclude that $\|u\|\|v\| \geq |\langle u, v\rangle|$, as claimed.

(c) Now take $V = C[0, 1]$, with the usual inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\, dx$. Take $g(x) = x$, so $\|g\|^2 = \int_0^1 x^2\, dx = 1/3$. The Cauchy-Schwartz inequality then says that $\langle f, g\rangle^2 \leq \|f\|^2\|g\|^2 = \|f\|^2/3$, or in other words

$$\left(\int_0^1 xf(x)\, dx\right)^2 \leq \frac{1}{3}\int_0^1 f(x)^2\, dx.$$

(d) Put

$$v_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad v_2 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \quad v_3 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \quad v_4 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

We apply the Gram-Schmidt procedure as follows:

$$\begin{aligned}u_1 &= v_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \\ \langle u_1, u_1\rangle &= 1^2 + 1^2 + 1^1 + 1^2 = 4 \\ \langle v_2, u_1\rangle &= 2 \\ u_2 &= v_2 - \frac{\langle v_2, u_1\rangle}{\langle u_1, u_1\rangle}u_1 \\ &= \begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \frac{2}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix} \\ \langle u_2, u_2\rangle &= \tfrac{1}{4}(1^2 + 1^2 + 1^2 + 1^2) = 1 \\ \langle v_3, u_1\rangle &= 1 \\ \langle v_3, u_2\rangle &= 1/2 \\ u_3 &= v_3 - \frac{\langle v_3, u_1\rangle}{\langle u_1, u_1\rangle}u_1 - \frac{\langle v_3, u_2\rangle}{\langle u_2, u_2\rangle}u_2 \\ &= \begin{bmatrix}0\\0\\0\\1\end{bmatrix} - \frac{1}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{1/2}{1}.\frac{1}{2}\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0\\0\\-1\\1\end{bmatrix} \\ \langle u_3, u_3\rangle &= \tfrac{1}{4}(0^2 + 0^2 + (-1)^2 + 1^2) = \tfrac{1}{2} \\ \langle v_4, u_1\rangle &= 1 \\ \langle v_4, u_2\rangle &= -\tfrac{1}{2} \\ \langle v_4, u_3\rangle &= 0 \\ u_4 &= v_4 - \frac{\langle v_4, u_1\rangle}{\langle u_1, u_1\rangle}u_1 - \frac{\langle v_4, u_2\rangle}{\langle u_2, u_2\rangle}u_2 - \frac{\langle v_4, u_3\rangle}{\langle u_3, u_3\rangle}u_3 \\ &= \begin{bmatrix}0\\1\\0\\0\end{bmatrix} - \frac{1}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{-1/2}{1}.\frac{1}{2}\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}.\end{aligned}$$

We conclude that the sequence

$$u_1, u_2, u_3, u_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ \frac{1}{2}\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \ \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \ \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

is an orthogonal sequence such that $\mathrm{span}(u_1, \ldots, u_i) = \mathrm{span}(v_1, \ldots, v_i)$ for $i = 1, \ldots, 4$.

(9) [0506 Q2]

(a) Let V and W be finite-dimensional vector spaces, and let $\phi\colon V \to W$ be a linear map. Prove that there is a number $r \geq 0$ and lists $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_m \in W$ such that

(i) $w_1, \ldots, w_m$ is a basis for W

(ii) $\phi(v_i) = w_i$ for $i = 1, \ldots, r$

(iii) $v_{r+1}, \ldots, v_n$ is a basis for $\ker(\phi)$.

Prove also that the list $v_1, \ldots, v_n$ is linearly independent. **(13 marks)**

(b) Find bases as in (a) for the following case: $V = M_2\mathbb{R}$, $W = \mathbb{R}[x]_{\leq 3}$ and

$$\phi(A) = [x^2, \ \ x]A\begin{bmatrix} 1 \\ x \end{bmatrix}.$$

(12 marks)

Solution:

(a) **This is a cut-down version of a theorem proved in lectures. I will tell the students that such questions may be set, and give them a short list of theorems that might be used.**
Choose a basis $w_1, \ldots, w_r$ for $\mathrm{image}(\phi) \leq W$ **[1]**. This is a linearly independent list in W **[1]**, so it can be extended to a list $\mathcal{W} = w_1, \ldots, w_m$ (for some $m \geq n$) that is a basis for W **[2]**. Next, for $i = 1, \ldots, r$ we note that $w_i \in \mathrm{image}(\phi)$, so we can choose $v_i \in V$ such that $\phi(v_i) = w_i$ **[1]**. This gives us elements $v_1, \ldots, v_r \in V$. Now choose a basis for $\ker(\phi)$ **[1]**, and label the elements as $v_{r+1}, \ldots, v_n$ say **[1]**. Now (i), (ii) and (iii) are satisfied. We must show that the list $\mathcal{V} = v_1, \ldots, v_n$ is linearly independent. Consider a relation $\lambda_1 v_1 + \ldots + \lambda_n v_n = 0$ **[1]**. We apply ϕ, which gives

$$(\lambda_1\phi(v_1) + \cdots + \lambda_r\phi(v_r)) + (\lambda_{r+1}\phi(v_{r+1}) + \cdots + \lambda_n\phi(v_n)) = 0.\mathbf{[1]}$$

In the first block of terms we have $\phi(v_i) = w_i$, and in the second block we have $\phi(v_i) = 0$. The equation therefore reduces to

$$\lambda_1 w_1 + \cdots + \lambda_r w_r = 0.\mathbf{[1]}$$

As the list $w_1, \ldots, w_r$ is a basis for $\mathrm{image}(\phi)$, it is linearly independent, so we must have $\lambda_1 = \cdots = \lambda_r = 0$ **[1]**. Thus, our original relation simplifies to $\lambda_{r+1}v_{r+1} + \cdots + \lambda_n v_n = 0$. As the list $v_{r+1}, \ldots, v_n$ is a basis for $\ker(\phi)$, it is linearly independent, so $\lambda_{r+1} = \cdots = \lambda_n = 0$ **[1]**, so our original relation was the trivial one. This shows that $\mathcal{V}$ is linearly independent **[1]**.

(b) Now consider the map $\phi\colon M_2\mathbb{R} \to \mathbb{R}[x]_{\leq 3}$ given by $\phi(A) = [x^2, \ x]A\left[\begin{smallmatrix} 1 \\ x \end{smallmatrix}\right]$, or equivalently

$$\phi\begin{bmatrix} a & b \\ c & d \end{bmatrix} = [x^2, \ x]\begin{bmatrix} a+bx \\ c+dx \end{bmatrix} = (a+bx)x^2 + (c+dx)x = bx^3 + (a+d)x^2 + cx.\mathbf{[2]}$$

From this it is clear that the list $w_1, w_2, w_3 = x^3, x^2, x$ is a basis for $\mathrm{image}(\phi)$ **[2]**. If we put $w_4 = 1$ then $w_1, \ldots, w_4$ is a basis for $\mathbb{R}[x]_{\leq 3}$ extending our basis for $\mathrm{image}(\phi)$ **[2]**. Now put

$$v_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad v_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ \mathbf{[2]}$$

Using the above formula for ϕ we see that $\phi(v_1) = x^3 = w_1$ and $\phi(v_2) = x^2 = w_2$ and $\phi(v_3) = x = w_3$ and $\phi(v_4) = 0$ **[2]**. More generally, the formula shows that $\phi\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is only zero when $b = c = 0$ and $d = -a$, which means that $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = a\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] = av_4$. This shows that v_4 is a basis for $\ker(\phi)$ **[2]**, so we have all the properties mentioned in (a).

(10) [0506R Q2]

(a) Let V and W be finite-dimensional vector spaces, and let $\phi\colon V \to W$ be a linear map. Prove that there is a number $r \geq 0$ and lists $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_m \in W$ such that

(i) $w_1, \ldots, w_m$ is a basis for W

(ii) $\phi(v_i) = w_i$ for $i = 1, \ldots, r$

(iii) $v_{r+1}, \ldots, v_n$ is a basis for $\ker(\phi)$.

Prove also that the list $v_1, \ldots, v_n$ is linearly independent. **(15 marks)**

(b) Find bases as in (a) for the following case: $V = \mathbb{R}[x]_{\leq 2}$, $W = \mathbb{R}^4$ and

$$\phi(f) = [f(0), f(1), f(0), f(1)]^T.$$

(10 marks)

Solution:

(a) **This is a cut-down version of a theorem proved in lectures, and is identical to the first half of a question on the June exam. I told the students that such questions might be set, and gave them a short list of theorems that might be used.**
Choose a basis $w_1, \ldots, w_r$ for $\text{image}(\phi) \leq W$ **[2]**. This is a linearly independent list in W **[1]**, so it can be extended to a list $\mathcal{W} = w_1, \ldots, w_m$ (for some $m \geq n$) that is a basis for W **[2]**. Next, for $i = 1, \ldots, r$ we note that $w_i \in \text{image}(\phi)$, so we can choose $v_i \in V$ such that $\phi(v_i) = w_i$ **[1]**. This gives us elements $v_1, \ldots, v_r \in V$. Now choose a basis for $\ker(\phi)$ **[2]**, and label the elements as $v_{r+1}, \ldots, v_n$ say **[1]**. Now (i), (ii) and (iii) are satisfied. We must show that the list $\mathcal{V} = v_1, \ldots, v_n$ is linearly independent. Consider a relation $\lambda_1 v_1 + \ldots + \lambda_n v_n = 0$ **[1]**. We apply ϕ, which gives

$$(\lambda_1\phi(v_1) + \cdots + \lambda_r\phi(v_r)) + (\lambda_{r+1}\phi(v_{r+1}) + \cdots + \lambda_n\phi(v_n)) = 0.\mathbf{[1]}$$

In the first block of terms we have $\phi(v_i) = w_i$, and in the second block we have $\phi(v_i) = 0$. The equation therefore reduces to

$$\lambda_1 w_1 + \cdots + \lambda_r w_r = 0.\mathbf{[1]}$$

As the list $w_1, \ldots, w_r$ is a basis for $\text{image}(\phi)$, it is linearly independent, so we must have $\lambda_1 = \cdots = \lambda_r = 0$ **[1]**. Thus, our original relation simplifies to $\lambda_{r+1}v_{r+1} + \cdots + \lambda_n v_n = 0$. As the list $v_{r+1}, \ldots, v_n$ is a basis for $\ker(\phi)$, it is linearly independent, so $\lambda_{r+1} = \cdots = \lambda_n = 0$ **[1]**, so our original relation was the trivial one. This shows that $\mathcal{V}$ is linearly independent **[1]**.

(b) Now consider the map $\phi\colon V = \mathbb{R}[x]_{\leq 2} \to \mathbb{R}^4 = W$ given by $\phi(f) = [f(0), f(1), f(0), f(1)]^T$, or equivalently

$$\phi(ax^2 + bx + c) = \begin{bmatrix} c \\ a+b+c \\ c \\ a+b+c \end{bmatrix} = c\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + (a+b)\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.\mathbf{[2]}$$

Thus, if we put $w_1 = [1,1,1,1]^T$ and $w_2 = [0,1,0,1]^T$ then the list w_1, w_2 is a basis for $\text{image}(\phi)$ **[2]**, which we can extend to a basis for all of $\mathbb{R}^4$ by taking $w_3 = [0,0,1,0]^T$ and

$w_4 = [0,0,0,1]^T$ **[2]**. If we put $v_1 = 1$ and $v_2 = x$ then we see that $\phi(v_i) = w_i$ for $i = 1, 2$ **[2]**. We also see from the above formulae that for a polynomial $f(x) = ax^2 + bx + c$ we have $\phi(f) = 0$ iff $a + b + c = c = 0$ iff $f(x) = a(x^2 - x)$ for some a, so the element $v_3 = x^2 - x$ gives a basis for $\ker(\phi)$ **[2]**. In summary, we have $r = 2$ and

$$v_1 = 1 \qquad v_2 = x \qquad v_3 = x^2 - x \qquad w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad w_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(11) [0607 Q2] Consider the linear map $\alpha\colon \mathbb{R}[x]_{\leq 2} \to M_2(\mathbb{R})$ given by

$$\alpha(f) = \begin{bmatrix} f(0) & f(1) \\ f'(0) & f'(1) \end{bmatrix}.$$

(a) Write down a basis $\mathcal{U}$ for $\mathbb{R}[x]_{\leq 2}$ and a basis $\mathcal{V}$ for $M_2(\mathbb{R})$. **(3 marks)**

(b) Find the matrix of α with respect to the bases $\mathcal{U}$ and $\mathcal{V}$. **(5 marks)**

(c) Show that α is injective. **(4 marks)**

(d) Give a basis for the image of α. **(4 marks)**

(e) Find a nonzero matrix $X = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ such that $\langle X, \alpha(x^i) \rangle = 0$ for $i = 0, 1, 2$. **(5 marks)** (Here we use the standard inner product for square matrices.)

(f) Show (by an explicit example) that α is not surjective. **(4 marks)**

Solution:

(a) **Similar to problem sheets** The obvious basis for $\mathbb{R}[x]_{\leq 2}$ consists of the polynomials $p_0(x) = 1$, $p_1(x) = x$ and $p_2(x) = x^2$.**[1]**The obvious basis for $M_2(\mathbb{R})$ consists of the matrices

$$E_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \qquad E_2 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \qquad E_3 = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right] \qquad E_4 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right].\mathbf{[2]}$$

(b) **Similar to problem sheets** We have

$$\begin{aligned}
\alpha(p_0) &= \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right] = 1.E_1 + 1.E_2 + 0.E_3 + 0.E_4 \\
\alpha(p_1) &= \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right] = 0.E_1 + 1.E_2 + 1.E_3 + 1.E_4 \\
\alpha(p_2) &= \left[\begin{smallmatrix} 0 & 1 \\ 0 & 2 \end{smallmatrix}\right] = 0.E_1 + 1.E_2 + 0.E_3 + 2.E_4\mathbf{[3]}
\end{aligned}$$

so the matrix of α with respect to our bases is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}.\mathbf{[2]}$$

(c) **Similar to problem sheets** From the above we see that

$$\alpha(a + bx + cx^2) = a\alpha(p_0) + b\alpha(p_1) + c\alpha(p_2) = \left[\begin{smallmatrix} a & a+b+c \\ b & b+2c \end{smallmatrix}\right].\mathbf{[2]}$$

Thus, if $\alpha(a + bx + cx^2) = 0$ we see that $a = a + b + c = b = b + 2c = 0$, which easily implies that $a = b = c = 0$. This shows that $\ker(\alpha) = \{0\}$ and thus that α is injective. **[2]**

(d) **Unseen** As α is injective, the matrices $\alpha(p_0) = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, $\alpha(p_1) = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $\alpha(p_2) = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 2 \end{smallmatrix}\right]$ form a basis for the image. **[4]**

(e) **Similar to problem sheets** We have

$$\langle X, \alpha(1)\rangle = \langle \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]\rangle = a + b$$
$$\langle X, \alpha(x)\rangle = \langle \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]\rangle = b + c + d$$
$$\langle X, \alpha(x^2)\rangle = \langle \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & 2 \end{smallmatrix}\right]\rangle = b + 2d\textbf{[2]}$$

We therefore must have $a + b = b + c + d = b + 2d = 0$, which gives $a = 2d$, $b = -2d$ and $c = d$, so $X = d\left[\begin{smallmatrix} 2 & -2 \\ 1 & 1 \end{smallmatrix}\right]$. **[2]** Here d is arbitrary so we can take $d = 1$ and so $X = \left[\begin{smallmatrix} 2 & -2 \\ 1 & 1 \end{smallmatrix}\right]$. **[1]**

(f) **Unseen** Observe that $\langle X, X\rangle = 2^2 + (-2)^2 + 1^2 + 1^2 = 10 \neq 0$, but $\langle X, A\rangle = 0$ for all A in the image of α (by part (d)). It follows that $X \notin \text{image}(\alpha)$, and thus that $\text{image}(\alpha) \neq M_2(\mathbb{R})$, so α is not surjective. **[4]**

(12) [0607R Q2] Define linear maps $\phi, \psi\colon M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by

$$\phi\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} a-c & a-c \\ b-d & b-d \end{smallmatrix}\right]$$
$$\psi(A) = \phi(\phi(A)).$$

(a) Write down a basis for $M_2(\mathbb{R})$. **(2 marks)**

(b) Find the matrix of ϕ with respect to your basis. **(5 marks)**

(c) Give a basis for $\ker(\phi)$. **(5 marks)**

(d) Give a formula for $\psi\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. **(4 marks)**

(e) Give a basis for $\text{image}(\psi)$. **(3 marks)**

(f) Show that $\text{image}(\psi) \leq \ker(\phi)$. **(3 marks)**

(g) What can you conclude about $\phi(\phi(\phi(A)))$? **(3 marks)**

Solution:

(a) The list

$$E_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \qquad E_2 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], \qquad E_3 = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right], \qquad E_4 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$$

is a basis for $M_2(\mathbb{R})$. **[2]**

(b) We have

$$\phi(E_1) = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right] = E_1 + E_2 + 0E_3 + 0E_4$$
$$\phi(E_2) = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right] = 0E_1 + 0E_2 + E_3 + E_4$$
$$\phi(E_3) = \left[\begin{smallmatrix} -1 & -1 \\ 0 & 0 \end{smallmatrix}\right] = -E_1 - E_2 + 0E_3 + 0E_4$$
$$\phi(E_4) = \left[\begin{smallmatrix} 0 & 0 \\ -1 & -1 \end{smallmatrix}\right] = 0E_1 + 0E_2 - E_3 - E_4\textbf{[3]}$$

The matrix of ϕ is thus

$$\left[\begin{smallmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{smallmatrix}\right].\textbf{[2]}$$

(c) Consider a matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. We have $A \in \ker(\phi)$ iff $a - c = 0 = b - d$ **[1]**iff $c = a$ and $d = b$ **[1]**, iff A has the form

$$A = \left[\begin{smallmatrix} a & b \\ a & b \end{smallmatrix}\right] = a\left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right] + b\left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right].\textbf{[1]}$$

It follows that the matrices $A_1 = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ and $A_2 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ give a basis for $\ker(\phi)$ **[2]**.

(d) We have $\phi\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right]$, where $a' = b' = a - c$ and $c' = d' = b - d$. Thus

$$\psi\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \phi\left[\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right] = \left[\begin{smallmatrix} a'-c' & a'-c' \\ b'-d' & b'-d' \end{smallmatrix}\right] = \left[\begin{smallmatrix} a-b-c+d & a-b-c+d \\ a-b-c+d & a-b-c+d \end{smallmatrix}\right] = (a-b-c+d)\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right].\textbf{[4]}$$

(e) From the above, we see that the matrix $A_3 = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ is a basis for image(ψ). **[3]**

(f) Every element $X \in \text{image}(\psi)$ has the form $X = tA_3$ for some t, but $A_3 = A_1 + A_2$, so $X = tA_1 + tA_2 \in \text{span}\{A_1, A_2\} = \ker(\phi)$. Thus image$(\psi) \leq \ker(\phi)$. **[3]**

(g) We have $\phi(\phi(\phi(A))) = \phi(\psi(A))$, and $\psi(A) \in \text{image}(\psi) \leq \ker(\phi)$, so this is just zero. Alternatively, we have

$$\phi(\psi(A)) = \phi\left((a-b-c+d)\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]\right) = (a-b-c+d)\phi\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] = 0.\textbf{[3]}$$

(13) [0708 Q2]

(a) Let V be a vector space, and let $\mathcal{V} = v_1, \ldots, v_n$ be a list of elements of V. Define what it means for $\mathcal{V}$ to (i) be linearly independent (ii) span V (iii) be a basis for V. **(6 marks)**

(b) In each of the following cases, say whether the given list is linearly independent, whether it spans, and whether it is a basis. Justify your answers, giving explicit counterexamples where appropriate. **(16 marks)**

(i) $V = \mathbb{R}^4$, $v_1 = \left[\begin{smallmatrix} 1 \\ 1 \\ 0 \\ 0 \end{smallmatrix}\right]$, $v_2 = \left[\begin{smallmatrix} 0 \\ 0 \\ 1 \\ 1 \end{smallmatrix}\right]$, $v_3 = \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \\ 1 \end{smallmatrix}\right]$.

(ii) $V = \mathbb{R}[x]_{\leq 2}$, $p_1(x) = (x-2)^2$, $p_2(x) = x^2$, $p_3(x) = (x+2)^2$.

(iii) $V = M_2(\mathbb{R})$, $A_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, $A_2 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & -1 \end{smallmatrix}\right]$, $A_3 = \left[\begin{smallmatrix} 0 & 0 \\ 1 & -1 \end{smallmatrix}\right]$, $A_4 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & -2 \end{smallmatrix}\right]$, $A_5 = \left[\begin{smallmatrix} 1 & 0 \\ 1 & -2 \end{smallmatrix}\right]$, $A_6 = \left[\begin{smallmatrix} 1 & 1 \\ 0 & -2 \end{smallmatrix}\right]$.

(iv) $V = \{f \in C^\infty(\mathbb{R}) \mid f'' = f\}$, $f_0(x) = \exp(x)$, $f_1(x) = \exp(-x)$, $f_2(x) = \sinh(x)$, $f_3(x) = \cosh(x)$. (Here you may quote standard facts about differential equations.)

(c) Give a list of four nonzero vectors in $\mathbb{R}^3$ such that the first three vectors in the list form a basis, but the last three do not. **(3 marks)**

Solution:

(a) Define $\mu_{\mathcal{V}} \colon \mathbb{R}^n \to V$ by $\mu_{\mathcal{V}}(\boldsymbol{\lambda}) = \sum_i \lambda_i v_i$. The list $\mathcal{V}$ is *linearly dependent* if there exists a nonzero $\boldsymbol{\lambda}$ with $\mu_{\mathcal{V}}(\boldsymbol{\lambda}) = 0$; otherwise, the list is *linearly independent* [**2**]. The list *spans* V if for each $v \in V$ there exists $\boldsymbol{\lambda} \in \mathbb{R}^n$ with $\mu_{\mathcal{V}}(\boldsymbol{\lambda}) = v$ [**2**]. The list is a *basis* for V if it is linearly independent and also spans [**2**]. **Bookwork. Formulations not involving $\mu_{\mathcal{V}}$ are also acceptable.**

(b) **Similar to problem sheets, lecture notes and past papers.** In each case there is one mark for independence, one mark for spanning, and two marks for justification.

(i) We have

$$\mu_{\mathcal{V}}(\boldsymbol{\lambda}) = \lambda_1 \left[\begin{smallmatrix} 1 \\ 1 \\ 0 \\ 0 \end{smallmatrix}\right] + \lambda_2 \left[\begin{smallmatrix} 0 \\ 0 \\ 1 \\ 1 \end{smallmatrix}\right] + \lambda_3 \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \lambda_1+\lambda_3 \\ \lambda_1 \\ \lambda_2 \\ \lambda_2+\lambda_3 \end{smallmatrix}\right].$$

This can only vanish if $\lambda_1 + \lambda_3 = \lambda_1 = \lambda_2 = \lambda_2 + \lambda_3 = 0$, which easily implies that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Thus $\mathcal{V}$ is linearly independent. However, if x is any of the vectors v_i then $x_1 - x_2 + x_3 - x_4 = 0$, so the same will be true for any $x \in \text{span}(\mathcal{V})$, so in particular the vector $\mathbf{e}_1 = [1, 0, 0, 0]^T$ does not lie in span$(\mathcal{V})$, so the list $\mathcal{V}$ does not span $\mathbb{R}^4$. **It is also acceptable to say that $\mathcal{V}$ is too short to span $\mathbb{R}^4$, or to use row-reduction.** It is therefore not a basis. [**4**]

(ii) We have

$$p_1(x) = x^2 - 4x + 4$$
$$p_2(x) = x^2$$
$$p_3(x) = x^2 + 4x + 4$$

so

$$x^2 = p_2(x)$$
$$x = (p_3(x) - p_1(x))/8$$
$$1 = (p_1(x) - 2p_2(x) + p_3(x))/8.$$

Thus $x^2, x, 1 \in \text{span}(\mathcal{P})$, and these monomials form a basis for $\mathbb{R}[x]_{\leq 2}$, so $\mathcal{P}$ spans $\mathbb{R}[x]_{\leq 2}$. As $\mathcal{P}$ has length three, which is the same as the dimension of $\mathbb{R}[x]_{\leq 2}$, we see that $\mathcal{P}$ is also linearly independent and thus a basis.[**4**]

A more equational proof is also acceptable. Some relevant formulae are given below.

$\mu_{\mathcal{P}}(\boldsymbol{\lambda}) = \lambda_1(x^2-4x+4)+\lambda_2 x^2+\lambda_3(x^2+4x+4) = (\lambda_1+\lambda_2+\lambda_3)x^2+(-4\lambda_1+4\lambda_3)x+(4\lambda_1+4\lambda_3)$.

Given a polynomial $f(x) = ax^2 + bx + c$, we have $\mu_{\mathcal{P}}(\boldsymbol{\lambda}) = f$ iff the following equations hold:

$$\lambda_1 + \lambda_2 + \lambda_3 = a \qquad \text{(A)}$$
$$-4\lambda_1 + 4\lambda_3 = b \qquad \text{(B)}$$
$$4\lambda_1 + 4\lambda_3 = c. \qquad \text{(C)}$$

These can be solved as follows:

$$\lambda_1 = (c - b)/8$$
$$\lambda_2 = a - c/4$$
$$\lambda_3 = (c + b)/8.$$

(iii) The list $\mathcal{A} = A_1, \dots, A_6$ is linearly dependent, because of the relation $A_1 + A_2 - A_6 = 0$. Also, for each matrix A_i, the sum of all the entries is zero. It follows that any matrix in $\text{span}(\mathcal{A})$ has the same property, so $I \notin \text{span}(\mathcal{A})$, so $\mathcal{A}$ is not a spanning set. It therefore cannot be a basis either. [**4**]

(iv) It is standard that any function $f(x)$ with $f''(x) = f(x)$ has the form $f(x) = a\,e^x + b\,e^{-x}$ for some constants a and b. Thus, $f = af_1 + bf_2 + 0f_3 + 0f_4 \in \text{span}(\mathcal{F})$, so $\mathcal{F}$ spans V. However, the relation $f_1 + f_2 - 2f_4 = 0$ shows that $\mathcal{F}$ is linearly dependent, and thus not a basis. [**4**]

(c) **Unseen.** There are many possible examples, such as the list

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}.\textbf{[3]}$$

(14) [0708R Q2] Let V be a finite-dimensional vector space over $\mathbb{R}$.

(a) Define what is meant by an *inner product* on V. **(5 marks)**

(b) State and prove the Cauchy-Schwartz inequality. (You need not discuss the case where it is actually an equality.) **(10 marks)**

(c) Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Prove that

$$\left(\int_0^1 x^2 f(x)\,dx\right)^2 \leq \frac{1}{5}\int_0^1 f(x)^2\,dx.$$

(4 marks)

(d) Find an orthogonal sequence u_1, u_2, u_3, u_4 in $\mathbb{R}^4$ such that $\operatorname{span}(u_1, \dots, u_i) = \operatorname{span}(v_1, \dots, v_i)$ for all i, where

$$v_1 = \begin{bmatrix} 1\\1\\1\\1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0\\0\\1\\1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0\\0\\0\\1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 0\\1\\0\\0 \end{bmatrix}$$

(6 marks)

Solution: This is a slight modification of a question from a past paper.

(a) **Bookwork.** An *inner product* on V is a rule that assigns a number $\langle u, v\rangle \in \mathbb{R}$ to each pair of elements $u, v \in V$ such that

(i) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$.**[1]**

(ii) $\langle tu, v\rangle = t\langle u, v\rangle$ for all $u, v \in V$ and $t \in \mathbb{R}$.**[1]**

(iii) $\langle u, v\rangle = \langle v, u\rangle$ for all $u, v \in V$. **[1]**

(iv) We have $\langle u, u\rangle \geq 0$ for all $u \in V$, **[1]**and $\langle u, u\rangle = 0$ iff $u = 0$. **[1]**

(b) **Bookwork.** Let V be a vector space with an inner product, and let u and v be elements of V. Then $|\langle u, v\rangle| \leq \|u\|\|v\|$.**[2]**

Proof: For any s and t we have

$0 \leq \|su - tv\|^2$**[1]** $= \langle su - tv, su - tv\rangle = s^2\langle u, u\rangle - 2st\langle u, v\rangle + t^2\langle v, v\rangle = s^2\|u\|^2 + t^2\|v\|^2 - 2st\langle u, v\rangle$.**[1]**

Now take $s = \|v\|^2$ and $t = \langle u, v\rangle$ **[2]** to get

$$0 \leq \|u\|^2\|v\|^4 + \langle u, v\rangle^2\|v\|^2 - 2\|v\|^2\langle u, v\rangle^2 = \|v\|^2(\|u\|^2\|v\|^2 - \langle u, v\rangle^2).\textbf{[1]}$$

If $v = 0$ then we have $|\langle u, v\rangle| = 0 = \|u\|\|v\|$ so the claim holds. **[1]**If $v \neq 0$ then $\|v\|^2 > 0$ so the above inequality will remain valid after dividing by $\|v\|^2$, giving $\langle u, v\rangle^2 \leq \|u\|^2\|v\|^2$. **[1]**We now take square roots (and note that $\sqrt{a^2} = |a|$) to get $|\langle u, v\rangle| \leq \|u\|\|v\|$, as claimed.**[1]**

(c) **Similar to problem sheets** Now take $V = C[0,1]$, with the usual inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$. Take $g(x) = x^2$, so $\|g\|^2 = \int_0^1 x^4\,dx = 1/5$ **[1]**. The Cauchy-Schwartz inequality **[1]**then says that $\langle f, g\rangle^2 \leq \|f\|^2\|g\|^2 = \|f\|^2/5$ **[1]**, or in other words

$$\left(\int_0^1 x^2 f(x)\,dx\right)^2 \leq \frac{1}{5}\int_0^1 f(x)^2\,dx.\textbf{[1]}$$

(d) **Similar to problem sheets** Put

$$v_1 = \begin{bmatrix} 1\\1\\1\\1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0\\0\\1\\1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0\\0\\0\\1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 0\\1\\0\\0 \end{bmatrix}$$

We apply the Gram-Schmidt procedure as follows:

$$\begin{aligned}
u_1 &= v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
\langle u_1, u_1 \rangle &= 1^2 + 1^2 + 1^2 + 1^2 = 4 \\
\langle v_2, u_1 \rangle &= 2[\mathbf{1}] \\
u_2 &= v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 [\mathbf{1}] \\
&= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} [\mathbf{1}] \\
\langle u_2, u_2 \rangle &= \tfrac{1}{4}(1^2 + 1^2 + 1^2 + 1^2) = 1 \\
\langle v_3, u_1 \rangle &= 1 \\
\langle v_3, u_2 \rangle &= 1/2 \\
u_3 &= v_3 - \frac{\langle v_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 [\mathbf{1}] \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1/2}{1} . \frac{1}{2} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} [\mathbf{1}] \\
\langle u_3, u_3 \rangle &= \tfrac{1}{4}(0^2 + 0^2 + (-1)^2 + 1^2) = \tfrac{1}{2} \\
\langle v_4, u_1 \rangle &= 1 \\
\langle v_4, u_2 \rangle &= -\tfrac{1}{2} \\
\langle v_4, u_3 \rangle &= 0 \\
u_4 &= v_4 - \frac{\langle v_4, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_4, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 - \frac{\langle v_4, u_3 \rangle}{\langle u_3, u_3 \rangle} u_3 \\
&= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-1/2}{1} . \frac{1}{2} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} . [\mathbf{1}]
\end{aligned}$$

We conclude that the sequence

$$u_1, u_2, u_3, u_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ \frac{1}{2} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \ \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \ \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

is an orthogonal sequence such that $\text{span}(u_1, \ldots, u_i) = \text{span}(v_1, \ldots, v_i)$ for $i = 1, \ldots, 4$.

(15) [Mock exam Q3] Define $\phi \colon \mathbb{R}[x]_{\leq 3} \to \mathbb{R}[x]_{\leq 3}$ by

$$\phi(f)(x) = x^3 f(4/x).$$

(a) Find the matrix of ϕ with respect to the usual basis of $\mathbb{R}[x]_{\leq 3}$.

(b) Hence or otherwise, find the eigenvalues of ϕ, and find a basis of $\mathbb{R}[x]_{\leq 3}$ consisting of eigenvectors for ϕ.

(c) Is ϕ injective?

(d) Is ϕ surjective?

Solution:

(a) We have

$$\begin{aligned}
\phi(1) &= x^3.1 && = 0.1 + 0.x + 0.x^2 + 1.x^3 \\
\phi(x) &= x^3.\frac{4}{x} && = 0.1 + 0.x + 4.x^2 + 0.x^3 \\
\phi(x^2) &= x^3.\left(\frac{4}{x}\right)^2 && = 0.1 + 16.x + 0.x^2 + 0.x^3 \\
\phi(x^3) &= x^3.\left(\frac{4}{x}\right)^3 && = 64.1 + 0.x + 0.x^2 + 0.x^3,
\end{aligned}$$

so the matrix of ϕ is

$$P = \begin{bmatrix} 0 & 0 & 0 & 64 \\ 0 & 0 & 16 & 0 \\ 0 & 4 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(b) The characteristic polynomial of P is the determinant of the matrix $tI - P$, which is

$$\det \begin{bmatrix} t & 0 & 0 & -64 \\ 0 & t & -16 & 0 \\ 0 & -4 & t & 0 \\ -1 & 0 & 0 & t \end{bmatrix} = t \det \begin{bmatrix} t & -16 & 0 \\ -4 & t & 0 \\ 0 & 0 & t \end{bmatrix} - (-64) \det \begin{bmatrix} 0 & t & -16 \\ 0 & -4 & t \\ -1 & 0 & 0 \end{bmatrix}$$
$$= t^2(t^2 - 64) + 64.(-1).(t^2 - 64) = (t^2 - 64)^2 = (t - 8)^2(t + 8)^2.$$

It follows that the eigenvalues of P (or of ϕ) are 8 and -8. Consider a function $f(x) = a + bx + cx^2 + dx^3$. We have $\phi(f) = 64d + 16cx + 4bx^2 + ax^3$, so $\phi(f) = 8f$ iff $64d = 8a$ and $16c = 8b$ and $4b = 8c$ and $a = 8d$, which reduces to $a = 8d$ and $b = 2c$, which means that f has the form $f(x) = d(x^3 + 8) + c(x^2 + 2x)$. It follows that $x^3 + 8$ and $x^2 + 2x$ are eigenvectors of eigenvalue 8. Similarly, we have $\phi(f) = -8f$ iff $64d = -8a$ and $16c = -8b$ and $4b = -8c$ and $a = -8d$, which reduces to $a = -8d$ and $b = -2c$, which means that f has the form $f(x) = d(x^3 - 8) + c(x^2 - 2x)$. It follows that $x^3 - 8$ and $x^2 - 2x$ are eigenvectors of eigenvalue -8. We thus have a list $\mathcal{V} = x^3 + 8, x^2 + 2x, x^3 - 8, x^2 - 2x$ of eigenvectors of ϕ, and this list is easily seen to be a basis of $\mathbb{R}[x]_{\leq 3}$.

(c) We see from (b) that 0 is not an eigenvalue of ϕ, so $\ker(\phi) = 0$, so ϕ is injective.

(d) The rank-nullity formula says that

$$\dim(\text{image}(\phi)) + \dim(\ker(\phi)) = \dim(\mathbb{R}[x]_{\leq 3}) = 4.$$

As $\ker(\phi) = 0$ this gives $\dim(\text{image}(\phi)) = 4$, so $\text{image}(\phi) = \mathbb{R}[x]_{\leq 3}$, so ϕ is surjective.

(16) [0506 Q3] Define linear maps $\phi, \psi \colon M_2\mathbb{R} \to M_2\mathbb{R}$ by

$$\phi \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}$$
$$\psi(A) = \phi(A) - 2A.$$

(a) Write down a basis for $M_2\mathbb{R}$. **(2 marks)**

(b) Find the matrix of ϕ with respect to your basis. **(4 marks)**

(c) Find the matrix of ψ with respect to your basis. **(3 marks)**

(d) Give bases for $\ker(\phi)$, $\text{image}(\phi)$ and $\ker(\psi)$. **(12 marks)**

(e) Using (d), give a basis for $M_2\mathbb{R}$ consisting of eigenvectors for ϕ. **(4 marks)**

Solution: This is all similar to questions on problem sheets, except perhaps for (e).

(a) The list

$$E_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \qquad E_2 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], \qquad E_3 = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right], \qquad E_4 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$$

is a basis for $M_2\mathbb{R}$. **[2]**

(b) We have

$$\begin{aligned}
\phi(E_1) &= \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right] = 1E_1 + 0E_2 + 1E_3 + 0E_4 \\
\phi(E_2) &= \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right] = 0E_1 + 1E_2 + 0E_3 + 1E_4 \\
\phi(E_3) &= \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right] = 1E_1 + 0E_2 + 1E_3 + 0E_4 \\
\phi(E_4) &= \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right] = 0E_1 + 1E_2 + 0E_3 + 1E_4 \textbf{[3]}
\end{aligned}$$

The matrix of ϕ is thus

$$\left[\begin{smallmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{smallmatrix}\right].\textbf{[1]}$$

(c) The simplest thing is just to say that the matrix of ψ is the matrix of ϕ minus twice the identity. Alternatively, we have

$$\psi \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} a+c & b+d \\ a+c & b+d \end{smallmatrix}\right] - \left[\begin{smallmatrix} 2a & 2b \\ 2c & 2d \end{smallmatrix}\right] = \left[\begin{smallmatrix} c-a & d-b \\ a-c & b-d \end{smallmatrix}\right]$$

so

$$\begin{aligned}
\psi(E_1) &= \left[\begin{smallmatrix} -1 & 0 \\ 1 & 0 \end{smallmatrix}\right] = (-1)E_1 + 0E_2 + 1E_3 + 0E_4 \\
\psi(E_2) &= \left[\begin{smallmatrix} 0 & -1 \\ 0 & 1 \end{smallmatrix}\right] = 0E_1 + (-1)E_2 + 0E_3 + 1E_4 \\
\psi(E_3) &= \left[\begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix}\right] = 1E_1 + 0E_2 + (-1)E_3 + 0E_4 \\
\psi(E_4) &= \left[\begin{smallmatrix} 0 & 1 \\ 0 & -1 \end{smallmatrix}\right] = 0E_1 + 1E_2 + 0E_3 + (-1)E_4 \textbf{[2]}
\end{aligned}$$

The matrix of ψ is thus

$$\left[\begin{smallmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{smallmatrix}\right].\textbf{[1]}$$

(d) Consider a matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. We have $A \in \ker(\phi)$ iff $a + c = 0 = b + d$ **[1]**iff A has the form

$$A = \left[\begin{smallmatrix} a & b \\ -a & -b \end{smallmatrix}\right] = a \left[\begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix}\right] + b \left[\begin{smallmatrix} 0 & 1 \\ 0 & -1 \end{smallmatrix}\right].\textbf{[1]}$$

It follows that $\left[\begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & -1 \end{smallmatrix}\right]$ is a basis for $\ker(\phi)$ **[2]**. Similarly, the formula in (c) shows that $\psi(A) = 0$ iff $c = a$ and $d = b$ **[1]**iff A has the form

$$A = \left[\begin{smallmatrix} a & b \\ a & b \end{smallmatrix}\right] = a \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right] + b \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right], \textbf{[1]}$$

so $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ is a basis for $\ker(\psi)$ **[2]**. Next, the formula

$$\phi \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} a+c & b+d \\ a+c & b+d \end{smallmatrix}\right]$$

shows that any matrix B in image(ϕ) has the form

$$B = \left[\begin{smallmatrix} p & q \\ p & q \end{smallmatrix}\right] = p \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right] + q \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right].\textbf{[1]}$$

Moreover, for any matrix B of the above form we have $B = \phi \left[\begin{smallmatrix} p & q \\ 0 & 0 \end{smallmatrix}\right]$, so $B \in$ image(ϕ) **[1]**. This means that image(ϕ) is precisely the set of matrices of the above form, so $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ is a basis for image(ϕ) **[2]**.

(e) The elements of $\ker(\phi)$ are eigenvectors for ϕ of eigenvalue 0 **[1]**, and the elements of $\ker(\psi)$ are eigenvectors of ϕ of eigenvalue 2 **[1]**. It follows that the list

$$\mathcal{E} = \left[\begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]$$

consists of eigenvectors of ϕ. It is easily seen to be a basis of $M_2\mathbb{R}$. **[2]**

(17) [0506R Q3] Define a linear map $\phi\colon M_2\mathbb{R} \to M_2\mathbb{R}$ by $\phi(A) = QA - AQ$, where $Q = \left[\begin{smallmatrix} 3 & 4 \\ 4 & -3 \end{smallmatrix}\right]$.

(a) Show that I and Q are eigenvectors for ϕ. (**Hint:** you do not need any elaborate calculation for this.) **(3 marks)**

(b) Give a formula for $\phi\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right)$ **(3 marks)**

(c) Write down a basis for $M_2\mathbb{R}$. **(2 marks)**

(d) Find the matrix of ϕ with respect to your basis. **(4 marks)**

(e) Find X such that X is an eigenvector of ϕ with eigenvalue 10, and show that X^T is an eigenvector of ϕ of eigenvalue -10. **(9 marks)**

(f) What is the matrix of ϕ with respect to the basis I, Q, X, X^T of $M_2\mathbb{R}$? **(4 marks)**

Solution:

(a) We have $\phi(I) = QI - IQ = Q - Q = 0$ **[1]**and $\phi(Q) = Q^2 - Q^2 = 0$ **[1]**, so I and Q are eigenvectors of eigenvalue 0 **[1]**.

(b)

$$\phi\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} 3 & 4 \\ 4 & -3 \end{smallmatrix}\right]\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] - \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\left[\begin{smallmatrix} 3 & 4 \\ 4 & -3 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 3a+4c & 3b+4d \\ 4a-3c & 4b-3d \end{smallmatrix}\right] - \left[\begin{smallmatrix} 3a+4b & 4a-3b \\ 3c+4d & 4c-3d \end{smallmatrix}\right] = \left[\begin{smallmatrix} -4b+4c & -4a+6b+4d \\ 4a-6c-4d & 4b-4c \end{smallmatrix}\right] \textbf{[3]}$$

(c) The list

$$E_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \qquad E_2 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], \qquad E_3 = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right], \qquad E_4 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$$

is a basis for $M_2\mathbb{R}$. **[2]**

(d) Using the formula in (b) we have

$$\begin{aligned}
\phi(E_1) &= \left[\begin{smallmatrix} 0 & -4 \\ 4 & 0 \end{smallmatrix}\right] = 0.E_1 - 4.E_2 + 4.E_3 + 0.E_4 \\
\phi(E_2) &= \left[\begin{smallmatrix} -4 & 6 \\ 0 & 4 \end{smallmatrix}\right] = -4.E_1 + 6.E_2 + 0.E_3 + 4.E_4 \\
\phi(E_3) &= \left[\begin{smallmatrix} 4 & 0 \\ -6 & -4 \end{smallmatrix}\right] = 4.E_1 + 0.E_2 - 6.E_3 - 4.E_4 \\
\phi(E_4) &= \left[\begin{smallmatrix} 0 & 4 \\ -4 & 0 \end{smallmatrix}\right] = 0.E_1 + 4.E_2 - 4.E_3 + 0.E_4 \textbf{[3]}
\end{aligned}$$

The matrix of ϕ is thus

$$M = \begin{bmatrix} 0 & -4 & 4 & 0 \\ -4 & 6 & 0 & 4 \\ 4 & 0 & -6 & -4 \\ 0 & 4 & -4 & 0 \end{bmatrix}.\textbf{[1]}$$

(c) Take $X = \left[\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right]$ **[1]**. We need $\phi(X) = 10X$ **[1]**, or in other words $\left[\begin{smallmatrix} -4x+4y & -4w+6x+4z \\ 4w-6y-4z & 4x-4y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 10w & 10x \\ 10y & 10z \end{smallmatrix}\right]$ **[1]**, or

$$\begin{aligned}
-10w - 4x + 4y &= 0 \\
-4w - 4x + 4z &= 0 \\
4w - 16y + 4z &= 0 \\
4x - 4y - 10z &= 0.\textbf{[1]}
\end{aligned}$$

From the first and last of these we get $w = -z$, and we can substitute this into the second and third equations to get $x = 2z$ and $y = -z/2$, so $X = \left[\begin{smallmatrix} -z & 2z \\ -z/2 & z \end{smallmatrix}\right]$. Here z is arbitrary so we take $z = -2$ to get $X = \left[\begin{smallmatrix} 2 & -4 \\ 1 & -2 \end{smallmatrix}\right]$ **[1]**.

We now have $X^T = \left[\begin{smallmatrix} 2 & 1 \\ -4 & -2 \end{smallmatrix}\right]$, so

$$\phi(X^T) = \left[\begin{smallmatrix} -4.1+4.(-4) & -4.2+6.1+4.(-2) \\ 4.2-6.(-4)-4.(-2) & 4.1-4.(-4) \end{smallmatrix}\right] = \left[\begin{smallmatrix} -20 & -10 \\ 40 & 20 \end{smallmatrix}\right] = -10X^T, \textbf{[3]}$$

so X^T is an eigenvector of eigenvalue -10. **[1]**

(d) We have

$$\begin{aligned}\phi(I) &= 0 = 0.I + 0.Q + 0.X + 0.X^T \\ \phi(Q) &= 0 = 0.I + 0.Q + 0.X + 0.X^T \\ \phi(X) &= 10X = 0.I + 0.Q + 10.X + 0.X^T \\ \phi(X^T) &= -10X^T = 0.I + 0.Q + 0.X - 10.X^T, \textbf{[2]}\end{aligned}$$

so the relevant matrix is

$$\begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&10&0 \\ 0&0&0&-10 \end{bmatrix}.\textbf{[2]}$$

(18) [0607 Q3]

(a) Define the standard inner product on the space $M_2(\mathbb{R})$. **(2 marks)**

(b) What is $\langle A, B\rangle$, where $A = \left[\begin{smallmatrix} 1&1\\0&1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1&0\\1&1 \end{smallmatrix}\right]$? **(2 marks)**

(c) Let R_θ denote the rotation matrix $\left[\begin{smallmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{smallmatrix}\right]$. Simplify $\langle R_\theta, R_\phi\rangle$, and thus describe when $\langle R_\theta, R_\phi\rangle = 0$. **(6 marks)**

(d) Put $V = \{X \in M_2(\mathbb{R}) \mid X \left[\begin{smallmatrix} 1\\1 \end{smallmatrix}\right] = 0\}$ and $W = \{Y \in M_2(\mathbb{R}) \mid Y \left[\begin{smallmatrix} 1\\-1 \end{smallmatrix}\right] = 0\}$.

(i) Find the general form for elements of V, and thus the dimension of V.**(3 marks)**

(ii) Find the general form for elements of W, and thus the dimension of W.**(2 marks)**

(iii) Show that every element of V is orthogonal to every element of W.**(1 marks)**

(iv) By comparing dimensions, prove that $V^\perp = W$.**(4 marks)**

(e) Find orthonormal bases for V and W. **(5 marks)**

Solution:

(a) **Bookwork.** The standard inner product on $M_2(\mathbb{R})$ is $\langle A, B\rangle = \text{trace}(AB^T)$. **[2]**

(b) In particular, if $A = \left[\begin{smallmatrix} 1&1\\0&1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1&0\\1&1 \end{smallmatrix}\right]$ we have $B^T = A$ so $AB^T = A^2 = \left[\begin{smallmatrix} 1&2\\0&1 \end{smallmatrix}\right]$ **[1]**so $\langle A, B\rangle = \text{trace}\left[\begin{smallmatrix} 1&2\\0&1 \end{smallmatrix}\right] = 2$ **[1]**.

(c) **Unseen.** We have

$$R_\theta R_\phi^T = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix} = \begin{bmatrix} \cos(\theta)\cos(\phi)+\sin(\theta)\sin(\phi) & \cos(\theta)\sin(\phi)-\sin(\theta)\cos(\phi) \\ \sin(\theta)\cos(\phi)-\cos(\theta)\sin(\phi) & \cos(\theta)\cos(\phi)+\sin(\theta)\sin(\phi) \end{bmatrix}.\textbf{[2]}$$

Taking traces, we get

$$\langle R_\theta, R_\phi\rangle = 2(\cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi)) = 2\cos(\theta - \phi).\textbf{[2]}$$

In particular, this means that $\langle R_\theta, R_\phi\rangle = 0$ iff $\theta - \phi$ has the form $(n + \frac{1}{2})\pi$ for some integer n. **[2]**

(d) **Similar to problem sheets.**

(i) For $X = \left[\begin{smallmatrix} w&x\\y&z \end{smallmatrix}\right] \in M_2(\mathbb{R})$ we have $X\left[\begin{smallmatrix} 1\\1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} w+x\\y+z \end{smallmatrix}\right]$, so $X \in V$ iff $x = -w$ and $z = -y$. **[1]**This means that

$$X = \left[\begin{smallmatrix} w&-w\\y&-y \end{smallmatrix}\right] = w\left[\begin{smallmatrix} 1&-1\\0&0 \end{smallmatrix}\right] + x\left[\begin{smallmatrix} 0&0\\1&-1 \end{smallmatrix}\right].\textbf{[1]}$$

It follows that V has dimension 2, with basis $\left[\begin{smallmatrix} 1&-1\\0&0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0&0\\1&-1 \end{smallmatrix}\right]$. **[1]**

(ii) Similarly, for a matrix $Y = \left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right] \in M_2(\mathbb{R})$ we have $Y\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} p-q \\ r-s \end{smallmatrix}\right]$, so $Y \in W$ iff $p = q$ and $r = s$. This means that

$$Y = \left[\begin{smallmatrix} p & p \\ r & r \end{smallmatrix}\right] = p\left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right] + q\left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right].\mathbf{[1]}$$

It follows that W also has dimension 2. **[1]**

(iii) We have

$$\langle X, Y\rangle = \operatorname{trace}(XY^T) = \operatorname{trace}\left(\left[\begin{smallmatrix} w & -w \\ y & -y \end{smallmatrix}\right]\left[\begin{smallmatrix} p & r \\ p & r \end{smallmatrix}\right]\right) = \operatorname{trace}\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] = 0.\mathbf{[1]}$$

(iv) Part (iii) tells us that $W \leq V^\perp$. **[1]**However, we also have $\dim(W) = 2$ and $\dim(V^\perp) = \dim(M_2(\mathbb{R})) - \dim(V) = 4 - 2 = 2 = \dim(W)$, so we must have $V^\perp = W$. **[3]**

(e) **Similar to problem sheets.** We have seen that $\left[\begin{smallmatrix} 1 & -1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 1 & -1 \end{smallmatrix}\right]$ is a basis for V **[1]**, and these two matrices are orthogonal **[1]**. It follows that $\frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & -1 \\ 0 & 0 \end{smallmatrix}\right], \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 0 & 0 \\ 1 & -1 \end{smallmatrix}\right]$ is an orthonormal basis for V **[1]**. Similarly, the list $\frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right], \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ is an orthonormal basis for W. **[2]**

(19) **[0607R Q3]** Let $\phi\colon V \to W$ be a linear map.

(a) Define what it means for ϕ be (i) injective; (ii) surjective; (iii) bijective. **(5 marks)**

(b) Define $\ker(\phi)$, and show that it is a subspace of V. **(5 marks)**

(c) Show that ϕ is injective iff $\ker(\phi) = 0$. **(7 marks)**

(d) Consider the linear map $\psi\colon \mathbb{R}[x]_{\leq 3} \to M_2(\mathbb{R})$ given by

$$\psi(p) = \begin{bmatrix} p(1) & p(-1) \\ p'(1) & p'(-1) \end{bmatrix}.$$

Show that ψ is injective. (You need not prove that it is linear.) **(4 marks)**

(e) Use the rank-nullity formula to deduce that ψ is also surjective. **(4 marks)**

Solution:

(a) **Bookwork.**

(i) ϕ is injective iff whenever $v, v' \in V$ and $\phi(v) = \phi(v')$ we have $v = v'$. **[2]**

(ii) ϕ is surjective iff for all $w \in W$ there exists some $v \in V$ with $\phi(v) = w$. **[2]**

(iii) ϕ is bijective iff it is both injective and surjective. **[1]**

(b) **Bookwork.** $\ker(\phi)$ is defined to be the set of all $v \in V$ such that $\phi(v) = 0_W$ **[2]**. We certainly have $\phi(0_V) = 0_W$, so $0_V \in \ker(\phi)$ **[1]**. Suppose we have $v, v' \in \ker(\phi)$ and $t, t' \in \mathbb{R}$. We then have

$$\phi(tv + t'v') = t\phi(v) + t'\phi(v') = t.0_W + t'.0_W = 0_W,$$

so $tv + t'v' \in \ker(\phi)$. **[2]** This proves that $\ker(\phi)$ is a subspace.

(c) **Bookwork.** Suppose that ϕ is injective. If $v \in \ker(\phi)$ then we have $\phi(v) = 0 = \phi(0)$ and so (by injectivity) $v = 0$. Thus $\ker(\phi) = 0$. **[3]**

Conversely, suppose that $\ker(\phi) = 0$. Suppose we have $v, v' \in V$ with $\phi(v) = \phi(v')$. Then $\phi(v - v') = \phi(v) - \phi(v') = 0 - 0 = 0$, so $v - v' \in \ker(\phi) = \{0\}$, so $v - v' = 0$, so $v = v'$. This shows that ϕ is injective. **[4]**

(d) **Similar to problem sheets.** Now consider the map $\psi\colon \mathbb{R}[x]_{\leq 3} \to M_2(\mathbb{R})$ given by

$$\psi(p) = \begin{bmatrix} p(1) & p(-1) \\ p'(1) & p'(-1) \end{bmatrix}.$$

More explicitly, if $p(x) = ax^3 + bx^2 + cx + d$ then

$$\psi(p) = \begin{bmatrix} a+b+c+d & -a+b-c+d \\ 3a+2b+c & 3a-2b+c \end{bmatrix}.\textbf{[1]}$$

If $p \in \ker(\psi)$ then $\psi(p) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so

$$a + b + c + d = 0 \quad \text{(A)}$$
$$-a + b - c + d = 0 \quad \text{(B)}$$
$$3a + 2b + c = 0 \quad \text{(C)}$$
$$3a - 2b + c = 0.\textbf{[1]} \quad \text{(D)}$$

By subtracting (C) and (D) we get $b = 0$. Similarly, by adding (A) and (B) we get $2b+2d = 0$ but $b = 0$ so $d = 0$. After substituting these values our equations become $a + c = 0$ and $3a+c = 0$, and we can subtract these to get $a = 0$ and thus also $c = 0$. As $a = b = c = d = 0$ we have $p(x) = 0$. **[1]**This shows that $\ker(\psi) = \{0\}$, so ψ is injective. **[1]**

(e) **Similar to problem sheets.** We have $\dim(\mathbb{R}[x]_{\leq 3}) = \dim(M_2(\mathbb{R})) = 4$. The rank-nullity formula says that

$$\dim(\ker(\psi)) + \dim(\text{image}(\psi)) = \dim(\mathbb{R}[x]_{\leq 3}) = 4.\textbf{[1]}$$

As $\ker(\psi) = 0$ **[1]**this gives $\dim(\text{image}(\psi)) = 4 = \dim(M_2(\mathbb{R}))$ **[1]**, so $\text{image}(\psi) = M_2(\mathbb{R})$**[1]**, so ψ is surjective.

(20) [0708 Q3]

(a) Suppose we have a finite-dimensional vector space U, and subspaces V and W of U. State a theorem about the existence of compatible bases for $V \cap W$, V, W and $V + W$. (You need not give a proof.) **(6 marks)**

(b) Deduce a formula relating the dimensions of $V \cap W$, V, W and $V + W$. **(2 marks)**

Now take $U = M_2(\mathbb{R})$, and consider the spaces

$$V = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R}) \mid a + b = c + d\}$$
$$W = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R}) \mid a + c = b + d\}.$$

(c) Find a basis A_1, A_2 for $V \cap W$ **(5 marks)**

(d) Find a matrix B such that A_1, A_2, B is a basis for V. You should justify your answer carefully, either directly or by consideration of dimensions. **(4 marks)**

(e) Find a matrix C such that A_1, A_2, C is a basis for W. Here you need not justify your answer. **(2 marks)**

(f) Show that $V + W = M_2(\mathbb{R})$, either directly or by consideration of dimensions. **(3 marks)**

(g) Write down a matrix X such that $\langle A, X\rangle = 0$ for all $A \in V$. (Here we use the standard inner product on $M_2(\mathbb{R})$.) **(3 marks)**

Solution:

(a) **Bookwork.** There exist vectors $u_1, \dots, u_p$, $v_1, \dots, v_q$ and $w_1, \dots, w_r$ (for some $p, q, r \geq 0$) **[2]** such that

- $u_1, \dots, u_p$ is a basis for $V \cap W$ **[1]**
- $u_1, \dots, u_p, v_1, \dots, v_q$ is a basis for V **[1]**
- $u_1, \dots, u_p, w_1, \dots, w_r$ is a basis for W **[1]**
- $u_1, \dots, u_p, v_1, \dots, v_q, w_1, \dots, w_r$ is a basis for $V + W$. **[1]**

(b) **Bookwork.** It follows that

$$\dim(V+W) = p+q+r = (p+r)+(q+r)-r = \dim(V)+\dim(W)-\dim(V\cap W).\textbf{[2]}$$

(c) **Similar to lecture notes and problem sheets.** $V \cap W$ is the set of matrices $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ satisfying $a+b-c-d=0$ and $a-b+c-d=0$ **[1]**. These equations easily reduce to $a=d$ and $b=c$ **[1]**, so $V \cap W$ is the set of matrices of the form

$$A = \left[\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right] = a\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] + b\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right].\textbf{[1]}$$

From this, we see that the matrices $A_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $A_2 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ give a basis for $V \cap W$. **[2]**

(d) **Similar to lecture notes and problem sheets.** Take $B = \left[\begin{smallmatrix} 0 & 0 \\ 1 & -1 \end{smallmatrix}\right]$ **[1]**. We find that $B \in V$ and A_1, A_2, B are linearly independent. They therefore span a subspace of V of dimension three, but V is a proper subspace of $M_2(\mathbb{R})$ and so has dimension at most three, so A_1, A_2, B must be a basis for V. More explicitly, a typical element $A \in V$ has the form $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with $a+b=c+d$, so $d = a+b-c$, so

$$A - aA_1 - bA_2 = \left[\begin{smallmatrix} a & b \\ c & a+b-c \end{smallmatrix}\right] - \left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right] - \left[\begin{smallmatrix} 0 & b \\ b & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 0 \\ c-b & b-c \end{smallmatrix}\right] = (c-b)B,$$

so $A = aA_1 + bA_2 + (c-b)B$. Thus V is spanned by A_1, A_2, B. **[3]**

(e) Take $C = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right]$. **[2]**

(f) **Unseen.** In general, if V and W are two subspaces of a vector space U, we have $\dim(V+W)+\dim(V\cap W) = \dim(V)+\dim(W)$. In our case, part (a) tells us that $\dim(V \cap W) = 2$, and parts (b) and (c) tell us that $\dim(V) = \dim(W) = 3$. It follows that $\dim(V+W) = 3+3-2 = 4$ **[2]**. However, $V+W$ is a subspace of the four-dimensional space $M_2(\mathbb{R})$, and a subspace of the same dimension as the total space must be equal to the total space, so $V+W = M_2(\mathbb{R})$ as required. **[1]**

(g) **Unseen.** Put $X = \left[\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right]$. Then for $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ we have $\langle A, X\rangle = a+b-c-d$, so $\langle A, X\rangle = 0$ iff $a+b=c+d$ iff $A \in V$. **[3]**

(21) [0708R Q3] Consider the linear map $\alpha\colon \mathbb{R}[x]_{\leq 2} \to M_2(\mathbb{R})$ given by

$$\alpha(f) = \begin{bmatrix} f(1) & f(2) \\ f'(1) & f'(2) \end{bmatrix}.$$

(a) Write down a basis $\mathcal{U}$ for $\mathbb{R}[x]_{\leq 2}$ and a basis $\mathcal{V}$ for $M_2(\mathbb{R})$. **(3 marks)**

(b) Find the matrix of α with respect to the bases $\mathcal{U}$ and $\mathcal{V}$. **(5 marks)**

(c) Show that α is injective. **(4 marks)**

(d) Give a basis for the image of α. **(4 marks)**

(e) Find a nonzero matrix $X = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ such that $\langle X, \alpha(x^i)\rangle = 0$ for $i = 0, 1, 2$. **(5 marks)**
(Here we use the standard inner product for square matrices.)

(f) Show (by an explicit example) that α is not surjective. **(4 marks)**

Solution: This is a slight modification of a question from a past paper.

(a) **Similar to problem sheets** The obvious basis for $\mathbb{R}[x]_{\leq 2}$ consists of the polynomials $p_0(x) = 1$, $p_1(x) = x$ and $p_2(x) = x^2$.**[1]**The obvious basis for $M_2(\mathbb{R})$ consists of the matrices

$$E_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \qquad E_2 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \qquad E_3 = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right] \qquad E_4 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right].\mathbf{[2]}$$

(b) **Similar to problem sheets** We have

$$\begin{aligned} \alpha(p_0) &= \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right] = 1.E_1 + 1.E_2 + 0.E_3 + 0.E_4 \\ \alpha(p_1) &= \left[\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right] = 1.E_1 + 2.E_2 + 1.E_3 + 1.E_4 \\ \alpha(p_2) &= \left[\begin{smallmatrix} 1 & 4 \\ 2 & 4 \end{smallmatrix}\right] = 1.E_1 + 4.E_2 + 2.E_3 + 4.E_4\mathbf{[3]} \end{aligned}$$

so the matrix of α with respect to our bases is

$$A = \left[\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{smallmatrix}\right].\mathbf{[2]}$$

(c) **Similar to problem sheets** From the above we see that

$$\alpha(a + bx + cx^2) = a\alpha(p_0) + b\alpha(p_1) + c\alpha(p_2) = \left[\begin{smallmatrix} a+b+c & a+2b+4c \\ b+2c & b+4c \end{smallmatrix}\right].\mathbf{[2]}$$

Thus, if $\alpha(a + bx + cx^2) = 0$ we see that

$$\begin{aligned} a + b + c &= 0 \\ a + 2b + 4c &= 0 \\ b + 2c &= 0 \\ b + 4c &= 0. \end{aligned}$$

By subtracting the last two equations we see that $c = 0$, and we can substitute this in the last equation to give $b = 0$. The first equation then gives $a = 0$ as well. This shows that $\ker(\alpha) = \{0\}$ and thus that α is injective. **[2]**

(d) **Unseen** As α is injective, the matrices $\alpha(p_0) = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, $\alpha(p_1) = \left[\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right]$ and $\alpha(p_2) = \left[\begin{smallmatrix} 1 & 4 \\ 2 & 4 \end{smallmatrix}\right]$ form a basis for the image. **[4]**

(e) **Similar to problem sheets** We have

$$\begin{aligned} \langle X, \alpha(1)\rangle &= \langle \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]\rangle = a + b \\ \langle X, \alpha(x)\rangle &= \langle \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right]\rangle = a + 2b + c + d \\ \langle X, \alpha(x^2)\rangle &= \langle \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 4 \\ 2 & 4 \end{smallmatrix}\right]\rangle = a + 4b + 2c + 4d\mathbf{[2]} \end{aligned}$$

We therefore must have

$$\begin{aligned} a + b &= 0 \\ a + 2b + c + d &= 0 \\ a + 4b + 2c + 4d &= 0. \end{aligned}$$

The first equation gives $b = -a$, using which we rewrite the other two as $-a + c + d = 0$ and $-3a + 2c + 4d = 0$. Subtracting three times the first of these from the second gives $c = d$, using which we get $a = 2d$. Thus X must have the form $X = \left[\begin{smallmatrix} 2d & -2d \\ d & d \end{smallmatrix}\right]$. **[2]** Here d is arbitrary so we can take $X = \left[\begin{smallmatrix} 2 & -2 \\ 1 & 1 \end{smallmatrix}\right]$. **[1]**

(f) **Unseen** Observe that $\langle X, X\rangle = 2^2+(-2)^2+1^2+1^2 = 10 \neq 0$, but $\langle X, A\rangle = 0$ for all A in the image of α (by part (d)). It follows that $X \notin \text{image}(\alpha)$, and thus that $\text{image}(\alpha) \neq M_2(\mathbb{R})$, so α is not surjective. **[4]**

(22) [**Mock exam Q4**] Define a linear map $\alpha\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}^3$ by

$$\alpha(p) = \left[\int_{-1}^{1} p(x)\,dx, \int_{-2}^{2} p(x)\,dx, \int_{-3}^{3} p(x)\,dx\right]^T$$

(a) Give a basis for $\mathbb{R}[x]_{\leq 2}$ and a basis for $\mathbb{R}^3$.

(b) Find the matrix of α with respect to your bases in (a).

(c) Find bases for $\ker(\alpha)$ and $\text{image}(\alpha)$.

(d) Show that

$$\text{image}(\alpha) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid 5x - 4y + z = 0 \right\}.$$

Solution:

(a) The list $\mathcal{P} = 1, x, x^2$ is a basis for $\mathbb{R}[x]_{\leq 2}$. The list $\mathcal{E} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a basis for $\mathbb{R}^3$.

(b) If $f(x) = a + bx + cx^2$ then $\int f(x)\,dx = F(x) = ax + bx^2/2 + cx^3/3$, so

$$\int_{-1}^{1} f(x)\,dx = F(1) - F(-1) = 2a + 2c/3$$
$$\int_{-2}^{2} f(x)\,dx = F(2) - F(-2) = 4a + 16c/3$$
$$\int_{-3}^{3} f(x)\,dx = F(3) - F(-3) = 6a + 54c/3,$$

so

$$\alpha(1) = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \quad \alpha(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \alpha(x^2) = \begin{bmatrix} 2/3 \\ 16/3 \\ 54/3 \end{bmatrix}.$$

The matrix of α with respect to our bases is thus

$$A = \begin{bmatrix} 2 & 0 & 2/3 \\ 4 & 0 & 16/3 \\ 6 & 0 & 54/3 \end{bmatrix}$$

(c) If $f(x) = a + bx + cx^2$ we have

$$\alpha(f) = [2a + 2c/3,\ 4a + 16c/3,\ 6a + 54c/3]^T.$$

For this to be zero we must have $2a + 2c/3 = 0$ and $4a + 16c/3 = 0$ and $6a + 54c/3 = 0$, or equivalently $a = -c/3$ and $a = -4c/3$ and $a = -3c$, which together give $a = 0$ and $c = 0$. However, b can be arbitrary. It follows that x is a basis for $\ker(\alpha)$. On the other hand, using the equation

$$\alpha(f) = 2a \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \frac{2c}{3} \begin{bmatrix} 1 \\ 8 \\ 27 \end{bmatrix},$$

we see that $[1, 2, 3]^T, [1, 8, 27]^T$ is a basis for $\text{image}(\alpha)$.

(d) Put

$$W = \left\{ \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix} \right] \mid 5x - 4y + z = 0 \right\}.$$

We claim that $W = \text{image}(\alpha)$. Indeed, we have

$$5 \times 1 - 4 \times 2 + 3 = 0$$
$$5 \times 1 - 4 \times 8 + 27 = 0,$$

so the vectors $[1, 2, 3]^T$ and $[1, 8, 27]^T$ lie in W. As these vectors span $\text{image}(\alpha)$, we see that $\text{image}(\alpha) \leq W$. On the other hand, W is a plane in $\mathbb{R}^3$ so it has dimension 2, which is the same as $\dim(\text{image}(\alpha))$, so $\text{image}(\alpha) = W$ as claimed.

(23) [0506 Q4] Let $\phi \colon V \to W$ be a linear map.

(a) Define what it means for ϕ be (i) injective; (ii) surjective; (iii) bijective. **(5 marks)**

(b) Define $\ker(\phi)$, and show that it is a subspace of V. **(5 marks)**

(c) Show that ϕ is injective iff $\ker(\phi) = 0$. **(7 marks)**

(d) Consider the linear map $\psi \colon \mathbb{R}[x]_{\leq 3} \to M_2\mathbb{R}$ given by

$$\psi(p) = \left[\begin{smallmatrix} p(0) & p(1) \\ p(-1) & p(2) \end{smallmatrix} \right].$$

Show that ψ is injective. (You need not prove that it is linear.) **(4 marks)**

(e) Use the rank-nullity formula to deduce that ψ is also surjective. **(4 marks)**

Solution:

(a) **Bookwork.**

(i) ϕ is injective iff whenever $v, v' \in V$ and $\phi(v) = \phi(v')$ we have $v = v'$. **[2]**

(ii) ϕ is surjective iff for all $w \in W$ there exists some $v \in V$ with $\phi(v) = w$. **[2]**

(iii) ϕ is bijective iff it is both injective and surjective. **[1]**

(b) **Bookwork.** $\ker(\phi)$ is defined to be the set of all $v \in V$ such that $\phi(v) = 0_W$ **[2]**. We certainly have $\phi(0_V) = 0_W$, so $0_V \in \ker(\phi)$ **[1]**. Suppose we have $v, v' \in \ker(\phi)$ and $t, t' \in \mathbb{R}$. We then have

$$\phi(tv + t'v') = t\phi(v) + t'\phi(v') = t.0_W + t'.0_W = 0_W,$$

so $tv + t'v' \in \ker(\phi)$. **[2]** This proves that $\ker(\phi)$ is a subspace.

(c) **Bookwork.** Suppose that ϕ is injective. If $v \in \ker(\phi)$ then we have $\phi(v) = 0 = \phi(0)$ and so (by injectivity) $v = 0$. Thus $\ker(\phi) = 0$. **[3]**

Conversely, suppose that $\ker(\phi) = 0$. Suppose we have $v, v' \in V$ with $\phi(v) = \phi(v')$. Then $\phi(v - v') = \phi(v) - \phi(v') = 0 - 0 = 0$, so $v - v' \in \ker(\phi) = \{0\}$, so $v - v' = 0$, so $v = v'$. This shows that ϕ is injective. **[4]**

(d) **Similar to problem sheets.** Now consider the map $\psi \colon \mathbb{R}[x]_{\leq 3} \to M_2\mathbb{R}$ given by

$$\psi(p) = \left[\begin{smallmatrix} p(0) & p(1) \\ p(-1) & p(2) \end{smallmatrix} \right].$$

If $p \in \ker(\psi)$ then $\psi(p) = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right]$, so $p(0) = p(1) = p(2) = p(3) = 0$ **[1]**. Thus p is a polynomial of degree 3 with at least 4 different roots; this can only happen if $p = 0$. More explicitly, suppose that $p(x) = a + bx + cx^2 + dx^3$. Then

$$\psi(p) = \left[\begin{smallmatrix} a & a+b+c+d \\ a-b+c-d & a+2b+4c+8d \end{smallmatrix} \right], \textbf{[1]}$$

so $\psi(p) = 0$ iff $a = 0$ and $a + b + c + d = 0$ and $a - b + c - d = 0$ and $a + 2b + 4c + 8d = 0$ [**1**]. Putting $a = 0$ in the remaining equations gives $b + c + d = 0$ and $-b + c - d = 0$ and $2b + 4c + 8d = 0$. By adding the first two of these we get $c = 0$, so $b + d = 0$ and $2b + 8d = 0$. These equations easily give $b = d = 0$ as well, so $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = 0$ [**1**]. Thus $\ker(\psi) = 0$ and so ψ is injective.

(e) **Similar to problem sheets.** We have $\dim(\mathbb{R}[x]_{\leq 3}) = \dim(M_2\mathbb{R}) = 4$. The rank-nullity formula says that

$$\dim(\ker(\psi)) + \dim(\text{image}(\psi)) = \dim(\mathbb{R}[x]_{\leq 3}) = 4.[\mathbf{1}]$$

As $\ker(\psi) = 0$ [**1**]this gives $\dim(\text{image}(\psi)) = 4 = \dim(M_2\mathbb{R})$ [**1**], so $\text{image}(\psi) = M_2\mathbb{R}$[**1**], so ψ is surjective.

(24) [0506R Q4] Let $\phi\colon V \to W$ be a linear map.

(a) Define what it means for ϕ be (i) injective; (ii) surjective. **(4 marks)**

(b) Define $\text{image}(\phi)$, and show that it is a subspace of W. **(5 marks)**

(c) Define $\ker(\phi)$, and show that if $\ker(\phi) = \{0\}$ then ϕ injective. **(5 marks)**

(d) State the rank-nullity formula. **(3 marks)**

(e) Consider the linear map $\psi\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}^2$ given by $\psi(f) = [f(2), f(3)]^T$. What are the dimensions of $\mathbb{R}[x]_{\leq 2}$, $\mathbb{R}^2$ and $\ker(\psi)$? **(5 marks)**

(f) Use the rank-nullity formula to deduce that ψ is surjective. **(3 marks)**

Solution:

(a) **Bookwork.**

(i) ϕ is injective iff whenever $v, v' \in V$ and $\phi(v) = \phi(v')$ we have $v = v'$. [**2**]

(ii) ϕ is surjective iff for all $w \in W$ there exists some $v \in V$ with $\phi(v) = w$. [**2**]

(b) **Bookwork.** $\text{image}(\phi) = \{w \in W \mid w = \phi(v) \text{ for some } v \in V\}$ [**1**]. As $0_W = \phi(0_V)$, we see that $0_W \in \text{image}(\phi)$. [**1**]If $w, w' \in \text{image}(\phi)$ and $t, t' \in \mathbb{R}$ then we can choose $v, v' \in V$ such that $w = \phi(v)$ and $w' = \phi(v')$, and then we find that $\phi(tv+t'v') = t\phi(v)+t'\phi(v') = tw+t'w'$, so $tw + t'w' \in \text{image}(\phi)$. This proves that $\text{image}(\phi)$ is a subspace of W. [**3**]

(c) **Bookwork.** $\ker(\phi) = \{v \in V \mid \phi(v) = 0\}$ [**1**]. Suppose that $\ker(\phi) = \{0\}$; we claim that ϕ is injective. Suppose that $v, v' \in V$ and $\phi(v) = \phi(v')$; we must show that $v = v'$ [**2**]. As $\phi(v) = \phi(v')$ we have $\phi(v - v') = \phi(v) - \phi(v') = 0$ [**1**], so $v - v' \in \ker(\phi) = \{0\}$, so $v - v' = 0$, so $v = v'$ as required [**1**].

(d) Let $\phi\colon V \to W$ be a linear map between finite-dimensional vector spaces. Then $\dim(\text{image}(\phi)) + \dim(\ker(\phi)) = \dim(V)$. [**3**]

(e) **Similar to problem sheets.** Now consider the linear map $\psi\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}^2$ given by $\psi(f) = [f(2), f(3)]^T$. The space $\mathbb{R}[x]_{\leq 2}$ has basis $1, x, x^2$ and so has dimension 3 [**1**]. The space $\mathbb{R}^2$ has basis $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ and so has dimension 2 [**1**]. We have $\psi(f) = 0$ iff $f(2) = f(3) = 0$ iff $f(x)$ is divisible by $(x-2)(x-3)$. If so then $f(x)$ must be a constant times $(x-2)(x-3)$ (as otherwise the degree would be too large). Thus $\ker(\psi)$ has basis $(x-2)(x-3)$ and thus dimension 1. [**3**]

(f) The rank-nullity formula now tells us that the space $\text{image}(\phi) \leq \mathbb{R}^2$ has dimension $3 - 1 = 2$ [**1**], which is the same as the dimension of $\mathbb{R}^2$ itself [**1**], so $\text{image}(\phi) = \mathbb{R}^2$, so ϕ is surjective [**1**].

(25) [0607 Q4]

(a) State and prove the Cauchy-Schwartz inequality. **(10 marks)**

(b) Find constants α and β such that $\int_{-1}^{1} f(x)\,dx = \alpha f(-1) + \beta f(0) + \alpha f(1)$ for all $f \in \mathbb{R}[x]_{\leq 2}$. **(6 marks)**

(c) Deduce that if $f \in \mathbb{R}[x]_{\leq 2}$ and $\int_{-1}^{1} f(t)^2\,dt = 1$, then $|f(-1) + 4f(0) + f(1)| \leq 3\sqrt{2}$. **(9 marks)**

(You may assume that the rule $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$ gives an inner product on $\mathbb{R}[x]_{\leq 2}$.)

Solution:

(a) **Bookwork.** Let V be a vector space with an inner product, and let u and v be elements of V. Then $|\langle u, v\rangle| \leq \|u\|\|v\|$.**[2]**

Proof: For any s and t we have

$0 \leq \|su - tv\|^2$**[1]** $= \langle su - tv, su - tv\rangle = s^2\langle u, u\rangle - 2st\langle u, v\rangle + t^2\langle v, v\rangle = s^2\|u\|^2 + t^2\|v\|^2 - 2st\langle u, v\rangle$.**[1]**

Now take $s = \|v\|^2$ and $t = \langle u, v\rangle$ **[2]** to get

$$0 \leq \|u\|^2\|v\|^4 + \langle u, v\rangle^2\|v\|^2 - 2\|v\|^2\langle u, v\rangle^2 = \|v\|^2(\|u\|^2\|v\|^2 - \langle u, v\rangle^2).\textbf{[1]}$$

If $v = 0$ then we have $|\langle u, v\rangle| = 0 = \|u\|\|v\|$ so the claim holds. **[1]**If $v \neq 0$ then $\|v\|^2 > 0$ so the above inequality will remain valid after dividing by $\|v\|^2$, giving $\langle u, v\rangle^2 \leq \|u\|^2\|v\|^2$. **[1]**We now take square roots (and note that $\sqrt{a^2} = |a|$) to get $|\langle u, v\rangle| \leq \|u\|\|v\|$, as claimed.**[1]**

(b) **Similar to problem sheets.** Consider a polynomial $f(x) = ax^2 + bx + c$. We then have

$$\int_{-1}^{1} f(x)\,dx = \left[ax^3/3 + bx^2/2 + cx\right]_{-1}^{1} = 2a/3 + 2c.\textbf{[2]}$$

On the other hand, we have

$$\alpha f(-1) + \beta f(0) + \alpha f(1) = \alpha(a - b + c) + \beta c + \alpha(a + b + c) = 2\alpha a + (2\alpha + \beta)c.\textbf{[2]}$$

For these to match up for all a, b and c we must have $2/3 = 2\alpha$ and $2 = 2\alpha + \beta$, which gives $\alpha = 1/3$ and $\beta = 4/3$. **[2]**

(c) **Unseen.** Part (b) tells us that $\langle f, 1\rangle = (f(-1) + 4f(0) + f(1))/3$, so $|f(-1) + 4f(0) + f(1)| = \langle f, 3\rangle$ **[2]**. The Cauchy-Schwartz inequality **[1]**tells us that this is at most $\|f\|\|3\|$ **[1]**. Here $\|f\|^2 = \int_{-1}^{1} f(t)^2\,dt$, and we are given that this is equal to one, so $\|f\| = 1$ **[1]**. We also have $\|3\|^2 = \int_{-1}^{1} 3^2\,dt = 18$ **[1]**and so $\|3\| = \sqrt{18} = 3\sqrt{2}$ **[1]**. Putting this together, we get $|f(-1) + 4f(0) + f(1)| \leq 3\sqrt{2}$ **[2]** as claimed.

(26) [0607R Q4] Let V be a vector space, and let $\mathcal{V} = v_1, \ldots, v_n$ be a list of elements of V.

(a) Define the map $\mu_{\mathcal{V}} \colon \mathbb{R}^n \to V$. **(3 marks)**

(b) Define, in terms of $\mu_{\mathcal{V}}$, what it means for $\mathcal{V}$ to be (i) linearly independent; (ii) a spanning set; (iii) a basis. **(6 marks)**

(c) Consider the polynomials

$$
\begin{aligned}
p_0(x) &= x^5 \\
p_1(x) &= 1 + x \\
p_2(x) &= x + x^2 \\
p_3(x) &= x^2 + x^3 \\
p_4(x) &= x^3 + x^4 \\
p_5(x) &= x^4 + x^5.
\end{aligned}
$$

Is the list $\mathcal{P} = p_0, \dots, p_4$ a basis for $\mathbb{R}[x]_{\leq 5}$? Justify your answer. **(7 marks)**

(d) Consider the list $\mathcal{A} = A_0, \dots, A_4$, where

$$
A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Prove that these are linearly independent. **(6 marks)**

(e) Find a linear relation between the following vectors **(3 marks)**

$$
\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_5 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}.
$$

Solution:

(a) The map $\mu_{\mathcal{V}} \colon \mathbb{R}^n \to V$ is defined by

$$\mu_{\mathcal{V}}([\lambda_1, \dots, \lambda_n]^T) = \lambda_1 v_1 + \cdots + \lambda_n v_n.\mathbf{[3]}$$

(b) The list $\mathcal{V}$ is linearly independent iff $\mu_{\mathcal{V}}$ is injective [**2**]. It spans V iff $\mu_{\mathcal{V}}$ is surjective [**2**]. It is a basis iff $\mu_{\mathcal{V}}$ is bijective [**2**].

(c) We have

$$
\begin{aligned}
\mu_{\mathcal{P}}(\boldsymbol{\lambda}) &= \lambda_0 x^5 + \lambda_1(1+x) + \lambda_2(x+x^2) + \lambda_3(x^2+x^3) + \lambda_4(x^3+x^4) + \lambda_5(x^4+x^5) \\
&= \lambda_1 + (\lambda_1+\lambda_2)x + (\lambda_2+\lambda_3)x^2 + (\lambda_3+\lambda_4)x^3 + (\lambda_4+\lambda_5)x^4 + (\lambda_0+\lambda_5)x^5.\mathbf{[2]}
\end{aligned}
$$

Thus, given a polynomial $f(x) = \sum_{i=0}^{5} a_i x^i$, we have $\mu_{\mathcal{P}}(\boldsymbol{\lambda}) = f$ iff the following equations are satisfied:

$$
\begin{aligned}
\lambda_1 &= a_0 \\
\lambda_1 + \lambda_2 &= a_1 \\
\lambda_2 + \lambda_3 &= a_2 \\
\lambda_3 + \lambda_4 &= a_3 \\
\lambda_4 + \lambda_5 &= a_4 \\
\lambda_0 + \lambda_5 &= a_5.\mathbf{[2]}
\end{aligned}
$$

It is easy to see that these have the unique solution

$$
\begin{aligned}
\lambda_0 &= a_5 - a_4 + a_3 - a_2 + a_1 - a_0 \\
\lambda_1 &= a_0 \\
\lambda_2 &= a_1 - a_0 \\
\lambda_3 &= a_2 - a_1 + a_0 \\
\lambda_4 &= a_3 - a_2 + a_1 - a_0 \\
\lambda_5 &= a_4 - a_3 + a_2 - a_1 + a_0.\mathbf{[2]}
\end{aligned}
$$

As this solution always exists and is unique, we see that $\mu_{\mathcal{P}}$ is a bijection and thus that $\mathcal{P}$ is a basis. [**1**]

(d) We have

$$\mu_{\mathcal{A}}(\boldsymbol{\lambda}) = \lambda_0 A_0 + \cdots + \lambda_4 A_4 = \left[\begin{smallmatrix} \lambda_0+\lambda_1+\lambda_2 & \lambda_1+\lambda_2 & \lambda_2 \\ \lambda_1+\lambda_2 & \lambda_1+\lambda_2+\lambda_3 & \lambda_2+\lambda_3 \\ \lambda_2 & \lambda_2+\lambda_3 & \lambda_2+\lambda_3+\lambda_4 \end{smallmatrix}\right].[\mathbf{2}]$$

For this to equal zero we would have to have

$$\begin{aligned} \lambda_0 + \lambda_1 + \lambda_2 &= 0 && (1) \\ \lambda_1 + \lambda_2 &= 0 && (2) \\ \lambda_2 &= 0 && (3) \\ \lambda_1 + \lambda_2 + \lambda_3 &= 0 && (4) \\ \lambda_2 + \lambda_3 &= 0 && (5) \\ \lambda_2 + \lambda_3 + \lambda_4 &= 0.[\mathbf{1}] && (6) \end{aligned}$$

From (3) we have $\lambda_2 = 0$, and we can substitute this in (2) and (5) to get $\lambda_1 = \lambda_3 = 0$. We can then substitute these values in (1) and (6) to get $\lambda_0 = \lambda_4 = 0$, so $\boldsymbol{\lambda} = 0$. **[2]** This shows that the list $\mathcal{A}$ has only the trivial linear relation, so it is linearly independent. **[1]**

(e) By inspection we have $3\mathbf{v}_1 - \mathbf{v}_3 - \mathbf{v}_4 = 0$. **[3]**

(27) [0708 Q4]

(a) Define the notion of an *inner product* on a finite-dimensional vector space over $\mathbb{R}$. **(5 marks)**

(b) Define what it means for a sequence of elements to be (i) *orthogonal*; (ii) *orthonormal*. **(3 marks)**

(c) Let V and W be vector spaces with inner products, and let $\phi\colon V \to W$ and $\psi\colon W \to V$ be linear maps. Define what it means for these maps to be *adjoint* to each other. **(2 marks)**

For the rest of this question, we use the spaces $V = M_2(\mathbb{R})$ and $W = \mathbb{R}[x]_{\leq 2}$, with the inner products

$$\begin{aligned} \langle A, B\rangle &= \text{trace}(A^T B) && \text{for } A, B \in V \\ \langle f, g\rangle &= f(-1)g(-1) + f(0)g(0) + f(1)g(1) && \text{for } f, g \in W. \end{aligned}$$

(You need not check that these are inner products.)

(d) Calculate $\langle x^i, x^j\rangle$ for $i, j = 0, \ldots, 2$. **(4 marks)**

(e) Using the Gram-Schmidt procedure or otherwise, find an orthonormal basis for W. **(6 marks)**

(f) Define $\phi\colon V \to W$ by $\phi(A) = \left[\begin{smallmatrix} x & 1 \end{smallmatrix}\right] A \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$, and let $\psi\colon W \to V$ be adjoint to ϕ. Calculate $\langle \phi \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], px^2 + qx + r\rangle$, and thus give a formula for $\psi(px^2 + qx + r)$. **(5 marks)**

Solution:

(a) **Bookwork.** An *inner product* on V is a rule that gives a number $\langle u, v\rangle \in \mathbb{R}$ for each $u, v \in V$ **[1]**, with the following properties:

(i) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$. **[1]**

(ii) $\langle tu, v\rangle = t\langle u, v\rangle$ for all $u, v \in V$ and $t \in \mathbb{R}$. **[1]**

(iii) $\langle u, v\rangle = \langle v, u\rangle$ for all $u, v \in V$. **[1]**

(iv) We have $\langle u, u\rangle \geq 0$ for all $u \in V$, and $\langle u, u\rangle = 0$ iff $u = 0$. **[1]**

(b) **Bookwork.** Consider a sequence $\mathcal{V} = v_1, \dots, v_n$ in a vector space V with inner product. We say that $\mathcal{V}$ is *orthogonal* if $\langle v_i, v_j\rangle = 0$ for all $i \neq j$ **[2]**, and *orthonormal* if it is orthogonal and also $\langle v_i, v_i\rangle = 1$ for all i. **[1]**

(c) **Bookwork.** Let V and W be vector spaces with inner products, and let $\phi\colon V \to W$ and $\psi\colon W \to V$ be linear maps. We say that ϕ and ψ are *adjoint* if for all $v \in V$ and all $w \in W$ we have $\langle \phi(v), w\rangle = \langle v, \psi(w)\rangle$. **[2]**

Some students will probably attempt to do parts (d) to (f) using either $\langle f, g\rangle = \int_0^1 fg$ or $\langle f, g\rangle = \int_{-1}^1 fg$. There will be an overall penalty of one point for that, together with the implicit penalty that the calculations become a little harder.

(d) **Similar to lecture notes and problem sheets.** For the inner product given, we have

$$\langle x^i, x^j\rangle = (-1)^{i+j} + 0^{i+j} + 1.$$

If $i = j = 0$ this gives 3. In all other cases, the second term is zero and we get 0 if $i + j$ is odd, and 2 if $i + j$ is even. Thus

$$\begin{aligned} \langle 1, 1\rangle &= 3 & \langle 1, x\rangle &= 0 & \langle 1, x^2\rangle &= 2 \\ \langle x, 1\rangle &= 0 & \langle x, x\rangle &= 2 & \langle x, x^2\rangle &= 0 \\ \langle x^2, 1\rangle &= 2 & \langle x^2, x\rangle &= 0 & \langle x^2, x^2\rangle &= 2. \mathbf{[4]} \end{aligned}$$

(e) **Similar to lecture notes and problem sheets.** We use the Gram-Schmidt procedure, starting with $u_1 = 1$ and $u_2 = x$ and $u_3 = x^2$. We then put

$$\begin{aligned} v_1 &= u_1 = 1 \\ v_2 &= u_2 - \frac{\langle u_2, v_1\rangle}{v_1, v_1} v_1 = x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} 1 = x - \frac{0}{3} = x \\ v_3 &= u_3 - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 \\ &= x^2 - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle} 1 - \frac{\langle x^2, x\rangle}{\langle x, x\rangle} x \\ &= x^2 - \frac{2}{3} 1 - \frac{0}{2} x = x^2 - 2/3. \mathbf{[4]} \end{aligned}$$

This gives an orthogonal basis. We find that $\|v_1\| = \sqrt{3}$ and $\|v_2\| = \sqrt{2}$ and

$$\|v_3\|^2 = \langle x^2 - 2/3, x^2 - 2/3\rangle = ((-1)^2 - 2/3)^2 + (0^2 - 2/3)^2 + (1^2 - 2/3)^2 = 6/9 = 2/3, \mathbf{[1]}$$

so $\|v_3\| = \sqrt{2/3}$. We then put $\hat{v}_i = v_i/\|v_i\|$, so

$$\begin{aligned} \hat{v}_1 &= 1/\sqrt{3} \\ \hat{v}_2 &= x/\sqrt{2} \\ \hat{v}_3 &= \sqrt{3/2}(x^2 - 2/3). \mathbf{[1]} \end{aligned}$$

This gives the required orthonormal basis.

(f) **Similar to lecture notes and problem sheets.** We have

$$\psi \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} x & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] = ax^2 + (b + c)x + d, \mathbf{[1]}$$

so

$$\begin{aligned}\langle \phi \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], px^2+qx+r\rangle &= \langle ax^2+(b+c)x+c, px^2+qx+r\rangle \\ &= (a-b-c+d)(p-q+r)+cr+(a+b+c+d)(p+q+r) \\ &= ap-aq+ar-bp+bq-br-cp+cq-cr+dp-dq+dr+cr+ \\ &\quad ap+aq+ar+bp+bq+br+cp+cq+cr+dp+dq+dr \\ &= (2p+2r)a+2qb+2qc+(2p+3r)d[\mathbf{2}] \\ &= \langle \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], \left[\begin{smallmatrix} 2p+2r & 2q \\ 2q & 2p+3r \end{smallmatrix}\right]\rangle.[\mathbf{1}]\end{aligned}$$

It follows that $\psi(px^2+qx+r) = \left[\begin{smallmatrix} 2p+2r & 2q \\ 2q & 2p+3r \end{smallmatrix}\right]$. **[1]**

(28) **[0708R Q4]** Let V be a vector space, and let $\mathcal{V} = v_1, \dots, v_n$ be a list of elements of V.

(a) Define the map $\mu_{\mathcal{V}} \colon \mathbb{R}^n \to V$. **(3 marks)**

(b) Define, in terms of $\mu_{\mathcal{V}}$, what it means for $\mathcal{V}$ to be (i) linearly independent; (ii) a spanning set; (iii) a basis. **(6 marks)**

(c) Consider the polynomials

$$\begin{aligned} p_0(x) &= x^5 \\ p_1(x) &= 1+x \\ p_2(x) &= x+x^2 \\ p_3(x) &= x^2+x^3 \\ p_4(x) &= x^3+x^4 \\ p_5(x) &= x^4+x^5. \end{aligned}$$

Is the list $\mathcal{P} = p_0, \dots, p_5$ a basis for $\mathbb{R}[x]_{\leq 5}$? Justify your answer. **(7 marks)**

(d) Consider the list $\mathcal{A} = A_0, \dots, A_4$, where

$$A_0 = \left[\begin{smallmatrix} 1&0&0\\0&0&0\\0&0&0 \end{smallmatrix}\right] \quad A_1 = \left[\begin{smallmatrix} 1&1&0\\1&1&0\\0&0&0 \end{smallmatrix}\right] \quad A_2 = \left[\begin{smallmatrix} 1&1&1\\1&1&1\\1&1&1 \end{smallmatrix}\right] \quad A_3 = \left[\begin{smallmatrix} 0&0&0\\0&1&1\\0&1&1 \end{smallmatrix}\right] \quad A_4 = \left[\begin{smallmatrix} 0&0&0\\0&0&0\\0&0&1 \end{smallmatrix}\right]$$

Prove that these are linearly independent. **(6 marks)**

(e) Find a linear relation between the following vectors **(3 marks)**

$$\mathbf{v}_1 = \begin{bmatrix} 1\\1\\1\\1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2\\1\\1\\2 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1\\1\\2\\2 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} 2\\2\\1\\1 \end{bmatrix} \qquad \mathbf{v}_5 = \begin{bmatrix} 1\\2\\1\\2 \end{bmatrix}.$$

Solution: This is a slight modification of a question from a past paper.

(a) The map $\mu_{\mathcal{V}} \colon \mathbb{R}^n \to V$ is defined by

$$\mu_{\mathcal{V}}([\lambda_1, \dots, \lambda_n]^T) = \lambda_1 v_1 + \cdots + \lambda_n v_n.[\mathbf{3}]$$

(b) The list $\mathcal{V}$ is linearly independent iff $\mu_{\mathcal{V}}$ is injective [**2**]. It spans V iff $\mu_{\mathcal{V}}$ is surjective [**2**]. It is a basis iff $\mu_{\mathcal{V}}$ is bijective [**2**].

(c) We have

$$\begin{aligned}\mu_{\mathcal{P}}(\boldsymbol{\lambda}) &= \lambda_0 x^5 + \lambda_1(1+x) + \lambda_2(x+x^2) + \lambda_3(x^2+x^3) + \lambda_4(x^3+x^4) + \lambda_5(x^4+x^5) \\ &= \lambda_1 + (\lambda_1+\lambda_2)x + (\lambda_2+\lambda_3)x^2 + (\lambda_3+\lambda_4)x^3 + (\lambda_4+\lambda_5)x^4 + (\lambda_0+\lambda_5)x^5.[\mathbf{2}]\end{aligned}$$

Thus, given a polynomial $f(x) = \sum_{i=0}^{5} a_i x^i$, we have $\mu_{\mathcal{P}}(\boldsymbol{\lambda}) = f$ iff the following equations are satisfied:

$$\begin{aligned} \lambda_1 &= a_0 \\ \lambda_1 + \lambda_2 &= a_1 \\ \lambda_2 + \lambda_3 &= a_2 \\ \lambda_3 + \lambda_4 &= a_3 \\ \lambda_4 + \lambda_5 &= a_4 \\ \lambda_0 + \lambda_5 &= a_5. \mathbf{[2]} \end{aligned}$$

It is easy to see that these have the unique solution

$$\begin{aligned} \lambda_0 &= a_5 - a_4 + a_3 - a_2 + a_1 - a_0 \\ \lambda_1 &= a_0 \\ \lambda_2 &= a_1 - a_0 \\ \lambda_3 &= a_2 - a_1 + a_0 \\ \lambda_4 &= a_3 - a_2 + a_1 - a_0 \\ \lambda_5 &= a_4 - a_3 + a_2 - a_1 + a_0. \mathbf{[2]} \end{aligned}$$

As this solution always exists and is unique, we see that $\mu_{\mathcal{P}}$ is a bijection and thus that $\mathcal{P}$ is a basis. **[1]**

(d) We have

$$\mu_{\mathcal{A}}(\boldsymbol{\lambda}) = \lambda_0 A_0 + \cdots + \lambda_4 A_4 = \begin{bmatrix} \lambda_0+\lambda_1+\lambda_2 & \lambda_1+\lambda_2 & \lambda_2 \\ \lambda_1+\lambda_2 & \lambda_1+\lambda_2+\lambda_3 & \lambda_2+\lambda_3 \\ \lambda_2 & \lambda_2+\lambda_3 & \lambda_2+\lambda_3+\lambda_4 \end{bmatrix}. \mathbf{[2]}$$

For this to equal zero we would have to have

$$\begin{aligned} \lambda_0 + \lambda_1 + \lambda_2 &= 0 && (1) \\ \lambda_1 + \lambda_2 &= 0 && (2) \\ \lambda_2 &= 0 && (3) \\ \lambda_1 + \lambda_2 + \lambda_3 &= 0 && (4) \\ \lambda_2 + \lambda_3 &= 0 && (5) \\ \lambda_2 + \lambda_3 + \lambda_4 &= 0. \mathbf{[1]} && (6) \end{aligned}$$

From (3) we have $\lambda_2 = 0$, and we can substitute this in (2) and (5) to get $\lambda_1 = \lambda_3 = 0$. We can then substitute these values in (1) and (6) to get $\lambda_0 = \lambda_4 = 0$, so $\boldsymbol{\lambda} = 0$. **[2]** This shows that the list $\mathcal{A}$ has only the trivial linear relation, so it is linearly independent. **[1]**

(e) By inspection we have $3\mathbf{v}_1 - \mathbf{v}_3 - \mathbf{v}_4 = 0$. **[3]**

(29) **[Mock exam Q5]** Let V be a vector space, and let $\mathcal{V} = v_1, \dots, v_n$ be a list of elements of V.

(a) Define the map $\mu_{\mathcal{V}} \colon \mathbb{R}^n \to V$.

(b) Define, in terms of $\mu_{\mathcal{V}}$, what it means for $\mathcal{V}$ to be (i) linearly independent; (ii) a spanning set; (iii) a basis.

(c) Consider the polynomials $p_i(x) = x^i + x^{i+1}$. Is $p_0, \dots, p_4$ a basis for $\mathbb{R}[x]_{\leq 5}$?

(d) Consider the matrices

$$A_0 = \begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1&1&0 \\ 1&1&0 \\ 0&0&0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1&1&1 \\ 1&1&1 \\ 1&1&1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0&0&0 \\ 0&1&1 \\ 0&1&1 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&1 \end{bmatrix}$$

Prove that these do not span the space V of all 3×3 symmetric matrices.

Solution:

(a) The map $\mu_{\mathcal{V}} \colon \mathbb{R}^n \to V$ is defined by

$$\mu_{\mathcal{V}}([\lambda_1, \dots, \lambda_n]^T) = \lambda_1 v_1 + \cdots + \lambda_n v_n.$$

(b) The list $\mathcal{V}$ is linearly independent iff $\mu_{\mathcal{V}}$ is injective. It spans V iff $\mu_{\mathcal{V}}$ is surjective. It is a basis iff $\mu_{\mathcal{V}}$ is bijective.

(c) Put $p_i(x) = x^i + x^{i+1} = x^i(1+x)$. Then $p_i(-1) = 0$ for all i, so $f(-1) = 0$ for all $f \in \operatorname{span}(p_0, \dots, p_4)$. In particular, the constant polynomial 1 does not lie in $\operatorname{span}(p_0, \dots, p_4)$, so $p_0, \dots, p_4$ does not span $\mathbb{R}[x]_{\leq 5}$ and so cannot be a basis.

(d) Consider the matrix $B = \left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right] \in V$. We claim that this does not lie in $\operatorname{span}(A_0, \dots, A_4)$. Indeed, we have

$$a_0 A_0 + \cdots + a_4 A_4 = \left[\begin{smallmatrix} a_0+a_1+a_2 & a_1+a_2 & a_2 \\ a_1+a_2 & a_1+a_2+a_3 & a_2+a_3 \\ a_2 & a_2+a_3 & a_2+a_3+a_4 \end{smallmatrix}\right].$$

For this to equal B we would have to have

$$\begin{aligned} a_0 + a_1 + a_2 &= 0 \\ a_1 + a_2 &= 0 \\ a_2 &= 1 \\ a_1 + a_2 + a_3 &= 0 \\ a_2 + a_3 &= 0 \\ a_2 + a_3 + a_4 &= 0. \end{aligned}$$

By subtracting the second and third equations we get $a_1 = -1$, but by subtracting the fourth and fifth equations we get $a_1 = 0$. This contradiction means that there is no list $a_0, \dots, a_4$ such that $\sum_i a_i A_i = B$, so $B \notin \operatorname{span}(A_0, \dots, A_4)$, so the list $A_0, \dots, A_4$ does not span V.

(30) [0506 Q5] Consider the vector space $U = \mathbb{R}[x]_{\leq 2}$ with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx.$$

(a) Given a polynomial $u = a + bx + cx^2$, calculate $\langle u, x^i \rangle$ for $i = 0, 1, 2$. **(3 marks)**

(b) Find an element $u \in U$ such that $\langle f, u \rangle = f(0)$ for all $f \in U$. **(6 marks)**

(c) By taking $f = u$, calculate $\|u\|$. **(2 marks)**

(d) State and prove the Cauchy-Schwartz inequality. (You need not discuss the case where it is actually an equality.) **(10 marks)**

(e) Deduce that

$$|f(0)| \leq \frac{3}{\sqrt{8}} \sqrt{\textstyle\int_{-1}^{1} f(x)^2\, dx}$$

for all $f \in U$. **(4 marks)**

Solution:

(a) **Straightforward use of definition.** We have

$$\begin{aligned}
\langle u,1\rangle &= \int_{-1}^{1} a+bx+cx^2\,dx = \left[ax+\tfrac{1}{2}bx^2+\tfrac{1}{3}cx^3\right]_{-1}^{1} \\
&= \left(a+\tfrac{b}{2}+\tfrac{c}{3}\right)-\left(-a+\tfrac{b}{2}-\tfrac{c}{3}\right) \\
&= 2a+\tfrac{2}{3}c \\
\langle u,x\rangle &= \int_{-1}^{1} ax+bx^2+cx^3\,dx = \left[\tfrac{1}{2}ax^2+\tfrac{1}{3}bx^3+\tfrac{1}{4}cx^4\right]_{-1}^{1} \\
&= \left(\tfrac{a}{2}+\tfrac{b}{3}+\tfrac{c}{4}\right)-\left(\tfrac{a}{2}-\tfrac{b}{3}+\tfrac{c}{4}\right) \\
&= \tfrac{2}{3}b \\
\langle u,x^2\rangle &= \int_{-1}^{1} ax^2+bx^3+cx^4\,dx = \left[\tfrac{1}{3}ax^3+\tfrac{1}{4}bx^4+\tfrac{1}{5}cx^5\right]_{-1}^{1} \\
&= \left(\tfrac{a}{3}+\tfrac{b}{4}+\tfrac{c}{5}\right)-\left(-\tfrac{a}{3}+\tfrac{b}{4}-\tfrac{c}{5}\right) \\
&= \tfrac{2}{3}a+\tfrac{2}{5}c\,\mathbf{[3]}
\end{aligned}$$

(b) **I have not yet written the relevant problem sheet, but will find something like this to put in.**
Consider an element $u = a + bx + cx^2 \in V$. We want $\langle u, f\rangle = f(0)$, so we must have

$$\begin{aligned}
1 &= \langle u,1\rangle = 2a+\tfrac{2}{3}c \\
0 &= \langle u,x\rangle = \tfrac{2}{3}b \\
0 &= \langle u,x^2\rangle = \tfrac{2}{3}a+\tfrac{2}{5}c\,\mathbf{[3]}
\end{aligned}$$

The second equation gives $b = d = 0$. The third equation gives $c = -\frac{5}{3}a$, which can be substituted into the first equation to give $1 = (2 - \frac{10}{9})a = \frac{8}{9}a$, so $a = \frac{9}{8}$ and $c = -\frac{5}{3}a = -\frac{15}{8}$ **[2]**. This gives

$$u = (9 - 15x^2)/8.\mathbf{[1]}$$

(c) **Unseen.** Taking $f = u$ gives $\langle u, u\rangle = u(0) = 9/8$, so $\|u\| = \sqrt{9/8} = 3/\sqrt{8}$. **[2]**

(d) **Bookwork.** The Cauchy-Schwartz inequality says that for $u, v \in U$ we have $|\langle u, v\rangle| \leq \|u\|\|v\|$. **[2]** To see this, first note that it is obviously true if $v = 0$, so we may assume that $v \neq 0$ and so $\|v\| > 0$. **[1]**Put $x = \langle v, v\rangle u - \langle u, v\rangle v$. **[2]** Then

$$\begin{aligned}
\|x\|^2 &= \langle x,x\rangle \\
&= \langle v,v\rangle^2\langle u,u\rangle - 2\langle v,v\rangle\langle u,v\rangle\langle u,v\rangle + \langle u,v\rangle^2\langle v,v\rangle \\
&= \langle v,v\rangle(\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2).\mathbf{[2]}
\end{aligned}$$

As $\langle v, v\rangle = \|v\|^2 > 0$, we can divide by this to get

$$\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2 = (\|x\|/\|v\|)^2.$$

This is a square, so it must be nonnegative, so $\langle u, u\rangle\langle v, v\rangle \geq \langle u, v\rangle^2$ **[2]**. As both sides are nonnegative this inequality remains valid when we take square roots. After noting that $\sqrt{t^2} = |t|$ for all $t \in \mathbb{R}$, we conclude that $\|u\|\|v\| \geq |\langle u, v\rangle|$, as claimed. **[1]**

(e) **Unseen.** The Cauchy-Schwartz inequality**[1]**says that $|\langle f, u\rangle| \leq \|u\|\|f\|$**[1]**, or in other words

$$|f(0)| \leq \frac{3}{\sqrt{8}}\|f\| = \frac{3}{\sqrt{8}}\sqrt{\textstyle\int_{-1}^{1} f(x)^2\,dx}.$$

[2]

(31) [0506R Q5] Consider the vector space $V = M_2\mathbb{R}$ with the usual inner product $\langle A, B\rangle = \text{trace}(A^T B)$. Let s and t be positive real numbers, and define $\phi\colon V \to \mathbb{R}$ by

$$\phi(A) = \left[\begin{smallmatrix} s & t \end{smallmatrix}\right] A \left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]$$

(a) Find a matrix P such that $\phi(A) = \langle P, A\rangle$ for all $A \in V$. **(5 marks)**

(b) Calculate $\|P\|$, simplifying your answer as much as possible. **(2 marks)**

(c) State and prove the Cauchy-Schwartz inequality. **(10 marks)**

(d) Deduce that $|\phi(A)| \leq s^2 + t^2$ for all $A \in V$ with $\|A\| \leq 1$. **(3 marks)**

(e) Find a matrix A such that $\|A\| = 1$ and and $\phi(A) = s^2 + t^2$. **(5 marks)**

Solution:

(a) If $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ then

$$\phi(A) = \left[\begin{smallmatrix} s & t \end{smallmatrix}\right] \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right] = \left[\begin{smallmatrix} s & t \end{smallmatrix}\right] \left[\begin{smallmatrix} as+bt \\ cs+dt \end{smallmatrix}\right] = s^2 a + stb + stc + t^2 d.\mathbf{[2]}$$

On the other hand, if $P = \left[\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right]$ then

$$\langle P, A\rangle = \text{trace}\left(\left[\begin{smallmatrix} w & y \\ x & z \end{smallmatrix}\right] \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right) = \text{trace}\begin{bmatrix} \scriptstyle wa+yc & \scriptstyle wb+yd \\ \scriptstyle xa+zc & \scriptstyle xb+zd \end{bmatrix} = wa + xb + yc + zd.\mathbf{[1]}$$

Thus, to have $\phi(A) = \langle P, A\rangle$ we must have $w = s^2$ and $x = y = st$ and $z = t^2$, so $P = \begin{bmatrix} s^2 & st \\ st & t^2 \end{bmatrix}$. **[2]**

(b) We now have $\|P\|^2 = s^4 + s^2t^2 + s^2t^2 + t^4$**[1]** $= (s^2 + t^2)^2$, so $\|P\| = s^2 + t^2$. **[1]**

(c) **Bookwork.** The Cauchy-Schwartz inequality says that for $u, v \in U$ we have $|\langle u, v\rangle| \leq \|u\|\|v\|$. **[2]** To see this, first note that it is obviously true if $v = 0$, so we may assume that $v \neq 0$ and so $\|v\| > 0$. **[1]**Put $x = \langle v, v\rangle u - \langle u, v\rangle v$. **[2]** Then

$$\begin{aligned} \|x\|^2 &= \langle x, x\rangle \\ &= \langle v, v\rangle^2 \langle u, u\rangle - 2\langle v, v\rangle\langle u, v\rangle\langle u, v\rangle + \langle u, v\rangle^2 \langle v, v\rangle \\ &= \langle v, v\rangle(\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2).\mathbf{[2]} \end{aligned}$$

As $\langle v, v\rangle = \|v\|^2 > 0$, we can divide by this to get

$$\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2 = (\|x\|/\|v\|)^2.$$

This is a square, so it must be nonnegative, so $\langle u, u\rangle\langle v, v\rangle \geq \langle u, v\rangle^2$ **[2]**. As both sides are nonnegative this inequality remains valid when we take square roots. After noting that $\sqrt{t^2} = |t|$ for all $t \in \mathbb{R}$, we conclude that $\|u\|\|v\| \geq |\langle u, v\rangle|$, as claimed. **[1]**

(d) **Unseen.** The Cauchy-Schwartz inequality**[1]**now tells us that $|\phi(A)| = |\langle P, A\rangle| \leq \|P\|\|A\| = (s^2 + t^2)\|A\|$ **[1]**. In particular, if $\|A\| \leq 1$ then $|\phi(A)| \leq s^2 + t^2$. **[1]**

(e) Now take $A = P/(s^2 + t^2) = P/\|P\|$ **[3]**. We then have $\|A\| = 1$ and $\phi(A) = \langle P, P/\|P\|\rangle = \|P\|^2/\|P\| = \|P\| = s^2 + t^2$ **[2]**.

(32) [0607 Q5] Let V be a vector space over $\mathbb{R}$.

(a) Define the map $\mu_{\mathcal{V}}\colon \mathbb{R}^n \to V$ (where $\mathcal{V} = v_1, \ldots, v_n$ is a list of elements of V). **(2 marks)**

(b) Show that any linear map $\phi\colon \mathbb{R}^n \to V$ has the form $\phi = \mu_{\mathcal{V}}$ for some list $\mathcal{V}$. **(5 marks)**

(c) Some of the following situations are possible, and some are not. For each situation that is possible, give an example. For each situation that is impossible, give a brief argument to show that it is impossible. **(8 marks)**

(i) A space V with a spanning list $\mathcal{A}$ of length 4 and a linearly independent list $\mathcal{B}$ of length 3.

(ii) A space V with a spanning list $\mathcal{A}$ of length 3 and a linearly independent list $\mathcal{B}$ of length 4.

(iii) A 3-dimensional space V with a list $\mathcal{V}$ of length 3 that is linearly independent but does not span.

(iv) A 3-dimensional space V with a list $\mathcal{V}$ of length 3 that is linearly dependent and does not span.

(d) Define what is meant by a *jump* in a sequence $\mathcal{V}$. **(3 marks)**

(e) Find the jumps in the following sequence:

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad v_5 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad v_6 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

(7 marks)

Solution:

(a) **Bookwork.** The map $\mu_{\mathcal{V}}$ is just given by $\mu_{\mathcal{V}}([\lambda_1, \dots, \lambda_n]^T) = \sum_i \lambda_i v_i$. **[2]**

(b) **Bookwork.** Let $\phi\colon \mathbb{R}^n \to V$ be a linear map. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$, so $\mathbf{x} = \sum_i x_i \mathbf{e}_i$ for all $\mathbf{x} \in \mathbb{R}^n$ **[1]**. Put $v_i = \phi(\mathbf{e}_i) \in V$ **[1]**and $\mathcal{V} = v_1, \dots, v_n$. We claim that $\phi = \mu_{\mathcal{V}}$ **[1]**. Indeed, for any $\mathbf{x} \in \mathbb{R}^n$ we have

$$\phi(\mathbf{x}) = \phi(\sum_i x_i \mathbf{e}_i) = \sum_i x_i \phi(\mathbf{e}_i) = \sum_i x_i v_i = \mu_{\mathcal{V}}(\mathbf{x})\mathbf{[2]}$$

as claimed.

(c) **Unseen.**

(i) An example is $V = \mathbb{R}^3$ with $\mathcal{A} = \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, 0$ and $\mathcal{B} = \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. **[2]**

(ii) This is impossible **[1]**by Steinitz's Lemma: any spanning list must be at least as long as any linearly independent list. **[1]**

(iii) This is impossible **[1]**: in a 3-dimensional space, a list of length three is independent iff it spans **[1]**.

(iv) An example is $V = \mathbb{R}^3$ with $\mathcal{V} = 0, 0, 0$.**[2]**

(d) **Bookwork.** Put $V_i = \text{span}(v_1, \dots, v_i)$ (with $V_0 = 0$). We then say that i is a *jump* if $v_i \notin V_{i-1}$. **[3]**

(e) **Similar to problem sheets.** As $v_1 = 0 \in V_0$ and $v_3 = 2v_2 \in V_2$ and $v_5 = v_3 - v_4 \in V_4$ and $v_6 = v_3 + v_4 \in V_5$, we see that 1, 3, 5 and 6 are not jumps **[3]**. As $V_1 = 0$ and $v_2 \neq 0$, we see that 2 is a jump **[2]**. It is also clear that V_3 is the set of vectors of the form $[t, t, t]^T$, and v_4 does not lie in that set, so 4 is a jump **[2]**. Thus, the set of jumps is precisely $\{2, 4\}$.

(33) [0607R Q5] Consider the vector space $V = M_2(\mathbb{R})$ with the usual inner product $\langle A, B \rangle = \text{trace}(A^T B)$. Define $\phi\colon V \to \mathbb{R}$ by

$$\phi(A) = \begin{bmatrix} 3 & 4 \end{bmatrix} A \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

(a) Find a matrix P such that $\phi(A) = \langle P, A\rangle$ for all $A \in V$. **(5 marks)**

(b) Calculate $\|P\|$ **(2 marks)**

(c) State and prove the Cauchy-Schwartz inequality. **(10 marks)**

(d) Deduce that $|\phi(A)| \leq 25$ for all $A \in V$ with $\|A\| \leq 1$. **(3 marks)**

(e) Find a matrix A such that $\|A\| = 1$ and and $\phi(A) = 25$. **(5 marks)**

Solution:

(a) If $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ then

$$\phi(A) = \left[\begin{smallmatrix} 3 & 4 \end{smallmatrix}\right]\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\left[\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 3 & 4 \end{smallmatrix}\right]\left[\begin{smallmatrix} 3a+4b \\ 3c+4d \end{smallmatrix}\right] = 9a + 12b + 12c + 16d.\mathbf{[2]}$$

On the other hand, if $P = \left[\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right]$ then

$$\langle P, A\rangle = \text{trace}\left(\left[\begin{smallmatrix} w & y \\ x & z \end{smallmatrix}\right]\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right) = \text{trace}\begin{bmatrix} wa+yc & wb+yd \\ xa+zc & xb+zd \end{bmatrix} = wa + xb + yc + zd.\mathbf{[1]}$$

Thus, to have $\phi(A) = \langle P, A\rangle$ we must have $w = 9$ and $x = y = 12$ and $z = 16$, so $P = \left[\begin{smallmatrix} 9 & 12 \\ 12 & 16 \end{smallmatrix}\right]$. **[2]**

(b) We now have $\|P\|^2 = 9^2 + 12^2 + 12^2 + 16^2 = 625$**[1]**, so $\|P\| = \sqrt{625} = 25$. **[1]**

(c) **Bookwork.** The Cauchy-Schwartz inequality says that for any inner product space U and any $u, v \in U$ we have $|\langle u, v\rangle| \leq \|u\|\|v\|$. **[2]** To see this, first note that it is obviously true if $v = 0$, so we may assume that $v \neq 0$ and so $\|v\| > 0$. **[1]**Put $x = \langle v, v\rangle u - \langle u, v\rangle v$. **[2]** Then

$$\begin{aligned} \|x\|^2 &= \langle x, x\rangle \\ &= \langle v, v\rangle^2\langle u, u\rangle - 2\langle v, v\rangle\langle u, v\rangle\langle u, v\rangle + \langle u, v\rangle^2\langle v, v\rangle \\ &= \langle v, v\rangle(\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2).\mathbf{[2]} \end{aligned}$$

As $\langle v, v\rangle = \|v\|^2 > 0$, we can divide by this to get

$$\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2 = (\|x\|/\|v\|)^2.$$

This is a square, so it must be nonnegative, so $\langle u, u\rangle\langle v, v\rangle \geq \langle u, v\rangle^2$ **[2]**. As both sides are nonnegative this inequality remains valid when we take square roots. After noting that $\sqrt{t^2} = |t|$ for all $t \in \mathbb{R}$, we conclude that $\|u\|\|v\| \geq |\langle u, v\rangle|$, as claimed. **[1]**

(d) **Unseen.** The Cauchy-Schwartz inequality**[1]**now tells us that $|\phi(A)| = |\langle P, A\rangle| \leq \|P\|\|A\| = 25\|A\|$ **[1]**. In particular, if $\|A\| \leq 1$ then $|\phi(A)| \leq 25$. **[1]**

(e) Now take $A = P/25 = P/\|P\| = \left[\begin{smallmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{smallmatrix}\right]$ **[3]**. We then have $\|A\| = 1$ and $\phi(A) = \langle P, P/\|P\|\rangle = \|P\|^2/\|P\| = \|P\| = 25$ **[2]**.

(34) [0708 Q5] Let V be a finite-dimensional vector space over $\mathbb{R}$, and let $\phi\colon V \to V$ be a linear map.

(a) Define the kernel of ϕ, and prove that it is a subspace of V. **(5 marks)**

(b) Define the terms *eigenvalue* and *eigenvector*, and show that if ϕ is not injective then 0 is an eigenvalue of ϕ. **(6 marks)**

For the rest of this question, we take $V = \mathbb{R}[x]_{\leq 2}$, and we define $\phi\colon V \to V$ by

$$\phi(f(x)) = (f(2) - f(0))x.$$

(c) Give a basis for V, and find the matrix of ϕ with respect to your basis. **(4 marks)**

(d) Find the trace and determinant of ϕ. **(3 marks)**

(e) Find a basis for V consisting of eigenvectors of ϕ. **(7 marks)**

Solution:

(a) **Bookwork.** The kernel of ϕ is the set $\ker(\phi) = \{v \in V \mid \phi(v) = 0\}$ **[1]**. We have $\phi(0) = 0$, so $0 \in \ker(\phi)$ **[1]**. If $u, v \in \ker(\phi)$ and $s, t \in \mathbb{R}$ then

$$\phi(su + tv) = s\phi(u) + t\phi(v) = s.0 + t.0 = 0,$$

so $su + tv \in \ker(\phi)$ **[3]**. This shows that $\ker(\phi)$ is a subspace of V.

(b) **Bookwork.** We say that a number $\lambda \in \mathbb{R}$ is an eigenvalue for ϕ if there is a nonzero element $v \in V$ with $\phi(v) = \lambda v$. Any such element v is called an eigenvector for ϕ of eigenvalue λ. **[2]**

If ϕ is not injective, then there must exist two elements $u, v \in V$ with $u \neq v$ but $\phi(u) = \phi(v)$ **[1]**. This means that the vector $w = u - v$ is nonzero and satisfies $\phi(w) = \phi(u) - \phi(v) = 0 = 0.w$ **[2]**, so w is a nonzero eigenvector of eigenvalue 0, so 0 is an eigenvalue of ϕ. **[1]**

(c) **Similar to lecture notes and problem sheets.** The obvious basis is $x^2, x, 1$ **[1]**. We have

$$\begin{aligned}\phi(ax^2 + bx + c) &= (4a + 2b)x \\ \phi(x^2) &= 0.x^2 + 4.x + 0.1 \\ \phi(x) &= 0.x^2 + 2.x + 0.1 \\ \phi(1) &= 0.x^2 + 0.x + 0.1\textbf{[2]}\end{aligned}$$

so the relevant matrix is

$$P = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.\textbf{[1]}$$

(d) **Similar to lecture notes and problem sheets.** The trace and determinant of ϕ are defined to be the trace and determinant of the corresponding matrix **[1]**(with respect to any basis), so $\operatorname{trace}(\phi) = \operatorname{trace}(P) = 0 + 2 + 0 = 2$ **[1]**and $\det(\phi) = \det(P) = 0$ **[1]**.

(e) **Similar to lecture notes and problem sheets.** The characteristic polynomial of ϕ is

$$\operatorname{char}(\phi)(t) = \det = \begin{bmatrix} t & 0 & 0 \\ -4 & t-2 & 0 \\ 0 & 0 & t \end{bmatrix} = t^2(t - 2).$$

Thus, the eigenvalues are 0 and 2. **[2]** Consider a polynomial $f(x) = ax^2 + bx + c$, so $\phi(f) = (4a + 2b)x$. Then f is an eigenvector of eigenvalue 0 iff $\phi(f) = 0$ iff $4a + 2b = 0$ iff f has the form $f(x) = ax^2 - 2ax + c = a(x^2 - 2x) + c.1$. Thus $x^2 - 2x$ and 1 are linearly independent eigenvectors of eigenvalue 0 **[2]**. Similarly, f is an eigenvector of eigenvalue 2 iff $\phi(f) = 2f$ iff $(4a + 2b)x = 2ax^2 + 2bx + 2c$ iff $a = c = 0$, so x is an eigenvector of eigenvalue 2. **[2]** Thus $1, x, x^2 - 2x$ is a basis consisting of eigenvectors. **[1]**

(35) [0708R Q5] Let V be a finite-dimensional vector space over $\mathbb{R}$, and let $\phi\colon V \to V$ be a linear map.

(a) Define the kernel of ϕ, and prove that it is a subspace of V. **(5 marks)**

(b) Define the terms *eigenvalue* and *eigenvector* (for linear maps, not for matrices). Show that if ϕ is not injective then 0 is an eigenvalue of ϕ. **(6 marks)**

For the rest of this question, we take $V = \mathbb{R}[x]_{\leq 2}$, and we define $\phi\colon V \to V$ by

$$\phi(f(x)) = (f(2) - 2f(1))x^2.$$

(c) Give a basis for V, and find the matrix of ϕ with respect to your basis. **(4 marks)**

(d) Find the trace and characteristic polynomial of ϕ. **(3 marks)**

(e) Find a basis for V consisting of eigenvectors of ϕ. **(7 marks)**

Solution: This is a slight modification of a question from the June exam.

(a) **Bookwork.** The kernel of ϕ is the set $\ker(\phi) = \{v \in V \mid \phi(v) = 0\}$ **[1]**. We have $\phi(0) = 0$, so $0 \in \ker(\phi)$ **[1]**. If $u, v \in \ker(\phi)$ and $s, t \in \mathbb{R}$ then

$$\phi(su + tv) = s\phi(u) + t\phi(v) = s.0 + t.0 = 0,$$

so $su + tv \in \ker(\phi)$ **[3]**. This shows that $\ker(\phi)$ is a subspace of V.

(b) **Bookwork.** We say that a number $\lambda \in \mathbb{R}$ is an eigenvalue for ϕ if there is a nonzero element $v \in V$ with $\phi(v) = \lambda v$. Any such element v is called an eigenvector for ϕ of eigenvalue λ. **[2]**

If ϕ is not injective, then there must exist two elements $u, v \in V$ with $u \neq v$ but $\phi(u) = \phi(v)$ **[1]**. This means that the vector $w = u - v$ is nonzero and satisfies $\phi(w) = \phi(u) - \phi(v) = 0 = 0.w$ **[2]**, so w is a nonzero eigenvector of eigenvalue 0, so 0 is an eigenvalue of ϕ. **[1]**

(c) **Similar to lecture notes and problem sheets.** The obvious basis is $x^2, x, 1$ **[1]**. We have

$$\begin{aligned}
\phi(ax^2 + bx + c) &= ((4a + 2b + c) - 2(a + b + c))x^2 = (2a - c)x^2 \\
\phi(x^2) &= 2.x^2 + 0.x + 0.1 \\
\phi(x) &= 0.x^2 + 0.x + 0.1 \\
\phi(1) &= -1.x^2 + 0.x + 0.1\mathbf{[2]}
\end{aligned}$$

so the relevant matrix is

$$P = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.\mathbf{[1]}$$

(d) **Similar to lecture notes and problem sheets.** The trace and characteristic polynomial of ϕ are defined to be the trace and characteristic polynomial of the corresponding matrix (with respect to any basis), so $\text{trace}(\phi) = \text{trace}(P) = 2 + 0 + 0 = 2$ **[1]**and

$$\text{char}(\phi)(t) = \det = \begin{bmatrix} t-2 & 0 & 1 \\ 0 & t & 0 \\ 0 & 0 & t \end{bmatrix} = t^2(t - 2).\mathbf{[2]}$$

(e) **Similar to lecture notes and problem sheets.** From the characteristic polynomial we see that the eigenvalues are 0 and 2. **[2]** Consider a polynomial $f(x) = ax^2 + bx + c$, so $\phi(f) = (2a - c)x^2$. Then f is an eigenvector of eigenvalue 0 iff $\phi(f) = 0$ iff $2a - c = 0$ iff f has the form $f(x) = ax^2 + bx + 2a = a(x^2 + 2) + bx$. Thus $x^2 + 2$ and x are linearly independent eigenvectors of eigenvalue 0 **[2]**. Similarly, f is an eigenvector of eigenvalue 2 iff $\phi(f) = 2f$ iff $(2a - c)x^2 = 2ax^2 + 2bx + 2c$ iff $b = c = 0$, so x^2 is an eigenvector of eigenvalue 2. **[2]** Thus $x^2 + 2, x, x^2$ is a basis consisting of eigenvectors. **[1]**