Vector spaces and Fourier Theory

Syllabus

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11. **Fourier theory:** in terms of inner product spaces.

Predefinition ??: A *vector space* (over $\mathbb{R}$) is a nonempty set V of things such that

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Example ??: The set $\mathbb{R}^3$ of all three-dimensional vectors is a vector space, because the sum of two vectors is a vector (eg $\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}4\\4\\4\end{bmatrix}$) and the product of a real number and a vector is a vector (eg $3\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}3\\6\\9\end{bmatrix}$). In the same way, the set $\mathbb{R}^2$ of two-dimensional vectors is also a vector space.

We generally use column vectors (rather than row vectors), as this makes formulae with matrix multiplication work better.

However, column vectors often fit awkwardly on the page, so we use the following notational device:

$$[1,\ 2,\ 3,\ 4]^T \qquad \text{means} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$[a+b,\ b+c,\ c+d,\ d+e,\ e+a]^T \qquad \text{means} \qquad \begin{bmatrix} a+b \\ b+c \\ c+d \\ d+e \\ e+a \end{bmatrix}$$

and so on.

The space $\mathbb{R}^n$

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Example ??: For any natural number n the set $\mathbb{R}^n$ of vectors of length n is a vector space. For example, the vectors $u = \begin{bmatrix} 1 & 2 & 4 & 8 & 16 \end{bmatrix}^T$ and $v = \begin{bmatrix} 1 & -2 & 4 & -8 & 16 \end{bmatrix}^T$ are elements of $\mathbb{R}^5$, with $u + v = \begin{bmatrix} 2 & 0 & 8 & 0 & 32 \end{bmatrix}^T$. We can even consider the set $\mathbb{R}^\infty$ of all infinite sequences of real numbers, which is again a vector space.

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Another trivial example is that $\mathbb{R}$ itself is a vector space (which can be thought of as $\mathbb{R}^1$).

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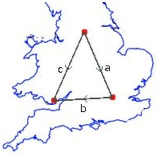
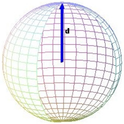
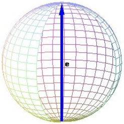
We then have $\mathbf{a}+\mathbf{b}=\mathbf{c}$ and $2\mathbf{d}=\mathbf{e}$.







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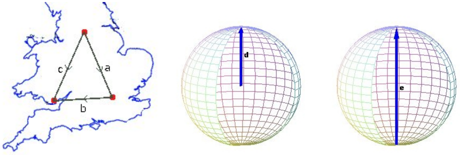

Once we have agreed on where our axes should point, and what units of length we should use, we can identify U with $\mathbb{R}^3$.

The space of functions

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For this to work properly, we must insist that $f(x)$ is defined for all x, and is a real number for all x; it cannot be infinite or imaginary. Thus the rules $p(x) = 1/x$ and $q(x) = \sqrt{x}$ do not define elements $p, q \in F(\mathbb{R})$.

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We actually have $g = 2f$, because f and g are defined as functions on $[1, 2]$, and $|x| = x$ for all $x \in [1, 2]$.

Spaces of matrices

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More generally still, for any n and m, the set $M_{n,m}\mathbb{R}$ of $n \times m$ matrices is a vector space, which can be identified with $\mathbb{R}^{nm}$.

The set of all lists

Example ??: Let L be the set of all finite lists of real numbers.

For example, the lists $\mathbf{a} = (10, 20, 30, 40)$ and $\mathbf{b} = (5, 6, 7)$ and $\mathbf{c} = (1, \pi, \pi^2)$ define three elements $\mathbf{a}, \mathbf{b}, \mathbf{c} \in L$.

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We cannot answer this unless we know what $\mathbf{a} + \mathbf{b}$ means.

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(Strictly speaking, the same is true of the expression $100\mathbf{b}$, but in that case there is only one reasonable possibility for what it should mean.)

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(If we used either of the other definitions of addition then things would still go wrong; details are left as an exercise.)

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In the course we will be content with an informal understanding of the phrase "all the usual algebraic rules", but for completeness, we will give an explicit list of axioms.

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Note that there are many rules that do not appear explicitly in the above list, such as the fact that $t(u + v - w/t) = tu + tv - w$, but it turns out that all such rules can be deduced from the ones listed. We will not discuss any such deductions.

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For example:

$$0_{\mathbb{R}^2} = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] \qquad\qquad 0_{M_2\mathbb{R}} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right].$$

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(d) $1 \neq 0$ (or equivalently, $K \neq \{0\}$).

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One can show that the ring $\mathbb{Z}/n\mathbb{Z}$ is a field if and only if n is a prime number.

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(2) $u + v = v + u$
(3) $u + (v + w) = (u + v) + w$
(4) $0u = 0$
(5) $1u = u$
(6) $(st)u = s(tu)$
(7) $(s + t)u = su + tu$
(8) $s(u + v) = su + sv$.

Example ??: Almost all our examples work over any field K.
$M_4\mathbb{Q} = \{4 \times 4 \text{ matrices with entries in } \mathbb{Q}\}$ is a vector space over $\mathbb{Q}$.

Definition ??: A *vector space* over a field K is a set V, together with an element $0 \in V$ and a definition of what it means to add elements of V or multiply them by elements of K, such that

(a) If u and v are elements of V, then $u + v$ is an also an element of V.

(b) If v is an element of V and t is an element of K, then $tv \in V$.

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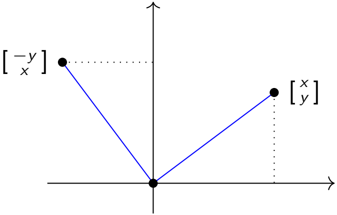

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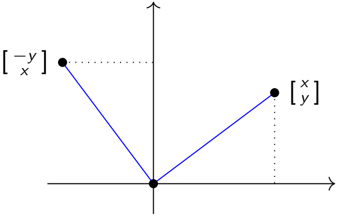

We thus have

$$\rho\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x' \\ y' \end{bmatrix}\right) = \rho\begin{bmatrix} x+x' \\ y+y' \end{bmatrix} = \begin{bmatrix} -y-y' \\ x+x' \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} + \begin{bmatrix} -y' \\ x' \end{bmatrix} = \rho\begin{bmatrix} x \\ y \end{bmatrix} + \rho\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\rho\left(t\begin{bmatrix} x \\ y \end{bmatrix}\right) = \rho\begin{bmatrix} tx \\ ty \end{bmatrix} = \begin{bmatrix} -ty \\ tx \end{bmatrix} = t\rho\begin{bmatrix} x \\ y \end{bmatrix},$$

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More general rotations

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Using this, we see that $\mathrm{rot}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map.

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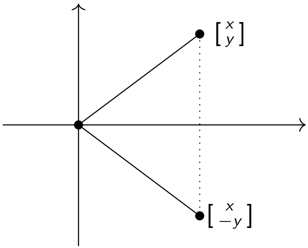
It is clear that the formula is $\tau \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ -y \end{smallmatrix}\right]$, and using this we see that τ is linear.



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Indeed, if $\mathbf{u} = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\mathbf{v} = \left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right]$ then $\theta(\mathbf{u}+\mathbf{v}) = 0$ but $\theta(\mathbf{u})+\theta(\mathbf{v}) = 1+1 = 2$.

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(where ρ and τ are as in Examples **??** and **??**).

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Given a vector $\mathbf{v} \in \mathbb{R}^n$, we can multiply A by $\mathbf{v}$ to get a vector $A\mathbf{v} \in \mathbb{R}^m$.

We can thus define $\phi_A \colon \mathbb{R}^n \to \mathbb{R}^m$ by $\phi_A(\mathbf{v}) = A\mathbf{v}$.

It is clear that $A(\mathbf{v} + \mathbf{v}') = A\mathbf{v} + A\mathbf{v}'$ and $At\mathbf{v} = tA\mathbf{v}$, so ϕ_A is a linear map.

We will see later that every linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ has this form.

In particular, if we put $R = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $T = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, we find that

$$R\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right] = \rho\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) \qquad\qquad T\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ -y \end{smallmatrix}\right] = \tau\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right)$$

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More generally, $\mathrm{rot}_\theta = \phi_{R_\theta}$ and $\mathrm{ref}_\theta = \phi_{T_\theta}$, where

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Using the obvious equations

$$\int_0^1 f(x) + g(x)dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx$$
$$\int_0^1 tf(x)dx = t\int_0^1 f(x)dx$$

we see that $I(f+g) = I(f) + I(g)$ and $I(tf) = t\,I(f)$, so I is a linear map.

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L(f+g) &= (f+g)'' + (f+g) = f'' + g'' + f + g \\
&= (f'' + f) + (g'' + g) = L(f) + L(g) \\
L(tf) &= (tf)'' + tf = t\,f'' + tf = tL(f).
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This shows that L is also linear. ○

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None of this is really restricted to 2×2 matrices. For any n we have a map $\mathrm{trace} \colon M_n\mathbb{R} \to \mathbb{R}$ given by $\mathrm{trace}(A) = \sum_{i=1}^{n} A_{ii}$, which is again linear. We also have a determinant map $\det \colon M_n\mathbb{R} \to \mathbb{R}$ which satisfies $\det(tI) = t^n$; this shows that det is not linear, except in the silly case where $n = 1$. ◯

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The map is not linear, because $\phi(I) = I$ and also $\phi(2I) = I$, so $\phi(2I) \neq 2\phi(I)$.

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Similarly, the space $M_{p,q}\mathbb{R}$ is isomorphic to $\mathbb{R}^{pq}$.

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We will be able to give more interesting examples of isomorphisms after we have learnt about subspaces. ◯

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We should therefore specify that addition in W is to be defined using the same rule as for V, and similarly for scalar multiplication. $\bigcirc$

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Conversely, suppose that (b) and (c) hold, and that $u, v \in W$ and $t, s \in \mathbb{R}$. Then condition (c) tells us that $tu \in W$, and similarly that $sv \in W$. Given these, condition (b) tells us that $tu + sv \in W$; we conclude that condition (d) holds, as required. $\bigcirc$

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Thus, conditions (a) and (d) in Remark **??** are satisfied, showing that W is a subspace as claimed. $\bigcirc$

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If we let f correspond to the vector $\begin{bmatrix} a_0 & \cdots & a_d \end{bmatrix}^T \in \mathbb{R}^{d+1}$, we get a one-to-one correspondence between $\mathbb{R}[x]_{\leq d}$ and $\mathbb{R}^{d+1}$. ○

More precisely, there is an isomorphism $\phi\colon \mathbb{R}^{d+1} \to \mathbb{R}[x]_{\leq d}$ given by

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Note that the list $0, 1, 2, 3$ has four entries (not three), and similarly, the list $0, 1, 2, \ldots, d$ has $d + 1$ entries (not d).

Even and odd functions

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Similarly, the set OF of odd functions is a subspace of $F(\mathbb{R})$. $\bigcirc$

Solutions of differential equations

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However, the sum of two solutions to the KdV equation does not satisfy the KdV equation, so the set of solutions is not a subspace of V.

The Wave and Heat equations are *linear*, but the KdV equation is not. ◯

Solutions of differential equations

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The motion of fluids and gasses is governed by the Navier-Stokes equation, which is nonlinear; because of this, massive supercomputers are needed for weather forecasting, climate modelling, and aircraft design.

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We will prove this for U_0 and U_4; the other cases are similar.

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So U_0 is a subspace. $\bigcirc$

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Thus U_4 is also a subspace.

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U_6 is not a subspace: if we put

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then $A, B \in U_6$ but $A + B = I \notin U_6$.

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This works for any $\mathbf{u} \in \mathbb{R}^3$, so $\mathbb{R}^3 = V + W$. $\bigcirc$

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$$\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \mathbf{v} = \frac{1}{12}\begin{bmatrix} 12x+8y+4z \\ 3x+2y+z \\ -6x-4y-2z \end{bmatrix} \qquad \mathbf{w} = \frac{1}{12}\begin{bmatrix} -8y-4z \\ -3x+10y-z \\ 6x+4y+14z \end{bmatrix}$$

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Subspaces of polynomials

Example ??: Take $U = \mathbb{R}[x]_{\leq 4}$ and

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In particular, the polynomial $f(x) = x$ does not lie in $V + W$, so $V + W \neq U$.○

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So $\ker(\phi)$ is a plane through the origin (and thus a two-dimensional subspace) and $\text{image}(\phi)$ is a line through the origin (and thus a one-dimensional subspace). $\bigcirc$

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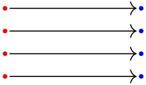

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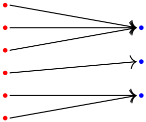

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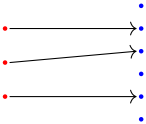

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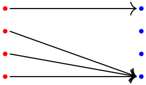

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- Recall that $\mathrm{image}(\phi)$ is the set of those $v \in V$ such that $v = \phi(u)$ for some $u \in U$. Thus $\mathrm{image}(\phi) = V$ iff *every* element $v \in V$ has the form $\phi(u)$ for some $u \in U$, which is precisely what it means for ϕ to be surjective. $\bigcirc$

Corollary ??: $\phi \colon U \to V$ is an isomorphism iff $\ker(\phi) = 0$ and $\mathrm{image}(\phi) = V$.

Direct sums

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Definition ??: Let V and W be vector spaces. We define $V \oplus W$ to be the set of pairs (v, w) with $v \in V$ and $w \in W$. Addition and scalar multiplication are defined in the obvious way:

$$(v, w) + (v', w') = (v + v', w + w')$$
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This makes $V \oplus W$ into a vector space, called the *direct sum* of V and W. We may sometimes use the notation $V \times W$ instead of $V \oplus W$.

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Example ??: An element of $\mathbb{R}^p \oplus \mathbb{R}^q$ is a pair $(\mathbf{x}, \mathbf{y})$, where $\mathbf{x}$ is a list of p real numbers, and $\mathbf{y}$ is a list of q real numbers. Such a pair is essentially the same thing as a list of $p + q$ real numbers, so $\mathbb{R}^p \oplus \mathbb{R}^q = \mathbb{R}^{p+q}$. ◯

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Internal direct sums

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Odd and even functions

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Two randomly-chosen vectors in $\mathbb{R}^2$ will generally not be parallel; it is an important special case if they happen to point in the same direction. Similarly, given three vectors u, v and w in $\mathbb{R}^3$, there will usually not be any plane that contains all three vectors. This means that we can get from the origin to any point by travelling a certain (possibly negative) distance in the direction of u, then a certain distance in the direction of v, then a certain distance in the direction of w. The case where u, v and w all lie in a common plane will have special geometric significance in any purely mathematical problem, and will often have special physical significance in applied problems. Our task in this section is to generalise these ideas, and study the corresponding special cases in an arbitrary vector space V. The abstract picture will be illuminating even in the case of $\mathbb{R}^2$ and $\mathbb{R}^3$.

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Otherwise, if the only relation is the trivial one, we say that the list $\mathcal{V}$ is *linearly independent*. ◯

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From this we see that $\lambda_1 = 0$, then from the equation $\lambda_1 + \lambda_2 = 0$ we see that $\lambda_2 = 0$, then from the equation $\lambda_1 + \lambda_2 + \lambda_3 = 0$ we see that $\lambda_3 = 0$.

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$$\begin{bmatrix} \lambda_1 \\ \lambda_1+\lambda_2 \\ \lambda_1+\lambda_2+\lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From this we see that $\lambda_1 = 0$, then from the equation $\lambda_1 + \lambda_2 = 0$ we see that $\lambda_2 = 0$, then from the equation $\lambda_1 + \lambda_2 + \lambda_3 = 0$ we see that $\lambda_3 = 0$. Thus, the only linear relation is the trivial one where $[\lambda_1, \lambda_2, \lambda_3]^T = [0, 0, 0]^T$

Example ??: Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

Then $\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 = 0$, so $[1, -2, 1]^T$ is a nontrivial linear relation, so the list $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linearly dependent.

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Example ??: Consider the polynomials $p_n(x) = (x+n)^2$, so

$$p_0(x) = x^2 \qquad p_1(x) = x^2 + 2x + 1$$
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(The entries in this list are the coefficients in the expansion of $(T-1)^3 = T^3 - 3T^2 + 3T - 1$; this is not a coincidence, but the explanation would take us too far afield.) ○

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Thus, the only linear relation is the trivial one, showing that $E_1, \ldots, E_4$ are linearly independent. ◯

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Thus, $\mathcal{V}$ is linearly independent iff $\ker(\mu_{\mathcal{V}}) = \{0\}$ iff $\mu_{\mathcal{V}}$ is injective (by Proposition **??**). ◯

The Wronskian

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$$WM(f_1, f_2, f_3, f_4) = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \\ f_1' & f_2' & f_3' & f_4' \\ f_1'' & f_2'' & f_3'' & f_4'' \\ f_1''' & f_2''' & f_3''' & f_4''' \end{bmatrix}.$$

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&= \exp .(-1) - \exp .(0) + \exp .(-1) \\
&= -2 \exp .
\end{aligned}
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and again to get

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The Wronskian and linear dependence

$$\begin{aligned}
\lambda_1 f_1(x) + \lambda_2 f_2(x) + \lambda_3 f_3(x) &= 0\\
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Corollary ??:
If $W(f_1, \ldots, f_n) \neq 0$, then $f_1, \ldots, f_n$ are linearly independent. □○

A Wronskian counterexample

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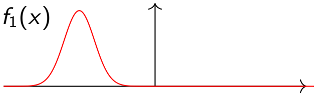
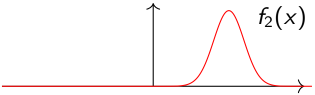
Remark ??: Consider a pair of smooth functions like this:





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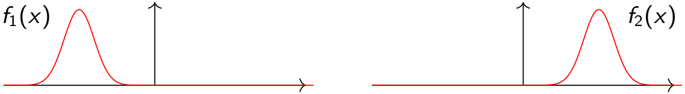

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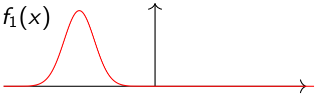

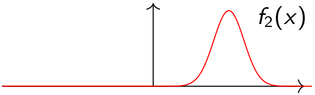

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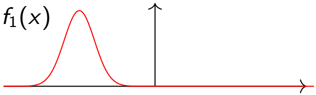

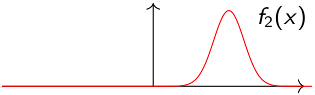

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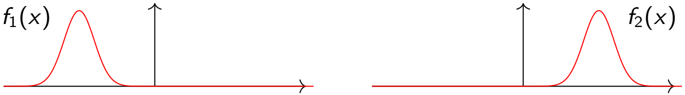

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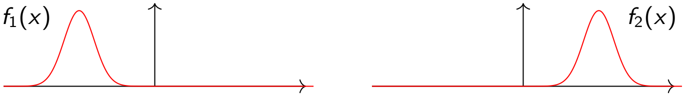

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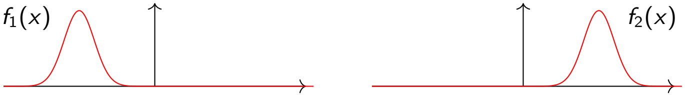

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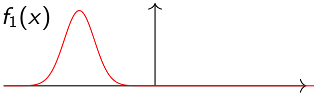

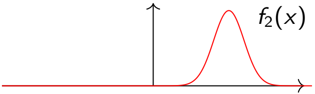

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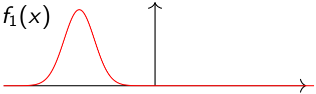

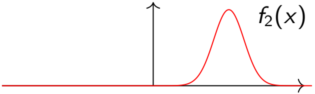

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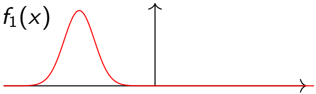

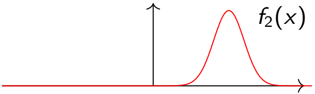

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A Wronskian counterexample

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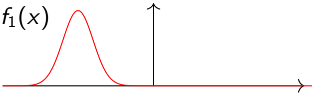

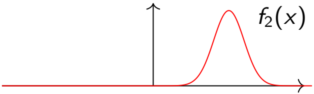

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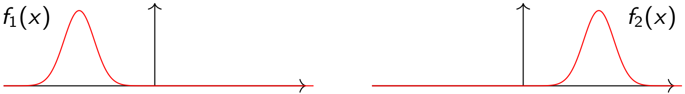

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Remark ??: Often V will be a subspace of some larger space U. If you are asked whether certain vectors $v_1, \dots, v_n$ span V, the *first* thing that you have to check is that they are actually elements of V. ○

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A spanning set for $\mathbb{R}^4$

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A spanning set for $\mathbb{R}^4$

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Consider an arbitrary vector $\mathbf{v} = \begin{bmatrix} a & b & c & d \end{bmatrix}^T \in \mathbb{R}^4$. We want to find p, q, r, s such that $\mathbf{v} = p\mathbf{u}_1 + q\mathbf{u}_2 + r\mathbf{u}_3 + s\mathbf{u}_4$, or equivalently

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = p \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + q \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p+q \\ p+q+r+s \\ p+q+r \\ p+r \end{bmatrix}, \text{ or}$$

$$p+q = a \ (1) \qquad p+q+r+s = b \ (2) \qquad p+q+r = c \ (3) \qquad p+r = d \ (4)$$

Subtracting (3) and (4) gives $q = c - d$; Subtracting (1) and(3) gives $r = c - a$; Subtracting (2) and(3) gives $s = b - c$; putting $q = c - d$ in (1) gives $p = a - c + d$.

$$(a-c+d)\mathbf{u}_1 + (c-d)\mathbf{u}_2 + (c-a)\mathbf{u}_3 + (b-c)\mathbf{u}_3 = \begin{bmatrix} a-c+d \\ a-c+d \\ a-c+d \\ a-c+d \end{bmatrix} + \begin{bmatrix} c-d \\ c-d \\ c-d \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c-a \\ c-a \\ c-a \end{bmatrix} + \begin{bmatrix} 0 \\ b-c \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \mathbf{v}$$

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$$\begin{aligned}
p_0(x) &= x^2 \\
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Thus for an arbitrary quadratic polynomial $f(x) = ax^2 + bx + c$, we have

$$\begin{aligned} f(x) &= ap_0(x) + \tfrac{1}{4}b(p_1(x) - p_{-1}(x)) + \tfrac{1}{8}c(p_2(x) + p_{-2}(x) - 2p_0(x)) \\ &= \tfrac{c}{8}p_{-2}(x) - \tfrac{b}{4}p_{-1}(x) + (a - \tfrac{c}{4})p_0(x) + \tfrac{b}{4}p_1(x) + \tfrac{c}{8}p_2(x). \bigcirc \end{aligned}$$

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Example ??: We will find a basis for the space V of antisymmetric 3×3 matrices. Such a matrix has the form

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$$\begin{aligned} s_0(x) &= (x^2 - 3x + 2)/2 \\ s_1(x) &= -x^2 + 2x \\ s_2(x) &= (x^2 - x)/2. \end{aligned}$$

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Using this, we find that $f = \lambda_0 t_0 + \lambda_1 t_1 + \lambda_2 t_2$, where $\lambda_i = \int_0^1 f(x)t_i(x)\,dx$.

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Thus, if we put $p(x) = x^3 - x$ and $q(x) = x^4 - 2x^2 + 1 = (x^2 - 1)^2$, then p, q is a basis for V. $\bigcirc$

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It is now convenient to consider the subspace $W = \{X \in V \mid T(X) = 0\}$, consisting of squares as above for which

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For such a square, we certainly have

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for some $q, a, b, d, e \in \mathbb{R}$. Applying T gives $9q = 0$ (because $T(A) = T(B) = T(D) = T(E) = 0$ and $T(Q) = 9$), and so $q = 0$. This leaves $aA + bB + dD + eE = 0$, and A, B, D and E are linearly independent, so $a = b = d = e = 0$ as well.

Next, the matrix $Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ lies in V but not in W (because $T(Q) = 9$). We claim that Q, A, B, D, E is a basis for V.

Indeed, given $X \in V$ we can put $t = T(X)/9$ and $Y = X - tQ$. We then have $Y \in V$ and $T(Y) = T(X) - tT(Q) = 0$, so $Y \in W$. As A, B, D, E is a basis for W, we see that $Y = aA + bB + dD + eE$ for some $a, b, d, e \in \mathbb{R}$. It follows that $X = tQ + Y = tQ + aA + bB + dD + eE$. This means that Q, A, B, D, E spans V.

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It is clear that $\mu_{v,w}$ is a linear map. ○

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This holds for all x and y, so $\phi = \mu_{v,w}$ as claimed. $\bigcirc$

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For any list $\mathcal{V} = v_1, \ldots, v_n$ of elements of V, we can define a linear map $\mu_{\mathcal{V}} \colon \mathbb{R}^n \to V$ by
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An example

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Put $\mathcal{A} = A_1, A_2, A_3$, where

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$$\mu_{\mathcal{A}} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a & a+b & a \\ a+b & a+b+c & a+b \\ a & a+b & a \end{bmatrix} = \phi \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

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Another example

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Consider the map $\phi\colon \mathbb{R}^3 \to \mathbb{R}[x]$ given by

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Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$

Corollary ??: Every linear map $\alpha\colon \mathbb{R}^n \to \mathbb{R}^m$ has the form ϕ_A (as in Example **??**) for some $m \times n$ matrix A (which is uniquely determined).

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Thus, a linear map $\alpha\colon \mathbb{R}^n \to \mathbb{R}^m$ is essentially the same thing as an $m \times n$ matrix. There are some things to check to see that this is compatible with Example **??**, but we shall not go through the details. $\bigcirc$

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This means that $\rho = \phi_R$, where

$$R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

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(Note incidentally that the matrices arising in this way are precisely the 3×3 antisymmetric matrices.) ◯

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Matrices for plane projections

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It is an exercise to check that $A^2 = A^T = A$ and $\det(A) = 0$. ○

Matrices for linear maps

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Remark ??: Often we consider the case where $W = V$ and so we have a map $\alpha\colon V \to V$, and $\mathcal{V}$ and $\mathcal{W}$ are bases for the same space. It is often natural to take $\mathcal{W} = \mathcal{V}$, but everything still makes sense even if $\mathcal{W} \neq \mathcal{V}$. ○

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Matrices for linear maps

Let V and W be finite-dimensional vector spaces, with bases $\mathcal{V} = v_1, \ldots, v_n$ and $\mathcal{W} = w_1, \ldots, w_m$ say. Let $\alpha \colon V \to W$ be a linear map. Then $\alpha(v_j)$ is an element of W, so it can be expressed (uniquely) in terms of the basis $\mathcal{W}$, say

$$\alpha(v_j) = a_{1j} w_1 + \cdots + a_{mj} w_m.$$

These numbers a_{ij} form an $n \times m$ matrix A, which we call *the matrix of α with respect to $\mathcal{V}$ and $\mathcal{W}$*.

Adapted bases for vector products

Example ??: Let $\mathbf{a}$ be a unit vector in $\mathbb{R}^3$, and define $\beta\colon \mathbb{R}^3 \to \mathbb{R}^3$ by

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$$\begin{aligned}
\beta(\mathbf{a}) &= 0 &&= 0\mathbf{a} + 0\mathbf{b} + 0\mathbf{c} \\
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$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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$$\begin{aligned} \phi(1) &= 1 \\ \phi(x) &= 1 + x \\ \phi(x^2) &= 1 + 2x + x^2 \\ \phi(x^3) &= 1 + 3x + 3x^2 + x^3, \end{aligned}$$

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or in other words

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\phi(x^0) &= 1.x^0 + 0.x^1 + 0.x^2 + 0.x^3 \\
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\end{aligned}
\qquad
A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The Vandermonde matrix

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so the matrix of ϕ with respect to the usual bases is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \end{bmatrix}.$$

The reverse maps

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Example ??: Define $\phi\colon \mathbb{R}^4 \to \mathbb{R}^4$ by

$$\phi \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix}.$$

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$$\phi \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix}.$$

The associated matrix (with respect to the standard basis) is

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

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we have $\phi(\cos) = -\frac{1}{\sqrt{2}}\sin + \frac{1}{\sqrt{2}}\cos$. It follows that the matrix of ϕ with respect to the basis $\{\sin, \cos\}$ is

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Matrix examples

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we have

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The matrix of ϕ is thus

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

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\psi(E_2) &= E_2 && = 0.E_1 + 1.E_2 + 0.E_3 + 0.E_4 \\
\psi(E_3) &= E_3 && = 0.E_1 + 0.E_2 + 1.E_3 + 0.E_4 \\
\psi(E_4) &= E_4 - I/2 = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} && = (-\tfrac{1}{2}).E_1 + 0.E_2 + 0.E_3 + \tfrac{1}{2}.E_4
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\end{aligned}$$

The matrix is thus

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}. \bigcirc$$

Reminders

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- Given a vector space V and a list $\mathcal{V} = v_1, \ldots, v_m$ of elements of V we define $\mu_{\mathcal{V}} \colon \mathbb{R}^m \to V$ by $\mu_{\mathcal{V}}(\boldsymbol{\lambda}) = \sum_i \lambda_i v_i$.

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- If $\mathcal{V}$ is a basis then $\mu_{\mathcal{V}}$ is an isomorphism.

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- Given a vector space V and a list $\mathcal{V} = v_1, \dots, v_m$ of elements of V we define $\mu_{\mathcal{V}} \colon \mathbb{R}^m \to V$ by $\mu_{\mathcal{V}}(\boldsymbol{\lambda}) = \sum_i \lambda_i v_i$.
- If $\mathcal{V}$ is a basis then $\mu_{\mathcal{V}}$ is an isomorphism.
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$$\begin{aligned} \alpha(v_1) &= a_{11} w_1 + a_{21} w_2 + a_{31} w_3 \\ \alpha(v_2) &= a_{12} w_1 + a_{22} w_2 + a_{32} w_3 \end{aligned} \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

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so

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This means precisely that C is the matrix of $\alpha\beta$ with respect to $\mathcal{U}$ and $\mathcal{W}$. □

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Consider the following bases of $\mathbb{R}[x]_{\leq 3}$:

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$$v_1' = 1.v_1 + 1.v_2 + 1.v_3 + 1.v_4$$
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An example

Consider the following bases of $\mathbb{R}[x]_{\leq 3}$:

$$\begin{aligned} v_1 &= x^3 \\ v_1' &= x^3 + x^2 + x + 1 \end{aligned} \qquad \begin{aligned} v_2 &= x^2 \\ v_2' &= x^3 + x^2 + x \end{aligned} \qquad \begin{aligned} v_3 &= x \\ v_3' &= x^3 + x^2 \end{aligned} \qquad \begin{aligned} v_4 &= 1 \\ v_4' &= x^3 \end{aligned}$$

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so the change of basis matrix is

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

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Consider the following bases of $M_2\mathbb{R}$:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

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so the change of basis matrix is

$$P = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

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For any $\mathbf{x} \in \mathbb{R}^n$, we have

$$\begin{aligned} \mu_{\mathcal{W}}(QA'\mathbf{x}) &= \mu_{\mathcal{W}'}(A'\mathbf{x}) && \text{(Lemma \textbf{??})} \\ &= \alpha(\mu_{\mathcal{V}'}(\mathbf{x})) && \text{(Proposition \textbf{??})} \end{aligned}$$

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\mu_{\mathcal{W}}(QA'\mathbf{x}) &= \mu_{\mathcal{W}'}(A'\mathbf{x}) && \text{(Lemma \textbf{??})} \\
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This shows that $\mu_{\mathcal{W}}((QA' - AP)\mathbf{x}) = 0$.

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For any $\mathbf{x} \in \mathbb{R}^n$, we have

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\mu_{\mathcal{W}}(QA'\mathbf{x}) &= \mu_{\mathcal{W}'}(A'\mathbf{x}) && \text{(Lemma ??)} \\
&= \alpha(\mu_{\mathcal{V}'}(\mathbf{x})) && \text{(Proposition ??)} \\
&= \alpha(\mu_{\mathcal{V}}(P\mathbf{x})) && \text{(Lemma ??)} \\
&= \mu_{\mathcal{W}}(AP\mathbf{x}) && \text{(Proposition ??).}
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This shows that $\mu_{\mathcal{W}}((QA' - AP)\mathbf{x}) = 0$. Moreover, $\mathcal{W}$ is linearly independent, so $\mu_{\mathcal{W}}$ is injective and has trivial kernel, so $(QA' - AP)\mathbf{x} = 0$.

Proposition ??: Let $\alpha\colon V \to W$ be a linear map.
Suppose we have two bases $\mathcal{V}$ and $\mathcal{V}'$ for V, with change-of basis matrix P and two bases $\mathcal{W}$ and $\mathcal{W}'$ for W, with change-of-basis matrix Q.
Let A be the matrix of α with respect to $\mathcal{V}$ and $\mathcal{W}$,
and let A' be the matrix with respect to $\mathcal{V}'$ and $\mathcal{W}'$. Then $A' = Q^{-1}AP$.

Proof: We actually prove that $QA' = AP$, which comes to the same thing.
For any $\mathbf{x} \in \mathbb{R}^n$, we have

$$\begin{aligned}
\mu_{\mathcal{W}}(QA'\mathbf{x}) &= \mu_{\mathcal{W}'}(A'\mathbf{x}) && \text{(Lemma \textbf{??})} \\
&= \alpha(\mu_{\mathcal{V}'}(\mathbf{x})) && \text{(Proposition \textbf{??})} \\
&= \alpha(\mu_{\mathcal{V}}(P\mathbf{x})) && \text{(Lemma \textbf{??})} \\
&= \mu_{\mathcal{W}}(AP\mathbf{x}) && \text{(Proposition \textbf{??})}.
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This shows that $\mu_{\mathcal{W}}((QA' - AP)\mathbf{x}) = 0$. Moreover, $\mathcal{W}$ is linearly independent, so $\mu_{\mathcal{W}}$ is injective and has trivial kernel, so $(QA' - AP)\mathbf{x} = 0$. This applies for *any* vector $\mathbf{x}$, so the matrix $QA' - AP$ must be zero, as claimed.

Change of basis

The upshot is that all parts of the following diagram commute:

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[rrr, "\phi_{A'}"] \arrow[dd, "\mu_{\mathcal{V}'}"'] \arrow[dr, "\phi_P"] & & & \mathbb{R}^m \arrow[dl, "\phi_Q"'] \arrow[dd, "\mu_{\mathcal{W}'}"] \\
& \mathbb{R}^n \arrow[r, "\phi_A"'] \arrow[dl, "\mu_{\mathcal{V}}"] & \mathbb{R}^m \arrow[dr, "\mu_{\mathcal{W}}"'] & \\
V \arrow[rrr, "\alpha"'] & & & W
\end{tikzcd}
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This shows that definitions are in fact basis-independent.

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This is much more complicated using B.

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We have $\operatorname{trace}(P) = 1 - a_1^2 + 1 - a_2^2 + 1 - a_3^2 = 3 - (a_1^2 + a_2^2 + a_3^2) = 2$.

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We shall not give proofs, however.

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If there is a nontrivial relation $\boldsymbol{\lambda}$, so that $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$ and $\lambda_k \neq 0$ for some k, then we define the *height* of $\boldsymbol{\lambda}$ to be the largest i such that $\lambda_i \neq 0$.

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For example, if $n = 6$ and $5v_1 - 2v_2 - 2v_3 + 3v_4 = 0$ then $[5, -2, -2, 3, 0, 0]^T$ is a nontrivial linear relation of height 4.

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Thus, in this example, with $i = 3$, we see that (a), (b) and (c) all hold.

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$$\begin{aligned} \lambda_i v_i &= -\lambda_1 v_1 - \cdots - \lambda_{i-1} v_{i-1} - \lambda_{i+1} v_{i+1} - \cdots - \lambda_n v_n \\ &= -\lambda_1 v_1 - \cdots - \lambda_{i-1} v_{i-1} - 0.v_{i+1} - \cdots - 0.v_n \\ &= -\lambda_1 v_1 - \cdots - \lambda_{i-1} v_{i-1} \\ v_i &= -\lambda_1 \lambda_i^{-1} v_1 - \cdots - \lambda_{i-1} \lambda_i^{-1} v_{i-1} \in V_{i-1}. \end{aligned}$$

so $v_i \in V_{i-1}$.

Jumps

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$$\begin{aligned} w =& \lambda_1 v_1 + \cdots + \lambda_{i-1} v_{i-1} + \lambda_i(\mu_1 v_1 + \cdots + \mu_{i-1} v_{i-1}) \\ =& (\lambda_1 + \lambda_i \mu_1) v_1 + (\lambda_2 + \lambda_i \mu_2) v_1 + \cdots + (\lambda_{i-1} + \lambda_i \mu_{i-1}) v_{i-1}. \end{aligned}$$

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If these three things are true, we say that i is a *jump*.

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If we write only the nontrivial terms, then the relation takes the form

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Proposition **??** therefore tells us that $v_{i_r} \in V_{i_r - 1}$

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If we write only the nontrivial terms, then the relation takes the form

$$\lambda_{i_1} v_{i_1} + \dots + \lambda_{i_r} v_{i_r} = 0$$

with $i_k \in I'$ for all k, and $\lambda_{i_k} \neq 0$ for all k, and $i_1 < \dots < i_r$.
This can be regarded as a nontrivial linear relation for $\mathcal{V}$, of height i_r.
Proposition **??** therefore tells us that $v_{i_r} \in V_{i_r - 1}$, which is impossible, as i_r is a jump.

Every spanning set contains a basis

Lemma ??: Let $\mathcal{V} = v_1, \dots, v_n$ be a list that spans a vector space V. Then some sublist $\mathcal{V}' \subseteq \mathcal{V}$ is a basis for V.

Proof: Put $I' = \{\text{jumps}\} = \{i \leq n \mid v_i \notin V_{i-1}\}$, and $\mathcal{V}' = \{v_i \mid i \in I'\}$.
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This contradiction shows that $\mathcal{V}'$ must be linearly independent, after all.

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Either way we have $V_i \leq V'$, which proves the induction step. We therefore have $V_i \leq V'$ for all $i \leq n$. In particular, we have $V_n \leq V'$. However, V_n is just $\mathrm{span}(\mathcal{V})$, and we assumed that $\mathcal{V}$ spans V, so $V_n = V$. This proves that $V \leq V'$, and it is clear that $V' \leq V$, so $V = V'$. This means that $\mathcal{V}'$ is a spanning list as well as being linearly independent

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(a) Suppose that i is a jump, so $i \in I'$. Then (by the definition of $\mathcal{V}'$) we have $v_i \in \mathcal{V}'$ and so $v_i \in V'$. As $V_i = V_{i-1} + \mathbb{R}v_i$ and $V_{i-1} \leq V'$ and $\mathbb{R}v_i \leq V'$, we conclude that $V_i \leq V'$.

(b) Suppose that i is not a jump, so $v_i \in V_{i-1}$ and so $V_i = V_{i-1}$. By the induction hypothesis we have $V_{i-1} \leq V'$, so $V_i \leq V'$.

Either way we have $V_i \leq V'$, which proves the induction step. We therefore have $V_i \leq V'$ for all $i \leq n$. In particular, we have $V_n \leq V'$. However, V_n is just $\mathrm{span}(\mathcal{V})$, and we assumed that $\mathcal{V}$ spans V, so $V_n = V$. This proves that $V \leq V'$, and it is clear that $V' \leq V$, so $V = V'$. This means that $\mathcal{V}'$ is a spanning list as well as being linearly independent, so it is a basis for V. $\square$

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Lemma ??: Let V be a vector space, and let $\mathcal{V} = v_1, \dots, v_n$ and $\mathcal{W} = w_1, \dots, w_m$ be finite lists of elements of $\mathcal{V}$ such that $\mathcal{V}$ spans V and $\mathcal{W}$ is linearly independent.

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$$\begin{aligned} y_k =& x_k - a_{ki}a_{pi}^{-1}x_p = a_{k1}v_1 + \cdots + a_{ki}v_i - a_{ki}a_{pi}^{-1}(a_{p1}v_1 + \cdots + a_{pi}v_i) \\ &= (a_{k1} - a_{ki}a_{pi}^{-1}a_{p1})v_1 + (a_{k2} - a_{ki}a_{pi}^{-1}a_{p2})v_2 + \cdots + (a_{ki} - a_{ki}a_{pi}{}^{-1}a_{pi})v_i. \end{aligned}$$

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In the last term, the coefficient $a_{ki} - a_{ki}a_{pi}^{-1}a_{pi}$ is zero, so y_k is actually a linear combination of $v_1, \ldots, v_{i-1}$

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Case (b): Suppose that for some x_j we have $a_{ji} \neq 0$.

It is harmless to reorder the x's, so for notational convenience we move this x_j to the end of the list, which means that $a_{pi} \neq 0$.

Now put $\alpha_k = a_{ki}a_{pi}^{-1}$ and $y_k = x_k - \alpha_k x_p$.

We will show that $y_1, \ldots, y_{p-1}$ is a linearly independent list in V_{i-1}. Assuming this, the induction hypothesis gives $p-1 \leq i-1$, so $p \leq i$ as required. First, we have

$$\begin{aligned} y_k =& x_k - a_{ki}a_{pi}^{-1}x_p = a_{k1}v_1 + \cdots + a_{ki}v_i - a_{ki}a_{pi}^{-1}(a_{p1}v_1 + \cdots + a_{pi}v_i) \\ &= (a_{k1} - a_{ki}a_{pi}^{-1}a_{p1})v_1 + (a_{k2} - a_{ki}a_{pi}^{-1}a_{p2})v_2 + \cdots + ({\color{red}a_{ki}} - {\color{red}a_{ki}}{\color{blue}a_{pi}}^{-1}{\color{blue}a_{pi}})v_i. \end{aligned}$$

In the last term, the coefficient $a_{ki} - a_{ki}a_{pi}^{-1}a_{pi}$ is zero, so y_k is actually a linear combination of $v_1, \ldots, v_{i-1}$, so $y_k \in V_{i-1}$.

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Proof: Let $\mathcal{V} = v_1, \dots, v_n$ be any basis; then $\mu_{\mathcal{V}} \colon \mathbb{R}^n \to V$ is an isomorphism.

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If we put $\alpha_i = \lambda_i + \mu_i$ we get

$$x = y + z = \alpha_1 u_1 + \cdots + \alpha_p u_p + \beta_1 v_1 + \cdots + \beta_q v_q + \gamma_1 w_1 + \cdots + \gamma_r w_r \in \operatorname{span}(\mathcal{X}).$$

It follows that $\operatorname{span}(\mathcal{X}) = V + W$.

Two subspaces

Two subspaces

Finally, suppose we have a linear relation

$$\alpha_1 u_1 + \cdots + \alpha_p u_p + \beta_1 v_1 + \cdots + \beta_q v_q + \gamma_1 w_1 + \cdots + \gamma_r w_r = 0.$$

Two subspaces

Finally, suppose we have a linear relation

$$\alpha_1 u_1 + \cdots + \alpha_p u_p + \beta_1 v_1 + \cdots + \beta_q v_q + \gamma_1 w_1 + \cdots + \gamma_r w_r = 0.$$

Put $y = \sum_i \alpha_i u_i + \sum_j \beta_j v_j$ and $z = \sum_k \gamma_k w_k$, so $y + z = 0$, so $z = -y$.

Two subspaces

Finally, suppose we have a linear relation

$$\alpha_1 u_1 + \cdots + \alpha_p u_p + \beta_1 v_1 + \cdots + \beta_q v_q + \gamma_1 w_1 + \cdots + \gamma_r w_r = 0.$$

Put $y = \sum_i \alpha_i u_i + \sum_j \beta_j v_j$ and $z = \sum_k \gamma_k w_k$, so $y + z = 0$, so $z = -y$. Now $y \in V$

Two subspaces

Finally, suppose we have a linear relation

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Put $y = \sum_i \alpha_i u_i + \sum_j \beta_j v_j$ and $z = \sum_k \gamma_k w_k$, so $y + z = 0$, so $z = -y$. Now $y \in V$, so z also lies in V, because $z = -y$.

Two subspaces

Finally, suppose we have a linear relation

$$\alpha_1 u_1 + \cdots + \alpha_p u_p + \beta_1 v_1 + \cdots + \beta_q v_q + \gamma_1 w_1 + \cdots + \gamma_r w_r = 0.$$

Put $y = \sum_i \alpha_i u_i + \sum_j \beta_j v_j$ and $z = \sum_k \gamma_k w_k$, so $y + z = 0$, so $z = -y$. Now $y \in V$, so z also lies in V, because $z = -y$. Also $z \in W$

Two subspaces

Finally, suppose we have a linear relation

$$\alpha_1 u_1 + \cdots + \alpha_p u_p + \beta_1 v_1 + \cdots + \beta_q v_q + \gamma_1 w_1 + \cdots + \gamma_r w_r = 0.$$

Put $y = \sum_i \alpha_i u_i + \sum_j \beta_j v_j$ and $z = \sum_k \gamma_k w_k$, so $y + z = 0$, so $z = -y$. Now $y \in V$, so z also lies in V, because $z = -y$. Also $z \in W$, so $z \in V \cap W$.

Two subspaces

Finally, suppose we have a linear relation

$$\alpha_1 u_1 + \cdots + \alpha_p u_p + \beta_1 v_1 + \cdots + \beta_q v_q + \gamma_1 w_1 + \cdots + \gamma_r w_r = 0.$$

Put $y = \sum_i \alpha_i u_i + \sum_j \beta_j v_j$ and $z = \sum_k \gamma_k w_k$, so $y + z = 0$, so $z = -y$. Now $y \in V$, so z also lies in V, because $z = -y$. Also $z \in W$, so $z \in V \cap W$. We know that $\mathcal{U}$ is a basis for $V \cap W$

Two subspaces

Finally, suppose we have a linear relation

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Two subspaces

Finally, suppose we have a linear relation

$$\alpha_1 u_1 + \cdots + \alpha_p u_p + \beta_1 v_1 + \cdots + \beta_q v_q + \gamma_1 w_1 + \cdots + \gamma_r w_r = 0.$$

Put $y = \sum_i \alpha_i u_i + \sum_j \beta_j v_j$ and $z = \sum_k \gamma_k w_k$, so $y + z = 0$, so $z = -y$. Now $y \in V$, so z also lies in V, because $z = -y$. Also $z \in W$, so $z \in V \cap W$. We know that $\mathcal{U}$ is a basis for $V \cap W$, so $z = \lambda_1 u_1 + \cdots + \lambda_p u_p$ for some $\lambda_1, \ldots, \lambda_p$. This means that

$$\lambda_1 u_1 + \cdots + \lambda_p u_p - \gamma_1 w_1 - \cdots - \gamma_r w_r = 0.$$

Two subspaces

Finally, suppose we have a linear relation

$$\alpha_1 u_1 + \cdots + \alpha_p u_p + \beta_1 v_1 + \cdots + \beta_q v_q + \gamma_1 w_1 + \cdots + \gamma_r w_r = 0.$$

Put $y = \sum_i \alpha_i u_i + \sum_j \beta_j v_j$ and $z = \sum_k \gamma_k w_k$, so $y + z = 0$, so $z = -y$. Now $y \in V$, so z also lies in V, because $z = -y$. Also $z \in W$, so $z \in V \cap W$. We know that $\mathcal{U}$ is a basis for $V \cap W$, so $z = \lambda_1 u_1 + \cdots + \lambda_p u_p$ for some $\lambda_1, \dots, \lambda_p$. This means that

$$\lambda_1 u_1 + \cdots + \lambda_p u_p - \gamma_1 w_1 - \cdots - \gamma_r w_r = 0.$$

We also know that $(u_1, \dots, u_p, w_1, \dots, w_r)$ is a basis for W

Two subspaces

Finally, suppose we have a linear relation

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We also know that $(u_1, \ldots, u_p, w_1, \ldots, w_r)$ is a basis for W, so the above gives $\lambda_1 = \cdots = \lambda_p = \gamma_1 = \cdots = \gamma_r = 0$.

Two subspaces

Finally, suppose we have a linear relation

$$\alpha_1 u_1 + \cdots + \alpha_p u_p + \beta_1 v_1 + \cdots + \beta_q v_q + \gamma_1 w_1 + \cdots + \gamma_r w_r = 0.$$

Put $y = \sum_i \alpha_i u_i + \sum_j \beta_j v_j$ and $z = \sum_k \gamma_k w_k$, so $y + z = 0$, so $z = -y$. Now $y \in V$, so z also lies in V, because $z = -y$. Also $z \in W$, so $z \in V \cap W$. We know that $\mathcal{U}$ is a basis for $V \cap W$, so $z = \lambda_1 u_1 + \cdots + \lambda_p u_p$ for some $\lambda_1, \ldots, \lambda_p$. This means that

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Finally, suppose we have a linear relation

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Put $y = \sum_i \alpha_i u_i + \sum_j \beta_j v_j$ and $z = \sum_k \gamma_k w_k$, so $y + z = 0$, so $z = -y$. Now $y \in V$, so z also lies in V, because $z = -y$. Also $z \in W$, so $z \in V \cap W$. We know that $\mathcal{U}$ is a basis for $V \cap W$, so $z = \lambda_1 u_1 + \cdots + \lambda_p u_p$ for some $\lambda_1, \ldots, \lambda_p$. This means that

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The list $(u_1, \ldots, u_p, v_1, \ldots, v_q)$ is a basis for V

Finally, suppose we have a linear relation

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The list $(u_1, \dots, u_p, v_1, \dots, v_q)$ is a basis for V, so the above gives $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 0$.

Finally, suppose we have a linear relation

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The list $(u_1, \ldots, u_p, v_1, \ldots, v_q)$ is a basis for V, so the above gives $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 0$. As all α's, β's and γ's are zero, we see that our original linear relation was trivial.

Finally, suppose we have a linear relation

$$\alpha_1 u_1 + \cdots + \alpha_p u_p + \beta_1 v_1 + \cdots + \beta_q v_q + \gamma_1 w_1 + \cdots + \gamma_r w_r = 0.$$

Put $y = \sum_i \alpha_i u_i + \sum_j \beta_j v_j$ and $z = \sum_k \gamma_k w_k$, so $y + z = 0$, so $z = -y$. Now $y \in V$, so z also lies in V, because $z = -y$. Also $z \in W$, so $z \in V \cap W$. We know that $\mathcal{U}$ is a basis for $V \cap W$, so $z = \lambda_1 u_1 + \cdots + \lambda_p u_p$ for some $\lambda_1, \ldots, \lambda_p$. This means that

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We also know that $(u_1, \ldots, u_p, w_1, \ldots, w_r)$ is a basis for W, so the above gives $\lambda_1 = \cdots = \lambda_p = \gamma_1 = \cdots = \gamma_r = 0$. Feeding this back into our original relation, we get $\alpha_1 u_1 + \cdots + \alpha_p u_p + \beta_1 v_1 + \cdots + \beta_q v_q = 0$.

The list $(u_1, \ldots, u_p, v_1, \ldots, v_q)$ is a basis for V, so the above gives $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 0$. As all α's, β's and γ's are zero, we see that our original linear relation was trivial. This shows that the list $\mathcal{X}$ is linearly independent

Finally, suppose we have a linear relation

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We also know that $(u_1, \ldots, u_p, w_1, \ldots, w_r)$ is a basis for W, so the above gives $\lambda_1 = \cdots = \lambda_p = \gamma_1 = \cdots = \gamma_r = 0$. Feeding this back into our original relation, we get $\alpha_1 u_1 + \cdots + \alpha_p u_p + \beta_1 v_1 + \cdots + \beta_q v_q = 0$.

The list $(u_1, \ldots, u_p, v_1, \ldots, v_q)$ is a basis for V, so the above gives $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 0$. As all α's, β's and γ's are zero, we see that our original linear relation was trivial. This shows that the list $\mathcal{X}$ is linearly independent, so it gives a basis for $V + W$ as claimed.

An example

An example

Put $U = M_3\mathbb{R}$ and

$$V = \{A \in U \mid \text{ all rows sum to } 0 \} = \{A \in U \mid A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$W = \{A \in U \mid \text{ all columns sum to } 0 \} = \{A \in U \mid [1,1,1]A = [0,0,0]\}$$

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Then $V \cap W$ is the set of all matrices of the form

$$A = \begin{bmatrix} a & b & -a-b \\ c & d & -c-d \\ -a-c & -b-d & a+b+c+d \end{bmatrix} = a \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

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It follows that the list

$$u_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, u_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, u_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

is a basis for $V \cap W$.

An example

$$u_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, u_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, u_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

An example

$$u_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix},\ u_2 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix},\ u_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix},\ u_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Now put

$$v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix},\ v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix},\ w_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},\ w_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

so $v_i \in V$ and $w_i \in W$.

An example

$$u_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, u_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, u_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

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so $v_i \in V$ and $w_i \in W$. A typical element of V has the form

$$A = \begin{bmatrix} a & b & -a-b \\ c & d & -c-d \\ e & f & -e-f \end{bmatrix}$$

An example

$$u_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, u_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, u_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

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so $v_i \in V$ and $w_i \in W$. A typical element of V has the form

$$A = \begin{bmatrix} a & b & -a-b \\ c & d & -c-d \\ e & f & -e-f \end{bmatrix} = au_1 + bu_2 + cu_3 + du_4 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e-a-c & f-b-d & a+b+c+d-e-f \end{bmatrix}$$

An example

$$u_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, u_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, u_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Now put

$$v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, w_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

so $v_i \in V$ and $w_i \in W$. A typical element of V has the form

$$A = \begin{bmatrix} a & b & -a-b \\ c & d & -c-d \\ e & f & -e-f \end{bmatrix} = au_1 + bu_2 + cu_3 + du_4 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e-a-c & f-b-d & a+b+c+d-e-f \end{bmatrix}$$

$$= au_1 + bu_2 + cu_3 + du_4 + (e-a-c)v_1 + (f-b-d)v_2.$$

An example

$$u_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, u_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, u_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Now put

$$v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, w_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

so $v_i \in V$ and $w_i \in W$. A typical element of V has the form

$$\begin{aligned} A = \begin{bmatrix} a & b & -a-b \\ c & d & -c-d \\ e & f & -e-f \end{bmatrix} &= au_1 + bu_2 + cu_3 + du_4 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e-a-c & f-b-d & a+b+c+d-e-f \end{bmatrix} \\ &= au_1 + bu_2 + cu_3 + du_4 + (e-a-c)v_1 + (f-b-d)v_2 . \end{aligned}$$

Using this, we see that $u_1, \ldots, u_4, v_1, v_2$ is a basis for V.

An example

$$u_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, u_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, u_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Now put

$$v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, w_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

so $v_i \in V$ and $w_i \in W$. A typical element of V has the form

$$\begin{aligned} A = \begin{bmatrix} a & b & -a-b \\ c & d & -c-d \\ e & f & -e-f \end{bmatrix} &= au_1 + bu_2 + cu_3 + du_4 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e-a-c & f-b-d & a+b+c+d-e-f \end{bmatrix} \\ &= au_1 + bu_2 + cu_3 + du_4 + (e-a-c)v_1 + (f-b-d)v_2 . \end{aligned}$$

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Remark ??: If we use bases as in the theorem, then the matrix of α with respect to those bases has the form

$$A = \left[\begin{array}{c|c} I_r & 0_{r,m-r} \\ \hline 0_{n-r,r} & 0_{n-r,m-r} \end{array}\right]$$

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$\mathcal{U} = u_1, \ldots, u_m$ a basis for U	$\mathcal{V} = v_1, \ldots, v_n$ a basis for V
$\alpha(u_i) = v_i$ for $1 \leq i \leq r$	$\alpha(u_i) = 0$ for $r < i \leq m$
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Corollary ??: If $\alpha \colon U \to V$ is a linear map then

$$\dim(\ker(\alpha)) + \dim(\mathrm{image}(\alpha)) = \dim(U).$$

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Proof: Choose bases as in the theorem. Then $\dim(U) = m$ and $\dim(\operatorname{image}(\alpha)) = r$ and

$$\dim(\ker(\alpha)) = |\{u_{r+1}, \ldots, u_m\}| = m - r.$$

The claim follows.

Adapted bases

$\mathcal{U} = u_1, \ldots, u_m$ a basis for U	$\mathcal{V} = v_1, \ldots, v_n$ a basis for V
(a) $\alpha(u_i) = v_i$ for $1 \leq i \leq r$	(b) $\alpha(u_i) = 0$ for $r < i \leq m$
(c) $u_{r+1}, \ldots, u_m$ a basis for $\ker(\alpha)$	(d) $v_1, \ldots, v_r$ a basis for $\text{image}(\alpha)$

Proof of Theorem ??:

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This gives us a list $u_1, \dots, u_r$ of elements of U; to these, we add vectors $u_{r+1}, \dots, u_m$ forming a basis for ker(α) (so that (b) and (c) are satisfied).

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Now everything is as claimed except that we have not shown that the list $\mathcal{U} = u_1, \dots, u_m$ is a basis for U.

Adapted bases

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As $u_{r+1}, \ldots, u_m$ is a basis for ker(α), these vectors are linearly independent, so the above relation must be trivial, so $\lambda_{r+1} = \cdots = \lambda_m = 0$. This shows that all the λ's are zero

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This is a linear relation between the vectors $v_1, \ldots, v_r$, but these form a basis for image(α), so this must be the trivial relation, so $\lambda_1 = \cdots = \lambda_r = 0$. This means that our original relation has the form

$$\lambda_{r+1} u_{r+1} + \cdots + \lambda_m u_m = 0$$

As $u_{r+1}, \ldots, u_m$ is a basis for ker(α), these vectors are linearly independent, so the above relation must be trivial, so $\lambda_{r+1} = \cdots = \lambda_m = 0$. This shows that all the λ's are zero, so the original relation was trivial.

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$$\begin{aligned} 0 &= \lambda_1 \alpha(u_1) + \cdots + \lambda_r \alpha(u_r) + \lambda_{r+1} \alpha(u_{r+1}) + \cdots + \lambda_m \alpha(u_m) \\ &= \lambda_1 v_1 + \cdots + \lambda_r v_r + \lambda_{r+1}.0 + \cdots + \lambda_m.0 \\ &= \lambda_1 v_1 + \cdots + \lambda_r v_r. \end{aligned}$$

This is a linear relation between the vectors $v_1, \ldots, v_r$, but these form a basis for image(α), so this must be the trivial relation, so $\lambda_1 = \cdots = \lambda_r = 0$. This means that our original relation has the form

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As $u_{r+1}, \ldots, u_m$ is a basis for ker(α), these vectors are linearly independent, so the above relation must be trivial, so $\lambda_{r+1} = \cdots = \lambda_m = 0$. This shows that all the λ's are zero, so the original relation was trivial. Thus, the vectors $u_1, \ldots, u_m$ are linearly independent.

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$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The characteristic polynomial is thus

$$\det(tI - P) = \det \begin{bmatrix} t-1 & -1 & -1 & -1 & -1 \\ 0 & t-1 & -2 & -3 & -4 \\ 0 & 0 & t-1 & -3 & -6 \\ 0 & 0 & 0 & t-1 & -4 \\ 0 & 0 & 0 & 0 & t-1 \end{bmatrix} = (t-1)^5$$

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&= (u_1 v_1 - t) \det \begin{bmatrix} u_2 v_2 - t & u_3 v_2 \\ u_2 v_3 & u_3 v_3 - t \end{bmatrix} - u_2 v_1 \det \begin{bmatrix} u_1 v_2 & u_3 v_2 \\ u_1 v_3 & u_3 v_3 - t \end{bmatrix} + u_3 v_1 \det \begin{bmatrix} u_1 v_2 & u_2 v_2 - t \\ u_1 v_3 & u_2 v_3 \end{bmatrix}
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Moreover, all our examples will rely heavily on the fact that $x^2 \geq 0$ for all $x \in \mathbb{R}$, and of course this ceases to be true if we work over $\mathbb{C}$. We will see in Section **??** how to fix things up in the complex case.

The standard inner product on $\mathbb{R}^n$

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^T \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 = \langle \mathbf{x}, \mathbf{y} \rangle.$$

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(There is a more careful proof in the notes.)

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$$\text{so } \langle A,B\rangle = \mathrm{trace}(AB^T) = \begin{matrix} a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} + \\ a_{21}b_{21} + a_{22}b_{22} + a_{23}b_{23} + \\ a_{31}b_{31} + a_{32}b_{32} + a_{33}b_{33} \end{matrix} = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}b_{ij}.$$

In other words $\langle A,B\rangle$ is the sum of the entries of A multiplied by the corresponding entries in B.

Inner products of matrices

(a) $\langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle$ (b) $\langle tu,v\rangle = t\langle u,v\rangle$
(c) $\langle u,v\rangle = \langle v,u\rangle$ (d) $\langle u,u\rangle \geq 0$, equality iff $u=0$

Example ??: We can define an inner product on the space $M_n\mathbb{R}$ by $\langle A,B\rangle = \mathrm{trace}(AB^T)$.

Consider for example the case $n=3$, so $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

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In other words $\langle A,B\rangle$ is the sum of the entries of A multiplied by the corresponding entries in B. Thus, if we identify $M_n\mathbb{R}$ with $\mathbb{R}^{n^2}$, our inner product on $M_n\mathbb{R}$ corresponds to the standard inner product on $\mathbb{R}^{n^2}$.

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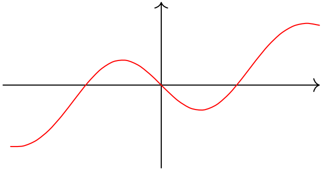

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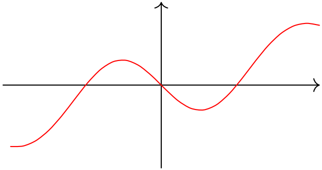

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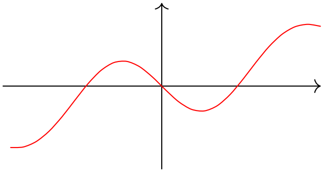

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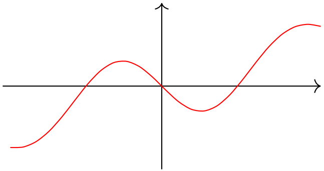

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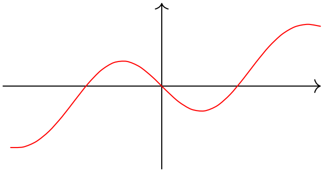

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Note that this only works because of the special form of the functions in V. For most functions f and g that you might think of, the integral $\int_{-\infty}^{\infty} f(x)g(x)\,dx$ will give an infinite or undefined answer. However, the function e^{-x^2} decays very rapidly to zero as $|x|$ tends to infinity, and one can check that this is enough to make the integral well-defined and finite when f and g are in V.

In fact, we have the formula

$$\langle x^n e^{-x^2/2}, x^m e^{-x^2/2} \rangle = \int_{-\infty}^{\infty} x^{n+m} e^{-x^2}\, dx$$

$$= \begin{cases} \frac{\sqrt{\pi}}{2^{n+m}} \frac{(n+m)!}{((n+m)/2)!} & \text{if } n+m \text{ is even} \\ 0 & \text{if } n+m \text{ is odd} \end{cases}$$

The Cauchy-Schwartz inequality

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If **v** and **w** are vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, you should be familiar with the fact that

$$\langle \mathbf{v}, \mathbf{w} \rangle = \|\mathbf{v}\| \, \|\mathbf{w}\| \, \cos(\theta),$$

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We would like to extend all this to arbitrary inner-product spaces.

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Theorem ??:
Let V be an inner product space over $\mathbb{R}$, and let v and w be elements of V.
Then $|\langle v, w\rangle| \leq \|v\|\,\|w\|$, with equality iff v and w are linearly dependent.

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Proof: Suppose that $v \in W \cap W^{\perp}$. As $v \in W^{\perp}$, we have $\langle v, w\rangle = 0$ for all $w \in W$. As $v \in W$, we can take $w = v$, which gives $\|v\|^2 = \langle v, v\rangle = 0$. This implies that $v = 0$, as required.

Orthogonal sequences

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Remark ??: If $\mathcal{V}$ is a strictly orthogonal sequence then we can define an orthonormal sequence $\hat{v}_1, \ldots, \hat{v}_n$ by $\hat{v}_i = v_i / \|v_i\|$.

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(Of course, these statements are only approximations. You can take it as an exercise to work out the size of the errors involved.)

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Lemma ??: Let $v_1, \dots, v_n$ be an orthogonal sequence, and put $v = v_1 + \cdots + v_n$. Then

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We can now take square roots to get the equation in the lemma.

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Remark ??: If the sequence $\mathcal{W}$ is orthonormal, then of course we have $\langle w_k, w_k \rangle = 1$ and the formula reduces to

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so $\langle w_i, v - \pi(v)\rangle = \langle w_i, v\rangle - \langle w_i, \pi(v)\rangle = 0$.

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so $\langle w_i, v - \pi(v)\rangle = \langle w_i, v\rangle - \langle w_i, \pi(v)\rangle = 0$. As this holds for all i, and the elements w_i span W, we see that $\langle w, v - \pi(v)\rangle = 0$ for all $w \in W$

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As the sequence $\mathcal{W}$ is orthogonal, we have $\langle w_i, w_j\rangle = 0$ for $j \neq i$, so only the i'th term in the above sum is nonzero. This means that

$$\langle w_i, \pi(v)\rangle = \lambda_i\langle w_i, w_i\rangle = \frac{\langle v, w_i\rangle}{\langle w_i, w_i\rangle}\langle w_i, w_i\rangle = \langle v, w_i\rangle = \langle w_i, v\rangle,$$

so $\langle w_i, v - \pi(v)\rangle = \langle w_i, v\rangle - \langle w_i, \pi(v)\rangle = 0$. As this holds for all i, and the elements w_i span W, we see that $\langle w, v - \pi(v)\rangle = 0$ for all $w \in W$, or in other words, that $v - \pi(v) \in W^{\perp}$, as claimed.

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Lemma ??: Let $v_1, \dots, v_n$ be an orthogonal sequence, and put $v = v_1 + \cdots + v_n$. Then

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Remark ??: If the sequence $\mathcal{W}$ is orthonormal, then of course we have $\langle w_k, w_k\rangle = 1$ and the formula reduces to

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&= \|x\|^2 + 0 + 0 + \|y\|^2
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Proof: Put $x = v - \pi(v)$, so $x \in W^{\perp}$. The distance from v to $\pi(v)$ is just $\|v - \pi(v)\| = \|x\|$. Now consider another point $w \in W$, with $w \neq \pi(v)$. The distance from v to w is just $\|v - w\|$; we must show that this is larger than $\|x\|$. Put $y = \pi(v) - w$, and note that $v - w = \pi(v) + x - w = x + y$. Note also that $y \in W$ (because $\pi(v) \in W$ and $w \in W$) and $x \in W^{\perp}$, so $\langle x, y\rangle = 0 = \langle y, x\rangle$. Finally, note that $y \neq 0$ and so $\|y\| > 0$. It follows that

$$\begin{aligned}\|v - w\|^2 &= \|x + y\|^2 = \langle x + y, x + y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= \|x\|^2 + 0 + 0 + \|y\|^2 > \|x\|^2.\end{aligned}$$

This shows that $\|v - w\| > \|x\| = \|v - \pi(v)\|$

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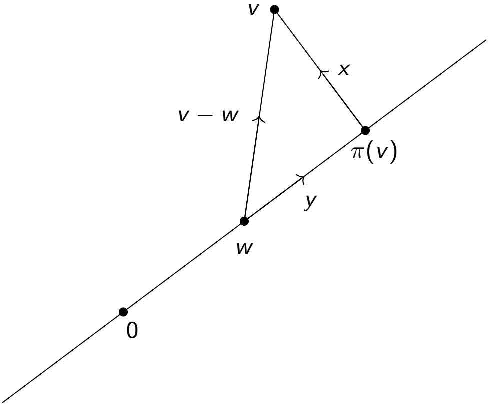

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The Gram-Schmidt procedure

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$$w_i = \sum_{j=1}^{i-1} \frac{\langle u_i, v_j \rangle}{\langle v_j, v_j \rangle} v_j \qquad v_i = u_i - w_i.$$

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This means that $v_1, \dots, v_i$ is a spanning set of the i-dimensional space U_i, so it must be a basis. $\square$

The Gram-Schmidt procedure

Corollary ??: If V and $\mathcal{U}$ are as above, then there is an *orthonormal* sequence $\hat{v}_1, \ldots, \hat{v}_n$ with $\text{span}(\hat{v}_1, \ldots, \hat{v}_i) = \text{span}(u_1, \ldots, u_i)$ for all i.

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Proof: Just find a strictly orthogonal sequence $v_1, \dots, v_n$ as in the Proposition, and put $\hat{v}_i = v_i / \|v_i\|$ as in Remark **??**.

An example

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Example ??: Consider the following elements of $\mathbb{R}^5$:

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It follows that $\langle v_2, v_2 \rangle = 3/2$

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We apply the Gram-Schmidt procedure to get an orthogonal basis for the space $U = \mathrm{span}(u_1, u_2, u_3, u_4)$. We have $v_1 = u_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix}^T$, so $\langle v_1, v_1 \rangle = 2$ and $\langle u_2, v_1 \rangle = 1$. Next, we have

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We apply the Gram-Schmidt procedure to get an orthogonal basis for the space $U = \mathrm{span}(u_1, u_2, u_3, u_4)$. We have $v_1 = u_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix}^T$, so $\langle v_1, v_1 \rangle = 2$ and $\langle u_2, v_1 \rangle = 1$. Next, we have

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In conclusion, we have

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$$v_3 = x^2 - \frac{\langle x^2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle x^2, v_2\rangle}{\langle v_2, v_2\rangle} v_2 = x^2 - \frac{2/3}{2} 1 = x^2 - 1/3.$$

A polynomial example

$$v_1 = 1 \qquad v_2 = x \qquad v_3 = x^2 - 1/3$$

We find that

$$\langle v_3, v_3 \rangle = \int_{-1}^{1} (x^2 - 1/3)^2 \, dx = \int_{-1}^{1} x^4 - \tfrac{2}{3}x^2 + \tfrac{1}{9} \, dx = \left[\tfrac{1}{5}x^5 - \tfrac{2}{9}x^3 + \tfrac{1}{9}x \right]_{-1}^{1} = 8/45.$$

The required orthonormal basis is thus given by

$$\begin{aligned} \hat{v}_1 &= v_1/\|v_1\| = 1/\sqrt{2} \\ \hat{v}_2 &= v_2/\|v_2\| = \sqrt{3/2}x \\ \hat{v}_3 &= v_3/\|v_3\| = \sqrt{45/8}(x^2 - 1/3). \end{aligned}$$

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It is not far from being an orthonormal basis: we have $\langle A_i, A_i \rangle = 2$ for all i, and when $i \neq j$ we have $\langle A_i, A_j \rangle = 0$ except for the case $\langle A_1, A_4 \rangle = 1$.

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The Gram-Schmidt procedure works out as follows:

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$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad A_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The Gram-Schmidt procedure works out as follows:

$$\begin{aligned} B_1 &= A_1 \\ B_2 &= A_2 - \frac{\langle A_2, B_1 \rangle}{\langle B_1, B_1 \rangle} B_1 \\ &= \\ &= \end{aligned}$$

A matrix example

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad A_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The Gram-Schmidt procedure works out as follows:

$$B_1 = A_1$$

$$B_2 = A_2 - \frac{\langle A_2, B_1 \rangle}{\langle B_1, B_1 \rangle} B_1 = A_2$$

$$=$$

$$=$$

A matrix example

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad A_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The Gram-Schmidt procedure works out as follows:

$$\begin{aligned} B_1 &= A_1 \\ B_2 &= A_2 - \frac{\langle A_2 , B_1 \rangle}{\langle B_1 , B_1 \rangle} B_1 = A_2 \\ B_3 &= A_3 - \frac{\langle A_3 , B_1 \rangle}{\langle B_1 , B_1 \rangle} B_1 - \frac{\langle A_3 , B_2 \rangle}{\langle B_2 , B_2 \rangle} B_2 \\ &= \\ &= \end{aligned}$$

A matrix example

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad A_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The Gram-Schmidt procedure works out as follows:

$$\begin{aligned}
B_1 &= A_1 \\
B_2 &= A_2 - \frac{\langle A_2, B_1 \rangle}{\langle B_1, B_1 \rangle} B_1 = A_2 \\
B_3 &= A_3 - \frac{\langle A_3, B_1 \rangle}{\langle B_1, B_1 \rangle} B_1 - \frac{\langle A_3, B_2 \rangle}{\langle B_2, B_2 \rangle} B_2 = A_3 \\
&= \\
&=
\end{aligned}$$

A matrix example

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad A_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The Gram-Schmidt procedure works out as follows:

$$\begin{aligned}
B_1 &= A_1 \\
B_2 &= A_2 - \frac{\langle A_2, B_1 \rangle}{\langle B_1, B_1 \rangle} B_1 = A_2 \\
B_3 &= A_3 - \frac{\langle A_3, B_1 \rangle}{\langle B_1, B_1 \rangle} B_1 - \frac{\langle A_3, B_2 \rangle}{\langle B_2, B_2 \rangle} B_2 = A_3 \\
B_4 &= A_4 - \frac{\langle A_4, B_1 \rangle}{\langle B_1, B_1 \rangle} B_1 - \frac{\langle A_4, B_2 \rangle}{\langle B_2, B_2 \rangle} B_2 - \frac{\langle A_4, B_3 \rangle}{\langle B_3, B_3 \rangle} B_3 \\
&=
\end{aligned}$$

A matrix example

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad A_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The Gram-Schmidt procedure works out as follows:

$$\begin{aligned}
B_1 &= A_1 \\
B_2 &= A_2 - \frac{\langle A_2, B_1 \rangle}{\langle B_1, B_1 \rangle} B_1 = A_2 \\
B_3 &= A_3 - \frac{\langle A_3, B_1 \rangle}{\langle B_1, B_1 \rangle} B_1 - \frac{\langle A_3, B_2 \rangle}{\langle B_2, B_2 \rangle} B_2 = A_3 \\
B_4 &= A_4 - \frac{\langle A_4, B_1 \rangle}{\langle B_1, B_1 \rangle} B_1 - \frac{\langle A_4, B_2 \rangle}{\langle B_2, B_2 \rangle} B_2 - \frac{\langle A_4, B_3 \rangle}{\langle B_3, B_3 \rangle} B_3 = A_4 - \frac{1}{2} B_1 \\
\\
&=
\end{aligned}$$

A matrix example

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad A_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The Gram-Schmidt procedure works out as follows:

$$\begin{aligned}
B_1 &= A_1 \\
B_2 &= A_2 - \frac{\langle A_2 , B_1 \rangle}{\langle B_1 , B_1 \rangle} B_1 = A_2 \\
B_3 &= A_3 - \frac{\langle A_3 , B_1 \rangle}{\langle B_1 , B_1 \rangle} B_1 - \frac{\langle A_3 , B_2 \rangle}{\langle B_2 , B_2 \rangle} B_2 = A_3 \\
B_4 &= A_4 - \frac{\langle A_4 , B_1 \rangle}{\langle B_1 , B_1 \rangle} B_1 - \frac{\langle A_4 , B_2 \rangle}{\langle B_2 , B_2 \rangle} B_2 - \frac{\langle A_4 , B_3 \rangle}{\langle B_3 , B_3 \rangle} B_3 = A_4 - \frac{1}{2} B_1 \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
&=
\end{aligned}$$

A matrix example

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad A_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The Gram-Schmidt procedure works out as follows:

$$\begin{aligned}
B_1 &= A_1 \\
B_2 &= A_2 - \frac{\langle A_2 , B_1 \rangle}{\langle B_1 , B_1 \rangle} B_1 = A_2 \\
B_3 &= A_3 - \frac{\langle A_3 , B_1 \rangle}{\langle B_1 , B_1 \rangle} B_1 - \frac{\langle A_3 , B_2 \rangle}{\langle B_2 , B_2 \rangle} B_2 = A_3 \\
B_4 &= A_4 - \frac{\langle A_4 , B_1 \rangle}{\langle B_1 , B_1 \rangle} B_1 - \frac{\langle A_4 , B_2 \rangle}{\langle B_2 , B_2 \rangle} B_2 - \frac{\langle A_4 , B_3 \rangle}{\langle B_3 , B_3 \rangle} B_3 = A_4 - \frac{1}{2} B_1 \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} \\
&=
\end{aligned}$$

A matrix example

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad A_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The Gram-Schmidt procedure works out as follows:

$$\begin{aligned}
B_1 &= A_1 \\
B_2 &= A_2 - \frac{\langle A_2, B_1 \rangle}{\langle B_1, B_1 \rangle} B_1 = A_2 \\
B_3 &= A_3 - \frac{\langle A_3, B_1 \rangle}{\langle B_1, B_1 \rangle} B_1 - \frac{\langle A_3, B_2 \rangle}{\langle B_2, B_2 \rangle} B_2 = A_3 \\
B_4 &= A_4 - \frac{\langle A_4, B_1 \rangle}{\langle B_1, B_1 \rangle} B_1 - \frac{\langle A_4, B_2 \rangle}{\langle B_2, B_2 \rangle} B_2 - \frac{\langle A_4, B_3 \rangle}{\langle B_3, B_3 \rangle} B_3 = A_4 - \frac{1}{2} B_1 \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} \\
B_5 &= A_5 - \frac{\langle A_5, B_1 \rangle}{\langle B_1, B_1 \rangle} B_1 - \frac{\langle A_5, B_2 \rangle}{\langle B_2, B_2 \rangle} B_2 - \frac{\langle A_5, B_3 \rangle}{\langle B_3, B_3 \rangle} B_3 - \frac{\langle A_5, B_4 \rangle}{\langle B_4, B_4 \rangle} B_4
\end{aligned}$$

A matrix example

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad A_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The Gram-Schmidt procedure works out as follows:

$$\begin{aligned}
B_1 &= A_1 \\
B_2 &= A_2 - \frac{\langle A_2 , B_1 \rangle}{\langle B_1 , B_1 \rangle} B_1 = A_2 \\
B_3 &= A_3 - \frac{\langle A_3 , B_1 \rangle}{\langle B_1 , B_1 \rangle} B_1 - \frac{\langle A_3 , B_2 \rangle}{\langle B_2 , B_2 \rangle} B_2 = A_3 \\
B_4 &= A_4 - \frac{\langle A_4 , B_1 \rangle}{\langle B_1 , B_1 \rangle} B_1 - \frac{\langle A_4 , B_2 \rangle}{\langle B_2 , B_2 \rangle} B_2 - \frac{\langle A_4 , B_3 \rangle}{\langle B_3 , B_3 \rangle} B_3 = A_4 - \frac{1}{2} B_1 \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} \\
B_5 &= A_5 - \frac{\langle A_5 , B_1 \rangle}{\langle B_1 , B_1 \rangle} B_1 - \frac{\langle A_5 , B_2 \rangle}{\langle B_2 , B_2 \rangle} B_2 - \frac{\langle A_5 , B_3 \rangle}{\langle B_3 , B_3 \rangle} B_3 - \frac{\langle A_5 , B_4 \rangle}{\langle B_4 , B_4 \rangle} B_4 = A_5 .
\end{aligned}$$

A matrix example

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B_4 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad B_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

A matrix example

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B_4 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad B_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

We have $\|B_4\| = \sqrt{3/2}$

A matrix example

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B_4 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad B_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

We have $\|B_4\| = \sqrt{3/2}$ and $\|B_i\| = \sqrt{2}$ for all other i.

A matrix example

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B_4 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad B_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

We have $\|B_4\| = \sqrt{3/2}$ and $\|B_i\| = \sqrt{2}$ for all other i. After noting that $(1/2)/\sqrt{3/2} = 1/\sqrt{6}$

A matrix example

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B_4 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad B_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

We have $\|B_4\| = \sqrt{3/2}$ and $\|B_i\| = \sqrt{2}$ for all other i. After noting that $(1/2)/\sqrt{3/2} = 1/\sqrt{6}$, it follows that the following matrices give an orthonormal basis for V:

$$\widehat{B}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \widehat{B}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \widehat{B}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

A matrix example

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B_4 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad B_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

We have $\|B_4\| = \sqrt{3/2}$ and $\|B_i\| = \sqrt{2}$ for all other i. After noting that $(1/2)/\sqrt{3/2} = 1/\sqrt{6}$, it follows that the following matrices give an orthonormal basis for V:

$$\widehat{B}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \widehat{B}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \widehat{B}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\widehat{B}_4 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

A matrix example

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B_4 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad B_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

We have $\|B_4\| = \sqrt{3/2}$ and $\|B_i\| = \sqrt{2}$ for all other i. After noting that $(1/2)/\sqrt{3/2} = 1/\sqrt{6}$, it follows that the following matrices give an orthonormal basis for V:

$$\widehat{B}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \widehat{B}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \widehat{B}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\widehat{B}_4 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \widehat{B}_5 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

A matrix example

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B_4 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad B_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

A matrix example

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B_4 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad B_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The closest point in V to P is $Q = \sum_{i=1}^{5} \frac{\langle P, B_i \rangle}{\langle B_i, B_i \rangle} B_i$.

A matrix example

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B_4 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad B_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The closest point in V to P is $Q = \sum_{i=1}^{5} \frac{\langle P, B_i \rangle}{\langle B_i, B_i \rangle} B_i$.

The relevant inner products are $\langle P, B_1 \rangle = \langle P, B_2 \rangle = \langle P, B_3 \rangle = 1$

A matrix example

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B_4 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad B_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The closest point in V to P is $Q = \sum_{i=1}^{5} \frac{\langle P, B_i \rangle}{\langle B_i, B_i \rangle} B_i$.

The relevant inner products are $\langle P, B_1 \rangle = \langle P, B_2 \rangle = \langle P, B_3 \rangle = 1$ and $\langle P, B_4 \rangle = -1/2$

A matrix example

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B_4 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad B_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The closest point in V to P is $Q = \sum_{i=1}^{5} \frac{\langle P, B_i \rangle}{\langle B_i, B_i \rangle} B_i$.

The relevant inner products are $\langle P, B_1 \rangle = \langle P, B_2 \rangle = \langle P, B_3 \rangle = 1$ and $\langle P, B_4 \rangle = -1/2$ and $\langle P, B_5 \rangle = 0$.

A matrix example

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B_4 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad B_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The closest point in V to P is $Q = \sum_{i=1}^{5} \frac{\langle P, B_i \rangle}{\langle B_i, B_i \rangle} B_i$.

The relevant inner products are $\langle P, B_1 \rangle = \langle P, B_2 \rangle = \langle P, B_3 \rangle = 1$ and $\langle P, B_4 \rangle = -1/2$ and $\langle P, B_5 \rangle = 0$.

Also $\langle B_1, B_1 \rangle = \langle B_2, B_2 \rangle = \langle B_3, B_3 \rangle = 2$

A matrix example

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Given an inner product, we will write $\|u\| = \sqrt{\langle u, u\rangle}$, and call this the *norm* of u. We say that u is a *unit vector* if $\|u\| = 1$.

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The above Hermitian form on $\mathbb{C}^n$ can then be rewritten as

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Moreover, this inequality is actually an equality iff $v \in \operatorname{span}(\mathcal{W})$. $\square$

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$$\langle \phi(v), w\rangle = \langle v, \psi(w)\rangle$$

for all $v \in V$ and $w \in W$.

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Example ??: Let A be an $n \times m$ matrix over $\mathbb{R}$, giving a linear map $\phi_A \colon \mathbb{R}^m \to \mathbb{R}^n$ by $\phi_A(\mathbf{v}) = A\mathbf{v}$. The transpose of A is then an $m \times n$ matrix A^T, giving a linear map $\phi_{A^T} : \mathbb{R}^n \to \mathbb{R}^m$. We claim that ϕ_{A^T} is adjoint to ϕ_A. This is easy to see using the formula $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$ as in Remark **??**.

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Cross products are anti self adjoint

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Fix a vector $\mathbf{a} = [a_1, a_2, a_3]^T \in \mathbb{R}^3$, and define $\alpha \colon \mathbb{R}^3 \to \mathbb{R}^3$ by $\alpha(\mathbf{x}) = \mathbf{a} \times \mathbf{x}$.

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It follows that $\alpha^T = \phi_{A^T} = \phi_{-A} = -\alpha$.

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Both limits here are zero, because the very rapid decrease of $e^{-x^2/2}$ wipes out the much slower increase of the polynomial terms.

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This can be done with Maple: entering

```
expand(int((a*x^2+b*x+c)*((30*p+30*q)*x^2 - (36*p+24*q)*x + (9*p+3*q)),x=0..1));
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gives $cp + aq + bq + cq$, as required.

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We will prove the complex case; the real case is similar but slightly easier.

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This shows that ψ is adjoint to ϕ, as required.

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De Moivre's theorem tells us that

$$\begin{aligned} e_n &= c_n + i\,s_n \\ s_n &= (e_n - e_{-n})/(2i) \\ c_n &= (e_n + e_{-n})/2. \end{aligned}$$

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This shows that $\mathcal{C}_n$ is an orthogonal sequence.

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Corollary ??: Using Proposition **??**, we deduce that

$$\pi_n(f) = \langle f, c_0 \rangle c_0 + 2\sum_{k=1}^{n}\langle f, c_k \rangle c_k + 2\sum_{k=1}^{n}\langle f, s_k \rangle s_k.$$

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By taking limits as n tends to infinity, we see that $\|f\|^2 = \sum_{k=-\infty}^{\infty} |\langle f, e_k\rangle|^2$.

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Similarly, using Corollary **??** and Proposition **??**, we see that

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We can again let n tend to infinity to see that

$$\|f\|^2 = |\langle f, c_0\rangle|^2 + 2\sum_{k=1}^{\infty}|\langle f, c_k\rangle|^2 + 2\sum_{k=1}^{\infty}|\langle f, s_k\rangle|^2.$$

Self-adjoint operators

Definition ??: Let V be a finite-dimensional vector space over $\mathbb{C}$. A *self-adjoint operator* on V is a linear map $\alpha\colon V \to V$ such that $\alpha^\dagger = \alpha$.

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The diagonalisation theorem

Theorem ??: If $\alpha\colon V \to V$ is a self-adjoint operator, then one can choose an orthonormal basis $\mathcal{V} = v_1, \ldots, v_n$ for V such that each v_i is an eigenvector of α.

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An example

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