VECTOR SPACES AND FOURIER THEORY — FURTHER EXERCISES

Exercise 1. Put

$$V = \{(a, b, c, d) \in \mathbb{R}^4 \mid a + b + c + d = 0\}$$

Show that each of the following lists of vectors is a basis for V.

- $u_1 = (1, 0, 0, -1)$, $u_2 = (0, 1, 0, -1)$, $u_3 = (0, 0, 1, -1)$
- $v_1 = (1, -1, 0, 0)$, $v_2 = (0, 1, -1, 0)$, $v_3 = (0, 0, 1, -1)$
- $w_1 = (1, 1, -1, -1)$, $w_2 = (1, -1, 1, -1)$, $w_3 = (1, -1, -1, 1)$

Solution: We first note that in every vector we have mentioned, the sum of the four coordinates is zero, so all our vectors lie in V.

(a) Suppose we have a vector $x = (a, b, c, d) \in V$. We then have $au_1 + bu_2 + cu_3 = (a, b, c, -a - b - c)$ but $d = -a - b - c$ (because $x \in V$) so $x = au_1 + bu_2 + cu_3$. This shows that u_1, u_2 and u_3 span V. Next, as the first three entries in $au_1 + bu_2 + cu_3$ are just a, b and c, the only way we can have $au_1 + bu_2 + cu_3 = 0$ is if $a = b = c = 0$. This shows that u_1, u_2 and u_3 are linearly independent, so they give a basis for V.

(b) Suppose again that $x = (a, b, c, d) \in V$. Then

$$\begin{aligned} av_1 + (a+b)v_2 + (a+b+c)v_3 &= (a, -a, 0, 0) + (0, a+b, -a-b, 0) + (0, 0, a+b+c, -a-b-c) \\ &= (a, b, c, -a-b-c) = x. \end{aligned}$$

This means that any $x \in V$ is a linear combination of v_1, v_2 and v_3, so these vectors span V. As V has dimension 3 (by part (a)) and our spanning set has size 3, it is automatically a basis. More explicitly, suppose that $pv_1 + qv_2 = rv_3 = 0$. This means that $(p, q - p, r - q, -r) = (0, 0, 0, 0)$, so $p = 0$ and $q = p$ and $r = q$ and $q = 0$, so $p = q = r = 0$. This shows that v_1, v_2 and v_3 are linearly independent, and so form a basis.

(c) Note that $u_1 = (w_1 + w_2)/2$ and $u_2 = (w_1 - w_3)/2$ and $u_3 = (w_2 - w_3)/2$. It follows that for $x = (a, b, c, d) \in V$ we have

$$x = au_1 + bu_2 + cu_3 = \tfrac{1}{2}(a + b + c)w_1 + \tfrac{1}{2}(a + c)w_2 - \tfrac{1}{2}(b + c)w_3,$$

so w_1, w_2 and w_3 span V. As V has dimension 3 and our spanning set has size 3, it is automatically a basis. More explicitly, suppose that $pw_1 + qw_2 + rw_3 = 0$. This means that

$$(p + q + r, p - q - r, -p + q - r, -p - q + r) = (0, 0, 0, 0),$$

so

$$\begin{aligned} p + q + r &= 0 \\ p - q - r &= 0 \\ -p + q - r &= 0 \\ -p - q + r &= 0 \end{aligned}$$

By adding the first equation to each of the other three, we see that $p = q = r = 0$. This shows that w_1, w_2 and w_3 are linearly independent, as claimed.

Exercise 2. Show that the following matrices give a basis for $M_2\mathbb{R}$:

$$W = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \qquad Y = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Solution: Suppose we have a matrix $P = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, which we want to write as a linear combination of W, X, Y and Z, say

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} = aW + bX + cY + dZ = \begin{bmatrix} a+b+c+d & a+b-c-d \\ a-b+c-d & a-b-c+d \end{bmatrix}$$

This is equivalent to the system of equations

$$\begin{aligned} p &= a+b+c+d \\ q &= a+b-c-d \\ r &= a-b+c-d \\ s &= a-b-c+d, \end{aligned}$$

which have the unique solution

$$\begin{aligned} a &= (p+q+r+s)/4 \\ b &= (p+q-r-s)/4 \\ c &= (p-q+r-s)/4 \\ d &= (p-q-r+s)/4. \end{aligned}$$

This means that P can be written in a unique way as a linear combination of W, X, Y and Z, so these matrices form a basis for $M_2\mathbb{R}$.

Exercise 3. Find p, q and r such that

$$\int_0^1 f(x)\,dx = pf(0) + qf(1/2) + rf(1)$$

for all $f(x) \in \mathbb{R}[x]_{\leq 2}$.

Solution: The answer is $p = r = 1/6$, $q = 2/3$.

Exercise 4. Put

$$\begin{aligned} V &= \{A \in M_3\mathbb{R} \mid A^T = A\} \\ W &= \{A \in M_3\mathbb{R} \mid \operatorname{trace}(A) = 0\} \end{aligned}$$

Show that $V + W = M_3\mathbb{R}$, and find a basis for $V \cap W$.

Solution: If $A \in M_3\mathbb{R}$, put $t = \operatorname{trace}(A)/3$ and $B = A - tI$. We have $\operatorname{trace}(I) = 3$ so $\operatorname{trace}(tI) = \operatorname{trace}(A)$, so $\operatorname{trace}(B) = 0$, so $B \in W$. We also have $tI \in V$ and $A = tI + B$ so $A \in V + W$. This shows that $V + W = M_3\mathbb{R}$.

Next, the matrices in V are those of the form

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$

Such a matrix lies in $V \cap W$ iff $f = -a - d$, so we have

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & -a-d \end{bmatrix}.$$

Now put

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad E_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad E_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

so $E_1, \dots, E_5 \in V \cap W$. Our previous equation for A can now be written

$$A = aE_1 + bE_2 + cE_3 + dE_4 + eE_5.$$

It follows that $E_1, \dots, E_5$ span $V \cap W$, and they are clearly linearly independent, so they form a basis.

Exercise 5. Define subspaces $V, W \leq \mathbb{R}^6$ as follows:

$$\begin{aligned} V &= \operatorname{span}((1,1,0,0,0,0),(1,1,1,1,0,0),(1,1,1,1,1,1)) \\ W &= \operatorname{span}((1,1,1,0,0,0),(0,0,0,1,1,1)). \end{aligned}$$

Find vectors u, v_1, v_2, w, x_1, x_2 such that

- $\{u\}$ is a basis for $V \cap W$
- $\{u, v_1, v_2\}$ is a basis for V
- $\{u, w\}$ is a basis for W
- $\{u, v_1, v_2, w\}$ is a basis for $V + W$
- $\{u, v_1, v_2, w, x_1, x_2\}$ is a basis for $\mathbb{R}^6$.

Solution: First, we note that

$$\begin{aligned} V &= \{t \in \mathbb{R}^6 \mid t_1 = t_2,\ t_3 = t_4,\ t_5 = t_6\} \\ W &= \{t \in \mathbb{R}^6 \mid t_1 = t_2 = t_3\ t_4 = t_5 = t_6\} \end{aligned}$$

so

$$\begin{aligned} V \cap W &= \{t \in \mathbb{R}^6 \mid t_1 = t_2 = \cdots = t_6\} \\ &= \operatorname{span}((1,1,1,1,1,1)) \end{aligned}$$

We therefore take $u = (1,1,1,1,1,1)$. If we put $v_1 = (1,1,0,0,0,0)$ and $v_2 = (0,0,1,1,0,0)$ then it is clear that $u_, v_1$ and v_2 are linearly independent and span V, so they form a basis. Similarly, if we put $w = (1,1,1,0,0,0)$ then u and w give a basis for W. It is automatic from this that $\{u, v_1, v_2, w\}$ is a basis for $V + W$. Finally, put $x_1 = (1,0,0,0,0,0)$ and $x_2 = (0,0,0,0,0,1)$. We then

$$au + bv_1 + cv_2 + dw + ex_1 + fx_2 = (a+b+d+e, a+b+d, a+c+d, a+c, a, a+f)$$

If this is zero then $a = 0$ (5th entry) so $c = f = 0$ (4th and 6th entries) so $d = 0$ (3rd entry) so $b = 0$ (2nd entry) so $e = 0$ (1st entry). This shows that our six vectors are linearly independent, so they form a basis for $\mathbb{R}^6$.

Exercise 6. Put $U = \{f \in C^\infty(\mathbb{R}) \mid D(D-1)(D-2)(D-3)f = 0\}$ and $V = \{f \in U \mid f(0) = 0\}$. Give a basis for V.

Solution: By standard theory of differential equations, we see that U is the set of functions of the form

$$f(x) = a_0 + a_1 e^x + a_2 e^{2x} + a_3 e^{3x}$$

for some $a_0, \dots, a_3 \in \mathbb{R}$. For such f we have $f(0) = a_0 + a_1 + a_2 + a_3$, so $f \in V$ iff we have $a_0 = -a_1 - a_2 - a_3$, which means that

$$f(x) = a_1(e^x - 1) + a_2(e^{2x} - 1) + a_3(e^{3x} - 1).$$

It follows that the functions $e^x - 1$, $e^{2x} - 1$ and $e^{3x} - 1$ give a basis for V.

Exercise 7. Let λ and ω be real numbers. Define functions $f_i \in C^\infty(\mathbb{R})$ by

$$\begin{aligned} f_1(x) &= e^{\lambda x}\sin(\omega x) \\ f_2(x) &= e^{\lambda x}\cos(\omega x) \\ f_3(x) &= xe^{\lambda x}\sin(\omega x) \\ f_4(x) &= xe^{\lambda x}\cos(\omega x). \end{aligned}$$

You may assume that these are linearly independent, so they form a basis for the space $V = \operatorname{span}(f_1, f_2, f_3, f_4)$. Show that $Df_i \in V$ for $i = 1, \dots, 4$, and write down the matrix for $D\colon V \to V$ with respect to our basis. Hence or otherwise, show that $((D-\lambda)^2 + \omega^2)^2$ acts as zero on V.

Solution: Using the product rule, we have

$$\begin{aligned}
f_1'(x) &= \lambda e^{\lambda x}\sin(\omega x) + \omega e^{\lambda x}\cos(\omega x) = \lambda f_1(x) + \omega f_2(x) \\
f_2'(x) &= \lambda e^{\lambda x}\cos(\omega x) - \omega e^{\lambda x}\sin(\omega x) = -\omega f_1(x) + \lambda f_2(x) \\
f_3'(x) &= e^{\lambda x}\sin(\omega x) + x\lambda e^{\lambda x}\sin(\omega x) + x\omega e^{\lambda x}\cos(\omega x) = f_1(x) + \lambda f_3(x) + \omega f_4(x) \\
f_4'(x) &= e^{\lambda x}\cos(\omega x) + x\lambda e^{\lambda x}\cos(\omega x) - x\omega e^{\lambda x}\sin(\omega x) = f_2(x) - \omega f_3(x) + \lambda f_4(x)
\end{aligned}$$

It follows that the matrix of D is

$$A = \begin{bmatrix} \lambda & -\omega & 1 & 0 \\ \omega & \lambda & 0 & 1 \\ 0 & 0 & \lambda & -\omega \\ 0 & 0 & \omega & \lambda \end{bmatrix}.$$

This means that

$$(A-\lambda I)^2 + \omega^2 I = \begin{bmatrix} 0 & -\omega & 1 & 0 \\ \omega & 0 & 0 & 1 \\ 0 & 0 & 0 & -\omega \\ 0 & 0 & \omega & 0 \end{bmatrix}^2 + \begin{bmatrix} \omega^2 & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -2\omega \\ 0 & 0 & 2\omega & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we see that $((A-\lambda I)^2 + \omega^2 I)^2 = 0$. Moreover, this is the matrix of the linear map $((D-\lambda)^2 + \omega)^2 \colon V \to V$, so we see that this map is zero as claimed.

Exercise 8. Define a map $T \colon M_3\mathbb{R} \to M_3\mathbb{R}$ by

$$T \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} b & c & f \\ a & e & i \\ d & g & h \end{bmatrix}$$

so the entries in the matrix get moved around like this:

$$\begin{tikzcd}
a \arrow[d] & b \arrow[l] & c \arrow[l] \\
d \arrow[d] & e & f \arrow[u] \\
g \arrow[r] & h \arrow[r] & i \arrow[u]
\end{tikzcd}$$

Find a basis for the kernel of $T-1$. Write down the matrix of T with respect to a suitable basis of $M_3\mathbb{R}$, and thus calculate the characteristic polynomial of T.

Solution: Consider a matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. We have $A \in \ker(T-1)$ iff $A = T(A)$, iff

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} b & c & f \\ a & e & i \\ d & g & h \end{bmatrix}$$

This means that $a=b$, $b=c$, $c=f$, $d=a$, $f=i$, $g=d$, $h=g$ and $i=h$, which just means that $a=b=c=d=f=g=h=i$ (but e can be different). In other words, we have

$$A = \begin{bmatrix} a & a & a \\ a & e & a \\ a & a & a \end{bmatrix} = a\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad E_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad E_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad E_7 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_8 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_9 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

These matrices form a basis for $M_3\mathbb{R}$, with the property that $T(E_1) = E_2$, $T(E_2) = E_3$, $T(E_3) = E_4$, $T(E_4) = E_5$, $T(E_5) = E_6$, $T(E_6) = E_7$, $T(E_7) = E_8$, $T(E_8) = E_1$ and $T(E_9) = E_9$. The matrix of T with respect to this basis is

$$U = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The characteristic polynomial of T is the determinant of $tI - U$, which is $(t^8 - 1)(t - 1)$.

Exercise 9. Let A be a 2×2 matrix over the reals. Define a map $\mu\colon M_2\mathbb{R} \to M_2\mathbb{R}$ by $\mu(X) = AX$. Find the matrix of M with respect to a suitable basis of $M_2\mathbb{R}$, and thus show that $\det(\mu) = \det(A)^2$.

Solution: Let A be $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The most convenient basis to use is as follows:

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

(It would be more usual to have E_2 and E_3 the other way around, but in this exercise that makes the picture a little less clear.) We then have

$$\begin{aligned} \mu(E_1) = AE_1 &= \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = aE_1 + cE_2 \\ \mu(E_2) = AE_2 &= \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix} = bE_1 + dE_2 \\ \mu(E_3) = AE_3 &= \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = aE_3 + cE_4 \\ \mu(E_4) = AE_4 &= \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} = bE_3 + cE_4. \end{aligned}$$

This means that the matrix of μ is

$$B = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}.$$

This gives

$$\det(B) = a \det \begin{bmatrix} d & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix} - b \det \begin{bmatrix} c & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix} = ad \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} - bc \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (ad - bc)^2 = \det(A)^2.$$

Exercise 10. Suppose we have vectors $a = (u, v, w)$ and $b = (x, y, z)$ in $\mathbb{R}^3$, with $a \neq 0 \neq b$ and $\langle a, b \rangle = 0$. Define matrices A, B and C by

$$A = \begin{bmatrix} u^2 & uv & uw \\ uv & v^2 & vw \\ uw & vw & w^2 \end{bmatrix} \qquad B = \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} \qquad C = A + B.$$

Show that $\operatorname{image}(C) = \operatorname{span}\{a, b\}$, and thus that $\operatorname{rank}(C) = 2$.

Solution: For any vector $p = (r, s, t)$ we have

$$Ar = \begin{bmatrix} u^2 & uv & uw \\ uv & v^2 & vw \\ uw & vw & w^2 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} u^2r + uvs + uwt \\ uvr + v^2s + vwt \\ uwr + vws + w^2t \end{bmatrix} = (ur + vs + wt) \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \langle a, p \rangle a.$$

Similarly, we have $Bp = \langle b, p \rangle b$, so

$$Cp = Ap + Bp = \langle a, p \rangle a + \langle b, p \rangle b \in \operatorname{span}\{a, b\}.$$

It follows that $\text{image}(C) \leq \text{span}\{a, b\}$. Now note that $a \neq 0$ so $\langle a, a \rangle > 0$ so we can take $p = a/\langle a, a \rangle$. As $\langle a, b \rangle = 0$, the above gives

$$Cp = \langle a, a/\langle a, a \rangle \rangle a = a,$$

which shows that $a \in \text{image}(C)$. Similarly we see that $b \in \text{image}(C)$, so any linear combination of a and b must also lie in $\text{image}(C)$, or in other words, $\text{image}(C) \leq \text{span}\{a, b\}$. We have already proved the reverse inclusion, so $\text{image}(C) = \text{span}\{a, b\}$. To complete the exercise we need to show that a and b are linearly independent (so they give a basis for $\text{image}(C)$, so $\text{rank}(C) = \dim(\text{image}(C)) = 2$). Consider a linear relation $\alpha a + \beta b = 0$. Taking the inner product with a gives

$$0 = \langle 0, a \rangle = \langle \alpha a + \beta b, a \rangle = \alpha \langle a, a \rangle + \beta \langle a, b \rangle = \alpha \|a\|^2.$$

As $a \neq 0$ we have $\|a\|^2 > 0$ and so we must have $\alpha = 0$. Similarly we have $\beta = 0$, so a and b are linearly independent, as required.

Exercise 11. For any real number a, we consider the matrix $A = \begin{bmatrix} a & a^2 & a^4 \\ a^2 & a^2 & a^4 \\ a^4 & a^4 & a^4 \end{bmatrix}$. Find the determinant of A, and factorise it. Using this as the first step, determine the rank of A for all a.

Solution: First, we have

$$\det(A) = a \det \begin{bmatrix} a^2 & a^4 \\ a^4 & a^4 \end{bmatrix} - a^2 \det \begin{bmatrix} a^2 & a^4 \\ a^4 & a^4 \end{bmatrix} + a^4 \det \begin{bmatrix} a^2 & a^2 \\ a^4 & a^4 \end{bmatrix} = (a - a^2)(a^6 - a^8) = a^7(1-a)^2(1+a).$$

If $a \notin \{0, 1, -1\}$ we see that $\det(A) \neq 0$ and so $\text{rank}(A) = 3$. If $a = 0$ then A is the zero matrix and $\text{rank}(A) = 0$. If $a = 1$ then every entry in A is equal to one, so the image of A is spanned by the vector $(1, 1, 1)$, so $\text{rank}(A) = 1$. If $a = -1$ then $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, so the first two columns are linearly independent but the third is equal to the second, which shows that $\text{rank}(A) = 2$. In conclusion, we have

$$\text{rank}(A) = \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{if } a = 1 \\ 2 & \text{if } a = -1 \\ 3 & \text{otherwise} \end{cases}$$

Exercise 12. Suppose that $u \in \mathbb{R}^3$ and $\|u\| = 1$. Define $\phi \colon \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$ by $\phi(x) = (\langle u, x \rangle, u \times x)$, and define $\psi \colon \mathbb{R} \times \mathbb{R}^3$ by $\psi(t, y) = tu - y$. Simplify $\psi(\phi(x))$, and deduce that $\ker(\phi) = 0$. Find $\ker(\psi)$.

Solution:

Exercise 13. Let A be a matrix of the form $\begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}$, with $0 < p, q < 1$. Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solution: The diagonal entries in D will be the eigenvalues of A, and the columns of P will be the corresponding eigenvectors. To find these, we note that the characteristic polynomial is

$$\begin{aligned} \det \begin{bmatrix} t-p & p-1 \\ q-1 & t-q \end{bmatrix} &= (t-p)(t-q) - (p-1)(q-1) = t^2 - pt - qt + pq - pq + p + q - 1 \\ &= t^2 - (p+q)t + (p+q-1) = (t-1)(t-p-q+1). \end{aligned}$$

The eigenvalues are thus 1 and $p+q-1$, so $D = \begin{bmatrix} p+q-1 & 0 \\ 0 & 1 \end{bmatrix}$. It is easy to see that $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ is an eigenvector of eigenvalue 1. Next, put

$$B = (p+q-1)I - A = \begin{bmatrix} q-1 & p-1 \\ q-1 & p-1 \end{bmatrix}.$$

We see that $B\left[\begin{smallmatrix}1-p\\ q-1\end{smallmatrix}\right]=\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]$, so $\left[\begin{smallmatrix}1-p\\ q-1\end{smallmatrix}\right]$ is an eigenvector of A with eigenvalue $p+q-1$. Our matrix P is thus $\begin{bmatrix}1-p & 1\\ q-1 & 1\end{bmatrix}$. As a check, we note that $\det(P)=2-p-q$ (which is nonzero, as $0<p,q<1$) and so

$$P^{-1}=\frac{1}{2-p-q}\begin{bmatrix}1 & -1\\ 1-q & 1-p\end{bmatrix}.$$

Exercise 14. Define

$$V=\{f\in\mathbb{R}[x]_{\leq 2}\mid\int_{-1}^{0}f(x)\,dx=0\}$$
$$W=\{f\in\mathbb{R}[x]_{\leq 2}\mid\int_{0}^{1}f(x)\,dx=0\}.$$

Find bases for V, W and $V\cap W$. Show that $V+W=\mathbb{R}[x]_{\leq 2}$.

Solution: We find that

$$V=\{ax^2+bx+c\mid a/3-b/2+c=0\}$$
$$W=\{ax^2+bx+c\mid a/3+b/2+c=0\}.$$

It follows that $\{1-3x^2,1+2x\}$ is a basis for V, and $\{1-3x^2,1-2x\}$ is a basis for W, and $\{1-3x^2\}$ is a basis for $V\cap W$.

Exercise 15. Put

$$V=\{A\in M_2\mathbb{R}\mid A\begin{bmatrix}1\\ 1\end{bmatrix}=0\}$$
$$W=\{A\in M_2\mathbb{R}\mid\begin{bmatrix}1 & 1\end{bmatrix}A=0\}.$$

Find bases for V, W and $V\cap W$.

Solution: Consider a matrix $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$. We have $A\in V$ iff $\begin{bmatrix}a+b\\ c+d\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$, or equivalently $b=-a$ and $d=-c$. In that case we have

$$A=\begin{bmatrix}a & -a\\ c & -c\end{bmatrix}=a\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}+c\begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix}.$$

Using this, we see that the matrices $P_1=\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}$ and $P_2=\begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix}$ give a basis for V. Similarly, we have $\begin{bmatrix}1 & 1\end{bmatrix}A=\begin{bmatrix}a+c & b+d\end{bmatrix}$, which vanishes iff A has the form

$$A=\begin{bmatrix}a & b\\ -a & -b\end{bmatrix}=a\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}+b\begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}.$$

Using this, we see that the matrices $Q_1=\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}$ and $Q_2=\begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}$ give a basis for W. Finally, we see that $A\in V\cap W$ iff $b=-a$ and $d=-c$ and $c=-a$ and $d=-b$, or equivalently $a=d=-b=-c$, which means that $A=a\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$. It follows that the single matrix $\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}=P_1-P_2=Q_1-Q_2$ gives a basis for $V\cap W$.

Exercise 16. For each $k>0$ we define $r_k\in C[0,1]$ by $r_k(x)=1/(x+k)$. Show that these are linearly independent in $C[0,1]$.

Solution: Find a good proof not using analytic continuation, and include some hints in the question.

Exercise 17. Put

$$\begin{aligned} V &= \{ \text{ symmetric } 3 \times 3 \text{ real matrices } \} \\ &= \{A \in M_3\mathbb{R} \mid A^T = A\} \\ W &= \{ \text{ homogeneous quadratic polynomials in } x, y, z\} \\ &= \{px^2 + qxy + rxz + sy^2 + tyz + uz^2 \mid p, \dots, u \in \mathbb{R}\}. \end{aligned}$$

Define $\phi \colon V \to W$ by

$$\phi(A) = \begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Show that this is an isomorphism.

Solution: Any element $A \in V$ has the form

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$

For such A we have

$$\phi(A) = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2$$

Now define $\psi \colon W \to V$ by

$$\psi(px^2 + qxy + rxz + sy^2 + tyz + uz^2) = \begin{bmatrix} p & q/2 & r/2 \\ q/2 & s & t/2 \\ r/2 & t/2 & u \end{bmatrix}.$$

We find that $\psi(\phi(A)) = A$ for all $A \in V$, and $\phi(\psi(Q)) = Q$ for all $Q \in W$, so ϕ and ψ are isomorphisms.

Exercise 18. Let V_0 be the set of functions $f(x)$ of the form

$$a_0 + a_1 \ln(x) + a_2 \ln(x)^2 + \cdots + a_n \ln(x)^n$$

for some $n \geq 0$ and some constants $a_0, \dots, a_n \in \mathbb{R}$. In other words, the functions in V_0 have the form $p(\ln(x))$ for some polynomial $p(t) \in \mathbb{R}[t]$. Next, let V_m be the set of functions of the form $x^m p(\ln(x))$ for some polynomial p, so for example, the function $f(x) = x^2(1 + \ln(x) + 9\ln(x)^4)$ is a typical element of V_2. Show that differentiation gives a homomorphism $D \colon V_m \to V_{m-1}$, which is an isomorphism except when $m = 0$.

Solution:

Exercise 19. Define a map

$$\phi \colon \mathbb{R}[x, y]_{\leq 2} \to \mathbb{R}[x]_{\leq 2} \times \mathbb{R}[x]_{\leq 2}$$

by

$$\phi(f)(x) = (f(x, x), f(x, -x)).$$

Find bases for the kernel and image of ϕ.

Solution: The space $\ker(\phi)$ is one-dimensional, with basis $\{x^2 - y^2\}$. The elements

$$(1, 1), (x, 0), (0, x), (x'0), (0, x^2)$$

give a basis for $\mathrm{image}(\phi)$.

Exercise 20. Define a map $H \colon \mathbb{R}[x, y]_{\leq 2} \to M_2\mathbb{R}$ (called the Hessian) by

$$H(u) = \begin{bmatrix} u_{xx}(0, 0) & u_{xy}(0, 0) \\ u_{xy}(0, 0) & u_{yy}(0, 0) \end{bmatrix}$$

Find bases for $\ker(H)$ and $\mathrm{image}(H)$.

Solution: Any element of $u \in \mathbb{R}[x,y]_{\leq 2}$ has the form

$$u(x,y) = a + bx + cy + dx^2 + exy + fy^2$$

for some $a, b, \ldots, f \in \mathbb{R}$. We then have

$$H(u) = \begin{bmatrix} 2d & e \\ e & 2f \end{bmatrix} = 2d \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 2f \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

It follows that the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ give a basis for $\operatorname{image}(H)$. In the above we have $H(u) = 0$ iff $d = e = f = 0$, or equivalently, u actually has degree ≤ 1. This means that $\{1, x, y\}$ is a basis for $\ker(H)$.

Exercise 21. Give a basis for the space

$$V = \{A \in M_3\mathbb{R} \mid A = A^T \text{ and } \operatorname{trace}(A) = 0\}.$$

Solution: Any matrix $A \in M_2\mathbb{R}$ lies in V iff it has the form

$$\begin{aligned} A &= \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & -a-d \end{bmatrix} \\ &= a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

for some $a, b, c, d \in \mathbb{R}$. It follows that the matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

span V, and they are clearly linearly independent, so they form a basis for V.

Exercise 22. Suppose we are given $u, v, q > 0$. Define a bilinear form on $\mathbb{R}[x]_{\leq 2}$ by

$$\langle f, g \rangle = uf(-q)g(-q) + vf(0)g(0) + uf(q)g(q).$$

Show that this is an inner product. Find u, v and q such that $\langle f, g \rangle = 9\int_{-1}^{1} f(t)g(t)\,dt$ for all $f, g \in \mathbb{R}[x]_{\leq 2}$.

Solution: The solution is $u = 5$, $v = 8$ and $q = \sqrt{3/5} = \sqrt{15}/5$. This can be found using Maple with the following steps:

```
b := (f,g) -> u*f(-q)*g(-q) + v * f(0)*g(0) + u*f(q)*g(q);
f := (t) -> a0+a1*t+a2*t^2;
g := (t) -> b0+b1*t+b2*t^2;
d := expand(b(f,g) - 9*int(f(t)*g(t),t=-1..1));
_EnvExplicit := true;
solve({coeffs(d,{a0,a1,a2,b0,b1,b2}),q>0},{u,v,q});
```

Exercise 23. Investigate the Haar basis.

Solution:

Exercise 24. Define a linear map $\phi\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}^3$ by

$$\phi(f) = \left[\, \int_0^1 f(x)\,dx,\ \int_1^2 f(x)\,dx,\ \int_2^3 f(x)\,dx \,\right]$$

Show that this is an isomorphism.

Solution: The effect of ϕ on the basis $1, x, x^2$ is as follows:

$$\begin{aligned} \phi(1) &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \\ \phi(x) &= \begin{bmatrix} 1/2 & 3/2 & 5/2 \end{bmatrix} \\ \phi(x^2) &= \begin{bmatrix} 1/3 & 3 & 19/3 \end{bmatrix} \end{aligned}$$

The matrix of ϕ with respect to the obvious bases is thus

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1 & 3/2 & 3 \\ 1 & 5/2 & 19/3 \end{bmatrix}.$$

By hand or by Maple, we can check that $\det(A) = 2 \neq 0$, so A is an invertible matrix, so ϕ is an isomorphism.

Exercise 25. For each of the following lists of vectors, say (with justification) whether they are linearly independent, whether they span $\mathbb{R}^3$, and whether they form a basis of $\mathbb{R}^3$. (If you understand the concepts involved, you should be able to do this by eye, without any calculation.)

(a) $\mathbf{u}_1 = \left[\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right]$, $\mathbf{u}_2 = \left[\begin{smallmatrix}2\\4\\6\end{smallmatrix}\right]$, $\mathbf{u}_3 = \left[\begin{smallmatrix}3\\6\\9\end{smallmatrix}\right]$.

(b) $\mathbf{v}_1 = \left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]$, $\mathbf{v}_2 = \left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right]$, $\mathbf{v}_3 = \left[\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right]$, $\mathbf{v}_4 = \left[\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right]$.

(c) $\mathbf{w}_1 = \left[\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right]$, $\mathbf{w}_2 = \left[\begin{smallmatrix}1\\0\\-1\end{smallmatrix}\right]$.

(d) $\mathbf{x}_1 = \left[\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right]$, $\mathbf{x}_2 = \left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right]$, $\mathbf{x}_3 = \left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]$.

Solution:

(a) These are linearly dependent, because of the relation $\mathbf{u}_2 - 2\mathbf{u}_1 = 0$. In each of the $\mathbf{u}_i$'s the y coordinate is twice the x coordinate. This will therefore also be true for anything in the span of the $\mathbf{u}_i$'s. In particular, the vector $\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]$ does not lie in the span, so the $\mathbf{u}_i$'s do not span all of $\mathbb{R}^3$. This means that they do not form a basis.

(b) Any list of four vectors in $\mathbb{R}^3$ is automatically linearly dependent (and so cannot form a basis). More specifically, the relation $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 - \mathbf{v}_4 = 0$ shows that the $\mathbf{v}_i$'s are dependent. These vectors span all of $\mathbb{R}^3$, because any vector $\mathbf{a} = \left[\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right] \in \mathbb{R}^3$ can be expressed as $\mathbf{a} = x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3 + 0\mathbf{v}_4$.

(c) A list of two vectors can only be linearly dependent if one is a multiple of the other, which is clearly not the case here, so $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly independent. Moreover, a list of two vectors can never span all of $\mathbb{R}^3$. More explicitly, $\mathbf{w}_1$ and $\mathbf{w}_2$ both have y-coordinate equal to zero, so the same is true of anything in the span of $\mathbf{w}_1$ and $\mathbf{w}_2$. In particular, $\left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right]$ is not in the span. This shows that $\mathbf{w}_1$ and $\mathbf{w}_2$ do not form a basis of $\mathbb{R}^3$.

(d) The vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly independent and span $\mathbb{R}^3$, so they form a basis. One way to see this is to write down the matrix $A = \left[\begin{smallmatrix}1&1&1\\1&1&0\\1&0&0\end{smallmatrix}\right]$ whose columns are $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$, and observe that it row-reduces almost instantly to the identity. Alternatively, we must show that for any vector $\mathbf{a} = \left[\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right] \in \mathbb{R}^3$, there are unique real numbers λ, μ, ν such that

$$\left[\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right] = \lambda \left[\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right] + \mu \left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right] + \nu \left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right].$$

This equation is equivalent to $\lambda + \mu + \nu = x$ and $\lambda + \mu = y$ and $\lambda = z$. It is easy to see that there is indeed a unique solution, namely $\lambda = z$ and $\mu = y - z$ and $\nu = x - y$.

Exercise 26. Define a map $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ by

$$\pi \left[\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right] = \left[\begin{smallmatrix}2x-y-z\\ \sqrt{3}(y-z)\end{smallmatrix}\right]$$

Investigate $\ker(\pi)$, $\operatorname{image}(\pi)$, π^*, $\pi\pi^*$, $\pi^*\pi$.

Solution: π is surjective, with kernel spanned by $[1, 1, 1]^T$. We have $\pi\pi^* = 6\iota_2$, and $\pi^*\pi = 6\iota_3 - 2\phi_J$, where $J_{ij} = 1$ for all i and j.